ORBIT EQUIVALENCE AND KAKUTANI EQUIVALENCE WITH STURMIAN SUBSHIFTS

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ABSTRACT. Using dimension group tools and Bratteli–Vershik representations of minimal Cantor systems we prove that a minimal Cantor system and a Sturmian subshift are topologically conjugate if and only if they are orbit equivalent and Kakutani equivalent.

1. PRELIMINARIES.

In the last decade concepts and techniques coming from the theory of C^*–algebras have been exhaustively used in topological dynamics in order to explain different phenomena appearing mainly in Cantor dynamical systems. In particular, those concepts together with the description of minimal Cantor systems by means of Bratteli–Vershik transformations [HPS][V1][V2], gave rise to a complete invariant of orbit and strong orbit equivalence for this class of maps [GPS][HPS]. In the same vein the authors of [BH] obtained new results about flow equivalence and orbit equivalence for non-minimal Cantor systems. In particular they obtained new conjugacy invariants for subshifts of finite type. The study of substitution systems and Toeplitz systems in this scope was undertaken in [F][DHS] and [GJ] respectively.

If we consider two (strong) orbit equivalent Cantor systems, their Bratteli-Vershik representation without considering the order is in some sense the same [GPS][HPS]. Therefore, we can ask which additional property could imply topological conjugacy, in other words how we recover the order. In this direction it is proved in [BT] that with a continuity condition over the cocycles involved in the orbit equivalence we get flip conjugacy. In general, (strong) orbit equivalence is not enough. It is known [O][Su] that in the same class of orbit equivalence we can have all possible entropies. In the case of odometers it is easy to show that orbit equivalence implies topological conjugacy.

Among the different conditions we can consider, Kakutani equivalence appears as a natural one which is intimately related to the order. In that case the systems can be represented by diagrams that are the same up to a finite number of edges and vertices.

In this paper we solve this question when one of the systems is a Sturmian subshift.

Theorem 1. *Let $0 < \overline{\alpha} < 1$ be an irrational number and (X, T) a minimal Cantor system. The Sturmian subshift $(\Omega_{\overline{\alpha}}, \sigma)$ and the system (X, T) are Kakutani and orbit equivalent if and only if they are topologically conjugate.*

The proof is based upon a detailed study of a Bratteli-Vershik representation of Sturmian systems. In that case we only need orbit equivalence and Kakutani equivalence because there are no infinitesimals in their dimension group. We remark that for two Sturmian systems only orbit equivalence is needed.

The paper is organized in three sections and one appendix. In the present section we give the background for what we will need later. The construction of a particular Bratteli–Vershik representation for Sturmian subshifts is done in Section 2. In Section 3 we prove Theorem 1. In the appendix we

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prove a matrix proposition needed for the proof of the main theorem and we compute the dimension group of a Sturmian subshift.

In what follows, we give some definitions and notation that will be used in the paper.

1.0.1. *Topological dynamical systems and subshifts.* A *topological dynamical system*, or just dynamical system, is a compact Hausdorff space X together with a homeomorphism $T : X \to X$. We use the notation (X,T). If X is a Cantor set we say that the system is Cantor. That is, X has a countable basis of closed and open sets and it has not isolated points. A dynamical system is *minimal* if all orbits are dense in X, or equivalently the only non trivial closed invariant set is X.

A particular class of Cantor systems is the class of *subshifts*. These systems are defined as follows. Take a finite set or alphabet A. The set $A^{\mathbb{Z}}$ consists of infinite sequences $(x_i)_{i\in\mathbb{Z}}$ with coordinates $x_i \in A$. With the product topology $A^{\mathbb{Z}}$ is a compact Hausdorff Cantor space. We define the *shift transformation* $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by $(\sigma(x))_i = x_{i+1}$ for any $x \in A^{\mathbb{Z}}$, $i \in \mathbb{Z}$. The pair $\left(A^{\mathbb{Z}}, \sigma\right)$ is called *fullshift*. A subshift is a pair (X, σ) where X is any σ–invariant closed subset of $A^{\mathbb{Z}}$. A classical procedure to construct subshifts is by considering the closure of the orbit under the shift of a single sequence $x \in A^{\mathbb{Z}}$, $\Omega(x) = \overline{\{\sigma^i(x) \mid i \in \mathbb{Z}\}}$.

Let $(x_i)_{i\in\mathbb{N}}$ be an element of $A^{\mathbb{N}}$. Another classical procedure is to consider the set $\Omega(x)$ of infinite sequences $(y_i)_{i\in\mathbb{Z}}$ such that for all $i \leq j$ there exists $k \geq 0$ such that $y_i y_{i+1} \cdots y_j = x_k x_{k+1} \cdots x_{k+j-i}$. In both cases we say that $(\Omega(x), \sigma)$ is the subshift generated by x.

In a minimal subshift any finite sequence of symbols appears with bounded gaps in any sequence of the system.

In this paper we consider two kinds of minimal subshifts: substitution subshifts and Sturmian subshifts. Let us first describe Sturmian subshifts.

Let $0 < \alpha < 1$ be an irrational number. We define the map $R_\alpha : [0,1[\to [0,1[$ by $R_\alpha(t) = t + \alpha$ (mod 1) and the map $I_\alpha : [0,1[\to \{0,1\}$ by $I_\alpha(t) = 0$ if $t \in [0, 1-\alpha[$ and $I_\alpha(t) = 1$ otherwise. Let $\Omega_\alpha = \overline{\left\{ (I_\alpha(R_\alpha^n(t)))_{n\in\mathbb{Z}} \;\middle|\; t \in [0,1[\right\}} \subset \{0,1\}^{\mathbb{Z}}$. The subshift (Ω_α, σ) is called *Sturmian subshift* (generated by α) and its elements are called *Sturmian sequences*. There exists a factor map (see [HM]) $\gamma : (\Omega_\alpha, \sigma) \to ([0,1[, R_\alpha)$ such that

(1) $\left|\gamma^{-1}(\{\beta\})\right| = 2$ if $\beta \in \{n\alpha \mid n \in \mathbb{Z}\}$ and
(2) $\left|\gamma^{-1}(\{\beta\})\right| = 1$ otherwise.

Let $\beta \in [0,1[$. It is well-known that $\Omega_\alpha = \Omega_\beta$ if and only if $\alpha = \beta$ and also that (Ω_α, σ) is a non-periodic uniquely ergodic minimal subshift. Sometimes we will write $(\Omega_\alpha, \sigma, \mu)$ instead of (Ω_α, σ) where μ is the unique ergodic measure of (Ω_α, σ). We give later a useful characterization of Sturmian subshifts to obtain Bratteli-Vershik representations of these systems. For more details and properties of Sturmian sequences and subshifts the reader can refer to [BS] and [HM].

A *substitution* is a map $\tau : A \to A^+$, where A^+ is the set of finite sequences with values in A. We associate to τ a $A \times A$ square matrix $M_\tau = (m_{a,b})_{a,b\in A}$ such that $m_{a,b}$ is the number of times that letter b appears in $\tau(a)$. We say that τ is *primitive* if M_τ is primitive, i.e. if some power of M_τ has strictly positive entries only. A substitution τ can be naturally extended by concatenation to A^+, $A^{\mathbb{N}}$ and $A^{\mathbb{Z}}$. We say that a subshift of $A^{\mathbb{Z}}$ is generated by the substitution τ if it is generated by a fixed point for τ in $A^{\mathbb{N}}$ (see [Q] for more details).

In this paper we are concerned with three notions of equivalence between dynamical systems. Let (X,T) and (Y,S) be dynamical systems. We say that they are *topologically conjugate* if there is a homeomorphism $\phi : X \to Y$ such that $\phi \circ T = S \circ \phi$. We say that they are *orbit equivalent (OE)* if there is a homeomorphism $\phi : X \to Y$ and integer functions $n : X \to \mathbb{Z}$ and $m : X \to \mathbb{Z}$ such that for any $x \in X$, $\phi \circ T^{n(x)}(x) = S \circ \phi(x)$ and $\phi \circ T(x) = S^{m(x)} \circ \phi(x)$. Now assume the systems are minimal, then the maps n, m are uniquely determined. Under this hypothesis we say that the systems are *strong orbit equivalent (SOE)* if the maps n, m have at most one point of discontinuity.

Finally, for Cantor systems, we say they are *Kakutani equivalent (KE)* if both have subsets that are closed and open *(clopen)* such that the corresponding induced systems are topologically conjugate.

1.0.2. *Bratteli–Vershik representations.* A *Bratteli Diagram* is an infinite graph (V, E) which consists of a vertex set V and an edge set E, both of which are divided into levels $V = V_0 \cup V_1 \cup \cdots$, $E = E_1 \cup E_2 \cup \cdots$ and all levels are pairwise disjoint. The set V_0 is a singleton $\{v_0\}$, and for $k \geq 1$, E_k is the set of edges joining vertices in V_{k-1} to vertices in V_k. It is also required that every vertex in V_k is the "end-point" of some edge in E_{k-1} for $k \geq 1$, and an "initial-point" of some edge in E_k for $k \geq 0$. We will say that the *level* k is the subgraph consisting of the vertices in $V_k \cup V_{k+1}$ and the edges E_{k+1} between these vertices. Level 0 will be called *hat* of the Bratteli diagram and it is uniquely determined by an integer vector $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_{|V_1|} \end{pmatrix} \in \mathbb{N}^{|V_1|}$, where each component represents the number of edges joining v_0 and a vertex of V_1.

We describe the edge set E_k using a $V_k \times V_{k-1}$ incidence matrix for which its (i, j)–entry is the number of edges in E_k joining vertex $j \in V_{k-1}$ with vertex $i \in V_k$.

An *ordered* Bratteli diagram $B = (V, E, \preceq)$ is a Bratteli diagram (V, E) together with a partial ordering $\preceq$ on E. Edges e and e' are comparable if and only if they have the same end-point.

Let $k < l$ in $\mathbb{N} \setminus \{0\}$ and let $E_{k,l}$ be the set of all paths in the graph joining vertices of V_{k-1} with vertices of V_l. The partial ordering of E induces another in $E_{k,l}$ given by $(e_k, \ldots, e_l) \prec (f_k, \ldots, f_l)$ if and only if there is $k \leq i \leq l$ such that $e_j = f_j$ for $i < j \leq l$ and $e_i \prec f_i$.

Given a strictly increasing sequence of integers $(m_n)_{n \geq 0}$ with $m_0 = 0$ we define the *contraction* of $B = (V, E, \preceq)$ (with respect to $(m_n)_{n \geq 0}$) as $\left((V_{m_n})_{n \geq 0}, (E_{m_n+1, m_{n+1}})_{n \geq 0}, \preceq\right)$, where $\preceq$ is the order induced in each set of edges $E_{m_n+1, m_{n+1}}$.

We say that an ordered Bratteli diagram is stationary if for any $k \geq 1$ the incidence matrix and order are the same (after labeling the vertices appropriately).

Given an ordered Bratteli diagram $(B = (V, E), \preceq)$ we define X_B as the set of infinite paths $(e_1, e_2, \cdots)$ starting in v_0 such that for all $i \geq 1$ the end-point of $e_i \in E_i$ is the initial-point of $e_{i+1} \in E_{i+1}$. We topologize X_B by postulating a basis of open sets, namely the family of *cylinder sets*

$$U(e_1, e_2, \ldots, e_k) = \{(f_1, f_2, \ldots) \in X_B \mid f_i = e_i, \text{ for } 1 \leq i \leq k\}.$$

Each $U(e_1, e_2, \ldots, e_k)$ is also closed, as is easily seen, and so we observe that X_B is a Cantor set.

When there is a unique $x = (x_1, x_2, \ldots) \in X_B$ such that x_i is maximal for any $i \geq 1$ and a unique $y = (y_1, y_2, \ldots) \in X_B$ such that y_i is minimal for any $i \geq 1$, we say that $(B = (V, E), \preceq)$ is a *properly ordered* Bratteli diagram. Call these particular points $x_{\max}$ and $x_{\min}$ respectively. In this case we can define a dynamic V_B over X_B called *Vershik map.* The map V_B is defined as follows: let $(e_1, e_2, \ldots) \in X_B \setminus \{x_{\max}\}$ and let $k \geq 1$ be the smallest integer so that e_k is not a maximal edge. Let f_k be the successor of e_k and $(f_1, \ldots, f_{k-1})$ be the unique minimal path in $E_{1,k-1}$ connecting v_0 with the initial point of f_k. We set $V_B(x) = (f_1, \ldots, f_{k-1}, f_k, e_{k+1}, \ldots)$ and $V_B(x_{\max}) = x_{\min}$. The dynamical system (X_B, V_B) is called *Bratteli-Vershik system* generated by $B = (V, E, \preceq)$. The dynamical system induced by any contraction of B is topologically conjugate to (X_B, V_B). In [HPS] it is proved that any minimal Cantor system (X, T) is topologically conjugate to a Bratteli-Vershik system (X_B, V_B). We say that (X_B, V_B) is a *Bratteli-Vershik representation* of (X, T).

1.0.3. *The notion of a Dimension Group.* Let (X, T) be a minimal Cantor system. Its dimension group is defined as $K^0(X, T) = C(X, \mathbb{Z}) / \partial_T C(X, \mathbb{Z})$, where $C(X, \mathbb{Z})$ is the countable additive Abelian group of continuous functions on X with values in $\mathbb{Z}$ and $\partial_T : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ is the coboundary operator $\partial_T(f) = f \circ T - f$. The *positive cone* of $K^0(X, T)$ is the set of equivalence classes of positive functions. We also distinguish an *order unit* [1] which is the equivalence class of the constant function equal to 1.

Let (V,E) be a Bratteli diagram and $(M_i)_{i\geq 0}$ be the corresponding incidence matrix of levels. We define $K_0(V,E)$ as the inductive limit of the system of ordered groups

$$\mathbb{Z} \xrightarrow{\mathbf{u}} \mathbb{Z}^{|V_1|} \xrightarrow{M_1} \mathbb{Z}^{|V_2|} \xrightarrow{M_2} \dots,$$

that is, $K_0(V,E) = \varinjlim \left(M_i, \mathbb{Z}^{|V_i|}\right)$. This group carries a natural order given by a cone $K_0(V,E)^+$. We also distinguish an order unit $\mathbf{1}$ which is the element of $K_0(V,E)^+$ corresponding to $1 \in \mathbb{Z} = \mathbb{Z}^{|V_0|}$. For more details we refer the reader to [GPS].

In [HPS, Th. 5.4, Cor. 6.3] it is proved that if (X,T) is a Cantor minimal system and (X_B, V_B) a Bratteli–Vershik representation of it, then the ordered groups with distinguished order units $K^0 = \left(K^0(X,T), K^0(X,T)^+, [1]\right)$ and $K_0 = \left(K_0(V,E), K_0(V,E)^+, \mathbf{1}\right)$ are isomorphic. In [GPS, Th. 2.1] it is proved that K^0 is a complete SOE invariant. They proved also that the quotient group $K^0/Inf(K^0)$ is a complete invariant of OE, where $Inf(K^0)$ is the subgroup of $K^0(X,T)$ consisting of elements $a \in K^0(X,T)$ such that $-\epsilon u \leq a \leq \epsilon u$ for all $0 < \epsilon \in \mathbb{Q}$.

In this paper we are particularly concerned with computations of dimension groups that are direct limits of a sequence of integer matrices in $GL(2,\mathbb{Z})$. That is

$$\mathbb{Z} \xrightarrow{\mathbf{u}} \mathbb{Z}^2 \xrightarrow{M_1} \mathbb{Z}^2 \xrightarrow{M_2} \dots,$$

where $M_i \in GL(2,\mathbb{Z})$ for $i \geq 1$. In this case and under some other conditions (see [ES]), the ordered group $\varinjlim \left(M_i, \mathbb{Z}^2, \mathbb{Z}^2_+\right)$ is isomorphic to $(\mathbb{Z}^2, P_\alpha)$ where

$$P_\alpha = \left\{ \left(\begin{array}{c} x \\ y \end{array} \right) \in \mathbb{Z}^2 \,\middle|\, x \cdot \alpha + y \geq 0 \right\}$$

for some $\alpha \in \mathbb{R}^+$.

We will say that a matrix $M \in GL(2,\mathbb{Z})$ is an *automorphism* of $\left(\mathbb{Z}^2, P_\alpha\right)$ if $M \cdot P_\alpha = P_\alpha$.

When an automorphism of $\left(\mathbb{Z}^2, P_\alpha\right)$ is induced by an automorphism of the dimension group of a Bratteli–Vershik system that possibly modifies its order unit from a positive vector $\left(\begin{array}{c} u_1 \\ u_2 \end{array} \right) > 0$ (see the notation below) to some other such vector, we say that such a matrix is a *unit keeping* automorphism of $\left(\mathbb{Z}^2, P_\alpha\right)$.

Finally, let us agree on some notation. The 2×2 identity matrix will be denoted by $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If M is a matrix with real entries, the notation $M \geq 0$ (respectively $M \leq 0$, $M > 0$, $M < 0$) will mean that all entries of M are ≥ 0 (respectively ≤ 0, > 0, < 0).

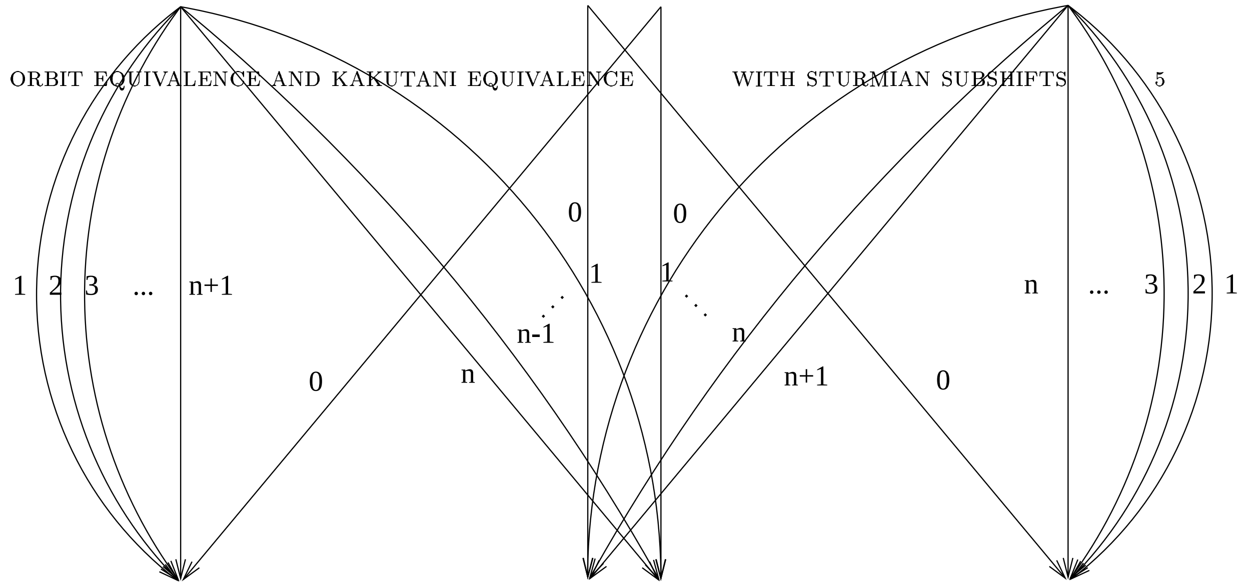
2. Bratteli-Vershik representations of Sturmian subshifts

A morphism $f : \{0,1\} \to \{0,1\}^* = \{0,1\}^+ \cup \{\epsilon\}$, where ϵ is the empty word, is called *Sturmian* if the image by f of each Sturmian sequence is a Sturmian sequence. In [MS] it is proved that a morphism is Sturmian if and only if it is an element of the free monoid $\mathcal{S}t$ generated by the morphisms E, ϕ and $\tilde{\phi}$ from $\{0,1\}$ to $\{0,1\}^*$, where

$$\begin{array}{l} E(0) = 1 \\ E(1) = 0 \end{array}, \quad \begin{array}{l} \phi(0) = 01 \\ \phi(1) = 0 \end{array}, \text{ and } \begin{array}{l} \tilde{\phi}(0) = 10 \\ \tilde{\phi}(1) = 0 \end{array}.$$

In the sequel the morphisms ρ_n and γ_n, $n \in \mathbb{N} \setminus \{0\}$, from $\{0,1\}$ to $\{0,1\}^*$ defined by

$$\begin{array}{l} \rho_n(0) = 01^{n+1} \\ \rho_n(1) = 01^n \end{array} \text{ and } \begin{array}{l} \gamma_n(0) = 10^{n+1} \\ \gamma_n(1) = 10^n \end{array}$$



(a) Ordering for $M^{(\gamma_n)}$ (b) Ordering for $M^{(\rho_n)}$

FIGURE 1.

will play a very important role. For $n \geq 1$, $\gamma_n = \left(\tilde{\phi}E\right)^{n-1}\tilde{\phi}\phi$ and $\rho_n = E\gamma_n$, therefore both belong to $\mathcal{S}t$. The following theorem is due to Hedlund and Morse [HM].

Theorem 2. *Let x be a Sturmian sequence. Then*

(1) There is $n \geq 1$ such that $x = \cdots v_{-1}v_0v_1\cdots$ where $(v_i)_{i\in\mathbb{Z}}$ is a sequence taking values in $\{01^{n+1}, 01^n\}$ or in $\{10^{n+1}, 10^n\}$.
(2) If $x = \rho_n(z)$ or $x = \gamma_n(z)$, for some $n \geq 1$ and $z \in \{0,1\}^{\mathbb{Z}}$, then z is Sturmian.

Proof. Assertion 1 follows from Theorem 7.1 in [HM] and Point 2 is Theorem 8.1 in [HM]. □

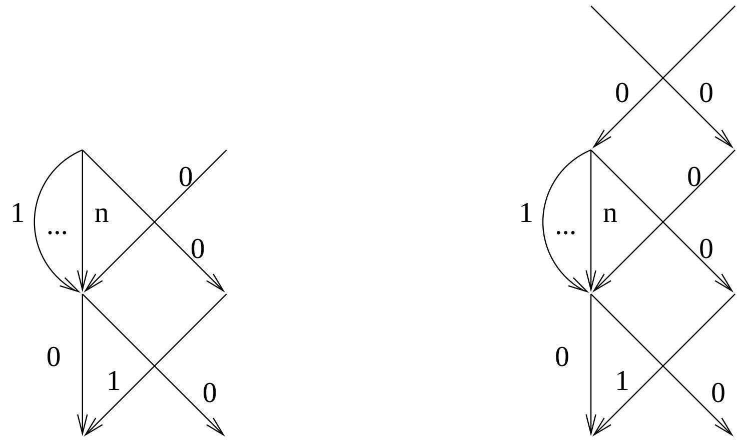
Let (X,σ) be a Sturmian subshift and $a \in \{0,1\}$, we set $[a] = \left\{ (x_i)_{i\in\mathbb{Z}} \in X \;\middle|\; x_0 = a \right\}$. This defines clopen sets of X.

Proposition 1. *Let (X,σ) be a Sturmian subshift. There exists a sequence $(\zeta_n)_{n\in\mathbb{N}}$ taking values in $\{\rho_1,\gamma_1,\rho_2,\gamma_2,\ldots\}$ such that*

(1) $y = \lim_{n\to+\infty} \zeta_1\cdots\zeta_n(00\cdots)$ *generates* (X,σ).
(2) $P_0 = \{[0],[1]\}$, *and for* $n \geq 1$, $P_n = \left\{\sigma^k\zeta_1\cdots\zeta_n([a]) \;\middle|\; 0 \leq k < |\zeta_1\cdots\zeta_n(a)|,\ a\in\{0,1\}\right\}$ *is a partition of X with the following properties:*
   (a) $\zeta_1\cdots\zeta_{n+1}([0]) \cup \zeta_1\cdots\zeta_{n+1}([1]) \subseteq \zeta_1\cdots\zeta_n([0]) \cup \zeta_1\cdots\zeta_n([1])$,
   (b) $P_n \prec P_{n+1}$ *as partitions,*
   (c) *the set* $\bigcap_{n\in\mathbb{N}} (\zeta_1\cdots\zeta_n([0]) \cup \zeta_1\cdots\zeta_n([1]))$ *consists of only one point,*
   (d) *the sequence of partitions* $(P_n)_{n\in\mathbb{N}}$ *spans the topology of* X.

Proof. Assertion 1 follows from Theorem 2 and assertion 2 comes from the fact (which can be proved by induction) that for all $n \in \mathbb{N}$ and all $x \in X$, x has a unique decomposition into a concatenation of elements of $\{\zeta_1\cdots\zeta_n(a) \mid a \in \{0,1\}\}$. □

Let (X,σ) be a Sturmian subshift and $(P_n)_{n\in\mathbb{N}}$ be the sequence of partitions given by Proposition 1. To such a sequence is associated an ordered Bratteli-Vershik diagram $B = (V,E,\preceq)$ which can be described as follows: For all $n \in \mathbb{N}\setminus 0$, V_n consists in two vertices and E_{n+1} is given by ζ_n and described in Figure 1, the hat is determined by P_0 and it is $\begin{pmatrix}1\\1\end{pmatrix}$, and this ordered Bratteli-Vershik diagram is isomorphic to (X,σ) (for more details see [HPS] or [DHS]).



(a) Ordering inducing $M^{(\gamma_n)}$ (b) Ordering inducing $M^{(\rho_n)}$

FIGURE 2.

3. PROOF OF THEOREM 1.

The proof of Theorem 1 will be the consequence of Proposition 2 stated below and the construction of the Bratteli–Vershik representation for Sturmian subshifts given in Section 2.

Proposition 2. *Let α be a positive quadratic irrational with periodic simple continued fraction expansion. There exists $M_\alpha \in GL(2,\mathbb{Z})$ such that if M is a unit keeping automorphism of $(\mathbb{Z}^2, P_\alpha)$, then there exists $k \in \mathbb{Z}$ such that $M = M_\alpha^k$.*

We will devote appendix A) to the proof of Proposition 2.

For the remaining of the section, we will consider a fixed Sturmian subshift $(\Omega_{\overline{\alpha}}, \sigma)$. From Proposition 1 there exists a sequence $(\zeta_n)_{n\in\mathbb{N}}$ in $\{\rho_1, \gamma_1, \rho_2, \gamma_2, \ldots\}$ such that $y = \lim_{n\to+\infty} \zeta_0\zeta_1\cdots\zeta_n(00\cdots)$ generates $(\Omega_{\overline{\alpha}}, \sigma)$.

For all $n \geq 1$ the matrices associated to γ_n and ρ_n are respectively

$$M^{(\gamma_n)} = \begin{bmatrix} n+1 & 1 \\ n & 1 \end{bmatrix} \quad \text{and} \quad M^{(\rho_n)} = \begin{bmatrix} 1 & n+1 \\ 1 & n \end{bmatrix}. \tag{1}$$

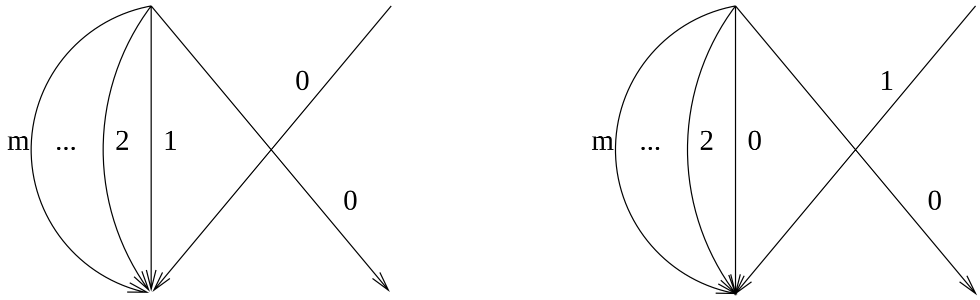
These matrices can be factored out as

$$M^{(\gamma_n)} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} n & 1 \\ 1 & 0 \end{bmatrix}, \; M^{(\rho_n)} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} n & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2}$$

Let us introduce the following notation: $N_n = \begin{bmatrix} n & 1 \\ 1 & 0 \end{bmatrix}$, for $n \in \mathbb{N}$. With this we can write the decompositions for $M^{(\gamma_n)}$ and $M^{(\rho_n)}$ as $M^{(\gamma_n)} = N_1 \cdot N_n$, $M^{(\rho_n)} = N_1 \cdot N_n \cdot N_0$.

We order the edges of the factor blocks in (2) as shown in Figure 2. These orderings are compatible with the ones the original matrices had, in the sense that when we contract we recover the orderings required for $M^{(\gamma_n)}$ and $M^{(\rho_n)}$.

In view of the discussion above, we conclude that a Bratteli–Vershik representation associated to $(\Omega_{\overline{\alpha}}, \sigma)$ can be obtained as a concatenation of blocks associated to matrices N_n, $n \geq 1$. Since $N_n \cdot N_0 \cdot N_m = N_{n+m}$, we can contract the diagram to obtain a new one in which the blocks are of the form N_d, with $d > 0$. (If the first matrix is N_0, we contract it with the vector $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ associated



(a) Ordering $O_m^{(0)}$ (b) Ordering $O_m^{(1)}$

FIGURE 3.

to the top edges in E_0, or hat, and the new diagram will have $\begin{pmatrix} u_2 \\ u_1 \end{pmatrix}$ as its hat and no N_0 block any longer.)

Let us analyze the order structure in this ordered Bratteli–Vershik diagram. Notice that a level with incidence matrix N_m can appear with two possible different orderings. We will use the notation $O_m^{(0)}$ to indicate a level with incidence matrix N_m ordered as shown in Figure 3 (a), and $O_m^{(1)}$ for a level with incidence matrix N_m ordered as shown in Figure 3 (b). If $m > 1$, the second ordering appears exactly when it comes from a level contraction. If $m = 1$, $O_1^{(1)}$ is the ordering for N_1 when it represents a bottom block in Figure 2, and $O_1^{(0)}$ the block on top of it.

Summarizing, to any Sturmian subshift $(\Omega_{\overline{\alpha}}, \sigma)$ we associate a Bratteli–Vershik representation whose incidence matrix and ordering for any level $k \geq 2$ are $N_{d_k}, O_{d_k}^{(i_k)}$, with $d_k > 0, i_k \in \{0,1\}$. We will call this representation *standard.*

The following technical lemma will be useful later.

Lemma 1. *Let* $\left((N_{d_k})_{k\geq 0}, \left(O_{d_k}^{(i_k)}\right)_{k\geq 0}\right)$ *and* $\left((N_{d_k})_{k\geq 0}, \left(O_{d_k}^{(j_k)}\right)_{k\geq 0}\right)$ *be two sequences of matrices and orderings coming from standard Bratteli–Vershik representations of Sturmian subshifts. Then,*

- *If* $(d_k)_{k\geq 0}$ *is not the constant sequence* $(111\ldots)$*, then* $i_k = j_k$ *for all* k *large enough.*
- *If* $(d_k)_{k\geq 0}$ *is* $(111\ldots)$*, then either* $i_k = j_k$ *for all* k *large enough, or* $i_k \neq j_k$ *for all* k *large enough, and in both cases* $i_{k+1} \neq i_k$ *for* k *large enough.*

Proof. Let us suppose there is $k > 0$ such that $i_k \neq j_k$. Without loss of generality we can suppose that $i_k = 0$ and $j_k = 1$.

First we assume that $d_k \neq 1$. Since $i_k = 0$ (this corresponds to a non contracted level) then $i_{k-1} = 1$ and the incidence matrix at level $k-1$ is N_1. On the other hand, since $j_k = 1$, level k for the second diagram is a contracted one, then the order associated to the previous level must be $j_{k-1} = 0$. Now, by the same argument we have that $j_{k-2} = 1$ with incidence matrix equal to N_1, which implies that $i_{k-2} = 0$ and the corresponding incidence matrix is also N_1. This way, we prove inductively that $d_0 = d_1 = \cdots = d_{k-1} = 1$, with $i_k = i_{k-2} = \cdots = 1 = j_{k-1} = j_{k-3} = \cdots$ and $i_{k-1} = i_{k-3} = \cdots = 0 = j_k = j_{k-2} = \cdots$. If $d_k = 1$, the same procedure implies that $d_0 = d_1 = \cdots = d_k = 1$ and similar conditions for the orderings in each level of both diagrams. □

We will now try to get more information about this Bratteli–Vershik diagram by studying its associated dimension group, and in particular we will be interested on the automorphisms of the group. For that we need the following proposition which proof is given in the appendix B).

Proposition 3. *Let $(\Omega_{\overline{\alpha}},\sigma)$ be a Sturmian subshift then $(K^0(\Omega_{\overline{\alpha}},\sigma),K^0(\Omega_{\overline{\alpha}},\sigma)^+)$ is isomorphic to $(\mathbb{Z}^2,P_{\frac{1-\overline{\alpha}}{\overline{\alpha}}})$ as ordered groups.*

Let K_0 be the dimension group of $(\Omega_{\overline{\alpha}},\sigma)$. From Proposition 3, K_0 is isomorphic to $\left(\mathbb{Z}^2,P_{\overline{\alpha}}\right)$. On the other hand, from the Bratteli–Vershik diagram one can compute this dimension group: $K_0 \cong \varinjlim \left(\mathbb{Z}^2,N_{d_k}\right)$. By [ES, Th. 3.2] we obtain that $K_0 \cong \left(\mathbb{Z}^2,P_{\alpha}\right)$, where the simple continued fraction expansion of α is eventually equal to the simple continued fraction expansion of $\overline{\alpha}$.

The following lemma implies that if $\overline{\alpha}$ is not a quadratic algebraic number, the identity is the only automorphism of the dimension group K_0. For the sake of completeness we will write the proof. Similar results appear in [S].

Lemma 2. *Let $\beta>0$ be a real number and $M=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in GL(2,\mathbb{Z})\setminus\{I_2\}$ be an automorphism of $\left(\mathbb{Z}^2,P_\beta\right)$. Then:*

(1) *β is a quadratic algebraic number.*
(2) *M does not have any column ≤ 0.*
(3) *If moreover β is irrational, then $b\neq 0,\ c\neq 0$ and the irreducible polynomial for β in $\mathbb{Q}[X]$ is $X^2+\dfrac{d-a}{b}X-\dfrac{c}{b}$.*

Proof. It is clear that if $M\cdot P_\beta=P_\beta$, then there exists an integer k such that the vector $M\cdot\begin{pmatrix}1\\-\beta\end{pmatrix}\in\mathbb{R}^2$ is equal to $k\begin{pmatrix}1\\-\beta\end{pmatrix}$. This implies that $(-a+b\beta)\cdot(-\beta)=(-c+d\beta)\cdot 1$, which is in turn equivalent to $b\beta^2+(d-a)\,\beta-c=0$. This proves 1. and 3..

To prove 2., assume that the first column of M is negative, then M would fail to be an automorphism of P_β, since it would send the vector $\begin{pmatrix}1\\0\end{pmatrix}\in P_\beta$ into $\begin{pmatrix}a\\c\end{pmatrix}\notin P_\beta$ (an identical argument works if the second column is negative). □

Let us study the case of α and $\overline{\alpha}$ quadratic irrationals. Thus, their continued fraction expansions are ultimately periodic, and since they are eventually equal, the period in both expansions is the same. Moreover, by Lemma 1, the sequence of orderings $\left(O^{(i_k)}_{d_k}\right)_{k\ge 0}$ is eventually periodic, with the same period. Without loss of generality we will suppose that α and the orderings are periodic: $\alpha=\left[\overline{d_0:d_1,\dots d_{T-1}}\right]$ (otherwise we multiply up all matrices in the Bratteli–Vershik diagram that appear before they become periodic, and we get a new "*hat*" and a periodic diagram), and T the length of the least period of α. Let us denote $M_\alpha=N_{d_{T-1}}\cdot\dots\cdot N_{d_1}\cdot N_{d_0}$.

We get,

Corollary 1. *Let $\overline{\alpha}=\left[c_0:c_1,\dots,c_p,\overline{d_0,d_1,\dots d_{T-1}}\right]$ be a quadratic irrational number and $\alpha=\left[\overline{d_0:d_1,\dots d_{T-1}}\right]$, where $d_0,d_1,\dots d_{T-1}>0$ and T is the length of a minimal period of $\overline{\alpha}$. Then,*

(1) *If $\alpha\neq\left[\overline{1}\right]$, then $(\Omega_{\overline{\alpha}},\sigma)$ is topologically conjugate to a stationary Bratteli-Vershik system with stationary incidence matrix $M_\alpha=N_{d_{T-1}}\cdot\dots\cdot N_{d_1}\cdot N_{d_0}$.*
(2) *If $\alpha=\left[\overline{1}\right]$, then $(\Omega_{\overline{\alpha}},\sigma)$ is topologically conjugate to a stationary Bratteli-Vershik system with stationary incidence matrix $M_\alpha=N_1\cdot N_1$ and order induced by $O^{(0)}_1$ followed by $O^{(1)}_1$.*
□

Before giving the proof of Theorem 1 let us remark that Proposition 16 in [DHS] and last corollary imply for a quadratic number α that $(\Omega_{\overline{\alpha}},\sigma)$ is a substitutive system. The converse is also true. In fact, if $(\Omega_{\overline{\alpha}},\sigma)$ is a substitutive system it is clear that there are non trivial automorphisms, then

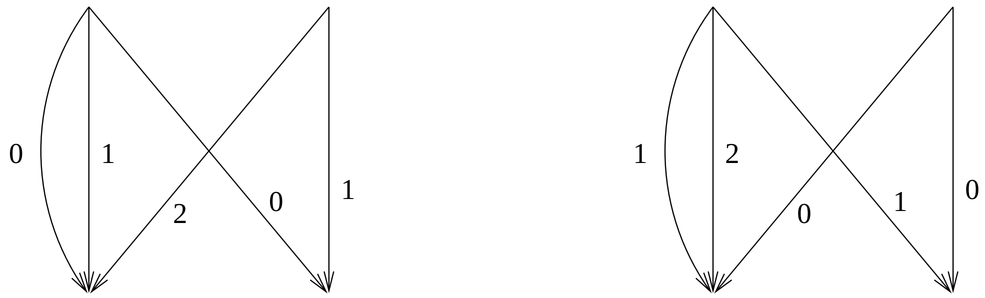


FIGURE 4. The two possible orders for N_1^2.

using Lemma 2 we conclude that $\overline{\alpha}$ is quadratic. This result is part of the folcklore but we do not know any reference to it. We are now ready to prove Theorem 1.

***Proof of Theorem* 1.** Let $0 < \overline{\alpha} < 1$ be an irrational number such that $(\Omega_{\overline{\alpha}}, \sigma)$ and (X, T) are Kakutani and orbit equivalent. From the Kakutani equivalence, given a Bratteli–Vershik representation of $(\Omega_{\overline{\alpha}}, \sigma)$, by deleting and adding a finite number of arrows to it, we get a representation of (X, T). Let $\alpha = [d_0 : d_1, d_2, \ldots]$ be the simple continued fraction expansions coming from the standard Bratteli–Vershik representations of the Sturmian system. By contracting both diagrams we will assume that they are the same, up to the corresponding hats $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ respectively.

It is easy to see that the unique infinitesimal of the ordered group $(\mathbb{Z}^2, P_\alpha)$ is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Consequently $(\Omega_{\overline{\alpha}}, \sigma)$ and (X, T) are strong orbit equivalent. It follows that there is a unit keeping automorphism between their dimension groups. Let M be this automorphism of $(\mathbb{Z}^2, P_\alpha)$ such that $M \cdot \mathbf{u} = \mathbf{v}$. If α is not a quadratic irrational, then by Lemma 2, $M = I_2$ and $\mathbf{u} = \mathbf{v}$, which implies that both representations are the same, and the systems topologically conjugate.

When α is quadratic we can assume it is periodic with expansion $\alpha = [\overline{d_0 : d_1, \ldots d_{T-1}}]$. Then, by Proposition 2 there is $k \in \mathbb{Z}$ such that $M = M_\alpha^k$. Thus, $M_\alpha^k \cdot \mathbf{u} = \mathbf{v}$. Without loss of generality we can assume $k \geq 0$.

We consider two cases. First assume $\alpha \neq [\overline{1}]$. Then by Corollary 1, Case 1, M_α is the stationary matrix in the Bratteli-Vershik representation of the system. Then by contracting the k first levels of the diagram with unit $\mathbf{u}$ we get the diagram of the system with unit $\mathbf{v}$. This proves they are topologically conjugate.

We suppose now that $\alpha = [\overline{1}]$. There exists an integer k such that $M = N_1^k$. On the other hand, by Corollary 1, Case 2, the stationary matrix of the Bratteli-Vershik representation of the system is $M_\alpha = N_1 \cdot N_1$. We contract the ordered Bratteli diagram starting with $\mathbf{v}$, to have a new ordered Bratteli diagram with $\mathbf{w} = M^k \mathbf{v} = \mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ as its hat and with stationary matrix $M_\alpha = N_1 N_1$. In this way we have two stationary ordered Bratteli diagrams, $B_{\mathbf{u}}$ and $B_{\mathbf{w}}$, which can only differ on the orderings of M_α. If the orderings are the same then the proof is finished, hence we can suppose that the orderings are given by Figure 4. Let B_1 and B_2 be respectively the stationary ordered Bratteli diagrams with the same incidence matrices and orderings as $B_{\mathbf{u}}$ and $B_{\mathbf{w}}$, for levels $k \geq 2$, but with hat $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. In [DHS] it is proved that (X_{B_1}, V_{B_1}) and (X_{B_2}, V_{B_2}) are respectively isomorphic to the subshifts (X_1, σ) and (X_2, σ) generated by the substitutions $\tau_1 : \{0,1\} \to \{0,1\}^*$ and $\tau_2 : \{0,1\} \to \{0,1\}^*$ defined by $\begin{matrix} \tau_1(0) = 100 \\ \tau_1(1) = 10 \end{matrix}$ and $\begin{matrix} \tau_2(0) = 001 \\ \tau_2(1) = 01. \end{matrix}$

Let $\phi : \{0,1\} \to \{ (0,i) \mid 0 \le i \le a-1 \} \cup \{ (1,i) \mid 0 \le i \le b-1 \}$ be the map defined by$\phi(0) = (0,0)\,(0,1)\cdots(0,a-1)$, $\phi(1) = (1,0)\,(1,1)\cdots(1,b-1)$.

Let $x \in X_1$ and $y \in X_2$. We see that the subshift (Y_1,σ) (resp. (Y_2,σ)) generated by $\phi(x)$ (resp. $\phi(y)$) is isomorphic to $(X_{B_{\mathbf{u}}}, V_{B_{\mathbf{u}}})$ (resp. $(X_{B_{\mathbf{w}}}, V_{B_{\mathbf{w}}})$). But we can prove that $X_1 = X_2$ (the proof is left to the reader), hence using the minimality of (X_1,σ) and (X_2,σ) and the fact that $\phi(X_i) \subseteq Y_i$, $i \in \{1,2\}$, it follows that $Y_1 = Y_2$ and that $(X_{B_{\mathbf{u}}}, V_{B_{\mathbf{u}}})$ is isomorphic to $(X_{B_{\mathbf{w}}}, V_{B_{\mathbf{w}}})$. This achieves the proof . □

Appendix.

A) Proof of Proposition 2.

The results of this section are closely related to the ones found in [S]. We write whole proofs here for the sake of completeness.

Lemma 3. *If $M \in GL(2,\mathbb{Z}) \setminus \{I_2\}$ is a unit keeping automorphism of $(\mathbb{Z}^2, P_\alpha)$, then either $M \ge 0$ or $M^{-1} \ge 0$.*

Proof. Since $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, $\det M = \pm 1$ and $M^{-1} = \det M \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Let us make a couple of remarks:

By Lemma 2, neither M nor M^{-1} (since M^{-1} is also an automorphism of $(\mathbb{Z}^2, P_\alpha)$) can have any column ≤ 0.

The matrix M can not have a row ≤ 0, otherwise, if for instance its first row was non-positive, M would send the unit $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} > 0$ into a vector which first coordinate would be $au + bv \le 0$, which is not possible by definition of a unit keeping automorphism.

It follows from the above remarks about the impossibility for M to have non-positive columns and rows that M has at most two non positive entries. If it has two of them, they must be either a and d, or b and c (and the other two entries must be > 0). But in view of the computation of M^{-1}, in that case all entries of this inverse have the same sign, which must be positive in view of a previous remark. If it has only one strictly positive entry, it is easy to see that the condition $|\det M| = 1$ can not be realized. This complete the proof. □

Let M be a unit keeping automorphism of $(\mathbb{Z}^2, P_\alpha)$. From Lemma 3, either M or its inverse must be positive. Without loss of generality we will assume that $M \ge 0$. Let $K_0(M) = (\mathbb{Z}^2, K^+(M))$ be the dimension group of M, that is to say,

$$K^+(M) = \left\{ \mathbf{v} \in \mathbb{Z}^2 \;\middle|\; M^k \mathbf{v} \ge 0, \text{ for some } k \in \mathbb{N} \right\}.$$

Lemma 4. *If $M \ge 0$ is an automorphism of $(\mathbb{Z}^2, P_\alpha)$ with $M \ne I_2$, then M is a primitive matrix and $K_0(M) = (\mathbb{Z}^2, P_\alpha)$.*

Proof. Calling $K^+ = K^+(M)$, the proof of the lemma consists on showing that $K^+ = P_\alpha$.

Notice first that it follows easily from Lemma 2 that M can have at most one 0 entry, and it would be in the diagonal, which implies that in any case $M^2 > 0$ and M is primitive.

Thus, by Perron–Frobenius theorem, M has an eigenvalue $\lambda_1 > 1$ associated to a strictly positive eigenvector $\mathbf{v}_1 > 0$. On the other hand, since $M \cdot P_\alpha = P_\alpha$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -\alpha \end{pmatrix}$ must be an eigenvector of M, and since $|\det M| = 1$, the eigenvalue $\lambda_2 = \frac{\det M}{\lambda_1}$ associated to $\mathbf{v}_2$ satisfies $|\lambda_2| < 1$.

Therefore any $\mathbf{v} \in \mathbb{Z}^2$ can be written in a unique way as $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$, with $x_1, x_2 \in \mathbb{R}$. Notice that for such a $\mathbf{v}$, $\mathbf{v} \in P_\alpha \Leftrightarrow x_1 \geq 0$. For any $k \in \mathbb{N}$ we get $M^k\mathbf{v} = \lambda_1^k x_1\mathbf{v}_1 + \lambda_2^k x_2\mathbf{v}_2$, and since $\mathbf{v}_1 > 0$, if $x_2 \neq 0$, $M^k\mathbf{v}$ will eventually become positive if and only if $x_1 > 0$. If $x_2 = 0$, the condition becomes $x_1 \geq 0$, and noticing that the only point $\mathbf{v}$ with integer coordinates on the line of equation $x_1 = 0$ is the origin, we conclude that $\mathbf{v} \in K^+ \Leftrightarrow x_1 \geq 0$. Thus $K^+ = P_\alpha$. □

Let us consider now a positive matrix $M \in GL(2, \mathbb{Z})$. From [ES, Lemma 4.1], $M = N_{c_l} \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_0}$, with $c_i \geq 0$, $i = 0, \ldots, l$. The following lemma tells us exactly what the dimension group for M is (not just a characterization up to isomorphism).

Lemma 5. *Let $M = N_{c_l} \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_0} \geq 0$, with $c_i \geq 0$, $i = 0, \ldots, l$, be a non negative invertible primitive 2×2 matrix, and let $\beta = [\overline{c_0 : c_1, \ldots, c_l}]$. Then M is an automorphism of $(\mathbb{Z}^2, P_\beta)$ and $K_0(M) = (\mathbb{Z}^2, P_\beta)$.*

Proof. Since $N_d \cdot N_0 \cdot N_{d'} = N_{d+d'}$ for all d, d', we can suppose that in the decomposition $M = N_{c_l} \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_0}$ all numbers c_i are strictly positive, except perhaps c_0 and/or c_l. Thus we have four cases, and in each of them we can compute the simple continued fraction expansion for the associated irrational number β:

(1) $c_0 \geq 1, c_l \geq 1$, $M = N_{c_l} \cdot \ldots \cdot N_{c_1} \cdot N_{c_0}$ and $\beta = [\overline{c_0 : c_1, \ldots, c_l}]$, $l \geq 0$.
(2) $c_0 = 0, c_l \geq 1$, $M = N_{c_l} \cdot \ldots \cdot N_{c_1} \cdot N_0$ and $\beta = [0 : c_1, \overline{c_2, \ldots, c_{l-1}, c_1 + c_l}]$, $l \geq 2$.
(3) $c_0 \geq 1, c_l = 0$, $M = N_0 \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_1} \cdot N_{c_0}$ and $\beta = [c_0 : \overline{c_1, \ldots, c_{l-2}, c_0 + c_{l-1}}]$, $l \geq 2$.
(4) $c_0 = c_l = 0$, $M = N_0 \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_1} \cdot N_0$ and $\beta = [0 : \overline{c_1, \ldots, c_{l-1}}]$, $l \geq 2$.

Notice that in cases 2 and 3, M fails to be primitive if $l = 1$.

Let us prove the lemma in Case 1, and the other three cases will follow from it later.

For $r_0, r_1, \ldots, r_m > 0$ in $\mathbb{R}$, the notation $r = [r_0 : r_1, \ldots, r_m]$ will stand for the positive real number r such that

$$r = r_0 + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cfrac{}{\ddots \; r_{m-1} + \cfrac{1}{r_m}}}}.$$

In an analogous way as in the theory of continued fractions presented in [HW], recursive matrix equations can be written for computing r. Namely, if we write $F_0 = N_{r_0}$, $F_k = \begin{bmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{bmatrix}$, and the recursion $F_k = N_{r_k} \cdot F_{k-1}$ for $1 \leq k \leq m$, one easily gets $F_m = N_{r_m} \cdot N_{r_{m-1}} \cdot \ldots \cdot N_{r_0}$, and $r = \frac{p_m}{q_m}$. Now, since $\beta = [\overline{c_0 : c_1, \ldots c_l}]$, then $\beta = c_0 + \cfrac{1}{c_1 + \cfrac{}{\ddots \; c_l + \cfrac{1}{\beta}}}$. In other words, $\beta = [c_0 : c_1, \ldots, c_l, \beta]$.

Therefore β will be of the form $\beta = \dfrac{x}{y}$, with the matrix $F_{l+1} = \begin{bmatrix} x & y \\ p_l & q_l \end{bmatrix}$ satisfying

$$\begin{bmatrix} x & y \\ p_l & q_l \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} p_l & q_l \\ p_{l-1} & q_{l-1} \end{bmatrix} \tag{3}$$

and $F_l = \begin{bmatrix} p_l & q_l \\ p_{l-1} & q_{l-1} \end{bmatrix}$ coming from the finite (rational) continued fraction $r = [c_0 : c_1, \ldots, c_l] = \frac{p_l}{q_l}$, which, incidentally, implies that $F_l = N_{c_l} \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_0} = M$. From the matrix equation (3), $\beta = \frac{x}{y} = \frac{p_l\beta + p_{l-1}}{q_l\beta + q_{l-1}}$, that is,

$$\beta^2 + \frac{q_{l-1} - p_l}{q_l}\beta - \frac{p_{l-1}}{q_l} = 0. \tag{4}$$

Notice that (4) defines the irreducible polynomial for β in $\mathbb{Q}[X]$, and since the constant coefficient is strictly negative, β is the only positive root of this equation.

Using the fact that $M \geq 0$ is primitive and following a similar argument as the one on the proof of Lemma 4, M has an associated ordered dimension group $K_0(M) = (\mathbb{Z}^2, P_\mu)$, $\mu \in \mathbb{R}^+ \setminus \mathbb{Q}$. Moreover P_μ can be computed as $P_\mu = \left\{ \mathbf{v} \in \mathbb{Z}^2 \mid M^k \mathbf{v} \geq 0, \text{ for some } k \in \mathbb{N} \right\}$, and therefore $M \cdot P_\mu = P_\mu$ (that is, M is an automorphism of $(\mathbb{Z}^2, P_\mu)$). Since $M = F_l = \begin{bmatrix} p_l & q_l \\ p_{l-1} & q_{l-1} \end{bmatrix}$, it follows from Lemma 2 that μ satisfies the equation $\mu^2 + \frac{q_{l-1} - p_l}{q_l}\mu - \frac{p_{l-1}}{q_l} = 0$. Since $\mu > 0$ is a root for (4), then $\mu = \beta$, and Case 1 is proved.

Let us deduce the Case 2 from what we just proved. Since $\beta = [0 : c_1, \overline{c_2, \ldots, c_{l-1}, c_1 + c_l}\,]$, we can write $\beta = 0 + \dfrac{1}{c_1 + \frac{1}{\mu}} = c_1 + \dfrac{1}{\mu} = \dfrac{c_1\mu + 1}{\mu}$, with $\mu = [\,\overline{c_2 : \ldots, c_{l-1}, c_1 + c_l}\,]$. Calling $M_\mu = N_{c_1 + c_l} \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_2} = N_{c_1} \cdot N_0 \cdot N_{c_l} \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_2}$ the matrix having μ as its associated irrational number, then $M = N_{c_l} \cdot \ldots \cdot N_{c_1} \cdot N_0 = (N_{c_1} \cdot N_0)^{-1} \cdot M_\mu \cdot (N_{c_1} \cdot N_0)$.

We know from Case 1 that $M_\mu \cdot \begin{pmatrix} 1 \\ -\mu \end{pmatrix}$ is parallel to $\begin{pmatrix} 1 \\ -\mu \end{pmatrix}$. Therefore $M \cdot (N_{c_1} \cdot N_0)^{-1} \cdot \begin{pmatrix} 1 \\ -\mu \end{pmatrix} = (N_{c_1} \cdot N_0)^{-1} \cdot M_\mu \cdot \begin{pmatrix} 1 \\ -\mu \end{pmatrix}$ is parallel to $(N_{c_1} \cdot N_0)^{-1} \begin{pmatrix} 1 \\ -\mu \end{pmatrix} = (c_1\mu + 1) \begin{pmatrix} 1 \\ -\frac{c_1\mu + 1}{\mu} \end{pmatrix}$, which means that $M \cdot \begin{pmatrix} 1 \\ -\beta \end{pmatrix}$ is parallel to $\begin{pmatrix} 1 \\ -\beta \end{pmatrix}$. Since M is positive, Case 2 is established. The remaining two cases are proved in a similar fashion.

□

We are ready to prove Proposition 2 now. If $M \neq I_2$ is a positive automorphism of $(\mathbb{Z}^2, P_\alpha)$, then by Lemma 4, M is primitive and $M \cdot P_\alpha = P_\alpha$. It follows from Lemma 5 that $M \cdot P_\beta = P_\beta$, with β the irrational number associated to M. But then it is clear that $\beta = \alpha$. Recall that $\alpha = [\,\overline{d_0 : d_1, \ldots d_{T-1}}\,]$, $M_\alpha = N_{d_{T-1}} \cdot \ldots \cdot N_{d_1} \cdot N_{d_0}$, with $d_i > 0$ for all $i = 0, \ldots, T-1$, and T is the minimal length of a period in the simple continued fraction expansion of α. Thus the simple continued fraction expansion for β is periodic and must be of the form of Case 1 in the proof of Lemma 5, that is to say $\beta = [\,\overline{c_0 : c_1, \ldots, c_l}\,]$ and $M = N_{c_l} \cdot \ldots \cdot N_{c_1} \cdot N_{c_0}$ with all $c_i > 0$. Finally, from the minimality of the period of α, $l + 1 = k \cdot T$ for some $k \in \mathbb{N}$, and $M = M_\alpha^k$.

B) Proof of Proposition 3.

We will make use of the following lemma which proof can be found in [H].

Lemma 6. *Let (X, T) be a minimal Cantor system and $f \in C(X, \mathbb{Z})$.*

1. *There exists $g \in C(X, \mathbb{Z})$ such that $f + g \circ T - g \geq 0$ if and only if for every $x \in X$ the sequence $(f(T^n x) + \cdots + f(Tx) + f(x))_{n \in \mathbb{N}}$ is bounded from below.*
2. *f is a coboundary if and only if for all $x \in X$ the sequence $\left(\sum_{i=0}^{n} f\left(T^i(x)\right)\right)_{n \in \mathbb{N}}$ is bounded.*

Let us also state a technical lemma:

Lemma 7. *Let $(\Omega_\alpha, \sigma, \mu)$ be a Sturmian subshift, then*

(1) [K] *For all clopen* U *of* Ω_α *and all* $x \in \Omega_\alpha$ *the sequence* $\left(\sum_{i=0}^{n-1}\left(\mathbf{1}_U\left(\sigma^i(x)\right)-\mu(U)\right)\right)_{n\in\mathbb{N}}$ *is bounded;*
(2) [HM]$\{\, \mu(U) \mid U$ *is a clopen set in* $\Omega_\alpha \,\} \subseteq \{\, m\alpha+n \mid m,n\in\mathbb{Z}\,\}$*;*

(For example we have $\mu([0]) = 1-\alpha$ and $\mu([1]) = \alpha$.)

We start the proof of Proposition 3 with some notations. For all $n\in\mathbb{N}$ let

$$L_n = \left\{\, x_i\cdots x_{i+n-1} \mid i\in\mathbb{Z},\ (x_n)_{n\in\mathbb{Z}}\in\Omega_\alpha \,\right\}$$

(this set is usually called *language* of Ω_α) and $Q_n = \{\, U(u) \mid u\in L_{2n+1}\,\}$ where for all $u = u_0\cdots u_{2n}\in L_{2n+1}$

$$U(u) = [u_0\cdots u_{n-1}.u_n u_{n+1}\cdots u_{2n}] = \left\{\, (y_n)_{n\in\mathbb{Z}}\in\Omega_\alpha \;\middle|\; y_{i-n}=u_i,\ 0\le i\le 2n \,\right\}.$$

It is classical that this last set is a clopen set, that Q_n is a partition and that $\cup_{n\in\mathbb{N}}Q_n$ is a basis for the topology Ω_α.

Let $f\in C(\Omega_\alpha,\mathbb{Z})$. There exists an integer n such that f is constant on each set of Q_n. Hence there exists $\{\, f_u \mid u\in L_{2n+1}\,\}\subseteq\mathbb{Z}$ such that $f=\sum_{u\in L_{2n+1}} f_u\mathbf{1}_{U(u)}$. From Lemma 7 there exists $p,q\in\mathbb{Z}$ such that $\sum_{u\in L_{2n+1}} f_u\mu(\{U(u)\}) = p\alpha+q$. Hence there exist two integers g_0 and g_1 (uniquely determined) such that $p\alpha+q = g_0(1-\alpha)+g_1\alpha = g_0\mu([0])+g_1\mu([1])$. We remark that g_0 and g_1 do not depend on Q_n is the sense that if f is constant on each clopen of Q_m, for some $m\in\mathbb{N}$, then $\sum_{u\in L_{2m+1}} f_u\mu(\{U(u)\}) = p\alpha+q$.

We define $g\in C(\Omega_\alpha,\mathbb{Z})$ by $g(x) = g_0\mathbf{1}_{[0]}(x)+g_1\mathbf{1}_{[1]}(x)$ for all $x\in\Omega_\alpha$. We now show that $\left(\sum_{i=0}^{N-1}(f-g)\left(\sigma^i(x)\right)\right)_{N\in\mathbb{N}}$ is bounded for all $x\in\Omega_\alpha$. Let $x\in\Omega_\alpha$, then

$$\sum_{i=0}^{N-1}(f-g)\left(\sigma^i(x)\right) =$$

$$\sum_{i=0}^{N-1}\left(\sum_{u\in L_{2n+1}}\left(f_u\mathbf{1}_{U(u)}\left(\sigma^i(x)\right)\right)-g_0\mu([0])-g_1\mu([1])\right) + \sum_{i=0}^{N-1}\left(g_0\mu([0])+g_1\mu([1])-g\left(\sigma^i(x)\right)\right).$$

Using Lemma 7 we clearly see that the second sum is bounded independently of N, and using the definition of g_0 and g_1 together with the same lemma it is not difficult to see that the first sum is also bounded independently of N. It follows from Lemma 6 that $f-g$ is a coboundary.

We set $\psi(f) = (g_0,g_1)$. It is not difficult to see that this defines a group homomorphism $\psi : C(\Omega_\alpha,\mathbb{Z})\to\mathbb{Z}^2$.

If $\psi(f) = 0$ then using 7 it is not difficult to prove that f if bounded and hence that $\mathrm{Ker}\psi = \beta C(\Omega_\alpha,\mathbb{Z})$, consequently $K^0(\Omega_\alpha,\mathbb{Z})$ is isomorphic to $\mathbb{Z}^2$. Moreover if f is positive then we obtain that $g_0(1-\alpha)+g_1\alpha\ge 0$, that is to say $(g_0,g_1)\in P_{\frac{1-\alpha}{\alpha}}$. And conversely if (a,b) belongs to $P_{\frac{1-\alpha}{\alpha}}$ Lemma 6 together with Lemma 7 show that the function $h = a\mathbf{1}_{[0]}+b\mathbf{1}_{[1]}$ is cohomologous to a positive function. Finally K^0 is isomorphic to $\left(\mathbb{Z}^2, P_{\frac{1-\alpha}{\alpha}}\right)$ as an ordered group.

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