BOUNDARY OF THE RAUZY FRACTAL SET IN $\mathbb{R} \times \mathbb{C}$ GENERATED BY $P(x) = x^4 - x^3 - x^2 - x - 1$

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Abstract. We study the boundary of the 3-dimensional Rauzy fractal $\mathcal{E} \subset \mathbb{R} \times \mathbb{C}$ generated by the polynomial $P(x) = x^4 - x^3 - x^2 - x - 1$. The finite automaton characterizing the boundary of $\mathcal{E}$ is given explicitly. As a consequence we prove that the set $\mathcal{E}$ has 18 neighboors where 6 of them intersect the central tile $\mathcal{E}$ in a point. Our construction shows that the boundary is generated by an iterated function system starting with 2 compact sets.

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1. Introduction

Consider $A = \{1, 2, 3\}$ as an alphabet. Let A^* be the set of finite words on A and $\sigma : A \to A^*$ be the map (called Tribonacci substitution) defined by

$$\sigma(1) = 12,\ \sigma(2) = 13,\ \sigma(3) = 1.$$

We extend σ to $A^{\mathbb{N}}$ by concatenation : $\sigma(a_0 \cdots a_n \ldots) = \sigma(a_0) \cdots \sigma(a_n) \ldots$. It is clear that σ has a unique fixed point u : $\sigma(u) = u \in A^{\mathbb{N}}$. The dynamical system associated to σ is the couple (Ω, S) where $S : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is the shift map $(S((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}})$ and Ω is the S-orbit closure of u : $\Omega = \overline{\{S^n u | n \in \mathbb{N}\}}$. It is well-known that (Ω, S) is minimal, uniquely ergodic and of zero entropy (see [Q87, F02] for more details).

In 1982, G. Rauzy [R82] studied the Tribonacci substitution σ. He proved that the dynamical system generated by σ is measure theoretically conjugate to an exchange of domains X_1, X_2, X_3 in a compact tile $X = X_1 \cup X_2 \cup X_3$. The set X is the classical two-dimensional Rauzy fractal. It has been extensively studied and is related to many topics : numeration systems [M00, M06, M05], geometrical representation of symbolic dynamical systems [AI01, AIS01, CS01, HZ98, M98, T06, S96], multidimensional continued fractions and simultaneous approximations [ABI02, CHM01, C02, HM06], self-similar tilings [A99, A00, AI01, P99] and Markov partitions of Hyperbolic automorphisms of the torus [KV98, M98, P99].

Among the main properties of the set X, let us recall it is compact, connected, its interior is simply connected, its boundary is fractal and it induces a periodic tiling of $\mathbb{R}^2$ ([R82]).

It is possible to associate such a fractal set to a large class of substitutions over an alphabet with d letters (called unimodular Pisot substitutions). Let us call them *Rauzy fractals*. P. Arnoux and S. Ito [AI01] (see also [CS01]) proved that the dynamical system associated to such a substitution σ is measure theoretically conjugate to an exchange of domains $X_1, \ldots, X_d$ in the Rauzy fractal $X_\sigma = X_1 \cup \ldots \cup X_d \subset \mathbb{R}^{d-1}$ provided that the "strong coincidence condition" is fulfilled. All

these sets X_σ are compact and generate periodic tilings and self-replicating tilings of $\mathbb{R}^{d-1}$.
There are different ways to define the Rauzy fractal associated to a given substitution σ over an alphabet of $d+1$ letters. One is through numeration systems.
Let $d \geq 2$ and $a_1, a_2, \dots, a_d$ be integers such that $a_1 \geq a_2 \geq \dots \geq a_d \geq 1$. Consider $A = \{1, 2, \dots, d+1\}$ as an alphabet. Let σ_d be the substitution defined by

$$\sigma_d(i) = \underbrace{11\dots1}_{a_i}(i+1) \text{ if } i \leq d \text{ and } \sigma_d(d+1) = 1.$$

We define the Rauzy fractal associated to σ_d as follows. Consider the sequence $(F_n)_{n\geq 0}$ defined by

$$F_{n+d+1} = a_1 F_{n+d} + a_2 F_{n+d-1} + \dots + a_d F_{n+1} + F_n, \ \forall n \geq 0,$$

with initial conditions (called *Parry conditions*)

$$F_0 = 1, \ F_n = a_1 F_{n-1} + \dots + a_n F_0 + 1, \ \forall\, 1 \leq n \leq d.$$

For any $n \in \mathbb{N}$, using the greedy algorithm, we have $n = \sum_{i=0}^{N} c_i F_i$ where the c_i's are integers satisfying

$$\sum_{i=0}^{k} c_i F_i < F_{k+1} \text{ for all } k \in \{0, 1 \dots, N-1\}.$$

We deduce that $(c_i)_{0 \leq i \leq N-1}$ belongs to $\mathcal{D}_{a_1,\dots,a_d}$, where $\mathcal{D}_{a_1,\dots,a_d}$ is the set of sequences $(\varepsilon_i)_{l \leq i \leq k}$, $l, k \in \mathbb{Z}$, such that for all $i \in \{l, l+1, \dots, \leq k\}$:

(1) $\varepsilon_i \in \{0, 1, \dots, a_1\}$,
(2) $\varepsilon_i \varepsilon_{i-1} \dots \varepsilon_{i-d} <_{lex} a_1 a_2 \dots a_d 1$ when $i \geq l+d$, and,
(3) $\varepsilon_i \varepsilon_{i-1} \dots \varepsilon_l 0^{d-i+l} <_{lex} a_1 a_2 \dots a_d 1$ when $l \leq i \leq l+d$,

where $<_{lex}$ is the usual lexicographic ordering. We set

$$\mathcal{D}^{\infty}_{a_1,\dots,a_d} = \{(\varepsilon_i)_{i \geq l}; l \in \mathbb{Z}, (\varepsilon_i)_{l \leq i \leq n} \in \mathcal{D}_{a_1,\dots,a_d}, \forall n \geq l\}.$$

Now, consider the following polynomial

$$P_{a_1,\dots a_d}(x) = x^{d+1} - a_1 x^d - a_2 x^{d-1} - \dots - a_d x - 1.$$

It can be checked that P has a root $\beta = \beta_1 \in]1, +\infty[$ and d roots with modulus less than 1. Let $\beta_1, \beta_2, \beta_3, \dots, \beta_r$ be the roots of P belonging to $\mathbb{R}$ and $\beta_{r+1}, \dots, \beta_{r+s}, \overline{\beta_{r+1}}, \dots, \overline{\beta_{r+s}}$ its complex roots. For all $i \in \mathbb{Z}$, we set

$$\alpha^i = (\beta_2^i, \dots, \beta_r^i, \beta_{r+1}^i, \dots, \beta_{r+s}^i).$$

We also put $\alpha^0 = 1 = (1, \dots, 1)$. Then, the Rauzy fractal associated to σ is the set $\mathcal{E}_{a_1,\dots a_d} \subset \mathbb{R}^{r-1} \times \mathbb{C}^s \approx \mathbb{R}^d$ defined by

$$\mathcal{E}_{a_1,\dots a_d} = \left\{ \sum_{i=d+1}^{+\infty} \varepsilon_i \alpha^i; (\varepsilon_i)_{i \geq d+1} \in \mathcal{D}^{\infty}_{a_1,\dots,a_d} \right\}.$$

The set $\mathcal{E}_{1,1} = X$ is the classical two-dimensional Rauzy fractal.
The structure of the boundary of Rauzy fractals has been first investigated by Ito and M. Kimura in [IK91]. They showed that the boundary of $\mathcal{E}_{1,1}$ is a Jordan curve generated by the Dekking method [D82] and they calculated its Hausdorff

dimension. Relating the boundary of $\mathcal{E}_{a_1,1}$ to the complex numbers having at least two expansions in base α, A. Messaoudi [M00, M05] constructed a finite automaton characterizing and generating this boundary. See also [ST10] for an other approach. As a consequence it permitted to parameterize the boundary of $\mathcal{E}_{a_1,1}$, to compute its Hausdorff dimension and to show it is a quasi circle.
In [T06], J. M. Thuswaldner studied the set $\mathcal{E}_{a_1,a_2}$. In particular, based on the self replicating tiling, he gave an explicit formula for the fractal dimension of the boundary of this set.
The purpose of this paper is to prove the following result.

Theorem 1. *The set $\mathcal{E}_{1,1,1} \subset \mathbb{R} \times \mathbb{C}$ has the following properties :*

(1) *There exists a finite automaton $\mathcal{A}$ with a unique initial state such that the following are equivalent :*
   (a) *z belongs to the boundary of $\mathcal{E}_{1,1,1}$;*
   (b) *there exist two infinite paths $(\epsilon_i)_{i\geq l}$ and $(\epsilon'_i)_{i\geq l}$ belonging to $\mathcal{D}^{\infty}_{1,1,1}$ such that $z = \sum_{i\geq l} \varepsilon_i \alpha^i$ and $(\varepsilon_i, \varepsilon'_i)_{i\geq l}$ is an infinite path in $\mathcal{A}$ beginning in the initial state;*

(2) *The set $\mathcal{E}_{1,1,1}$ tiles $\mathbb{R} \times \mathbb{C}$ and has exactly 18 neighboors and 6 of them intersect the central tile $\mathcal{E}_{1,1,1}$ in a point;*

(3) *The boundary of $\mathcal{E}_{1,1,1}$ is $\bigcup_{i=1}^{18} X_i$ where $X_i,\ i = 1, \ldots, 6$ are singletons, and for all $i \in [7, 18]$, there exist affine functions $f_{ij},\ j = 1, \ldots, m_i$ and $g_{ij},\ j = 1, \ldots, n_i$ from $\mathbb{R} \times \mathbb{C}$ to itself such that*

$$X_i = \bigcup_{j=1}^{m_i} f_{ij}(X_7) \bigcup \bigcup_{j=1}^{n_i} g_{ij}(X_8).$$

For a graphic representation of $\mathcal{E}_{1,1,1}$, see the Annexe section (the colored image is available at http://www.mathinfo.u-picardie.fr/fdurand/publications.html).

2. Notations, definitions and background

2.1. **β-expansions.** Let $\beta > 1$ be a real number. A β-representation of a non-negative real number x is an infinite sequence $(x_i)_{i\leq k},\ x_i \in \mathbb{Z}^+ = [0, +\infty[$, such that

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots + x_1\beta + x_0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \cdots .$$

where k is an integer. It is denoted by

$$x = x_k x_{k-1} \ldots x_1 x_0 . x_{-1} x_{-2} \ldots .$$

A particular β-representation, called the β-expansion, is computed by the "greedy algorithm" (see [P60]): denote by $\lfloor y \rfloor$ and $\{y\}$ respectively the integer part and the fractional part of a number y. There exists $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Let $x_k = \lfloor x/\beta^k \rfloor$ and $r_k = \{x/\beta^k\}$. Then for $i < k$, put $x_i = \lfloor \beta r_{i+1} \rfloor$ and $r_i = \{\beta r_{i+1}\}$. We get

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots$$

If $k < 0$ ($x < 1$), we put $x_0 = x_{-1} = \cdots = x_{k+1} = 0$. If an expansion ends by infinitely many zeros, it is said to be finite, and the ending zeros are omitted.
The digits x_i belong to the set $A = \{0, \cdots, \beta - 1\}$ if β is an integer, or to the set $A = \{0, \cdots, \lfloor \beta \rfloor\}$ if β is not an integer. The β-expansion of every positive real number x is the lexicographically greatest among all β-representations of x.

We denote by $\text{Fin}(\beta)$ the set of numbers which have finite greedy β-expansion. Let $N \in \mathbb{Z}$, we denote by $\text{Fin}_N(\beta)$ the set of numbers x such that in their β-expansion $(x_i)_{i\leq k}$, $x_i = 0$ for all $i < N$. We will sometimes denote a β-expansion $x_n \cdots x_k$, $n \geq k$ by $(x_i)_{k\leq i\leq n}$. We put

$$E_\beta = \{(x_i)_{i\geq k}; k \in \mathbb{Z},\ \forall n \geq k,\ (x_i)_{k\leq i\leq n} \text{ is a finite } \beta\text{-expansion}\}.$$

In the case where β is the dominant root of the polynomial $P_{a_1,\ldots a_d}$, it is known (see [FS92]) that $E_\beta = \mathcal{D}^\infty_{a_1,\ldots,a_d} = \{(\varepsilon_i)_{i\geq l}; l \in \mathbb{Z}, (\varepsilon_i)_{l\leq i\leq n} \in \mathcal{D}_{a_1,\ldots,a_d}, \forall n \geq l\}$. We will need the two following classical lemmas.

Lemma 2 ([P60]). *Let $x_n \cdots x_0$ and $y_m \cdots y_0$ be two β-expansions. Then, the following are equivalent*

- $\sum_{i=0}^n x_i\beta^i < \sum_{i=0}^m y_i\beta^i$,
- $x_n \cdots x_0 <_{lex} y_m \cdots y_0$,

where $<_{lex}$ is the lexicographical order.

Lemma 3 ([FS92]). *If $\beta = \beta_1$, then $\mathbb{Z}[\beta] \cap [0, +\infty[\subset \text{Fin}(\beta)$.*

2.2. **Boundary of $\mathcal{E}_{a_1,\ldots,a_d}$.** The coordinates of α have modulus strictly less than 1. Moreover, Lemma 3 and Theorem 2 of [A99] imply that 0 belongs to the interior of the central tile $\mathcal{E}_{a_1,\ldots,a_d}$. Hence, for all $z \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ there exists $k \in \mathbb{N}$ such that $\alpha^k z \in \mathcal{E}_{a_1,\ldots,a_d}$. Then, all $z \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ can be written as follows $z = \sum_{i=l}^\infty \varepsilon_i\alpha^i$, where $l \in \mathbb{Z}$ and $(\varepsilon_i)_{i\geq l} \in \mathcal{D}^\infty_{a_1,\ldots,a_d}$. The sequence $(\varepsilon_i)_{i\geq l}$ is called α-expansion of z. We should remark that these α-expansions are not unique : some z can have many different α-expansions. In [M05] it is proven that the points belonging to the boundary of $\mathcal{E}_{a_1,\ldots,a_d}$ have at least two different α-expansions. These points are characterized by the following proposition which is a straightforward consequence of a result due to W. Thurston [T90] (see also [M05]).

Proposition 4. *There exists a finite automaton B such that for all distinct elements of $\mathcal{D}^\infty_{a_1,\ldots,a_d}$, $(b_i)_{i\geq l}$ and $(c_i)_{i\geq l}$, the following are equivalent :*

- $\sum_{i=l}^\infty b_i\alpha^i = \sum_{i=l}^\infty c_i\alpha^i$
- $((b_i, c_i))_{i\geq l}$ *is recognizable by B (i.e an infinite path in B beginning in the initial state).*

The proof of this result does not give explicitly the states of the automaton. In [M98] is given an algorithm that gives these states for $\mathcal{E}_{1,1}$. In [M06], they were given for $\mathcal{E}_{a_1,1}$ where $a_1 \geq 2$.

3. Characterization of the boundary of $\mathcal{E}_{1,1,1}$

In the sequel we suppose $d = 3$ and $a_1 = a_2 = a_3 = 1$, and $P(x) = P_{1,1,1}(x) = x^4 - x^3 - x^2 - x - 1 = (x - \beta_1)(x - \beta_2)(x - \beta_3)(x - \overline{\beta_3})$ where $\beta_1, \beta_2, \beta_3$ are defined in Section 1. Approximations of these numbers are $\beta = \beta_1 = 1.9275\ldots$, $\beta_2 = -0.7748\ldots$ and $\beta_3 = -0.0763\cdots + i0.8147\ldots$. We recall that we defined for all $i \in \mathbb{Z}$, $\alpha^i = (\beta_2^i, \beta_3^i)$.
In this situation

$$\mathcal{D} = \mathcal{D}_{1,1,1} = \{(\varepsilon_i)_{l\le i\le n}; l,n\in\mathbb{Z}, \varepsilon_i\in\{0,1\}, \varepsilon_i\varepsilon_{i-1}\varepsilon_{i-2}\varepsilon_{i-3}\neq 1111, l\le i\le n\},$$
$$\mathcal{D}^\infty = \mathcal{D}^\infty_{1,1,1} = \{(\varepsilon_i)_{i\ge l}; l\in\mathbb{Z}, (\varepsilon_i)_{l\le i\le n}\in\mathcal{D}_{1,1,1}, n\ge l\} \text{ and}$$
$$\mathcal{E} = \mathcal{E}_{1,1,1} = \left\{\sum_{i=4}^{+\infty}\varepsilon_i\alpha^i; (\varepsilon_i)_{i\ge 4}\in\mathcal{D}^\infty\right\}$$
$$= \left\{\sum_{i=4}^{+\infty}\varepsilon_i\alpha^i; \varepsilon_i\in\{0,1\}, \varepsilon_i\varepsilon_{i-1}\varepsilon_{i-2}\varepsilon_{i-3}\neq 1111, i\ge 4\right\}.$$

An important and known result is:

Theorem 5. *The set $\mathcal{E}$ is compact, connected and generates a periodic tiling of $\mathbb{R}\times\mathbb{C}$ with group periods $G = \mathbb{Z}\alpha^0 + \mathbb{Z}\alpha + \mathbb{Z}\alpha^2$:*

$$\mathbb{R}\times\mathbb{C} = \bigcup_{p\in G}(\mathcal{E}+p),$$

and the intersection of the interior of $(\mathcal{E}+p)$ with $(\mathcal{E}+q)$ is empty whenever $p\neq q$, $p,q\in G$. Morever the boundary of $\mathcal{E}$ is of zero measure and is equal to the union of all $\mathcal{E}_p$, $p\in G$ where $\mathcal{E}_p = \mathcal{E}\cap(\mathcal{E}+p)$.

Proof. The proof can be deduced from [R82] (done in case of cubic Rauzy fractal), see also [CS01]. For clarity, we will give the proof in the Annexe section. □

3.1. **Definition of the automaton recognizing the points with at least two expansions.** In the sequel we proceed to the construction of the automaton $\mathcal{A}$ that characterizes the boundary of $\mathcal{E}$. This characterization will be proven in Section 3.2.

The set of states of the automaton $\mathcal{A}$ is

$$S = \left\{\pm\sum_{i=0}^{3}c_i\alpha^i; c_0c_1c_2c_3\neq 1111, c_i\in\{0,1\}, 0\le i\le 3\right\} \bigcup \left\{\pm(\alpha^{-1}+1+\alpha^2), \pm(\alpha^{-2}+\alpha^{-1}+\alpha), \pm(\alpha^{-3}+\alpha^{-2}+1+\alpha^3)\right\}.$$

Let s and t be two states. The set of edges is the set of $(s,(a,b),t)\in S\times\{0,1\}^2\times S$ satisfying $t = \frac{s}{\alpha} + (a-b)\alpha^3$. The set of initial states is $\{0\}$ and the set of states is S. A path (resp. infinite path) of $\mathcal{A}$ is a sequence $(a_n,b_n)_{k\le n\le l}$ (resp. $(a_n,b_n)_{n\ge k}$) such that there exists a sequence $(e_n)_{k\le n\le l+1}$ (resp. $(e_n)_{n\ge k}$) of elements of S for which $(e_n,(a_n,b_n),e_{n+1})$ belongs to S for all $n\in\{k,k+1,\dots,l+1\}$ (resp. $n\ge k$). We say it starts in the initial state when $e_k = 0$. The automaton is explicitly defined in the Annexe at the end of this paper.

Let us explain the behavior of this automaton. Let $\varepsilon = (\varepsilon_i)_{i\ge l}$ and $\varepsilon' = (\varepsilon'_i)_{i\ge l}$ belonging to $\mathcal{D}^\infty$, $x = \sum_{i=l}^{\infty}\varepsilon_i\alpha^i$ and $y = \sum_{i=l}^{\infty}\varepsilon'_i\alpha^i$. For all $k\ge l$ we set

$$A_k(\varepsilon,\varepsilon') = \alpha^{-k+3}\sum_{i=l}^{k}(\varepsilon_i-\varepsilon'_i)\alpha^i \tag{1}$$

In Subsection 3.2 we will prove that $x=y$ if and only if all the A_k, $k\ge l$, belong to S. But as, for all $k\ge l$, we have

$$(2) \qquad A_{k+1}(\varepsilon,\varepsilon') = \frac{A_k(\varepsilon,\varepsilon')}{\alpha} + (\varepsilon_{k+1} - \varepsilon'_{k+1})\alpha^3,$$

this means that $x = y$ if and only if

$$(0, (\varepsilon_l, \varepsilon'_l), A_l(\varepsilon,\varepsilon'))\left((A_k(\varepsilon,\varepsilon'), (\varepsilon_{k+1}, \varepsilon'_{k+1}), A_{k+1}(\varepsilon,\varepsilon'))\right)_{k\geq l}$$

is an infinite sequence of edges of S starting in the initial state. And, this is equivalent to say that $(\varepsilon_i, \varepsilon'_i)_{i\geq l}$ is an infinite path of $\mathcal{A}$ starting in the initial state. Let us give an example on how we can use this automaton to obtain information about the digits of x and y. Let s be the smallest integer such that $\varepsilon_s \neq \varepsilon'_s$. Hence $A_i(\varepsilon,\varepsilon') = 0$ for $i \in \{l, \cdots, s-1\}$. Suppose $\varepsilon_s > \varepsilon'_s$, that is $\varepsilon_s = 1$ and $\varepsilon'_s = 0$. Then, $A_s = \alpha^3$. ¿From (2) we deduce $A_{s+1}(\varepsilon,\varepsilon') = \alpha^2 + (\varepsilon_{s+1} - \varepsilon'_{s+1})\alpha^3$ which should belong to S. Hence $A_{s+1}(\varepsilon,\varepsilon') = \alpha^2 \in S$ if $\varepsilon_{s+1} = \varepsilon'_{s+1}$, and, $A_{s+1}(\varepsilon,\varepsilon') = \alpha^2 + \alpha^3 \in S$ if $(\varepsilon_{s+1}, \varepsilon'_{s+1}) = (1,0)$. Hence, $(\alpha^3, (1,0), \alpha^2 + \alpha^3)$, $(\alpha^3, (0,0), \alpha^2)$ and $(\alpha^3, (1,1), \alpha^2)$ are edges coming from the state α^3. Let us explain why $(\alpha^3, (0,1), \alpha^2 - \alpha^3)$ is not an edge, and hence why we cannot have $(\varepsilon_{s+1}, \varepsilon'_{s+1}) = (0,1)$. We should have that $\alpha^2 - \alpha^3 = -\alpha^{-1} - 1 - \alpha$ belongs to S. Then β should satisfy the same equality. Hence $\beta^{-1} + 1 + \beta$ should belong to

$$\left\{\sum_{i=0}^{3} c_i\beta^i; c_0c_1c_2c_3 \neq 1111, c_i \in \{0,1\}, 0 \leq i \leq 3\right\}$$
$$\bigcup \left\{(\beta^{-1} + 1 + \beta^2), (\beta^{-2} + \beta^{-1} + \beta), (\beta^{-3} + \beta^{-2} + 1 + \beta^3)\right\},$$

which is not possible by Lemma 2.

3.2. **Characterization of the points with at least two expansions.**

Lemma 6. *Let* $(\varepsilon_i)_{i\geq 0}, (\varepsilon'_i)_{i\geq 0} \in \mathcal{D}^\infty$. *Then,*

$$\left|\sum_{i=0}^{+\infty}(\varepsilon_i - \varepsilon'_i)\beta_2^i\right| \leq \frac{1}{1+\beta_2}, \quad \left|\sum_{i=0}^{+\infty}(\varepsilon_i - \varepsilon'_i)\beta_3^i\right| \leq \frac{C}{1-|\beta_3|^6}.$$

where $C = \max\left\{\left|\sum_{i=0}^{5}(c_i - d_i)\beta_3^i\right|; (c_i)_{0\leq i\leq 5} \in \mathcal{D}, (d_i)_{0\leq i\leq 5} \in \mathcal{D}\right\}$.

Proof. The second inequality is easy to establish. For the first inequality, as $-1 < \beta_2 < 0$, all sequences $(c_i)_{i\geq 0}$ which terms are 0 or 1 satisfy the following inequality :

$$\frac{\beta_2}{1-\beta_2^2} = \sum_{i=0}^{+\infty}\beta_2^{2i+1} \leq \sum_{i=0}^{+\infty}c_i\beta_2^i \leq \sum_{i=0}^{+\infty}\beta_2^{2i} = \frac{1}{1-\beta_2^2}.$$

This achieves the proof. □

For all $\varepsilon = (\varepsilon_i)_{i\geq l}$ and $\varepsilon' = (\varepsilon'_i)_{i\geq l}$ belonging to $\mathcal{D}^\infty$, we set

$$S(\varepsilon,\varepsilon') = \{A_k(\varepsilon,\varepsilon'); k \geq l\} = \left\{\alpha^{-k+3}\sum_{i=l}^{k}(\varepsilon_i - \varepsilon'_i)\alpha^i; k \geq l\right\}.$$

Proposition 7. *Let* $x = \sum_{i=l}^{\infty} \varepsilon_i \alpha^i$, $y = \sum_{i=l}^{\infty} \varepsilon'_i \alpha^i$, *where* $\varepsilon = (\varepsilon_i)_{i \geq l}$ *and* $\varepsilon' = (\varepsilon'_i)_{i \geq l}$ *belong to* $\mathcal{D}^\infty$. *Then,* $x = y$ *if and only if the set* $S(\varepsilon, \varepsilon')$ *is finite. Moreover*

$$S(\varepsilon, \varepsilon') \subset S = \left\{ \pm \sum_{i=0}^{3} c_i \alpha^i ; (c_i)_{0 \leq i \leq 3} \in \mathcal{D} \right\} \bigcup \left\{ \pm(\alpha^{-1} + 1 + \alpha^2), \pm(\alpha^{-2} + \alpha^{-1} + \alpha), \pm(\alpha^{-3} + \alpha^{-2} + 1 + \alpha^3) \right\}$$

and

$$S = \bigcup_{(\varepsilon, \varepsilon') \in \Delta} S(\varepsilon, \varepsilon'),$$

where $\Delta = \left\{ ((\varepsilon_i)_{i \geq l}, (\varepsilon'_i)_{i \geq l}) \in \mathcal{D}^\infty \times \mathcal{D}^\infty ; \sum_{i=l}^{\infty} \varepsilon_i \alpha^i = \sum_{i=l}^{\infty} \varepsilon'_i \alpha^i \right\}$.

Proof. It is easy to establish that if $S(\varepsilon, \varepsilon')$ is finite then $x = y$. Let us prove the reciprocal. Let $x = \sum_{i=l}^{\infty} \varepsilon_i \alpha^i = \sum_{i=l}^{\infty} \varepsilon'_i \alpha^i = y$ with $\varepsilon = (\varepsilon_i)_{i \geq l}$ and $\varepsilon' = (\varepsilon'_i)_{i \geq l}$ belonging to $\mathcal{D}^\infty$. Let us prove that $A_k = A_k(\varepsilon, \varepsilon')$ belongs to S for all $k \geq l$. As $x = y$, for all $k \geq l$, we have

$$A_k = \sum_{i=k+1}^{\infty} (\varepsilon'_i - \varepsilon_i) \alpha^{i-k+3} = \sum_{i=4}^{\infty} (\varepsilon'_{i+k-3} - \varepsilon_{i+k-3}) \alpha^i. \tag{3}$$

Let us fix $k \geq l$ and assume $A_k \neq 0$. ¿From (1), we deduce there exist $n, p, q, r \in \mathbb{Z}$ such that

$$A_k = n\alpha^3 + p\alpha^2 + q\alpha + r. \tag{4}$$

But $n\beta^3 + p\beta^2 + q\beta + r$ or $-(n\beta^3 + p\beta^2 + q\beta + r)$ belongs to $\mathbb{Z}[\beta] \cap \mathbb{R}^+$, which is contained in $\mathrm{Fin}(\beta)$ (see Lemma 3). We deduce there exists $(c_i)_{s \leq i \leq m} \in \mathcal{D}$ such that $c_m = 1$ and

$$n\beta^3 + p\beta^2 + q\beta + r = \pm \sum_{i=s}^{m} c_i \beta^i. \tag{5}$$

We suppose it is equal to $\sum_{i=s}^{m} c_i \beta^i$. The other case can be treated in the same way. As β, β_2 and β_3 are algebraically conjugate, from (1), (4) and (5) we have

$$\beta^{-k+3} \sum_{i=l}^{k} \varepsilon_i \beta^i = \beta^{-k+3} \sum_{i=l}^{k} \varepsilon'_i \beta^i + \sum_{i=s}^{m} c_i \beta^i. \tag{6}$$

¿From Lemma 2, $\beta^{-k+3} \sum_{i=l}^{k} \varepsilon_i \beta^i < \beta^4$, consequently $m \leq 3$. Setting $c_i = 0$ for $i > m$, we have

$$A_k = \sum_{i=s}^{3} c_i \alpha^i. \tag{7}$$

Remark that if $s \geq 0$ then A_k belongs to S. Hence we suppose $s \leq -1$.
Suppose $s = -1$ and $c_{-1} = 1$. Let us show that A_k is equal to $\alpha^{-1} + 1 + \alpha^2$ and consequently belongs to S. In order to do so, we show that the other cases are

not possible. Using Lemma 6 and (3), the first entry of (A_k), $(A_k)_1$, should satisfy $|(A_k)_1| \leq \beta_2^4(1+\beta_2)^{-1}$ which is less than $a = 1.6004$. This excludes the following points : $\alpha^{-1}+\alpha+\alpha^2+\alpha^3$, $\alpha^{-1}+\alpha+\alpha^3$, $\alpha^{-1}+\alpha$ and $\alpha^{-1}+\alpha^3$ because the absolute value of their first entries is greater than the value below it in the following array :

$\beta_2^{-1}+\beta_2+\beta_2^2+\beta_2^3$	$\beta_2^{-1}+\beta_2+\beta_2^3$	$\beta_2^{-1}+\beta_2$	$\beta_2^{-1}+\beta_2^3$
1.9	2.5	2.0	1.7

In the same way we should have $|(A_k)_2| \leq C|\beta_3|^4(1-|\beta_3|^6)^{-1}$ which is less than $b = 1,8120$. This excludes the following points : $\alpha^{-1}+1+\alpha^3$, $\alpha^{-1}+\alpha^2+\alpha^3$ and $\alpha^{-1}+1+\alpha^2+\alpha^3$, because the absolute value of their second entries is greater than the value below it in the following array :

$\beta_3^{-1}+1+\beta_3^3$	$\beta_3^{-1}+\beta_3^2+\beta_3^3$	$\beta_3^{-1}+1+\beta_3^2+\beta_3^3$
2.0	1.9	1.9

In order to exclude the other cases, except $\frac{1}{\alpha}+1+\alpha^2$, we used (2) to compute A_{k+i}, $i \geq 1$. Let us explain the strategy. Suppose neither $(A_k)_1$ nor $(A_k)_2$ is greater than respectively a and b. Then, we compute A_{k+1} using (2). We have three possible values : $\frac{A_k}{\alpha}$, $\frac{A_k}{\alpha}+\alpha^3$ and $\frac{A_k}{\alpha}-\alpha^3$. To check that A_k does not belong to S, it suffices to show that for all these values, either the first entry or the second is respectively greater than a or b. If it is not the case, for each value that does not satisfy this (both entries are less than, respectively, a and b) we apply again this strategy. Applying this just once we show that $\frac{1}{\alpha}+1+\alpha+\alpha^3$ does not belong to S. The values of the relevant entries are in the following array and should be read in the following way : The value (1.9 for example) below a relevant entry of A_{k+1} (resp. $\frac{1}{\beta_2^2}+\frac{1}{\beta_2}+1+\beta_2^2$) is greater than the absolute value of the relevant entry : $|\frac{1}{\beta_2^2}+\frac{1}{\beta_2}+1+\beta_2^2| > 1.9$.

A_k	$\frac{1}{\alpha}+1+\alpha+\alpha^3$		
A_{k+1}	$\frac{1}{\beta_2^2}+\frac{1}{\beta_2}+1+\beta_2^2$	$\frac{1}{\beta_3^2}+\frac{1}{\beta_3}+1+\beta_3^2+\beta_3^3$	$\frac{1}{\beta_2^2}+\frac{1}{\beta_2}+1+\beta_2^2-\beta_2^3$
	1.9	1.9	2.4

For the following case, $\frac{1}{\alpha}+1$, we need to apply the strategy twice because for $A_{k+1} = \frac{1}{\beta_3^2}+\frac{1}{\beta_3}-\beta_3^3$ both entries are respectively less than a and b.

A_k	$\frac{1}{\alpha}+1$		
A_{k+1}	$\frac{1}{\beta_3^2}+\frac{1}{\beta_3}$	$\frac{1}{\beta_3^2}+\frac{1}{\beta_3}+\beta_3^3$	
	1.83	2.0	
A_{k+2}	$\frac{1}{\beta_3^3}+\frac{1}{\beta_3^2}-\beta_3^2$	$\frac{1}{\beta_3^3}+\frac{1}{\beta_3^2}-\beta_3^2+\beta_3^3$	$\frac{1}{\beta_3^3}+\frac{1}{\beta_3^2}-\beta_3^2-\beta_3^3$
	2.1	1.63	2.7

For the case $A_k = \frac{1}{\alpha}+1+\alpha$ we need two steps because at the first one both $\frac{1}{\beta_2^2}+\frac{1}{\beta_2}+1$ and $\frac{1}{\beta_2^2}+\frac{1}{\beta_2}+1+\beta_3^3$ have entries less than, respectively, a and b.

A_k	$\frac{1}{\alpha}+1+\alpha$		
A_{k+1}			$\frac{1}{\beta_2^2}+\frac{1}{\beta_2}+1-\beta_2^3$
			1.84
A_{k+2}	$\frac{1}{\beta_2^3}+\frac{1}{\beta_2^2}+\frac{1}{\beta_2}$	$\frac{1}{\beta_2^3}+\frac{1}{\beta_2^2}+\frac{1}{\beta_2}+\beta_2^3$	$\frac{1}{\beta_3^3}+\frac{1}{\beta_3^2}+\frac{1}{\beta_3}-\beta_3^3$
	1.77	2.23	1.818
	$\frac{1}{\beta_3^3}+\frac{1}{\beta_3^2}+\frac{1}{\beta_3}+\beta_3^2$	$\frac{1}{\beta_2^3}+\frac{1}{\beta_2^2}+\frac{1}{\beta_2}+\beta_2^2+\beta_2^3$	$\frac{1}{\beta_3^3}+\frac{1}{\beta_3^2}+\frac{1}{\beta_3}+\beta_3^2-\beta_3^3$
	1.86	1.63	2.24

For the three following cases, $\frac{1}{\alpha}+\alpha^2$, $\frac{1}{\alpha}$ and $\frac{1}{\alpha}+\alpha+\alpha^2$, we need three steps.

A_k	$\frac{1}{\alpha}+\alpha^2$		
A_{k+1}	$\frac{1}{\beta_3^2}+\beta_3$		$\frac{1}{\beta_3^2}+\beta_3-\beta_3^3$
	1.89		2.34
A_{k+2}	$\frac{1}{\beta_3^3}+1+\beta_3^2$		$\frac{1}{\beta_3^3}+1+\beta_3^2-\beta_3^3$
	1.83		2.26
A_{k+3}	$\frac{1}{\beta_3^4}+\frac{1}{\beta_3}+\beta_3+\beta_3^2$	$\frac{1}{\beta_3^4}+\frac{1}{\beta_3}+\beta_3+\beta_3^2+\beta_3^3$	$\frac{1}{\beta_2^4}+\frac{1}{\beta_2}+\beta_2+\beta_2^2-\beta_2^3$
	1.818	2.32	1.77

A_k	$\frac{1}{\alpha}$		
A_{k+1}	$\frac{1}{\beta_2^2}$		$\frac{1}{\beta_2^2}-\beta_2^3$
	1.66		2.13
A_{k+2}		$\frac{1}{\beta_2^3}+\beta_2^2+\beta_2^3$	$\frac{1}{\beta_3^3}+\beta_3^2-\beta_3^3$
		2.01	2.17
A_{k+3}	$\frac{1}{\beta_3^4}+\beta_3$	$\frac{1}{\beta_3^4}+\beta_3+\beta_3^3$	$\frac{1}{\beta_3^4}+\beta_3-\beta_3^3$
	2.00	2.21	1.92

A_k	$\frac{1}{\alpha}+\alpha+\alpha^2$		
A_{k+1}	$\frac{1}{\beta_2^2}+1+\beta_2$		$\frac{1}{\beta_2^2}+1+\beta_2-\beta_2^3$
	1.89		2.35
A_{k+2}	$\frac{1}{\beta_2^3}+\frac{1}{\beta_2}+1+\beta_2^2$	$\frac{1}{\beta_2^3}+\frac{1}{\beta_2}+1+\beta_2^2+\beta_2^3$	
	1.84	2.30	
A_{k+3}	$\frac{1}{\beta_2^4}+\frac{1}{\beta_2^2}+\frac{1}{\beta_2}+\beta_2$ $-\beta_2^2$	$\frac{1}{\beta_3^4}+\frac{1}{\beta_3^2}+\frac{1}{\beta_3}+\beta_3$ $-\beta_3^2+\beta_3^3$	$\frac{1}{\beta_2^4}+\frac{1}{\beta_2^2}+\frac{1}{\beta_2}+\beta_2$ $-\beta_3^2-\beta_3^3$
	1.77	1.818	2.23

Hence the only possible A_k (with $c_{-1}=1$) is $\frac{1}{\alpha}+1+\alpha^2$.
Suppose now $s\leq -2$ and $c_s=1$. It is useful for the sequel to remark that $U=(u_i)_{i\geq s}=(c_s,c_{s+1},\dots,c_2,c_3,\varepsilon_{k+1},\varepsilon_{k+2},\varepsilon_{k+3},\dots)$ belongs to $\mathcal{D}^\infty$. Indeed, if $c_3=0$, it is clear. If $c_3=1$ and $c_2=0$ then by (6), $\varepsilon_k=1$. Hence $\varepsilon_{k+1}\varepsilon_{k+2}\varepsilon_{k+3}\neq 111$ and U belongs to $\mathcal{D}^\infty$. The other cases can be treated in the same way.
Using (3) and (7) we obtain

$$u=\sum_{i=s}^{3}c_i\alpha^i+\sum_{i=4}^{\infty}\varepsilon_{i+k-3}\alpha^i=\sum_{i=4}^{\infty}\varepsilon'_{i+k-3}\alpha^i=v.$$

We set $V = (\varepsilon'_{i+k-3})_{i\geq s}$ where $\varepsilon'_{i+k-3} = 0$ when $s \leq i \leq 3$. Then (U, V) belongs to Δ, $A_{s+4}(U, V) = c_s\alpha^{-1} + c_{s+1} + c_{s+2}\alpha + c_{s+3}\alpha^2 + c_{s+4}\alpha^3$ and $A_3(U, V) = \sum_{i=s}^{3} c_i\alpha^i$. Doing what we did for x and y to u and v we obtain that $A_{s+4}(U, V) = \alpha^{-1} + 1 + \alpha^2$. Let us show that for all $n \geq s + 5$, $A_n(U, V)$ belongs to

$$\mathcal{C} = \left\{\pm(\alpha^{-2} + \alpha^{-1} + \alpha), \pm(\alpha^{-3} + \alpha^{-2} + 1 + \alpha^3)\right\}.$$

This will imply that A_k belongs to S for all $k \geq l$. We have that $A_{s+5}(U, V)$ belongs to

$$\left\{\frac{1}{\alpha^2} + \frac{1}{\alpha} + \alpha, \frac{1}{\alpha^2} + \frac{1}{\alpha} + \alpha + \alpha^3, \frac{1}{\alpha^2} + \frac{1}{\alpha} + \alpha - \alpha^3\right\}.$$

The third one can be excluded because $\frac{1}{\beta_3^2} + \frac{1}{\beta_3} + \beta_3 - \beta_3^3 \geq 1.85$. We proceed as before to exclude the second element :

A_k	$\frac{1}{\alpha^2} + \frac{1}{\alpha} + \alpha + \alpha^3$		
A_{k+1}	$\frac{1}{\beta_3^3} + \frac{1}{\beta_3^2} + 1 + \beta_3^2$		$\frac{1}{\beta_3^3} + \frac{1}{\beta_3^2} + 1 + \beta_3^2 - \beta_3^3$
	2.00		2.55
A_{k+2}	$\frac{1}{\beta_3^4} + \frac{1}{\beta_3^3} + \frac{1}{\beta_3} + \beta_3$	$\frac{1}{\beta_3^4} + \frac{1}{\beta_3^3} + \frac{1}{\beta_3} + \beta_3 + \beta_3^3$	$\frac{1}{\beta_3^4} + \frac{1}{\beta_3^3} + \frac{1}{\beta_3} + \beta_3 - \beta_3^3$
	2.45	2.54	2.47

Consequently, $A_{s+5}(U, V) = \frac{1}{\alpha^2} + \frac{1}{\alpha} + \alpha$. We deduce $A_{s+6}(U, V) = \frac{1}{\alpha^3} + \frac{1}{\alpha^2} + 1 + \alpha^3$ because $\frac{1}{\beta_3^3} + \frac{1}{\beta_3^2} + 1 > 2.03$ and $\frac{1}{\beta_3^3} + \frac{1}{\beta_3^2} + 1 - \beta_3^3 > 2.56$. Once again, $A_{s+7}(U, V) = \frac{1}{\alpha^4} + \frac{1}{\alpha^3} + \frac{1}{\alpha} + \alpha^2 - \alpha^3$ because $\frac{1}{\beta_3^4} + \frac{1}{\beta_3^3} + \frac{1}{\beta_3} + \beta_3^2 > 1.85$ and $\frac{1}{\beta_3^4} + \frac{1}{\beta_3^3} + \frac{1}{\beta_3} + \beta_3^2 + \beta_3^3 > 2.16$. But an easy computation leads to $\frac{1}{\alpha^4} + \frac{1}{\alpha^3} + \frac{1}{\alpha} + \alpha^2 - \alpha^3 = -(\frac{1}{\alpha^2} + \frac{1}{\alpha} + \alpha) = -A_{s+5}(U, V)$. Then continuing in the same way we can check $A_n(U, V) = -A_{n+2}(U, V)$ and $A_n \in \mathcal{C}$ for $n \geq s + 5$. As $3 \geq s + 5$, we obtain that $A_k(\varepsilon, \varepsilon') = A_3(U, V)$ belongs to $\mathcal{C}$. Thus $S(\varepsilon, \varepsilon')$ is included in S.

To complete the proof we should show that each element of S belongs to $\Gamma = \cup_{(\varepsilon,\varepsilon')\in\Delta} S(\varepsilon, \varepsilon')$.

Remark that if A_k belongs to Γ then $-A_k$ also belongs to Γ. Consequently it is sufficient to consider the cases where $A_k = \sum_{i=0}^{3} c_i\alpha^i$ with $(c_i)_{0\leq i\leq 3} \in \mathcal{D}$ or $A_k = \alpha^{-1} + 1 + \alpha^2$, $\alpha^{-2} + \alpha^{-1} + \alpha$ or $\alpha^{-3} + \alpha^{-2} + 1 + \alpha^3$.

Notice that we have

$$\begin{aligned}
-\alpha^3 &= \sum_{i=1}^{+\infty}(\alpha^{4i} + \alpha^{4i+1} + \alpha^{4i+2}) = 1 + \alpha + \alpha^2 + \sum_{i=1}^{+\infty}(\alpha^{4i+1} + \alpha^{4i+2} + \alpha^{4i+3}) \\
&= \alpha + \alpha^2 + \alpha^4 + \sum_{i=1}^{+\infty}(\alpha^{4i+2} + \alpha^{4i+3} + \alpha^{4i+4}) \\
&= \alpha^2 + \alpha^4 + \alpha^5 + \sum_{i=1}^{+\infty}(\alpha^{4i+3} + \alpha^{4i+4} + \alpha^{4i+5}).
\end{aligned}$$

Hence, $1 + \alpha + \alpha^2$, $\alpha + \alpha^2$ and α^2 belong to Γ. Multiplying by α we deduce $\alpha + \alpha^2 + \alpha^3$, $\alpha^2 + \alpha^3$ and α^3 belong to Γ. Now subtracting $-\alpha^2$ we obtain $1 + \alpha$ and α belong to Γ. We have that 1 belongs to Γ because $\sum_{i=1}^{+\infty}\alpha^{4i} = 1 + \sum_{i=1}^{+\infty}\alpha^{4i+1}$. Now, $1 + \alpha^2$ belongs to Γ because

$$\sum_{i=2}^{\infty}\alpha^{2i} = 1+\alpha^2+\sum_{i=2}^{\infty}\alpha^{2i+1}.$$

Multiplying by α we deduce $\alpha+\alpha^3$ belongs to Γ. Because

$$\alpha^{-3}+\alpha^{-2}+1+\alpha^3+\sum_{i=1}^{\infty}(\alpha^{4i+2}+\alpha^{4i+3}) = \sum_{i=1}^{\infty}(\alpha^{4i}+\alpha^{4i+1}), \tag{8}$$

we obtain that $\alpha^{-3}+\alpha^{-2}+1+\alpha^3$ belongs to Γ. Multiplying (8) by, respectively, α and α^2 we obtain, respectively, that $\alpha^{-2}+\alpha^{-1}+\alpha$ and $\alpha^{-1}+1+\alpha^2$ belong to Γ. From

$$\alpha^4+\sum_{i=1}^{\infty}\alpha^{4i+3} = 1+\alpha^3+\sum_{i=1}^{\infty}\alpha^{4i+1} = 1+\alpha+\alpha^3+\sum_{i=1}^{\infty}\alpha^{4i+2}$$

it is clear $1+\alpha^3$ and $1+\alpha+\alpha^3$ belong to Γ.
The equality

$$\alpha^4+\sum_{i=1}^{\infty}\alpha^{4i+2} = 1+\alpha^2+\alpha^3+\sum_{i=1}^{\infty}\alpha^{4i+1}$$

implies that $1+\alpha^2+\alpha^3$ belongs to Γ and achieves the proof. □

Proposition 8. *Let $\mathcal{A}$ be the automaton defined in Subsection 3.1. Then, for all $(\varepsilon_i)_{i\geq l}$ and $(\varepsilon'_i)_{i\geq l}$ belonging to $\mathcal{D}^\infty$ the following assertions are equivalent :*

- $\sum_{i\geq l}\varepsilon_i\alpha^i = \sum_{i\geq l}\varepsilon'_i\alpha^i$;
- $(\varepsilon_i,\varepsilon'_i)_{i\geq l}$ *is an infinite path in $\mathcal{A}$ beginning in the initial state.*

Proof. Let $x=\sum_{i\geq l}\varepsilon_i\alpha^i$ and $y=\sum_{i\geq l}\varepsilon'_i\alpha^i$. By Proposition 7 and the definition of the automaton (see subsection 3.1), we deduce that $x=y$ if and only if

$$(0,(\varepsilon_l,\varepsilon'_l),A_l(\varepsilon,\varepsilon'))\left((A_k(\varepsilon,\varepsilon'),(\varepsilon_{k+1},\varepsilon'_{k+1}),A_{k+1}(\varepsilon,\varepsilon'))\right)_{k\geq l}$$

is an infinite sequence of edges of S starting in the initial state. And, this is equivalent to say that $(\varepsilon_i,\varepsilon'_i)_{i\geq l}$ is an infinite path of $\mathcal{A}$ starting in the initial state. □

This proposition proves the first part of Theorem 1.

Corollary 9. *Let $(\varepsilon_i)_{i\geq l}$ be an element of $\mathcal{D}^\infty$ and $(\varepsilon'_i)_{l\leq i\leq m}$ an element of $\mathcal{D}$ with $l,m\in\mathbb{Z}$ such that $\sum_{i=l}^{+\infty}\varepsilon_i\alpha^i=\sum_{i=l}^{m}\varepsilon'_i\alpha^i$. Then $\varepsilon'_i=0$ for all $i>m$ and $\varepsilon_i=\varepsilon'_i$ for all $l\leq i\leq m$.*

3.3. **Neighboors of $\mathcal{E}$.** Here we prove that the set $\mathcal{E}$ has 18 neighboors where 6 of them have an intersection with $\mathcal{E}$ reduced to a singleton, and that the boundary can be generated by just 2 subregions.

Lemma 10. *Let $(\varepsilon_i)_{i\geq 4}$ and $(\varepsilon'_i)_{i\geq l}$ be two elements of $\mathcal{D}^\infty$ such that $\sum_{i=4}^{\infty}\varepsilon_i\alpha^i=\sum_{i=l}^{\infty}\varepsilon'_i\alpha^i$, where $l<4$ and $\varepsilon'_l=1$, then $\varepsilon'_l\alpha^l+\varepsilon'_{l+1}\alpha^{l+1}\cdots+\varepsilon'_3\alpha^3$ belongs to S. In particular $l\geq -3$ and*

$$\varepsilon'_l\alpha^l + \cdots + \varepsilon'_3\alpha^3 = \alpha^{-3} + \alpha^{-2} + 1 + \alpha^3 \qquad \textit{if } l = -3,$$
$$\varepsilon'_l\alpha^l + \cdots + \varepsilon'_3\alpha^3 = \alpha^{-2} + \alpha^{-1} + \alpha \qquad \textit{if } l = -2,$$
$$\varepsilon'_l\alpha^l + \cdots + \varepsilon'_3\alpha^3 = \alpha^{-1} + 1 + \alpha^2 \qquad \textit{if } l = -1.$$

Proof. Let $(\varepsilon_i)_{i\geq 4}$ and $(\varepsilon'_i)_{i\geq l}$ be two elements of $\mathcal{D}^\infty$ such that $\sum_{i=4}^{\infty}\varepsilon_i\alpha^i = \sum_{i=l}^{\infty}\varepsilon'_i\alpha^i$ where $l < 4$ and $\varepsilon'_l = 1$. ¿From Proposition 7, for all $l \leq i \leq 3$, $\varepsilon'_l\alpha^i + \varepsilon'_{l+1}\alpha^{i+1}\cdots + \varepsilon'_{l-i+3}\alpha^3$ belongs to S. In particular, for $i = l$, we obtain the result. □

Lemma 11. *Let $u \in S$. Then, there exist $(\varepsilon_i)_{i\geq 4}$ and $(\varepsilon'_i)_{i\geq 4}$ belonging to $\mathcal{D}^\infty$ such that $\sum_{i=4}^{\infty}\varepsilon_i\alpha^i = u + \sum_{i=4}^{\infty}\varepsilon'_i\alpha^i$.*

Proof. This comes from Proposition 7 and the identity (3). □

In our context, Lemma 2 in [M05] can be formulated in the following way.

Lemma 12. *Let $x \in \mathbb{R}\times\mathbb{C}$, then x belongs to the boundary of $\mathcal{E}$ if and only if there exists $l \leq 3$ such that $x = \sum_{i=4}^{+\infty}\epsilon_i\alpha^i = \sum_{i=l}^{+\infty}\epsilon'_i\alpha^i$, where $(\epsilon_i)_{i\geq 4}$ and $(\epsilon'_i)_{i\geq l}$ belong to $\mathcal{D}^\infty$, and, $\epsilon'_l \neq 0$.*

Theorem 13. *The boundary of $\mathcal{E}$ is the union of the 18 non empty regions $\mathcal{E}(u)$, $u \in \{a, -a; a \in A\}$, whose pairwise intersections have measure zero, where*

$$\mathcal{E}(u) = \mathcal{E} \cap (\mathcal{E} + u) \textit{ and}$$

$$A = \{1,\ 1+\alpha,\ 1+\alpha^2,\ 1+\alpha+\alpha^2,\ \alpha^{-3}+\alpha^{-2}+1+\alpha^3 = 1+2\alpha+\alpha^2,$$
$$\alpha,\ \alpha+\alpha^2,\ \alpha^2,\ \alpha^{-2}+\alpha^{-1}+\alpha = -1+\alpha^2\}.$$

Proof. Let u be an element of A, then u is a state of the automaton $\mathcal{A}$. ¿From Lemma 11, there exist $(\varepsilon_i)_{i\geq 4}$ and $(\varepsilon'_i)_{i\geq 4}$ belonging to $\mathcal{D}^\infty$ such that $\sum_{i=4}^{\infty}\varepsilon_i\alpha^i = u + \sum_{i=4}^{\infty}\varepsilon'_i\alpha^i$. Thus, from Theorem 5, $\mathcal{E}\cap(\mathcal{E}+u)$ is not empty and with measure zero. It will be useful to check that

$$\alpha^{-3}+\alpha^{-2}+1+\alpha^3 = 1+2\alpha+\alpha^2 \text{ and } \alpha^{-2}+\alpha^{-1}+\alpha = -1+\alpha^2.$$

Consequently, Theorem 5 implies $\bigcup_{u\in A}\mathcal{E}(u)\cup\mathcal{E}(-u)$ is contained in the boundary of $\mathcal{E}$.

Now, let z be an element of the boundary of $\mathcal{E}$, then by Lemma 12 there exist two elements of $\mathcal{D}^\infty$, $(\varepsilon_i)_{i\geq 4}$ and $(\varepsilon'_i)_{i\geq l}$, $l \in \mathbb{Z}$, $l < 4$ such that $z = \sum_{i=4}^{+\infty}\varepsilon_i\alpha^i = \sum_{i=l}^{+\infty}\varepsilon'_i\alpha^i$. We can suppose $\varepsilon_l = 1$. Let us consider the following four cases.

Suppose $l = -3$. ¿From Lemma 10, we deduce that $z \in \mathcal{E}(\alpha^{-3}+\alpha^{-2}+1+\alpha^3)$.

Suppose $l = -2$. ¿From Lemma 10, we deduce that $z \in \mathcal{E}(\alpha^{-2}+\alpha^{-1}+\alpha)$.

Suppose $l = -1$. ¿From Lemma 10, we deduce that $z = \sum_{i=4}^{+\infty}\varepsilon_i\alpha^i = \alpha^{-1}+1+\alpha^2+\sum_{i=4}^{+\infty}\varepsilon'_i\alpha^i$. Proposition 8 implies that $t = (0,1)(0,1)(0,0)(0,1)(0,0)(\varepsilon_4,\varepsilon'_4)\ldots$ is an infinite path of the automaton $\mathcal{A}$ starting at the initial state. Using the automaton, we see that $t = (0,1)(0,1),(0,0)(0,1)(0,0)awww\ldots$, where $a = (1,1)$ or $(0,0)$ and $w = (0,1)(1,0)(1,0)(0,1)$. Consequently, $z = \alpha^{-1}+1+\alpha^2+\alpha^4+\alpha^5+\sum_{i=2}^{\infty}(\alpha^{4i}+\alpha^{4i+1})$ or $z = \alpha^{-1}+1+\alpha^2+\alpha^5+\sum_{i=2}^{\infty}(\alpha^{4i}+\alpha^{4i+1})$. Thus,

$z=-\alpha-\alpha^2+\alpha^6+\sum_{i=2}^{\infty}(\alpha^{4i}+\alpha^{4i+1})$ or $z=-1-2\alpha-\alpha^2+\sum_{i=1}^{\infty}(\alpha^{4i}+\alpha^{4i+1})$, and, $z\in\mathcal{E}(-\alpha-\alpha^2)\cup\mathcal{E}(-1-2\alpha-\alpha^2)$.

Suppose $l\geq 0$. Then $z=\sum_{i=4}^{+\infty}\varepsilon_i\alpha^i=\varepsilon'_0+\varepsilon'_1\alpha+\varepsilon'_2\alpha^2+\varepsilon'_3\alpha^3+\sum_{i=4}^{+\infty}\varepsilon'_i\alpha^i$.
If $\varepsilon'_3=0$, then $z\in\mathcal{E}(u)$ where $u=\varepsilon'_0+\varepsilon'_1\alpha+\varepsilon'_2\alpha^2$.
If $\varepsilon'_3=1$ and $\varepsilon'_4=0$, then $z=(\varepsilon'_0-1)+(\varepsilon'_1-1)\alpha+(\varepsilon'_2-1)\alpha^2+\alpha^4+\sum_{i=5}^{+\infty}\varepsilon'_i\alpha^i\in\mathcal{E}(u)$ where $u=(\varepsilon'_0-1)+(\varepsilon'_1-1)\alpha+(\varepsilon'_2-1)\alpha^2$.
Now suppose $\varepsilon'_3=\varepsilon'_4=1$ and $\varepsilon'_5=0$. Then $\varepsilon'_1=0$ or $\varepsilon'_2=0$, and, $z=\varepsilon'_0+(\varepsilon'_1-1)\alpha+(\varepsilon'_2-1)\alpha^2+\alpha^5+\sum_{i=6}^{+\infty}\varepsilon'_i\alpha^i$. Hence :

- If $\varepsilon'_0=0$, then $z\in\mathcal{E}(u)$ where $u=(\varepsilon'_1-1)\alpha+(\varepsilon'_2-1)\alpha^2$.
- If $\varepsilon'_0=1$, then $t=(0,1)(0,\varepsilon'_1)(0,\varepsilon'_2)(0,1)(\varepsilon_4,1)(\varepsilon_5,0)\ldots$ is an infinite path in the automaton beginning in the initial state. This implies that $t=(0,1)(0,1)(0,0)(0,1)(1,1)(0,0)ww\ldots$ where $w=(0,1)(1,0)(1,0)(0,1)$. Hence $z=1+\alpha+\alpha^3+\alpha^4+\alpha^6+\sum_{i=2}^{\infty}(\alpha^{4i+1}+\alpha^{4i+2})$. Thus $z+\alpha^{-2}+\alpha^{-1}+\alpha=\alpha^5+\alpha^6+\sum_{i=2}^{\infty}(\alpha^{4i+1}+\alpha^{4i+2})$ and z belongs to $\mathcal{E}(-\alpha^{-2}-\alpha^{-1}-\alpha)$.

If $\varepsilon'_3=\varepsilon'_4=\varepsilon'_5=1$, then $\varepsilon'_2=\varepsilon'_6=0$. Hence $z=\varepsilon'_0+\varepsilon'_1\alpha-\alpha^2+\alpha^6+\sum_{i=7}^{+\infty}\varepsilon'_i\alpha^i$.

- If $\varepsilon'_0=\varepsilon'_1=0$, then $z\in\mathcal{E}(-\alpha^2)$.
- When $\varepsilon'_0+\varepsilon'_1=1$, there is no infinite path in the automaton starting in the initial state and beginning with $(0,\varepsilon'_0)(0,\varepsilon'_1)(0,0)(0,1)(\varepsilon_4,1)(\varepsilon_5,1)$.
- Hence it remains to consider the case : $\varepsilon'_0+\varepsilon'_1=2$. But it is easy to check that this implies $\varepsilon'_6=1$ which is not possible.

This ends the proof. □

This theorem, together with the remark before the acknowledgements proves the second part of Theorem 1.
Using the automaton given in the Annexe, we deduce the following result which is the third and last part of Theorem 1.

Theorem 14. *Let $X=\mathcal{E}(1+\alpha+\alpha^2)$ and $Y=\mathcal{E}(1+\alpha)$. Then,*

a) $\mathcal{E}(1)=1+\alpha X$,
b) $\mathcal{E}(\alpha^2)=-\frac{1}{\alpha}-1-\alpha+\frac{X}{\alpha}$,
c) $\mathcal{E}(1+\alpha^2)=\left\{\frac{\alpha^4}{1-\alpha^2}\right\}$,
d) $\mathcal{E}(\alpha^{-2}+\alpha^{-1}+\alpha)=\left\{\frac{\alpha^5+\alpha^6}{1-\alpha^4}\right\}$,
e)$\mathcal{E}(\alpha^{-3}+\alpha^{-2}+1+\alpha^3)=\left\{\frac{\alpha^4+\alpha^5}{1-\alpha^4}\right\}$
f) $\mathcal{E}(\alpha)=f_0(X)\cup f_1(X)\cup f_1(Y)$

g) $\mathcal{E}(\alpha+\alpha^2)=g_0(X)\cup g_1(X)\cup g_1(Y)\cup g_2(Y)\cup g_3(Y)$, *where*

$f_0(z)=\alpha+\alpha^2z$, $f_1(z)=\alpha+\alpha^4+\alpha^2z$, $g_0(z)=\alpha^5+\alpha^4z$,
$g_1(z)=\alpha^5+\alpha^6+\alpha^4z$, $g_2(z)=\alpha z$, $g_3(z)=\alpha^4+\alpha z$,

h) $X=\bigcup_{i=0}^{4}h_i(X)\cup h_1(Y)\cup h_3(Y)$ *and i)* $Y=\bigcup_{i=5}^{11}h_i(Y)\cup\bigcup_{i=12}^{17}h_i(X)$, *where*

$$\begin{array}{ll}
h_0(z)=\alpha^4+\alpha^4 z, & h_1(z)=\alpha^4+\alpha^6+\alpha^4 z\\
h_2(z)=\alpha^4+\alpha^5+\alpha^4 z, & h_3(z)=\alpha^4+\alpha^5+\alpha^6+\alpha^4 z,\\
h_4(z)=1+\alpha+\alpha^2+\alpha^7+\alpha^5 z, & h_5(z)=h_2(z)\\
h_6(z)=\alpha^4+\alpha^7+\alpha^4 z, & h_7(z)=\alpha^4+\alpha^8+\alpha^9+\alpha^7 z,\\
h_8(z)=h_0(z), & h_9(z)=\alpha^4+\alpha^5+\alpha^7+\alpha^4 z,\\
h_{10}(z)=\alpha^4+\alpha^5+\alpha^8+\alpha^9+\alpha^7 z, & h_{11}(z)=1+\alpha+\alpha^6+\alpha^7+\alpha^5 z\\
h_{12}(z)=\alpha^4+\alpha^8+\alpha^7 z, & h_{13}(z)=h_7(z),\\
h_{14}(z)=\alpha^4+\alpha^5+\alpha^8+\alpha^7 z, & h_{15}(z)=h_{10}(z)\\
h_{16}(z)=1+\alpha+\alpha^6+\alpha^5 z, & h_{17}(z)=h_{11}(z),
\end{array}$$

Proof. a) The set $1+\alpha X$ is clearly included in $1+\alpha\mathcal{E}$. Moreover it is easy to check that $1+\alpha X$ is a subset of $\alpha^4+\alpha\mathcal{E}$ which is included in $\mathcal{E}$. Hence $1+\alpha X\subset\mathcal{E}(1)$. On the other hand, let $z\in\mathcal{E}(1)$. Then, there exist $(\varepsilon_i)_{i\geq 4}$ and $(\varepsilon'_i)_{i\geq 4}$ in $\mathcal{D}^\infty$ such that $z=1+\sum_{i\geq 4}\varepsilon_i\alpha^i=\sum_{i\geq 4}\varepsilon'_i\alpha^i$. ¿From Proposition 8, the finite path $(1,0)(0,0)(0,0)(0,0)(\varepsilon_4,\varepsilon'_4)$ $(\varepsilon_5,\varepsilon'_5)$ is a finite path in the automaton $\mathcal{A}$ starting at the initial state. Following this path in the automaton we deduce $(\varepsilon_4,\varepsilon'_4)=(0,1)$ and $(\varepsilon_5,\varepsilon'_5)=(1,0)$. It gives $z=1+\alpha^5+\alpha^2 w=\alpha^4+\alpha^2 w'$ where $w,w'\in\mathcal{E}$. Consequently $\mathcal{E}(1)\subset(1+\alpha\mathcal{E})\cap(\alpha^4+\alpha\mathcal{E})=1+\alpha(\mathcal{E}\cap(1+\alpha+\alpha^2+\mathcal{E}))=1+\alpha X$.
b) We have $1+\alpha+\alpha^2+\alpha\mathcal{E}(\alpha^2)=(\alpha^4+\alpha\mathcal{E})\cap(1+\alpha+\alpha^2+\alpha\mathcal{E})\subset\mathcal{E}\cap(1+\alpha+\alpha^2+\mathcal{E})=X$. Hence $\mathcal{E}(\alpha^2)\subset-\frac{1}{\alpha}-1-\alpha+\frac{X}{\alpha}$. To prove the other inclusion, let $z\in X$. Then by the automaton we deduce that $z=1+\alpha+\alpha^2+\alpha w=\alpha^4+\alpha w'$, $w,w'\in\mathcal{E}$. Hence $-\frac{1}{\alpha}-1-\alpha+\frac{z}{\alpha}=w=\alpha^2+w'$ and $-\frac{1}{\alpha}-1-\alpha+\frac{X}{\alpha}\subset\mathcal{E}(\alpha^2)$.
c) Let $z\in\mathcal{E}(1+\alpha^2)$: $z=\sum_{i\geq 4}\varepsilon_i\alpha^i=1+\alpha^2+\sum_{i\geq 4}\varepsilon'_i\alpha^i$, $(\varepsilon_i)_{i\geq 4}$, $(\varepsilon'_i)_{i\geq 4}\in\mathcal{D}^\infty$. Proposition 8 and the automaton show that $(0,1)(0,0)(0,1)(0,0)(\varepsilon_4,\varepsilon'_4)\ldots$ is an infinite path starting in the initial state and $(\varepsilon_i,\varepsilon'_i)_{i\geq 4}$ is equal to $uuu\ldots$ where $u=(1,0)(0,1)$. Then, $z=\alpha^4+\alpha^6+\alpha^8+\cdots=1+\alpha^2+\alpha^5+\alpha^7+\alpha^9+\ldots$ and $\mathcal{E}(1+\alpha^2)=\alpha^4(1-\alpha^2)^{-1}$.
d) Let $z\in\mathcal{E}(\alpha^{-2}+\alpha^{-1}+\alpha)$. ¿From Proposition 8 and using the automaton we deduce that $z=\alpha^{-2}+\alpha^{-1}+\alpha+\alpha^4+\sum_{i=2}^\infty(\alpha^{4i-1}+\alpha^{4i})=\sum_{i=1}^\infty(\alpha^{4i+1}+\alpha^{4i+2})$. Hence $\mathcal{E}(\alpha^{-2}+\alpha^{-1}+\alpha)=(\alpha^5+\alpha^6)(1-\alpha^4)^{-1}$.
e) Proposition 8 and the automaton give the result.
f) Let $z\in\mathcal{E}(\alpha)$. Then, there exist $(\varepsilon_i)_{i\geq 4}$ and $(\varepsilon'_i)_{i\geq 4}$ in $\mathcal{D}^\infty$ such that $z=\sum_{i\geq 4}\varepsilon_i\alpha^i=\alpha+\sum_{i\geq 4}\varepsilon'_i\alpha^i$. ¿From Proposition 8, $(0,0)(0,1)(0,0)(0,0)(\varepsilon_4,\varepsilon'_4)\ldots$ is a path in the automaton starting in the initial state. Hence, $(\varepsilon_4,\varepsilon'_4)(\varepsilon_5,\varepsilon'_5)(\varepsilon_6,\varepsilon'_6)$ belongs to $\{(0,0),(1,1),(0,1)\}(1,0)(0,1)$. Consequently, z belongs to the union of $(\alpha^5+\alpha^2\mathcal{E})\cap(\alpha+\alpha^2\mathcal{E})$, $(\alpha^4+\alpha^5+\alpha^2\mathcal{E})\cap(\alpha+\alpha^4+\alpha^2\mathcal{E})$ and $(\alpha^5+\alpha^2\mathcal{E})\cap(\alpha+\alpha^4+\alpha^2\mathcal{E})$ which is equal to $f_0(X)\cup f_1(X)\cup f_1(Y)$. Hence $\mathcal{E}(\alpha)=f_0(X)\cup f_1(X)\cup f_1(Y)$.
g) Let $z\in\mathcal{E}(\alpha+\alpha^2)$. Then, there exist $(\varepsilon_i)_{i\geq 4}$ and $(\varepsilon'_i)_{i\geq 4}$ in $\mathcal{D}^\infty$ such that $z=\sum_{i\geq 4}\varepsilon_i\alpha^i=\alpha+\alpha^2+\sum_{i\geq 4}\varepsilon'_i\alpha^i$. ¿From Proposition 8, the infinite path $(0,0)(0,1)(0,1)(0,0)(\varepsilon_4,\varepsilon'_4)\ldots$ is a path in the automaton starting in the initial state. Hence, we either have

(1) $((\varepsilon_i,\varepsilon'_i))_{4\leq i\leq 7}\in(0,1)(1,0)\{(0,0),(1,1),(1,0)\}(0,1)$,
(2) $(\varepsilon_4,\varepsilon'_4)\in\{(0,0),(1,1)\}$, or
(3) $((\varepsilon_i,\varepsilon'_i))_{i\geq 4}\in(0,1)\{(0,0)(0,0),(0,0)(1,1),(1,1)(0,0)\}ww\ldots$,

where $w=(0,1)(1,0)(1,0)(0,1)$. This means that z belongs to

$$
\begin{array}{ll}
 & \big((\alpha^5+\alpha^4\mathcal{E})\cap(\alpha+\alpha^2+\alpha^4+\alpha^7+\alpha^4\mathcal{E})\big)\\
\cup & \big((\alpha^5+\alpha^6+\alpha^4\mathcal{E})\cap(\alpha+\alpha^2+\alpha^4+\alpha^6+\alpha^7+\alpha^4\mathcal{E})\big)\\
\cup & \big((\alpha^5+\alpha^6+\alpha^4\mathcal{E})\cap(\alpha+\alpha^2+\alpha^4+\alpha^7+\alpha^4\mathcal{E})\big)\\
\cup & \big((\alpha\mathcal{E}\cap(\alpha+\alpha^2+\alpha\mathcal{E})\big)\cup\big((\alpha^4+\alpha\mathcal{E})\cap(\alpha+\alpha^2+\alpha^4+\alpha\mathcal{E})\big)\\
\cup & \{z_1,z_2,z_3\}\\
= & g_0(X)\cup g_1(X)\cup g_1(Y)\cup g_2(Y)\cup g_3(Y)\cup\{z_1,z_2,z_3\}
\end{array}
$$

where $z_1=\sum_{i=2}^{+\infty}(\alpha^{4i}+\alpha^{4i+1})=\alpha+\alpha^2+\alpha^4+\alpha^7+\sum_{i=2}^{+\infty}(\alpha^{4i+2}+\alpha^{4i+3})$, $z_2=\alpha^6+z_1$ and $z_3=\alpha^5+z_1$. We can also check that $(1,0)(1,0)(0,0)(0,0)uuu\dots$, where $u=(0,1)(1,1)(1,0)(1,0)$, is an infinite path of the automaton starting in the initial state. Consequently, $z_1=\alpha^4+\alpha^5+\sum_{i=2}^{+\infty}(\alpha^{4i+1}+\alpha^{4i+2})$ and

$$z_1\in\big((\alpha^4+\alpha\mathcal{E})\cap(\alpha+\alpha^2+\alpha^4+\alpha\mathcal{E})\big)=g_3(Y).$$

Moreover, it shows that z_2 belongs to $g_3(Y)$. In the same way, $z_1=\alpha^5+\alpha^6+\alpha^8+\sum_{i=2}^{+\infty}(\alpha^{4i+3}+\alpha^{4i+4})$. Thus, $z_3=2\alpha^5+\alpha^6+\alpha^8+\sum_{i=2}^{+\infty}(\alpha^{4i+3}+\alpha^{4i+4})=\alpha^5+\sum_{i=2}^{+\infty}(\alpha^{4i}+\alpha^{4i+1})$. But $2\alpha^5+\alpha^6=\alpha+\alpha^2+\alpha^7$, consequently z_3 belongs to $\big(\alpha\mathcal{E})\cap(\alpha+\alpha^2+\alpha\mathcal{E})\big)=g_2(Y)$.

h) Let $z\in X=\mathcal{E}(1+\alpha+\alpha^2)$. Then, there exist $(\varepsilon_i)_{i\geq4}$ and $(\varepsilon'_i)_{i\geq4}$ in $\mathcal{D}^\infty$ such that $z=\sum_{i\geq4}\varepsilon_i\alpha^i=1+\alpha+\alpha^2+\sum_{i\geq4}\varepsilon'_i\alpha^i$. ¿From Proposition 8, we necessarily have $(\varepsilon_4,\varepsilon'_4)=(1,0)$ and one of the following situations :

(1) $((\varepsilon_i,\varepsilon'_i))_{i\geq5}\in(1,0)\{(0,0),(1,1)\}ww\dots$ where $w=(0,1)(1,0)$;
(2) $((\varepsilon_i,\varepsilon'_i))_{i\geq5}\in(0,1)\{(0,0),(1,1)\}ww\dots$ where $w=(0,1)(1,0)(1,0)(0,1)$;
(3) $(\varepsilon_i,\varepsilon'_i)_{5\leq i\leq8}\in\{(0,0),(1,1)\}^2(0,1)(1,0)$;
(4) $(\varepsilon_i,\varepsilon'_i)_{5\leq i\leq9}=(1,0)(1,0)(0,1)(1,0)(0,1)$;
(5) $(\varepsilon_i,\varepsilon'_i)_{5\leq i\leq8}\in\{(0,0),(1,1)\}(1,0)(0,1)(1,0)$.

This means z belongs to $\bigcup_{i=0}^4 h_i(X)\cup h_1(Y)\cup h_3(Y)\cup\{x_1,x_2,x_3,x_4\}$ where

$$
\begin{aligned}
x_1 &=\alpha^4+\alpha^5+\sum_{i=4}^{+\infty}\alpha^{2i}=1+\alpha+\alpha^2+\sum_{i=3}^{+\infty}\alpha^{2i+1},\\
x_2 &=x_1+\alpha^6,\\
x_3 &=\alpha^4+\sum_{i=2}^{+\infty}(\alpha^{4i}+\alpha^{4i+1})=1+\alpha+\alpha^2+\alpha^5+\alpha^7+\sum_{i=2}^{+\infty}(\alpha^{4i+2}+\alpha^{4i+3}),\\
x_4 &=x_3+\alpha^6,
\end{aligned}
$$

$$
\begin{aligned}
h_0(X) &=(\alpha^4+\alpha^4\mathcal{E})\cap(1+\alpha+\alpha^2+\alpha^7+\alpha^4\mathcal{E}),\\
h_1(X) &=(\alpha^4+\alpha^6+\alpha^4\mathcal{E})\cap(1+\alpha+\alpha^2+\alpha^6+\alpha^7+\alpha^4\mathcal{E}),\\
h_2(X) &=(\alpha^4+\alpha^5+\alpha^4\mathcal{E})\cap(1+\alpha+\alpha^2+\alpha^5+\alpha^7+\alpha^4\mathcal{E}),\\
h_3(X) &=(\alpha^4+\alpha^5+\alpha^6+\alpha^4\mathcal{E})\cap(1+\alpha+\alpha^2+\alpha^5+\alpha^6+\alpha^7+\alpha^4\mathcal{E}),\\
h_4(X) &=(\alpha^4+\alpha^5+\alpha^6+\alpha^8+\alpha^5\mathcal{E})\cap(1+\alpha+\alpha^2+\alpha^7+\alpha^5\mathcal{E}),\\
h_1(Y) &=(\alpha^4+\alpha^6+\alpha^4\mathcal{E})\cap(1+\alpha+\alpha^2+\alpha^7+\alpha^4\mathcal{E})\text{ and}\\
h_3(Y) &=(\alpha^4+\alpha^5+\alpha^6+\alpha^4\mathcal{E})\cap(1+\alpha+\alpha^2+\alpha^5+\alpha^7+\alpha^4\mathcal{E}).
\end{aligned}
$$

We easily can check (using Proposition 8 and the automaton) that

$$x_1 = x_3 = \alpha^4 + \sum_{i=2}^{+\infty}(\alpha^{4i} + \alpha^{4i+1}),$$

and thus $x_1 \in h_0(X)$, $x_2 \in h_1(X)$, and $x_2 = x_4$, which concludes the proof of h).
i) Let $z \in X = \mathcal{E}(1+\alpha)$. Then, there exist $(\varepsilon_i)_{i\geq 4}$ and $(\varepsilon'_i)_{i\geq 4}$ in $\mathcal{D}^\infty$ such that $z = \sum_{i\geq 4}\varepsilon_i\alpha^i = 1 + \alpha + \sum_{i\geq 4}\varepsilon'_i\alpha^i$. ¿From Proposition 8, we necessarily have $(\varepsilon_4, \varepsilon'_4) = (1,0)$ and one of the following situations :

(1) $((\varepsilon_i, \varepsilon'_i))_{i\geq 5} \in \{(0,0),(1,1)\}(0,1)^2\{(0,0),(1,1)\}^2 ww\dots$;
(2) $((\varepsilon_i, \varepsilon'_i))_{i\geq 5} \in (1,0)\{(0,0),(1,1)\}^2(1,0)(0,1)ww\dots$;
(3) $((\varepsilon_i, \varepsilon'_i))_{5\leq i\leq 7} \in \{(0,0),(1,1)\}(0,1)\{(0,0),(1,1)\}$;
(4) $((\varepsilon_i, \varepsilon'_i))_{5\leq i\leq 8} \in \{(0,0),(1,1)\}(0,1)^2(1,0)\{(0,0),(1,1),(1,0)\}(0,1)(1,0)$;
(5) $((\varepsilon_i, \varepsilon'_i))_{5\leq i\geq 8} \in (1,0)(0,1)\{(0,0),(1,1),(0,1)\}(1,0)(0,1)$.

where $w = (0,1)(1,0)(1,0)(0,1)$. Hence z belongs to

$$\left(\bigcup_{i=5}^{11} h_i(Y)\right) \cup \left(\bigcup_{i=12}^{17} h_i(X)\right) \cup \{y_i; 1\leq i\leq 8\},$$

where

$$y_1 = \alpha^4 + \sum_{i=2}^{+\infty}(\alpha^{4i+3} + \alpha^{4i+4}) = 1 + \alpha + \alpha^6 + \alpha^7 + \alpha^{10} + \sum_{i=3}^{+\infty}(\alpha^{4i+1} + \alpha^{4i+2}),$$

$$y_2 = y_1 + \alpha^9,\ y_3 = y_1 + \alpha^8,\ y_4 = y_1 + \alpha^5,\ y_5 = y_1 + \alpha^5 + \alpha^9.$$

$$y_6 = y_1 + \alpha^5 + \alpha^8 = 1 + \alpha + \sum_{i=2}^{+\infty}(\alpha^{4i+1} + \alpha^{4i+2}),\ y_7 = y_6 + \alpha^7,\ y_8 = y_6 + \alpha^6,$$

$$h_5(Y) = (\alpha^4 + \alpha^5 + \alpha^4\mathcal{E}) \cap (1 + \alpha + \alpha^5 + \alpha^6 + \alpha^4\mathcal{E}),$$
$$h_6(Y) = (\alpha^4 + \alpha^7 + \alpha^4\mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^7 + \alpha^4\mathcal{E}),$$
$$h_7(Y) = (\alpha^4 + \alpha^8 + \alpha^9 + \alpha^7\mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^7 + \alpha^{10} + \alpha^7\mathcal{E}),$$
$$h_8(Y) = (\alpha^4 + \alpha^4\mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^4\mathcal{E}),$$
$$h_9(Y) = (\alpha^4 + \alpha^5 + \alpha^7 + \alpha^4\mathcal{E}) \cap (1 + \alpha + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^4\mathcal{E}),$$
$$h_{10}(Y) = (\alpha^4 + \alpha^5 + \alpha^8 + \alpha^9 + \alpha^7\mathcal{E}) \cap (1 + \alpha + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^{10} + \alpha^7\mathcal{E}),$$
$$h_{11}(Y) = (\alpha^4 + \alpha^5 + \alpha^8 + \alpha^5\mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^7 + \alpha^5\mathcal{E}),$$
$$h_{12}(Y) = (\alpha^4 + \alpha^8 + \alpha^7\mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^7 + \alpha^{10} + \alpha^7\mathcal{E}),$$
$$h_{13}(X) = (\alpha^4 + \alpha^8 + \alpha^9 + \alpha^7\mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^7 + \alpha^9 + \alpha^{10} + \alpha^7\mathcal{E}),$$
$$h_{14}(X) = (\alpha^4 + \alpha^5 + \alpha^8 + \alpha^7\mathcal{E}) \cap (1 + \alpha + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^{10} + \alpha^7\mathcal{E}),$$
$$h_{15}(X) = (\alpha^4 + \alpha^5 + \alpha^8 + \alpha^9 + \alpha^7\mathcal{E}) \cap (1 + \alpha + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^9 + \alpha^{10} + \alpha^7\mathcal{E}),$$
$$h_{16}(X) = (\alpha^4 + \alpha^5 + \alpha^8 + \alpha^5\mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^5\mathcal{E}),$$
$$h_{17}(X) = (\alpha^4 + \alpha^5 + \alpha^7 + \alpha^8 + \alpha^5\mathcal{E}) \cap (1 + \alpha + \alpha^6 + \alpha^7 + \alpha^5\mathcal{E}),$$

Let us prove that for each integer $i \in \{1, \ldots, 8\}$, there exists $j \in \{5, \ldots, 11\}$ or $k \in \{12, \ldots, 17\}$ such that y_i belongs to $h_j(X)$ or to $h_k(Y)$.
Indeed, since $y_1 = \alpha^4 + \alpha^3 z_1$ (see case g)), then $y_1 \in (\alpha^4 + \alpha^7 + \alpha^4\mathcal{E}) \cap (2\alpha^4 + \alpha^5 + \alpha^7 + \alpha^4\mathcal{E}) = h_6(Y)$. We deduce that y_2 and y_3 belong also to $h_6(Y)$, and, y_4 and y_5 belong to $h_9(Y)$.
Using the automaton we can verify that $y_6 = 1+\alpha+\alpha^5+\alpha^6+\sum_{i=2}^{+\infty}(\alpha^{4i+2}+\alpha^{4i+3})$. Hence, y_6 belongs to $1 + \alpha + \alpha^5 + \alpha^6 + \alpha^4\mathcal{E}$. But it also belongs to $\alpha^4 + \alpha^5 + \alpha^4\mathcal{E}$. Thus $y_6 \in h_5(Y)$ and $y_7 \in h_9(Y)$.
We have $y_8 = y_6 + \alpha^6 \in (1 + \alpha + \alpha^6 + \alpha^5\mathcal{E})$. On the other hand we can check using the automaton that $y_8 = \alpha^4 + \alpha^5 + \alpha^8 + \sum_{i=5}^{+\infty} \alpha^{2i}$, hence $y_8 \in (\alpha^4 + \alpha^5 + \alpha^8 + \alpha^5\mathcal{E})$ and y_8 belongs to $h_{16}(X)$. □

Remarks and comments. There are points which have at least 6 expansions in base α. For example:

$$\begin{aligned}
\alpha + \sum_{i=2}^{+\infty} \alpha^{2i} &= \sum_{i=1}^{+\infty}(\alpha^{4i} + \alpha^{4i+1}) \\
&= 1 + \alpha + \alpha^2 + \sum_{i=2}^{\infty} \alpha^{2i+1} \\
&= 1 + \alpha + \sum_{i=1}^{\infty}(\alpha^{4i+1} + \alpha^{4i+2})) \\
&= \alpha + \alpha^2 + \alpha^4 + \sum_{i=1}^{\infty}(\alpha^{4i+3} + \alpha^{4i+4}) \\
&= \alpha^{-3} + \alpha^{-2} + 1 + \alpha^3 + \sum_{i=1}^{\infty}(\alpha^{4i+2} + \alpha^{4i+3}).
\end{aligned}$$

We address the two following questions :

(1) Can you parameterize the boundary of $\mathcal{E}_{1,1,1}$?
(2) Does this boundary be homeomorphic to the sphere ?

The technics used in this work can be used to study $\mathcal{E}_{a_1,a_2,\ldots,a_d}$ with the assumption that $a_1 \geq a_2 \geq \cdots \geq a_d \geq 1$.

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4. Annexe

In the sequel we prove Theorem 5, show the Rauzy fractal and its automaton. We will need several intermediate results.

4.1. **Proof of Theorem 5.**

Lemma 15. *Let $i \geq 4$, then $\beta^i = G_i\beta^3 + (G_{i-1} + G_{i-2} + G_{i-3})\beta^2 + (G_{i-1} + G_{i-2})\beta + G_{i-1}$ where $(G_i)_{i\geq 0}$ is the sequence defined by: $G_0 = G_1 = G_2 = 0, G_3 = 1, G_n = G_{n-1} + G_{n-2} + G_{n-3}$ for all $i \geq 4$. In particular for all $(\varepsilon_i)_{4\leq i\leq N} \in \mathcal{D}$, $\sum_{i=4}^{N} \varepsilon_i\beta^i = n\beta^3 + a_n\beta^2 + b_n\beta + c_n$ where $n = \sum_{i=4}^{N} \varepsilon_i G_i$, $a_n = \sum_{i=4}^{N} \varepsilon_i(G_{i-1} + G_{i-2} + G_{i-3})$, $b_n = \sum_{i=4}^{N} \varepsilon_i(G_{i-1} + G_{i-2})$ and $c_n = \sum_{i=4}^{N} \varepsilon_i G_{i-1}$.*

Proof. It is left to the reader. □

Proposition 16. $\mathbb{R}\times\mathbb{C}=\bigcup_{p\in\mathbb{Z}\alpha^0+\mathbb{Z}\alpha+\mathbb{Z}\alpha^2}(\mathcal{E}+p)$.

Proof. ¿From Lemma 15 and Proposition 1 in [A99]), we know that the set $E=\{n\alpha^3+p\alpha^0+q\alpha+r\alpha^2, n\in\mathbb{N},\ p,q,r\in\mathbb{Z}\}$ is dense in $\mathbb{R}\times\mathbb{C}$.
Let $z\in\mathbb{R}\times\mathbb{C}$ and $\varepsilon>0$, then there exist a positive integer N such that for all integer $k\geq N,\ |z-z_k|<\varepsilon$ where

$$z_k=n_k\alpha^3+p_k\alpha^0+q_k\alpha+r_k\alpha^2, (n_k,p_k,q_k,r_k)\in\mathbb{N}\times\mathbb{Z}^3\ \forall k\geq N.$$

On the other hand, we can write every integer n_k in base G_n (by using greedy algorithm) as $n_k=\sum_{i=4}^N\varepsilon_iG_i$ where $(\varepsilon_i)_{4\leq i\leq N}\in\mathcal{D}$. Therefore by Lemma 15, there exists $t_k=a_{n_k}\alpha^2+b_{n_k}\alpha+c_{n_k}\alpha^0\in G=\mathbb{Z}\alpha^0+\mathbb{Z}\alpha+\mathbb{Z}\alpha^2$ such that $x_k=n_k\alpha^3+t_k\in\mathcal{E}$. We deduce that for all $k\geq N,\ |x_k-z_k|\leq|x_k-z|+|z-z_k|<\varepsilon+|z|+M$ where $M=\max\{|x|,\ x\in\mathcal{E}\}$. Since for all $k\geq N,\ x_k-z_k$ belongs to G, which is a discrete group, then there exists an increasing sequence $(k_i)_{i\geq1}$ of integers such that for all i, $x_{k_i}-z_{k_i}=y_0$ where $y_0=p\alpha^0+q\alpha+r\alpha^2$ is an element of G. Since for all $i,\ x_{k_i}=z_{k_i}+y_0$ belongs to $\mathcal{E}$ and $\mathcal{E}$ is a compact set, we deduce that $z+y_0\in\mathcal{E}$. Thus we are done. □

Proposition 17. *For all* $u,v\in\mathbb{Z}\alpha^0+\mathbb{Z}\alpha+\mathbb{Z}\alpha^2$, *we have* $u=v$ *whenever* $\mathrm{Int}\,((\mathcal{E}+u))\cap(\mathcal{E}+v)\neq\emptyset$.

Proof. We proceed by contradiction. Assume that there exist integers $p,q,r\in\mathbb{Z}$ and an element $z=\sum_{i=4}^{+\infty}\varepsilon_i\alpha^i$ of $\mathcal{E}$ such that $z+p\alpha^0+q\alpha+r\alpha^2\in\mathrm{Int}(\mathcal{E})$. Then there exists an integer $n_0\geq0$ such that for all $n\geq n_0$

(9) $$\sum_{i=4}^{n}\varepsilon_i\alpha^i+p\alpha^0+q\alpha+r\alpha^2\in\mathcal{E}\ .$$

Case 1 : The set $\{i\geq4,\ \varepsilon_i\neq0\}$ is infinite.
Since $\beta>1$, there exists an integer $N\geq n_0$ such that $\sum_{i=4}^N\varepsilon_i\beta^i+p+q\beta+r\beta^2>0$. By Lemma 3, we deduce that

(10) $$\sum_{i=4}^{N}\varepsilon_i\beta^i+p+q\beta+r\beta^2=\sum_{i=l}^{M}d_i\beta^i$$

where $(d_i)_{l\leq i\leq M}\in\mathcal{D}$ and $l,M\in\mathbb{Z}$. ¿From (9) and (10), we obtain that $\sum_{i=l}^M d_i\alpha^i=\sum_{i=4}^\infty e_i\alpha^i\in\mathcal{E}$, for some $(e_i)_{i\geq4}\in\mathcal{D}$. Corollary 9 implies that there exists an integer $K\leq M$ verifying $e_i=0$ for all $i\geq K$. Therefore

(11) $$\sum_{i=4}^{N}\varepsilon_i\beta^i+p+q\beta+r\beta^2=\sum_{i=4}^{K}e_i\beta^i\ .$$

Lemma 15 gives that $m\beta^3+(r+a_m)\beta^2+(q+b_m)\beta+(p+c_m)=l\beta^3+a_l\beta^2+b_l\beta+c_l$, where $m=\sum_{i=4}^N\varepsilon_iG_i$ and $l=\sum_{i=4}^K e_iG_i$. Thus $l=m$ and $\varepsilon_i=e_i$ for all i (because of the unicity of representation in base G_n) and finally $p=q=r=0$.

Case 2 : The set $\{i\geq4,\ \varepsilon_i\neq0\}$ is finite.
Let $N=\max\{i\geq4,\ \varepsilon_i\neq0\}$. If $\sum_{i=4}^N\varepsilon_i\beta^i+p+q\beta+r\beta^2\geq0$, then we are done using the same argument as in Case 1.
Now, assume that $\sum_{i=4}^N\varepsilon_i\beta^i+p+q\beta+r\beta^2<0$. We have $\sum_{i=4}^N\varepsilon_i\alpha^i+p+q\alpha+r\alpha^2=\sum_{i=4}^{+\infty}d_i\alpha^i$.

Since $\sum_{i=4}^{N} \varepsilon_i \alpha^i$ is an interior point of $\mathcal{E}$ (see [A99]), we deduce that there exists a nonnegative integer M such that $-p - q\alpha - r\alpha^2 + \sum_{i=4}^{M} d_i \alpha^i = \sum_{i=4}^{+\infty} e_i \alpha^i \in \mathcal{E}$. Since $-p - q\beta - r\beta^2 + \sum_{i=4}^{M} d_i \beta^i > 0$ we deduce that $-p - q\alpha - r\alpha^2 + \sum_{i=4}^{M} d_i \alpha^i = \sum_{i=l}^{k} f_i \alpha^i = \sum_{i=4}^{+\infty} e_i \alpha^i$ where $(f_i)_{l \le i \le k} \in \mathcal{D}$ and $l, k \in \mathbb{Z}$.
¿From Corollary 9 , there exists an integer L such that $e_i = 0$ for all $i \geq L$ and by an argument used in Case 1 we obtain $p = q = r = 0$. □

Proposition 18. *The boundary of $\mathcal{E}$ has Lebesgue measure zero and is equal to the union of all $\mathcal{E}_p$, $p \in G$, where $\mathcal{E}_p = \mathcal{E} \cap (\mathcal{E} + p)$.*

Proof. Let z be an element of $\partial\mathcal{E} = \overline{\mathcal{E}} \setminus \mathrm{Int}\mathcal{E}$, the boundary of $\mathcal{E}$. There exists a sequence $(z_n)_{n\geq 0}$ such that $\lim z_n = z$ and for all n, $z_n \notin \mathcal{E}$. Then by Proposition 16, there exists a sequence $(p_n)_{n\geq 0}$ of elements of $G = \mathbb{Z}\alpha^0 + \mathbb{Z}\alpha + \mathbb{Z}\alpha^2 \backslash \{0\}$ such that for all n, $z_n \in (\mathcal{E} + p_n)$ with $p_n \in G = \mathbb{Z}\alpha^0 + \mathbb{Z}\alpha + \mathbb{Z}\alpha^2 \backslash \{0\}$. Hence the sequence $(p_n)_{n\geq 0}$ is bounded. Since G is a discrete group, we deduce that $(p_n)_{n\geq 0}$ is a finite sequence. Consequently there exists $p \in \mathbb{Z}\alpha^0 + \mathbb{Z}\alpha + \mathbb{Z}\alpha^2$ such that $z \in \mathcal{E} \cap (\mathcal{E} + p)$. Thus, $\partial\mathcal{E}$ is included in $\bigcup_{p\in G} \mathcal{E}_p$.
On the other hand, if $z \in \mathcal{E} \cap (\mathcal{E} + p)$, $p \in G \backslash \{0\}$, then by Proposition 17, $z \notin \mathrm{Int}(\mathcal{E})$. Hence $z \in \partial\mathcal{E}$. The fact that the boundary has measure zero is proven in [A02]. □

4.2. **The Rauzy fractal.** Here is a two-dimensional image of the Rauzy fractal in $\mathbb{R} \times \mathbb{C}$ generated by $P(x) = x^4 - x^3 - x^2 - x - 1$.

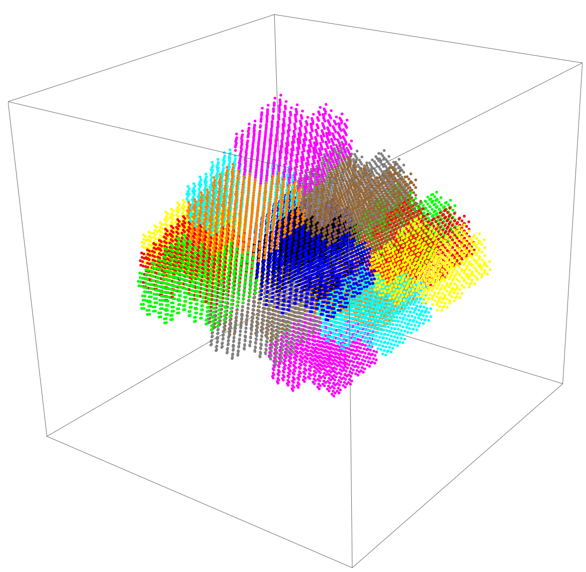
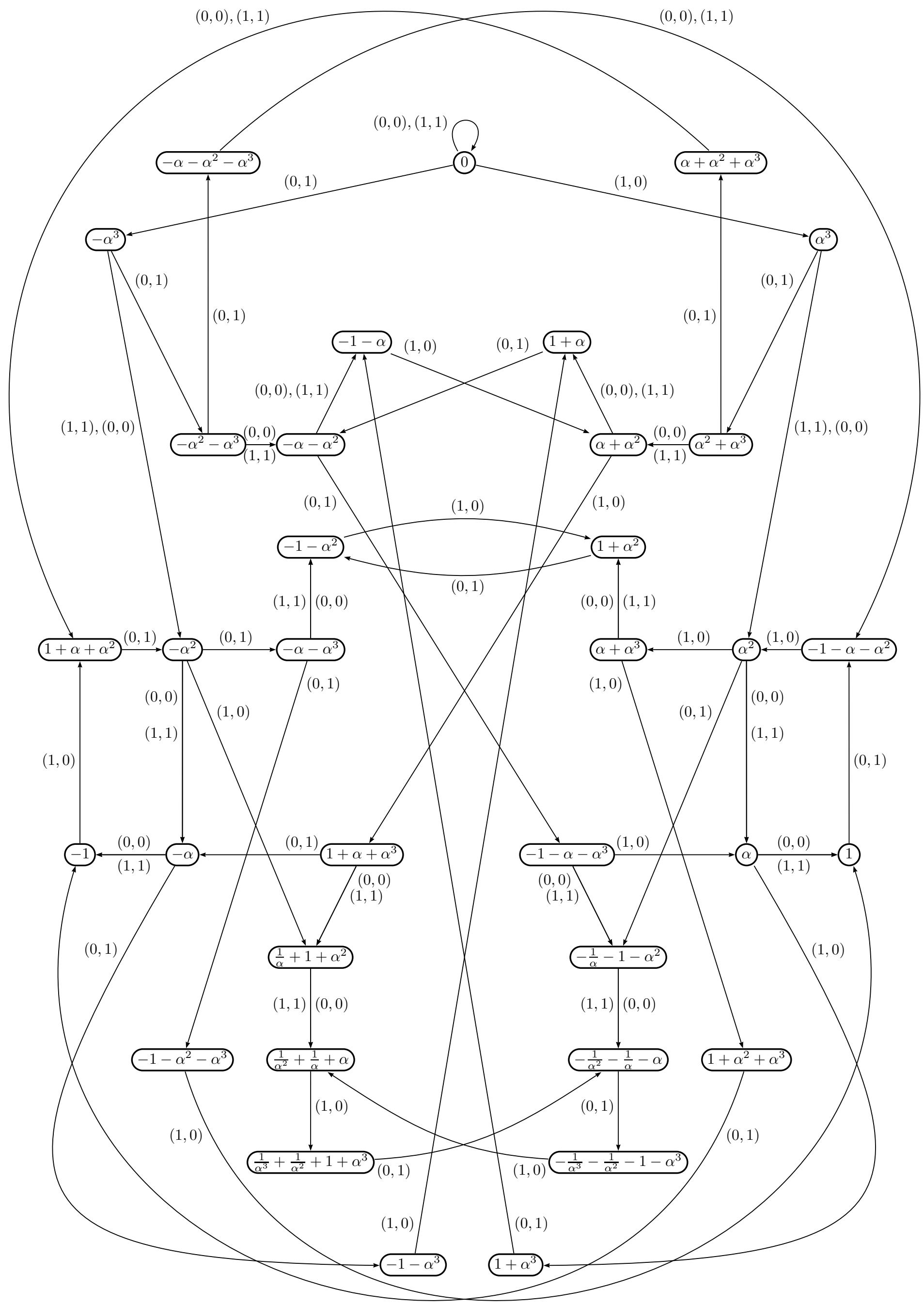


FIGURE 1. Rauzy fractal in $\mathbb{R} \times \mathbb{C}$ generated by $P(x) = x^4 - x^3 - x^2 - x - 1$

4.3. **The automaton.** Here is the automaton built in Section 3.



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