Combinatorics, Automata and Number Theory

CANT

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Combinatorics on Bratteli diagrams and dynamical systems

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The aim of this chapter is to show how Bratteli diagrams are used to study topological dynamical systems. We illustrate their wide range of applications through classical notions: invariant measures, entropy, expansivity, representation theorems, strong orbit equivalence, eigenvalues of the Koopman operator.

6.1 Introduction

In 1972 O. Bratteli (Bratteli 1972) introduced special infinite graphs – subsequently called *Bratteli diagrams* – which conveniently encoded the successive embeddings of an ascending sequence $(A_n)_{n\geq 0}$ of finite-dimensional semi-simple algebras over $\mathbb{C}$ ("multi-matrix algebras"). The sequence $(A_n)_{n\geq 0}$ determines a so-called approximately finite-dimensional (AF) C^*-algebra. Bratteli proved that the equivalence relation on Bratteli diagrams generated by the operation of telescoping is a complete isomorphism invariant for AF-algebras.

From a different direction came the extremely fruitful idea of A. M. Vershik (Vershik 1985) to associate dynamics (*adic transformations*) with Bratteli diagrams (*Markov compacta*) by introducing a lexicographic ordering on the infinite paths of the diagram. By a careful refining of Vershik's construction, R. H. Herman, I. F. Putnam and C. F. Skau (Herman, Putnam, and Skau 1992) succeeded in showing that every minimal Cantor dynamical system is isomorphic to a Bratteli-Vershik dynamical system.

This chapter will give the details of this isomorphism and present some developments.

In this chapter all the dynamical systems (X, T) we consider are such that T is a homeomorphism. We thus work with the two-sided *orbit* of $x \in X$, that is, $\{T^n x \mid n \in \mathbb{Z}\}$.

6.2 Cantor dynamical systems

The notion of dynamical system has been defined 1.6.2 in two contexts: topological and measure-theoretic. Below we specify a special class of topological dynamical systems with respect to the space where it is defined.

We say that (X, T) is a *Cantor dynamical system* whenever X is a Cantor space, that is, X is non-empty, without isolated points, compact, totally disconnected, and metrisable. We recall that all Cantor spaces are homeomorphic and have a countable basis of their topology consisting of open sets that are also closed. These sets are usually called *clopen.*

Let (X, T) and (Y, S) be two dynamical systems. We say that (Y, S) is a *factor* of (X, T) if there is a continuous and onto map $\varphi : X \to Y$ such that $\varphi \circ T = S \circ \varphi$. Note that the term "factor" is used here with a completely different meaning as the meaning it has in Chapter 4 for instance. Then, φ is called *factor map.* If φ is one-to-one we say that it is an *isomorphism* (it is also called a *conjugacy*), and that (X, T) and (Y, S) are *(topologically) isomorphic.*

Let (X, T) be a minimal Cantor dynamical system and $U \subseteq X$ be a clopen set. Let $T_U : U \to U$ be the map defined by, for $x \in U$,

$$T_U(x) = T^{r_U(x)}(x), \text{ where } r_U(x) = \inf\{n > 0 \mid \ T^n(x) \in U\}.$$

As (X, T) is minimal, the *first entrance time map to* U, $r_U : X \to \mathbb{Z}$, is well defined and continuous. The pair (U, T_U) is a minimal Cantor dynamical system. We say that (U, T_U) is the *induced dynamical system of* (X, T), and $T_U : U \to U$ the *induced map*, with respect to U.

6.3 Bratteli diagrams

In this section we define the notion of *Bratteli diagram* and of *Bratteli-Vershik dynamical systems.*

6.3.1 Basics on Bratteli diagrams

Definition 6.3.1 A *Bratteli diagram* is an infinite directed graph (V, E) where the *vertex* set V and the *edge* set E can be partitioned into finite sets

$$V = V(0) \cup V(1) \cup V(2) \cup \cdots \quad \text{and} \quad E = E(1) \cup E(2) \cup \cdots$$

with the following properties:

(i) $V(0) = \{v(0)\}$ is a one-point set,
(ii) $r(E(n)) \subseteq V(n),\ s(E(n)) \subseteq V(n-1), n = 1, 2, \ldots,$

where $r : E \to V$ is called the *range map* and $s : E \to V$ the *source map*. They satisfy $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V(0)$.

It is convenient to give a diagrammatic presentation of the Bratteli diagram with $V(n)$ the vertices at (horizontal) level n, and $E(n)$ the edges (downward directed) connecting the vertices at level $n-1$ with those at level n. Also, if $\mathrm{Card}(V(n-1)) = t(n-1)$ and $\mathrm{Card}(V(n)) = t(n)$, then $E(n)$ determines a $t(n) \times t(n-1)$ *incidence matrix* $\mathbf{M}(n)$ (see Figure 6.1).

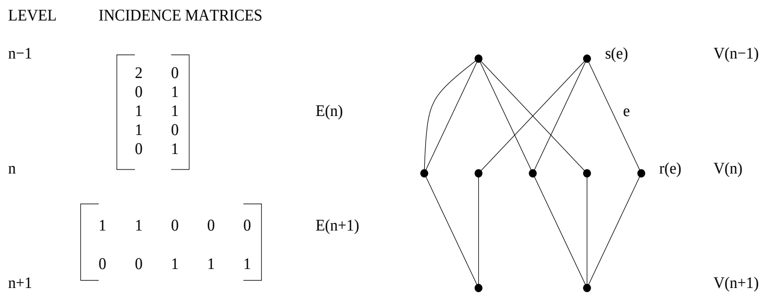

Fig. 6.1. Diagrammatic representation between the levels $n-1$ and $n+1$.

We say that two Bratteli diagrams (V, E) and (V', E') are *isomorphic* whenever there exists a pair of bijections $f : V \to V'$, preserving the gradings, and $g : E \to E'$, intertwining the respective source and range maps:

$$s' \circ g = f \circ s \text{ and } r' \circ g = f \circ r\ .$$

Let $k, l \in \mathbb{Z}_{\geq 0}$ with $k < l$ and let $E(k+1) \circ E(k+2) \circ \cdots \circ E(l)$ denote the set of paths from $V(k)$ to $V(l)$. Specifically,

$$E(k+1) \circ \cdots \circ E(l)$$

$$= \{(e_{k+1}, \ldots, e_l) \mid e_i \in E(i), k+1 \leq i \leq l, r(e_i) = s(e_{i+1}), k+1 \leq i \leq l-1\}\ .$$

Remark that the incidence matrix of $E(k+1) \circ \cdots \circ E(l)$ is $\mathbf{M}(l) \cdots \mathbf{M}(k+1)$. We define $r(e_{k+1}, \ldots, e_l) := r(e_l)$ and $s(e_{k+1}, \ldots, e_l) := s(e_{k+1})$.

Given a Bratteli diagram (V, E) and a sequence

$$m_0 = 0 < m_1 < m_2 < \ldots$$

in $\mathbb{Z}^+$, we define the *telescoping* of (V, E) to $\{m_n \mid n \in \mathbb{N}\}$ as the new Bratteli diagram (V', E'), where $V'(n) = V(m_n)$ and $E'(n) = E(m_{n-1} + 1) \circ \cdots \circ E(m_n)$ and the range and source maps are as above (see Figure 6.2).

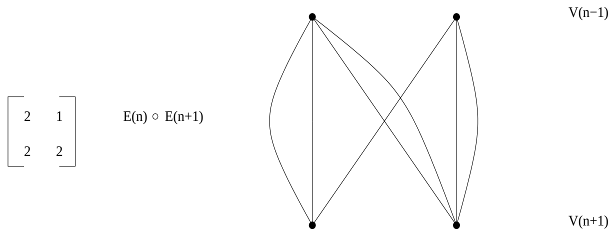

Fig. 6.2. Telescoping between the levels $n-1$ and $n+1$ in the diagram of Figure 6.1.

We say that (V, E) is a *simple Bratteli diagram* if there exists a telescoping (V', E') of (V, E) such that the incidence matrices of (V', E') have only non-zero entries at each level.

We let $\sim$ denote the *equivalence relation on Bratteli diagrams* generated by isomorphism and telescoping. It is not hard to show that $(V^1, E^1) \sim (V^2, E^2)$ if, and only if, there exists a Bratteli diagram (V, E) such that telescoping (V, E) to odd levels $0 < 1 < 3 < \ldots$ yields a telescoping of either (V^1, E^1) or (V^2, E^2), and telescoping (V, E) to even levels $0 < 2 < 4 < \ldots$ yields a telescoping of the other.

We say that a Bratteli diagram B has a *simple hat* whenever it has only simple edges between the top vertex and the first level.

6.3.2 Ordered Bratteli diagrams

Definition 6.3.2 An *ordered Bratteli diagram* $(V, E, \geq)$ is a Bratteli diagram (V, E) together with a partial order $\geq$ on E such that edges e, e' in E are *comparable* if, and only if, $r(e) = r(e')$, in other words, we have a linear order on each set $r^{-1}(\{v\})$, where v belongs to $V \setminus V(0)$ (see Figure 6.3).

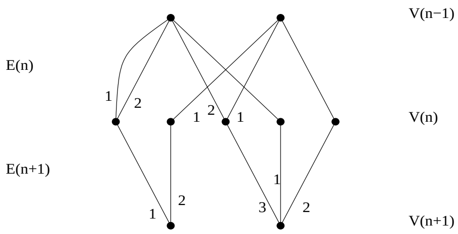

Fig. 6.3. Order on the diagram of Figure 6.1.

Note that if $(V, E, \geq)$ is an ordered Bratteli diagram and $k < l$ in $\mathbb{Z}^+$, then the set $E(k+1) \circ E(k+2) \circ \cdots \circ E(l)$ of paths from $V(k)$ to $V(l)$ may be given an induced (*lexicographic*) order as follows:

$$(e_{k+1}, e_{k+2}, \dots, e_l) > (f_{k+1}, f_{k+2}, \dots, f_l)$$

if, and only if, for some i with $k+1 \le i \le l$, $e_j = f_j$ for $i < j \le l$ and $e_i > f_i$. It is a simple observation that if $(V, E, \geq)$ is an ordered Bratteli diagram and (V', E') is a telescoping of (V, E) as defined above, then with the induced order $\geq'$, $(V', E', \geq')$ is again an ordered Bratteli diagram. We say that $(V', E', \geq')$ is a *telescoping* of $(V, E, \geq)$ (see Figure 6.4).

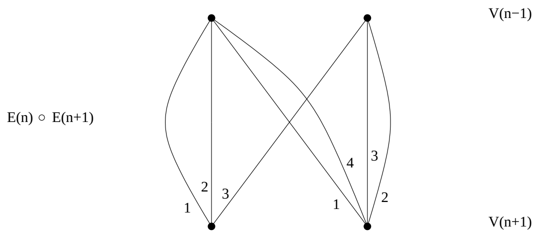

Fig. 6.4. Telescoping of the diagram of Figure 6.3.

Again there is an obvious notion of isomorphism between ordered Bratteli diagrams. Let $\approx$ denote the equivalence relation on ordered Bratteli diagrams generated by isomorphism and by telescoping. One can show that $B^1 \approx B^2$, where $B^1 = (V^1, E^1, \geq^1)$, $B^2 = (V^2, E^2, \geq^2)$, if, and only if, there exists an ordered Bratteli diagram $B = (V, E, \geq)$ such that telescoping B to odd levels $0 < 1 < 3 < \dots$ yields a telescoping of either B^1 or B^2, and telescoping B to even levels $0 < 2 < 4 < \dots$ yields a telescoping of the other. This is analogous to the situation for the equivalence relation $\sim$ on Bratteli diagrams as we discussed above. We write $B^1 \sim B^2$ to say $(V^1, E^1) \sim (V^2, E^2)$.

The following notion will be important when we will deal with Bratteli diagrams and subshifts. Fix $n \geq 1$ and let us consider $V(n-1)$ and $V(n)$ as alphabets. For every letter $a \in V(n)$, consider the ordered list $(e_1, \dots, e_k)$ of edges of $E(n)$ which range at a, and let $(a_1, \dots, a_k)$ be the ordered list of the labels of the sources of these edges. This defines a morphism $a \mapsto a_1 \cdots a_k$ from $V(n)^*$ to $V(n-1)^*$ we call *the morphism read on* $E(n)$. For example in Figure 6.3 the morphism we read on:

- $E(n)$ is $\tau_n : \ 0 \mapsto AA, \ 1 \mapsto B, \ 2 \mapsto BA, \ 3 \mapsto A, \ 4 \mapsto B,$
- $E(n+1)$ is $\tau_{n+1} : \ a \mapsto 01, \ b \mapsto 342,$

and on Figure 6.4 the morphism we read on $E(n) \circ E(n+1)$ is $\sigma : a \mapsto AAB$, $b \mapsto ABBA$. We can check we of course that we have $\sigma = \tau_n \circ \tau_{n+1}$.

6.3.3 Dynamics for ordered Bratteli diagrams

Let $B = (V, E, \geq)$ be an ordered Bratteli diagram. Let X_B denote the associated *infinite path space*, *i.e.*,

$$X_B = \{(e_1, e_2, \ldots) \mid e_i \in E(i), r(e_i) = s(e_{i+1}), i = 1, 2, \ldots\} \ .$$

We exclude trivial cases and assume henceforth that X_B is an infinite set. Two paths in X_B are said to be *cofinal* if they have the same tails, *i.e.*, the edges agree from a certain level. We endow X_B with a topology by postulating a basis of open sets, namely the family of *cylinder sets*

$$[e_1, e_2, \ldots, e_k]_B = \{(f_1, f_2, \ldots) \in X_B \mid f_i = e_i, 1 \leq i \leq k\} \ .$$

Each $[e_1, \ldots, e_k]$ is also closed, as is easily seen. When it will be clear from the context we will write $[e_1, \ldots, e_k]$ instead of $[e_1, \ldots, e_k]_B$. Endowed with this topology, we call X_B the *Bratteli compactum* associated with $B = (V, E, \geq)$. Let d_B be the distance on X_B defined by $d_B((e_n)_n, (f_n)_n) = \frac{1}{2^k}$ where $k = \inf\{i \mid e_i \neq f_i\}$. It clearly coincides with the topology of the cylinder sets.

If (V, E) is a simple Bratteli diagram, then X_B has no isolated points, and so is a Cantor space.

Let $x = (e_1, e_2, \ldots)$ be an element of X_B. We will call e_n the nth label of x and denote it by $x(n)$. We let $X_B^{\max}$ denote those elements x of X_B such that $x(n)$ is a maximal edge for all n and $X_B^{\min}$ the analogous set for the minimal edges. It is not difficult to show that $X_B^{\max}$ and $X_B^{\min}$ are non-empty.

Definition 6.3.3 The ordered Bratteli diagram $B = (V, E, \geq)$ is *properly ordered* if it is simple and if $X_B^{\max}$ and $X_B^{\min}$ both are a one point set: $X_B^{\max} = \{x_{\max}\}$ and $X_B^{\min} = \{x_{\min}\}$.

We can now define a map $V_B : X_B \to X_B$, called the *Vershik map* (or the *lexicographic map*), associated with the properly ordered Bratteli diagram $B = (V, E, \geq)$. We call the resulting pair (X_B, V_B) a *Bratteli-Vershik dynamical system.* It is a Cantor dynamical system. Note that B being simple, (X_B, V_B) is minimal.

We let $V_B(x_{\max}) = x_{\min}$. If $x = (e_1, e_2, \ldots) \neq x_{\max}$, let k be the smallest number such that e_k is not a maximal edge. Let f_k be the successor of e_k (and so $r(e_k) = r(f_k)$). Define $V_B(x) = y = (f_1, \ldots, f_{k-1}, f_k, e_{k+1}, e_{k+2}, \ldots)$, where $(f_1, \ldots, f_{k-1})$ is the minimal edge in $E(1) \circ E(2) \circ \cdots \circ E(k-1)$ with range equal to $s(f_k)$.

In the sequel BV will refer to *Bratteli-Vershik.*

6.4 The Bratteli-Vershik model theorem

In what follows (X, T) will always refer to a minimal Cantor dynamical system. We will give all the details of the main result of (Herman, Putnam, and Skau 1992) saying that (X, T) can be topologically realised as a BV-dynamical system. We recall that A. M. Vershik obtained in (Vershik 1985) such a result in a measure-theoretic context.

6.4.1 Existence of Kakutani-Rohlin partitions

A *clopen partition* $\mathcal{P}$ of a set X is a partition whose elements (also called *atoms*) are clopen sets. Note that X being compact these partitions are finite.

Definition 6.4.1 A *Kakutani-Rohlin partition* of the minimal Cantor dynamical system (X, T) is a clopen partition $\mathcal{P}$ of the form:

$$\mathcal{P} = \{T^j C_k \mid k \in V,\ 0 \leq j < h_k\} \ ,$$

where V is a finite set, C_k is a clopen set and h_k is a positive integer. The *tower* k of $\mathcal{P}$ is $\{T^j C_k \mid 0 \leq j < h_k\}$, the set $T^j C_k$ is the jth *level* of the tower k and the *base* of $\mathcal{P}$ is the set $C := \cup_{k \in V} C_k$. The *height* of the tower k is h_k.

Figure 6.6 illustrates the notion of tower. In the sequel we will refer to such a partition as a KR-partition. Let us show such partitions can be chosen to be finer than any given clopen partition. We follow the details given in (Putnam 1989).

Proposition 6.4.2 *Let $\mathcal{Q}$ be a clopen partition of X and C be a clopen set. Then, there exist a clopen partition C_1, ..., C_t, of C and some integers $(h_i)_{1 \leq i \leq t}$ such that*

$$\mathcal{P} = \left\{T^j C_i \mid 0 \leq j < h_i, 1 \leq i \leq t\right\}$$

is finer than $\mathcal{Q}$.

Proof The first entrance time map r_C is continuous and C is compact hence it takes finitely many values: $r_1, r_2, \ldots, r_{t'}$. For all $i \in [1, t']$, we define the clopen set $C'_i = T^{-r_i} C$. Then, the collection

$$\mathcal{P}' = \left\{T^j C'_i \mid 0 \leq j < r_i, 1 \leq i \leq t'\right\}$$

is a clopen partition of X but it is not necessarily finer than $\mathcal{Q}$. Let $\mathcal{Q}' = \{P' \cap Q \mid P' \in \mathcal{P}', Q \in \mathcal{Q}\}$. It suffices to find $\mathcal{P}$ finer than $\mathcal{Q}'$. Let Q'

be an atom of $\mathcal{Q}'$. There exists a unique pair (i_0, j_0) with $i_0 \in [1, t']$ and $j_0 \in [0, r_{i_0} - 1]$ such that $Q' \subseteq T^{j_0} C'_{i_0}$. We divide the tower i_0 into two new towers and obtain a new KR-partition $\mathcal{P}''$ with $t' + 1$ towers:

$$\begin{array}{rl} \mathcal{P}'' = & \left\{T^j C'_i \mid 0 \leq j < r_i, i \in [1, t'] \setminus \{i_0\}\right\} \\ \bigcup & \left\{T^j \left(Q'\right) \mid -j_0 \leq j < r_{i_0} - j_0\right\} \\ \bigcup & \left\{T^j \left(\left(T^{j_0} C'_{i_0}\right) \setminus Q'\right) \mid -j_0 \leq j < r_{i_0} - j_0\right\} \ . \end{array}$$

We repeat this procedure with the new partition $\mathcal{P}''$ and a new atom of $\mathcal{Q}'$. There are finitely many steps and it ends with the KR-partition we are looking for. □

The following theorem is fundamental to represent minimal Cantor dynamical systems by BV-dynamical systems.

Theorem 6.4.3 *Let $x \in X$. There exists a sequence of KR-partitions $(\mathcal{P}(n))_n$ with*

$$\mathcal{P}(n) := \left\{T^j B_i(n) \mid 0 \leq j < h_i(n),\ 1 \leq i \leq t(n)\right\}$$

satisfying

(KR1) $\bigcap_n \cup_{1 \leq i \leq t(n)} B_i(n) = \{x\}$*,*
(KR2) $\mathcal{P}(n+1)$ *is finer than* $\mathcal{P}(n)$ *for all* n*,*
(KR3) $\bigcup_n \mathcal{P}(n)$ *generates the topology of* X*.*

Proof We start by choosing a decreasing sequence of clopen sets $(C(n))_n$, whose intersection is $\{x\}$, and an increasing sequence of partitions $(\mathcal{P}'(n))_n$ generating the topology. We apply Proposition 6.4.2 to $\mathcal{Q} = \mathcal{P}'(1)$ and $C = C(1)$. We obtain $\mathcal{P}(1)$. We continue applying Proposition 6.4.2 inductively, for $n \geq 2$, to $C = C(n)$ and

$$\mathcal{Q} = \mathcal{P}'(n) \vee \mathcal{P}(n-1) = \{P' \cap P \mid P' \in \mathcal{P}'(n), B \in \mathcal{P}(n-1)\}$$

to obtain $\mathcal{P}(n)$. This achieves the proof. □

6.4.2 From Kakutani-Rohlin partitions to Bratteli-Vershik representations

In the present section we use Theorem 6.4.3 to obtain the BV-representation of minimal Cantor dynamical systems.

Definition 6.4.4 The properly ordered Bratteli diagram $B = (V, E, \geq)$ is a *BV-representation* of (X, T) if (X_B, V_B) is isomorphic to (X, T).

Let $\left(\mathcal{P}(n) = \left\{T^j B_i(n) \mid 0 \le j < h_i(n), 1 \le i \le t(n)\right\}\right)_n$ be a sequence of KR-partitions of (X, T) satisfying (KR1), (KR2) and (KR3). We also suppose that $\mathcal{P}(0) = \{X\}$. Hence $t(0) = 1$, $h_1(0) = 1$ and $B_1(0) = X$.

Let $V(n) = \{(n, 1), \ldots, (n, t(n))\}$, for $n \ge 0$, and $E(n)$ be the set of quadruples (n, t', t, j) satisfying

$$T^j B_t(n) \subseteq B_{t'}(n-1) \tag{6.1}$$

for $1 \le t' \le t(n-1)$, $1 \le t \le t(n)$, $0 \le j \le h_t(n) - 1$ and $n \ge 1$. The range and source maps are given by

$$s(n, t', t, j) = (n-1, t') \text{ and } r(n, t', t, j) = (n, t) \ . \tag{6.2}$$

Two edges $e_1 = (n_1, t'_1, t_1, j_1)$ and $e_2 = (n_2, t'_2, t_2, j_2)$ are comparable whenever $n_1 = n_2$ and $t_1 = t_2$. In this case we define $e_1 \ge e_2$ if $j_1 \ge j_2$. It is straightforward to verify that $(V, E, \ge)$ is an ordered Bratteli diagram. It is useful to remark, from (6.2), that $((n, t'_n, t_n, j_n))_n$ is an infinite path of $(V, E, \ge)$ if, and only if, $t_{n-1} = t'_n$ for all $n \ge 1$. Hence the paths of the Bratteli diagram have the form $((n, t_{n-1}, t_n, j_n))_n$. Note that (n, t_{n-1}, t_n, j_n) is a minimal edge if, and only if, $j_n = 0$ and is maximal if, and only if, $j_n = h_{t_n}(n) - h_{t_{n-1}}(n-1)$.

For example suppose that $(\mathcal{P}(n))_n$ is a sequence of KR-partitions satisfying (KR1), (KR2) and (KR3) such that

(i) $\mathcal{P}(1) = \{B_1(1), TB_1(1), B_2(1), TB_2(1), T^2B_2(1)\}$ and
(ii) $\mathcal{P}(2) = \left\{T^j B_i(2) \mid 0 \le j < h_i(2), 1 \le i \le t(2)\right\}$ with
  (a) $t(2) = 3$, $h_1(2) = 9$, $h_2(2) = 4$, $h_3(2) = 7$,
  (b) $B_1(2) \subseteq B_1(1)$, $T^2B_1(2) \subseteq B_1(1)$, $T^4B_1(2) \subseteq B_2(1)$, $T^7B_1(2) \subseteq B_1(1)$,
  (c) $B_2(2) \subseteq B_1(1)$, $T^2B_2(2) \subseteq B_1(1)$,
  (d) $B_3(2) \subseteq B_2(1)$, $T^3B_3(2) \subseteq B_1(1)$, $T^5B_3(2) \subseteq B_1(1)$.

This is summarised in Figure 6.5.

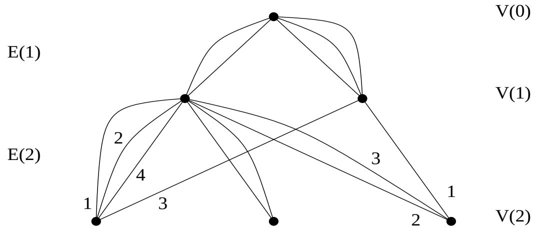

Fig. 6.5. Diagrammatic representation of $\mathcal{P}(0)$, $\mathcal{P}(1)$ and $\mathcal{P}(2)$.

This can be compared to the representation with towers in Figure 6.6.

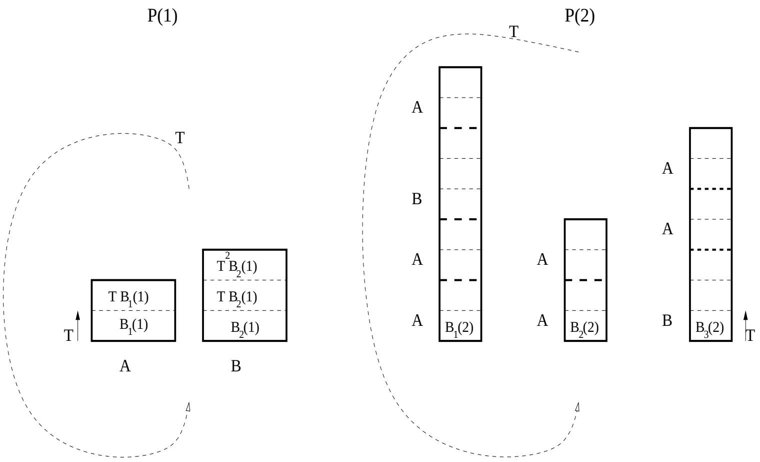

Fig. 6.6. The partition $\mathcal{P}(1)$ consists of two towers called A and B. The dynamics T acts vertically except for the last levels where it goes back to the base. The partition $\mathcal{P}(2)$ can be seen as a concatenation of "vertical pieces" of the towers A and B.

Lemma 6.4.5 *Under the assumptions of this section, the following are equivalent:*

(i) $((n, t_{n-1}, t_n, j_n))_n$ *is an infinite path of* $(V, E, \geq)$,
(ii) $\bigcap_{n\geq 1} T^{\sum_{i=1}^{n} j_i} B_{t_n}(n) \neq \emptyset$ *with* (n, t_{n-1}, t_n, j_n) *satisfying* (6.1),
(iii) $\mathrm{Card}(\bigcap_{n\geq 1} T^{\sum_{i=1}^{n} j_i} B_{t_n}(n)) = 1$ *with* (n, t_{n-1}, t_n, j_n) *satisfying* (6.1).

Moreover, for such an infinite path we have $0 \leq \sum_{i=1}^{n} j_i \leq h_{t_n}(n) - 1$.

Proof [Sketch] (From (i) we deduce (ii) using Baire's Theorem. From (ii) we deduce (iii) because the sequence of partitions generates the topology. The last implication is easy to establish). □

We can now state and prove the BV-representation theorem.

Theorem 6.4.6 (Herman, Putnam, and Skau 1992) *There exists a properly ordered Bratteli diagram* $B = (V, E, \geq)$ *such that* (X, T) *is isomorphic to* (X_B, V_B).

Moreover, any contraction of B *yields a BV-representation of* (X, T) *and some of them have incidence matrices with positive entries.*

Proof Let $(\mathcal{P}(n))_n$ and $B = (V, E, \geq)$ be as defined above. Let us show that $X_B^{\min}$ consists of a single path. Let $(n, t_{n-1}, t_n, j_n)_n$ be an infinite path of $X_B^{\min}$. The edges comparable to (n, t_{n-1}, t_n, j_n) are the edges of the form (n, t, t_n, j) and exactly one of them is $(n, t, t_n, 0)$ for some t. It is clearly a minimal edge. Hence $j_n = 0$ for all n. But

$$\emptyset \neq \bigcap_n T^0 B_{t_n}(n) \subseteq \bigcap_n B(n)$$

which consists of a single point. Consequently $((n, t_{n-1}, t_n, 0))_n$ is the unique path of $X_B^{\min}$. The proof for $X_B^{\max}$ is left as an exercise.

Consider the map $\varphi : X_B \to X$ defined by

$$\varphi((n, t_{n-1}, t_n, j_n)_n) = x \text{ where } \{x\} = \bigcap_{n\geq 1} T^{\sum_{i=1}^n j_i} B_{t_n}(n) \ .$$

It is well defined (Lemma 6.4.5) and is a homeomorphism (see Exercise 6.7). Note that $(\mathcal{P}(n))_n$ being a decreasing sequence of partitions, we also have $\{x\} = \bigcap_{n\geq N} T^{\sum_{i=1}^n j_i} B_{t_n}(n)$ for all N.

It remains to show that it commutes with the dynamics. Let $e = (e_n)_n$ be an infinite path of X_B with $e_n = (n, t_{n-1}, t_n, j_n)$. Suppose that e is not the maximal path. Then there exists n_0 such that $V_B(x) = e'_1 \cdots e'_{n_0-1} e'_{n_0} e_{n_0+1} e_{n_0+2} \cdots$ where $e'_n = (n, t'_{n-1}, t'_n, j'_n)$, with $j'_n = 0$, $1 \leq n \leq n_0 - 1$ and $e'_{n_0} = (n_0, t'_{n_0-1}, t_{n_0}, j'_{n_0})$. Note that, for $1 \leq n \leq n_0 - 1$, the edges e_n being maximal we have $j_n = h_{t_n}(n) - h_{t_{n-1}}(n-1)$. Moreover,

$$\begin{aligned} T^{h_{t_n}(n) - h_{t_{n-1}}(n-1)} B_{t_n}(n) &\subseteq B_{t_{n-1}}(n-1) \text{ for all } 1 \leq n \leq n_0 - 1, \\ T^{j_{n_0}} B_{t_{n_0}}(n_0) &\subseteq B_{t_{n_0-1}}(n_0 - 1), \\ T^{j'_{n_0}} B_{t_{n_0}}(n_0) &\subseteq B_{t'_{n_0-1}}(n_0 - 1) \ . \end{aligned}$$

From the definition of V_B we deduce $j'_{n_0} = j_{n_0} + h_{t_{n_0-1}}(n_0 - 1)$. Hence $\sum_{1\leq n\leq n_0} j'_n = j_{n_0} + h_{t_{n_0-1}}(n_0 - 1)$ and

$$\sum_{1\leq n\leq n_0} j_n = j_{n_0} + \sum_{1\leq n\leq n_0} h_{t_n}(n) - h_{t_{n-1}}(n-1) = j_{n_0} + h_{t_{n_0-1}}(n_0 - 1) - 1 \ .$$

Suppose now that e is the maximal path. Let $x_{\min}$ be the minimal path of B. Then we have to prove that $\varphi(x_{\min}) = T(\varphi(e))$. But as $\mathcal{P}(0) = \{X\}$, we have $h_{t_0}(0) = 1$ and consequently

$$\begin{aligned} T(\varphi(\{e\})) &= T\left(\bigcap_{n\geq 1} T^{\sum_{i=1}^n h_{t_i}(i) - h_{t_{i-1}}(i-1)} B_{t_n}(n) \right) \\ &= \bigcap_{n\geq 1} T^{h_{t_n}(n)} B_{t_n}(n) \subseteq \bigcap_{n\geq 1} \bigcup_{1\leq i\leq t(n)} B_i(n) = \{\varphi(x_{\min})\} \ . \end{aligned}$$

This achieves the main part of the proof. The last part is left as an exercise. ∎

6.4.3 Kakutani equivalence

Definition 6.4.7 The minimal Cantor dynamical systems (X,T) and (Y,S) are *Kakutani equivalent* if they have (up to isomorphism) a common induced system, *i.e.*, there exist clopen sets $U \subseteq X$ and $V \subseteq Y$ such that the induced systems (X_U, T_U) and (Y_V, S_V) are isomorphic.

Let us relate Kakutani equivalence to Bratteli diagrams.

If $B = (V, E, \geq)$ is a properly ordered Bratteli diagram we may change B into a new properly ordered Bratteli diagram $B' = (V', E', \geq')$ by making a finite change, *i.e.*, by adding and/or removing any finite number of edges (vertices), and then making arbitrary choices of linear orderings of the edges meeting at the same vertex (for a finite number of vertices). So B and B' are cofinally identical, *i.e.*, they only differ on finite initial portions. (Observe that this defines an equivalence relation on the family of properly ordered Bratteli diagrams.) We have the following nice characterisation of the Kakutani equivalence.

Theorem 6.4.8 (Herman, Putnam, and Skau 1992) *Let (X_B, V_B) be the BV-dynamical system associated with the properly ordered Bratteli diagram $B = (V, E, \geq)$. Then the minimal Cantor dynamical system (X,T) is Kakutani equivalent to (X_B, V_B) if, and only if, (X,T) is isomorphic to $(X_{B'}, V_{B'})$, where $B' = (V', E', \geq')$ is obtained from B by a finite change as described above.*

An interesting consequence of this result is the following. Let U be a clopen set of (X_B, V_B). It is a finite union of cylinder sets. We can suppose that they all have the same length, *i.e.*, for some n, $U = \cup_{p \in P}[p]_B$ where P is a set of paths from level n to level 0. To obtain a BV-representation of the induced system on U it suffices to take the properly ordered Bratteli diagram B' which consists of all the paths starting with an element of P endowed with the induced ordering. It is not to much work to prove that the induced system on U is isomorphic to $(X_{B'}, V_{B'})$.

The following result was proved in (Holton and Zamboni 1999). It was also obtained in (Durand 1998b) but not for all cylinders, only for those coming from the prefixes of the fixed points.

Theorem 6.4.9 *Minimal substitution subshifts have finitely many induced subshifts on cylinders up to isomorphism.*

This can be considered as a symbolic counterpart to a theorem of M. Boshernitzan and C. R. Carroll (see (Boshernitzan and Carroll 1997)) which states that an interval exchange transformation defined over a quadratic field has only finitely many induced systems (with respect to some inducing procedure).

According to what has been said before, it is not difficult to show Theorem 6.4.9 does not hold for clopen sets instead of cylinders.

6.5 Examples of BV-models

In this section we give examples of BV-representations for some classical dynamical systems.

6.5.1 Odometers

Let (p_n) be a strictly increasing sequence such that p_n divides p_{n+1} for all n. We endow $\prod_{n=0}^{+\infty} \mathbb{Z}/p_n\mathbb{Z}$ with the product topology of the discrete topologies. The set of (p_n)-adic integers is the inverse limit

$$\mathbb{Z}_{(p_n)} = \left\{ (x_n) \in \prod_{n=0}^{+\infty} \mathbb{Z}/p_n\mathbb{Z} \mid x_n \equiv x_{n+1} \mod p_n \right\} .$$

We endow $\mathbb{Z}_{(p_n)}$ with the induced topology. It is a compact topological ring (see Exercise 6.4). A base of its topology is given by the sets

$$[a_0, a_1, \ldots, a_m] := \{(x_n) \in \mathbb{Z}_{(p_n)} \mid x_i = a_i, 0 \leq i \leq m\} .$$

When $p_n = p^n$ for all n, it defines the classical ring of p-adic integers $\mathbb{Z}_p$. Let $R : \mathbb{Z}_{(p_n)} \to \mathbb{Z}_{(p_n)}$ be the map $x \mapsto x + 1$. The pair $(\mathbb{Z}_{(p_n)}, R)$ is called *odometer in base* (p_n). It is a minimal dynamical system.

For all n, we set $B_1(n) = [0^{n-1}]$, $h(n) = p_n$ and

$$\mathcal{P}(n) := \{R^j B(n) \mid 0 \leq j \leq h(n) - 1\} .$$

Then $(\mathcal{P}(n))_n$ is a sequence of KR-partitions satisfying (KR1), (KR2) and (KR3) (see Exercise 6.9) Remark that $R^j B(n) = [j_0 j_1 \cdots j_{n-1}]$ where $j_i = j \mod p_i$. The edges of the BV-representation of $(\mathbb{Z}_{(p_n)}, R)$ given in Subsection 6.4.2 are of the form $(n, 1, 1, lp_{n-1})$, $0 \leq l \leq q_n - 1 = \frac{p_n}{p_{n-1}} - 1$.

For example if $p_1 = 2$, $p_2 = 10$ and $p_3 = 30$, the first three levels are given in Figure 6.7.

6.5.2 Substitutions

The next result (Theorem 6.5.1 below) was first proven in (Forrest 1997). The proofs given in that paper are mostly of existential nature and do not state a method to compute effectively the BV-representation associated with substitution systems. Another proof was given in

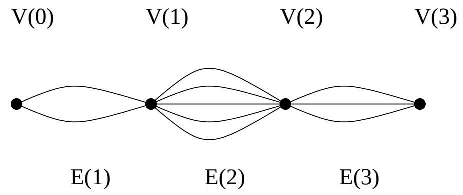

Fig. 6.7. The three first levels of the BV representation of $(\mathbb{Z}_{(p_n)}, R)$.

(Durand, Host, and Skau 1999) that provides such an algorithm. We first need a new definition. Let us note that we work in all this section with bi-infinite words and two-sided shifts.

6.5.2.1 The representation theorem

A Bratteli diagram (V, E) is *stationary* if there exists k such that $k = \mathrm{Card}(V(n))$ for all n, and if (by an appropriate labelling of the vertices) the incidence matrices between level n and $n+1$ are the same $k \times k$ matrix $\mathbf{M}$ for all $n = 1, 2, \ldots$. In other words, beyond level 1 the diagram repeats itself. (Clearly we may label the vertices in $V(n)$ as $v(n, a_1), \cdots, v(n, a_k)$, where $A = \{a_1, \ldots, a_k\}$ is a set of k distinct symbols.)

The ordered Bratteli diagram $B = (V, E, \geq)$ is *stationary* if (V, E) is stationary, and the ordering on the edges with range $v(n, a_i)$ is the same as the ordering on the edges with range $v(m, a_i)$ for $m, n = 2, 3, \ldots$ and $i = 1, \ldots, k$. In other words, beyond level 1 the diagram with the ordering repeats itself.

Theorem 6.5.1 *The family $\mathcal{B}$ of Bratteli-Vershik systems associated with stationary, properly ordered Bratteli diagrams is (up to isomorphism) the disjoint union of the family of substitution minimal systems and the family of stationary odometer systems. Furthermore, the correspondence in question is given by an explicit and algorithmic effective construction.*

In the sequel we describe the algorithm that, starting with a subshift generated by a primitive substitution, gives a stationary BV-representation.

Let B be a stationary ordered Bratteli diagram. The morphism read on $E(n)$ is constant from $n \geq 2$. We call it *the substitution read on B*.

We recall that a subshift is said *periodic* if there exist $x \in X$ and an integer k such that $X = \{x, Sx, \ldots, S^k x = x\}$. Otherwise it is said *aperiodic*.

Proposition 6.5.2 (Durand, Host, and Skau 1999) *Let B be a sta-*

tionary, properly ordered Bratteli diagram with a simple hat, and let $\sigma : A^* \to A^*$ *be the substitution read on* B.

(i) *If* (X_σ, S) *is not periodic, then it is isomorphic to* (X_B, V_B).
(ii) *If* (X_σ, S) *is periodic, then* (X_B, V_B) *is isomorphic to an odometer in base* $(qp^n)_n$, $p, q \in \mathbb{N}$.

6.5.2.2 Combinatorics on words for Bratteli diagrams

Definition 6.5.3 A substitution σ on the alphabet A is *proper* if there exist an integer $p > 0$ and two letters $r, l \in A$ such that, for every $a \in A$, r is the last letter of $\sigma^p(a)$ and l is the first letter of $\sigma^p(a)$.

A proper substitution has exactly one fixed point (in $A^{\mathbb{N}}$ and in $A^{\mathbb{Z}}$). Remark that the substitution read on a stationary properly ordered Bratteli diagram is proper. Consequently, in the light of Proposition 6.5.2, it suffices to find a proper substitution ζ such that (X_ζ, S) is isomorphic to (X_σ, S) to have a stationary BV-representation of a substitution subshift (X_σ, S). In order to find such a substitution we will need the following proposition from (Durand, Host, and Skau 1999) which is a modification of an unpublished result of G. Rauzy. See also Theorem 4.6.1. It has an interest outside of the scope of BV-representations of substitution subshifts. Before stating Proposition 6.5.4, we first need to define the notion of circular code.

Let A be an alphabet, and C a finite subset of A^+. We recall that the set C is a *code* if every word $u \in A^+$ admits at most one decomposition as a concatenation of elements of C. The code C is said to be *circular* if for all words

$$w_1, \ldots, w_j,\ w,\ w'_1, \ldots, w'_k \in C,\ s \in A^+ \text{ and } t \in A^*$$

that satisfy

$$w = ts \text{ and } w_1 \ldots w_j = sw'_1 \ldots w'_k t\ ,$$

then t is the empty word. Note that it follows that $j = k+1$, $w_{i+1} = w'_i$ for $1 \le i \le k$ and $w_1 = s$. One of the interests of circular codes in the present context is that they display a unique decomposition property for sequences. Suppose indeed that C is a circular code on the alphabet A, and that some $x \in A^{\mathbb{Z}}$ can be decomposed as a concatenation of words $(w_k)_{k\in\mathbb{Z}}$ belonging to C, *i.e.*,

$$x = \ldots w_{-3}w_{-2}w_{-1}.w_0w_1w_2\ldots$$

Then this decomposition is unique.

Proposition 6.5.4 *Let* $y \in R^{\mathbb{N}}$ *be a fixed point of a primitive substitution*

τ on the alphabet R, A an alphabet, $\varphi\colon R^* \to A^*$ a non-erasing morphism, $x = \varphi(y)$, and (X, S) the subshift spanned by x.

There exist a primitive substitution ζ on an alphabet B, an admissible fixed point z of ζ and a map $\theta\colon B \to A$ such that:

(i) $\theta(z) = x$,
(ii) *If φ is injective and $\varphi(R)$ is a circular code, then θ is an isomorphism from (X_ζ, S) to (X, S),*
(iii) *If τ is proper, then ζ is proper.*

Proof The proof below is very simple, but the notations are, in an unavoidable way, a bit heavy. By substituting τ by a power of itself if needed, we can assume that $|\tau(j)| \geq |\varphi(j)|$ for all $j \in R$. We define:

- an alphabet B by $B = \{(j,p) \mid j \in R,\ 1 \leq p \leq |\varphi(j)|\}$,
- a map $\theta\colon B \to A$ by $\theta(j,p) = (\varphi(j))_p$,
- a map $\gamma : R \to B^+$ by $\gamma(j) = (j,1)(j,2)\cdots(j,|\varphi(j)|)$.

Clearly $\theta \circ \gamma = \varphi$. We define a substitution ζ on B as follows. For j in R and $1 \leq p \leq |\varphi(j)|$, we set

$$\zeta(j,p) = \begin{cases} \gamma\Big((\tau(j))_p\Big) & \text{if} \quad 1 \leq p < |\varphi(j)| \\ \gamma\Big((\tau(j))_{[\![\varphi(j),\tau(j)]\!]}\Big) & \text{if} \quad p = |\varphi(j)| \ . \end{cases}$$

Hence, for every $j \in R$, $\zeta\left(\gamma(j)\right) = \zeta(j,1)\cdots\zeta(j,|\varphi(j)|) = \gamma\left(\tau(j)\right)$, *i.e.*,

$$\zeta \circ \gamma = \gamma \circ \tau \tag{6.3}$$

and it follows that

$$\zeta^n \circ \gamma = \gamma \circ \tau^n \text{ for all } n \geq 0 \ .$$

We claim that ζ is primitive. Let n be an integer such that b occurs in $\tau^n(a)$ for all $a, b \in R$. Let (j,p) and (k,q) belong to B. By construction, $\zeta(j,p)$ contains $\gamma\left(\tau(j)_p\right)$ as a factor, thus $\zeta^{n+1}(j,p)$ contains $\zeta^n\left(\gamma\left(\tau(j)_p\right)\right) = \gamma\left(\tau^n\left(\tau(j)_p\right)\right)$ as a factor. By the choice of n, k occurs in $\tau^n\left(\tau(j)_p\right)$, thus $\gamma(k)$ is a factor of $\gamma\left(\tau^n\left(\tau(j)_p\right)\right)$, and also of $\zeta^{n+1}(j,p)$. As (k,q) is a letter of $\gamma(k)$, (k,q) occurs in $\zeta^{n+1}(j,p)$ and our claim is proved.

Let $z = \gamma(y)$. By (6.3) we get $\zeta(z) = \gamma(\tau(y)) = \gamma(y) = z$, and z is a fixed point of ζ. By construction, z is uniformly recurrent, thus it is an admissible fixed point of ζ. Moreover, $\theta(z) = \theta(\gamma(y)) = \varphi(y) = x$, and (i) is proved.

Proof of (ii). As θ commutes with the shift and maps z to x, and by minimality of the subshifts, it maps X_ζ onto X. It remains to prove that $\theta : X_\zeta \to X$ is one-to-one. Let $\alpha \in X$. By definition of X, there exist

$t \in X_\tau$ and an integer p, with $0 \leq p < |\varphi(t_0)|$, such that $\alpha = S^p\varphi(t)$. Let β be an element of X_ζ with $\theta(\beta) = \alpha$. By definition of γ, there exist some $\delta \in X_\tau$ and some integer q, with $0 \leq q < |\gamma(\delta_0)|$, such that $\beta = S^q\gamma(\delta)$. It follows that $S^q\varphi(\delta) = \theta(\beta) = \alpha = S^p\varphi(t)$. As $0 \leq q < |\gamma(\delta_0)| = |\varphi(\delta_0)|$ by construction of γ, since $\varphi(R)$ is a circular code and since φ is injective, it follows that $\delta = t$ and $q = p$, thus $\beta = S^p\gamma(t)$: β is uniquely determined by α, and θ is one-to-one.

Proof of (iii). Let $l \in R$ be the letter such that l is the first letter of $\tau(k)$ for every $k \in R$. Let $(j,p) \in B$, and $k = \tau(j)_p$. By definition of ζ, the first letter of $\zeta(j,p)$ is $(k,1)$, and the first letter of $\zeta^2(j,p)$ is the first letter of $\zeta(k,1)$, *i.e.*, $(l,1)$. By the same method, if r is the last letter of $\tau(k)$ for every $k \in R$, then the last letter of $\zeta^2(j,p)$ is $(r, |\varphi(r)|)$ for every $(j,p) \in B$. □

In (Vershik and Livshits 1992) the authors showed that when σ is a primitive substitution then the subshift it generates can be represented (in a measure-theoretic sense) by an ordered Bratteli diagram B where σ is the substitution we read on B. For example, in the case of the *Morse substitution* $a \mapsto ab$, $b \mapsto ba$ the Bratteli diagram they consider is given in Figure 6.8.

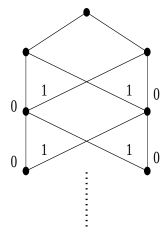

Fig. 6.8. The Bratteli diagram for the Thue–Morse substitution.

It is clear that it has two maximal and two minimal paths. Hence this representation does not fit our settings.

Given a primitive substitution σ on the alphabet A such that (X_σ, S) is aperiodic, let us describe how to construct a primitive proper substitution ζ such that (X_σ, S) is isomorphic to (X_ζ, S). With the techniques used below, we do not need to suppose primitivity. We only need σ to generate a minimal subshift. For example, this includes the Chacon substitution $0 \mapsto 0010$ and $1 \mapsto 1$. An illustration of the construction given below is given in Section 6.5.2.4.

6.5.2.3 Return words

In order to find ζ we need to introduce the notion of return words. For more details, the reader is referred *e.g.*, (Durand 1998b), (Durand 2000), (Durand 2003). We define an *occurrence of* $u.v$ *in the bi-infinite word* x to be an integer n such that $x_{[\![n-|u|,n+|v|-1]\!]} = uv$. A finite word w on A is a *return word to* $u.v$ in x if there exist two consecutive occurrences j, k of $u.v$ in x such that $w = x_{[\![j,k-1]\!]}$.

Let $x \in A^{\mathbb{Z}}$ be one of the fixed points of σ. We write $r = x_{-1}$ and $l = x_0$, so that r is the last letter of $\sigma(r)$ and l the first letter of $\sigma(l)$.

We denote by $L(\sigma)$ the set of finite words appearing in X_σ and we call it the *language* of σ. Let $\mathcal{R}$ be the set of return words to $r.l$. By minimality, it is a finite set. We set $R = \{1, \dots, \mathrm{Card}(\mathcal{R})\}$ and let $\varphi : R \to \mathcal{R}$ be the bijection defined as follows: let $\mathcal{R}$ be ordered according to the rank of first occurrence in $x_{[0,+\infty)}$, and $\varphi(k)$ defined to be the kth element of $\mathcal{R}$ for this order. Let z be the unique element of $R^{\mathbb{Z}}$ such that $\varphi(z) = x$.

We define now a substitution τ on the alphabet R. Let $j \in R$, and $w = w_1 \dots w_k = \varphi(j) \in \mathcal{R}$. We have $rwl \in L(\sigma)$, $w_1 = l$ and $w_k = r$. As $\sigma(x) = x$, the finite word $\sigma(rwl) = \sigma(r)\sigma(w)\sigma(l)$ belongs also to $L(\sigma)$. But r is the last letter of $\sigma(r)$, and l the first of $\sigma(l)$, consequently, $r\sigma(w)l$ belongs to $L(\sigma)$. Note also that the first letter of $\sigma(w)$ is l and its last letter r. Then, rl is a prefix and also a suffix of $r\sigma(w)l$. Therefore, the finite word $\sigma(w)$ appears in x between two occurrences of $r.l$, thus it is a concatenation of return words, *i.e.*, it belongs to $\varphi(R^+)$. Then, there exists a unique finite word $u \in R^+$ such that $\sigma(w) = \varphi(u)$.

We define $\tau(j) = u$. This defines a substitution τ on the alphabet R, characterised by

$$\varphi \circ \tau = \sigma \circ \varphi \ .$$

It follows that $\varphi \circ \tau^n = \sigma^n \circ \varphi$ for each $n \geq 0$.

The first element in the decomposition of $x_{[0,+\infty)}$ in return words is $\varphi(1)$, *i.e.*, $\varphi(1)l$ is a prefix of $x_{[0,+\infty)}$. Let n be large enough for $|\sigma^n(l)| > |\varphi(1)|$. As $\sigma^n(l)$ and $\varphi(1)l$ are both prefixes of $x_{[0,+\infty)}$, $\varphi(1)l$ is a prefix of $\sigma^n(l)$. Let $j \in R$, and $w = \varphi(j)$. As l is the first letter of w, $\sigma^n(l)$ is a prefix of $\sigma^n(w)$, and so does $\varphi(1)l$. It follows that $\varphi(1)$ is the first element in the decomposition of $\sigma^n(w) = \varphi(\tau^n(j))$ in a concatenation of return words, *i.e.*, 1 is the first letter of $\tau^n(j)$.

Let $m = z_{-1}$. The word $r\varphi(m)$ is a suffix of $x_{(-\infty,-1]}$, and the same argument shows that, for every large enough n and every $j \in R$, m is the last letter of $\tau^n(j)$. This implies that τ is proper.

Let $k > 0$ be an occurrence of $r.l$ large enough for every return word $w \in \mathcal{R}$ to appear in the decomposition of $x_{[0,k)}$, *i.e.*, for every $j \in R$ to occur in the finite word $u \in R^+$ defined by $\varphi(u) = x_{[0,k)}$. Let n be so large that $|\sigma^n(l)| > k$. Let $i, j \in R$. As above, $x_{[0,k)}l$ is a prefix of $\sigma^n(l)$, which is a prefix of $\sigma^n(\varphi(i)) = \varphi(\tau^n(i))$. Thus u is a prefix of $\tau^n(i)$, and j occurs in $\tau^n(i)$, hence τ is primitive. Moreover,

$$\varphi(\tau(z)) = \sigma(\varphi(z)) = \sigma(x) = x = \varphi(z) ,$$

thus $\tau(z) = z$ by the unique decomposition property, and z is the unique fixed point of τ. As $\varphi(z) = x$ is not periodic, z is not periodic.

Proposition 6.5.4 gives, using Proposition 6.5.2, the substitution ζ we are looking for: (X_σ, S) is isomorphic to (X_B, V_B) where B has a simple hat and ζ is read on B.

6.5.2.4 An example: the Chacon substitution

Let us illustrate this on the non-primitive case given by the Chacon substitution σ. For more details about this substitution, see (Ferenczi 1995).

Let $x = \sigma^\omega(0.0)$. This is the *Chacon word*. It is uniformly recurrent. Using the return words to 0.0 we see that $x = \varphi(y)$ where $\varphi : \{a, b, c\}^* \to \{0, 1\}^*$ is defined by $\varphi(a) = 0$, $\varphi(b) = 010$ and $\varphi(c) = 01010$, and $y = \tau^\omega(b.a)$ where τ is defined by $\tau(a) = ab$, $\tau(b) = acb$ and $\tau(c) = accb$. According to the proof of Proposition 6.5.4 we need to take τ^2 instead of τ and we take

(i) $B = \{(a, 1), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3), (c, 4), (c, 5)\}$,
(ii) $\theta : B \to \{0, 1\}$ given by the following table

α	$(a,1)$	$(b,1)$	$(b,2)$	$(b,3)$	$(c,1)$	$(c,2)$	$(c,3)$	$(c,4)$	$(c,5)$
$\theta(\alpha)$	0	0	1	1	0	1	0	1	0

(iii) $\gamma : \{a, b, c\} \to B^+$ defined by

$$\gamma(a) = (a, 1),\ \gamma(b) = (b, 1)(b, 2)(b, 3),\ \gamma(c) = (c, 1) \cdots (c, 5) ,$$

(iv) the substitution $\zeta : B^* \to B^*$ defined by the following tables

α	$(a,1)$	$(b,1)$	$(b,2)$	$(b,3)$
$\zeta(\alpha)$	$\gamma(\tau(a))$	$\gamma(a)$	$\gamma(b)$	$\gamma(accbacb)$

α	$(c,1)$	$(c,2)$	$(c,3)$	$(c,4)$	$(c,5)$
$\zeta(\alpha)$	$\gamma(a)$	$\gamma(b)$	$\gamma(a)$	$\gamma(c)$	$\gamma(cbaccbacb)$

A BV-representation of the Chacon subshift (*i.e.*, generated by x) is isomorphic to (X_B, V_B) where B a stationary properly ordered Bratteli diagram with a simple hat such that ζ is the substitution we read on it.

6.5.2.5 Generality in the Chacon example

Let $\sigma : A \to A^*$ be a substitution and consider the sets

$$L(\sigma) = \{w \in A^* \mid w \text{ is a factor of some } \sigma^n(a), a \in A, n \in \mathbb{N}\} \text{ and}$$
$$\Omega(\sigma) = \{x \in A^{\mathbb{Z}} \mid x_i x_{i+1} \cdots x_j \in L(\sigma), i, j \in \mathbb{Z}\} \ .$$

For $\Omega(\sigma)$ to be non-empty, it is necessary and sufficient that there exists $e \in A$ with

$$\lim_{n \to +\infty} |\sigma^n(e)| = +\infty \ . \tag{6.4}$$

Without lost of generality we suppose that all letters of A have an occurrence in some element of $\Omega(\sigma)$ and that

$$\forall a \in A,\ \exists n \in \mathbb{N},\ |\sigma^n(a)|_a \geq 1 \ . \tag{6.5}$$

In the paper (Damanik and Lenz 2006) the authors propose to call *substitution subshift* any pair $(\Omega(\sigma), S)$ where σ satisfies (6.4), (6.5) and $L(\sigma) = L(\Omega(\sigma))$, where $L(\Omega(\sigma))$ is the set of factors of elements of $\Omega(\sigma)$. They prove the following result (the definition of linearly recurrent subshift is given in Section 6.5.3.

Theorem 6.5.5 (Damanik and Lenz 2006) *Let $(\Omega(\sigma), S)$ be a substitution subshift. Then the following are equivalent.*

(i) *There exists $e \in A$ satisfying* (6.4) *which occurs with bounded gaps and furthermore, σ is assumed to satisfy* (6.5).
(ii) *$(\Omega(\sigma), S)$ is minimal.*
(iii) *$(\Omega(\sigma), S)$ is linearly recurrent.*

A fourth equivalent statement could be added: $(\Omega(\sigma), S)$ is uniquely ergodic.

The longest part of the proof given by D. Damanik and D. Lenz is to prove that (ii) implies (iii). The techniques we used before permit to give an alternative proof and even to prove a bit more.

Proposition 6.5.6 *Let $(\Omega(\sigma), S)$ be a substitution subshift. Then, it is minimal if, and only if, it is isomorphic to some $(\Omega(\tau), S)$ where τ is a primitive proper substitution.*

6.5.3 Linearly recurrent subshifts

In what follows we show that a subshift is linearly recurrent if, and only if, it has a BV-representation where the incidence matrices have positive entries and belong to a finite set.

Before we need to introduce some new definitions.

Let x be an element of $A^{\mathbb{Z}}$. As we will manipulate return words, it is important to observe that a finite word $w \in A^+$ is a return word to $u.v$ in x if, and only if,

(i) uwv has an occurrence in x, and
(ii) v is a prefix of wv and u is a suffix of uw, and
(iii) the finite word uwv contains exactly two occurrences of the finite word uv.

We denote by $\mathcal{R}_{x,u.v}$ the set of return words to $u.v$ in x. If u is the empty word ε, then we speak of return words to v instead of the return words to $u.v$ and we set $\mathcal{R}_{x,v} = \mathcal{R}_{x,u.v}$. When it will be clear from the context we will refer to $\mathcal{R}_{u.v}$ in place of $\mathcal{R}_{x,u.v}$.

Definition 6.5.7 We say that $x \in A^{\mathbb{Z}}$ is *linearly recurrent* (LR) (with constant $K \in \mathbb{N}$) if it is uniformly recurrent and if for all u having an occurrence in x and all return words w to u in x we have $|w| \leq K|u|$. A subshift is *linearly recurrent* whenever it is the shift orbit closure of a linearly recurrent word.

From Theorem 6.5.5 we know that all minimal substitution subshifts are linearly recurrent.

Remark that all words of a LR subshift are LR with the same constant. Here are some important properties proved in (Durand, Host, and Skau 1999).

Proposition 6.5.8 *Let x be a non-periodic word and suppose that it is linearly recurrent with constant K. Then:*

(i) *The number of distinct factors of length n of x is less than or equal to Kn.*
(ii) *x is $(K+1)$-power free: u^{K+1} has an occurrence in x if, and only if, u is the empty word.*
(iii) *For all u having an occurrence in x and for all $w \in \mathcal{R}_u$ we have $\frac{1}{K}|u| < |w|$.*
(iv) *If u has an occurrence in x then $|\mathcal{R}_u| \leq K(K+1)^2$.*

We suppose now that x is a uniformly recurrent word. It is easy to see that for all $u, v \in L(x)$ the set $\mathcal{R}_{x,u.v}$ is finite.

It will be convenient to label the return words. We enumerate the elements w of $\mathcal{R}_{x,u.v}$ in the order of the first appearance of uwv in $x_{[-|u|,+\infty)}$. This defines a bijective map $\Theta_{x,u.v} : R_{x,u.v} \to \mathcal{R}_{x,u.v} \subset A^+$ where

$R_{x,u.v} = \{1, \ldots, \mathrm{Card}(\mathcal{R}_{x,u.v})\}$, and $u\Theta_{x,u.v}(k)v$ is the kth finite word of the type uwv ($w \in \mathcal{R}_{x,u.v}$) that occurs in $x_{[-|u|,+\infty)}$. We consider $R_{x,u.v}$ as an alphabet. The map $\Theta_{x,u.v}$ defines a morphism from $R_{x,u.v}$ to A^* and the set $\Theta_{x,u.v}(R^*_{x,u.v})$ consists of all concatenations of return words to $u.v$.

In (Durand, Host, and Skau 1999) is proved the following result we will used in the sequel.

Proposition 6.5.9 *The map* $\Theta_{x,u.v} : R^+_{x,u.v} \to A^+$ *is one-to-one.*

The following result characterises linearly recurrent subshifts in terms of BV-representations. We first need to recall that a dynamical system (X, T), endowed with the distance d, is *expansive* if there exists ε such that for all pairs of points (x, y), $x \neq y$, there exists n with $d(T^n x, T^n y) \geq \varepsilon$. We say that ε is a *constant of expansivity* of (X, T). It is easy to show that the subshifts are expansive and that, in fact, every expansive dynamical system is isomorphic to a subshift (see (Kůrka 2003) for example).

Theorem 6.5.10 *A subshift is linearly recurrent if, and only if, it has an expansive BV-representation satisfying:*

(i) *its incidence matrices have positive entries and belong to a finite set of matrices,*
(ii) *for all* $n \geq 1$ *the substitution read on* $E(n)$ *is proper.*

Proof Let (X, S) be a LR subshift. The periodic case is trivial hence we suppose that (X, S) is not periodic. It suffices to construct a sequence of KR-partitions having the desired properties.

From Proposition 6.5.8 there exists an integer K such that for all u occurring in some $x \in X$ and all $w \in \mathcal{R}_u$, we have

$$\frac{|u|}{K} \leq |w| \leq K|u| \ .$$

We set $\alpha = K^2(K+1)$. Let $x = (x_n)_n$ be an element of X. For each non-negative integer n, we set $u_n = x_{-\alpha^n} \cdots x_{-2}x_{-1}$, $v_n = x_0 x_1 \cdots x_{\alpha^n - 1}$, $\mathcal{R}_n = \mathcal{R}_{x,u_n.v_n}$, $R_n = R_{x,u_n.v_n}$ and $\Theta_n = \Theta_{x,u_n.v_n}$.

Now define for all n

$$\mathcal{P}(n) = \left\{ S^j[u_n.wv_n] \mid w \in \mathcal{R}_n,\ 0 \leq j < |w| \right\} \ .$$

We claim $(\mathcal{P}(n))_n$ is a sequence of KR-partitions having the desired properties. We leave the details to the reader. They can be found in (Durand 2003).

Now let B be a properly ordered Bratteli diagram satisfying (i) and (ii).

Let ε be a constant of expansivity of X_B. Let n_0 be a level of B such that all cylinders $[e_1, \ldots, e_{n_0-1}]$ are included in a ball of radius $\varepsilon/2$. For any vertex $v \in V(n_0-1)$ let h_v denote the number of edges from v to $V(0)$. Now consider the alphabet

$$A = \{(v,j) \mid v \in V(n_0-1),\ 0 \leq j < h_v\} ,$$

the map

$$\begin{array}{llll} C: & X_B & \to & A \\ & (e(n))_n & \mapsto & (r(e_{n_0-1}), j) , \end{array}$$

if $(e(n))_{1\leq n\leq n_0-1}$ is the jth finite path of B from $r(e_{n_0-1})$ to $V(0)$ with respect to the order on B, and finally define

$$\begin{array}{llll} \varphi: & X_B & \to & A^{\mathbb{Z}} \\ & x & \mapsto & (C \circ V_B^n(x))_{n\in\mathbb{Z}} . \end{array}$$

We clearly have that (X_B, V_B) is isomorphic to (Ω, S) where $\Omega = \varphi(X_B)$ and S is the shift on A. It remains to show that (Ω, S) is LR.

Let $K = \sup_{n\geq n_0} \max_{v\in V(n)} \sum_{v'\in V(n-1)} \mathbf{M}(n)_{v,v'}$. Condition (i) implies that K is finite. Let $L = \max_{v,v'\in V(n_0-1)} \frac{h_v}{h_{v'}}$.

Let $v \in V(n_0-1)$ and w be a return word to $u = (v,0)(v,1)\cdots,(v,h_v-1)$. Due to Condition (i) and Condition (ii) we have

$$|w| \leq 2K \max_{v'\in V(n_0-1)} h_{v'} \leq 2KL|u| . \tag{6.6}$$

Now for all $n \geq n_0$, let $\tau_n : V(n) \to V(n-1)^*$ be the substitution read on $E(n)$. We set $W = \{(v,0)(v,1)\cdots,(v,h_v-1) \mid v \in V(n_0-1)\}$ and we define the morphism $\sigma : V(n_0-1) \to A^*$ by $\sigma(v) = (v,0)(v,1)\cdots,(v,h_v-1)$.

It is clear that all the elements of Ω are concatenations of finite words belonging to W. They are also concatenations of finite words belonging to $\sigma\circ\tau_{n_0}(V(n_0))$, and more generally, of finite words belonging to $\sigma\circ\tau_{n_0}\circ\cdots\circ\tau_n(V(n))$ for all $n \geq n_0$. As for (6.6), we can prove that all return words to some elements of $\sigma\circ\tau_{n_0}\circ\cdots\circ\tau_n(V(n))$ satisfy the same inequality.

Now let u be any non-empty finite word appearing in some word of Ω and w be a return word to u. There exists n such that

$$\max_{v\in V(n)} |\sigma\circ\tau_{n_0}\circ\cdots\circ\tau_n(v)| \leq |u| < \max_{v\in V(n+1)} |\sigma\circ\tau_{n_0}\cdots\circ\tau_{n+1}(v)| .$$

Then u is a factor of some $\sigma\circ\tau_{n_0}\circ\cdots\circ\tau_{n+1}(vv')$, v and v' belonging to $V(n+1)$. From Condition (i) and Condition (ii), we deduce vv' is a factor of some $\tau_{n+2}\circ\tau_{n+3}(v'')$, $v'' \in V(n+3)$. Then, u is a factor of $\sigma\circ\tau_{n_0}\circ\cdots\circ\tau_{n+3}(v'')$. Consequently

$$|w| \leq 2KL|\sigma\circ\tau_{n_0}\circ\cdots\circ\tau_{n+3}(v'')| \leq 2KL^4|u|$$

and (Ω, S) is LR. □

Let us call *linearly recurrent* any Cantor dynamical systems having a BV-representation satisfying (i) and (ii) in Theorem 6.5.10. In fact more can be said about these dynamical systems but we need the following theorem proved in (Downarowicz and Maass 2008). It can be seen as an extension of Proposition 6.5.2. We say that a minimal Cantor dynamical system (X,T) has *topological rank* k if k is the smallest integer such that (X,T) has a BV-representation (X_B, V_B) with the sequence of number of vertices $(\mathrm{Card}(V(n)))_n$ bounded by k. When such a k does not exist, we say that it has *infinite topological rank*. Of course, linearly recurrent BV-dynamical systems have finite topological rank. A topological dynamical system (X,T), endowed with the distance d, is said *equicontinuous* whenever

$$\forall \varepsilon, \exists \delta > 0, \ \sup_{n\in\mathbb{Z}} d(T^n x, T^n y) < \varepsilon \text{ if } d(x,y) < \delta \ .$$

Theorem 6.5.11 *Let (X,T) be a minimal Cantor dynamical system with topological rank $k \in \mathbb{N}$. Then, (X,T) is expansive if, and only if, $k \geq 2$. Otherwise it is equicontinuous.*

Let us take the notation of the proof of Theorem 6.5.10. We call σ_{n_0} the morphism σ and A_{n_0} the alphabet A. Let X_{n_0} be the subset of $A_{n_0}^{\mathbb{Z}}$ consisting of all the words x such that for all i,j, $x_i x_{i+1} \cdots x_j$ is a factor of $\sigma_{n_0} \circ \tau_{n_0} \cdots \circ \tau_n(v)$ for some $n \in \mathbb{N}$ and $v \in V(n)$. It can be checked that (X_{n_0}, S) is a minimal subshift.

Corollary 6.5.12 *Let (X_B, V_B) be a BV-dynamical system with finite topological rank. Then, (X_B, V_B) is expansive if, and only if, there exists n_0 such that (X_{n_0}, S) is not periodic.*

Moreover, if the cylinders $[e_1, \ldots, e_{n_0-1}]$ of X_B are all included in balls of radius $\frac{\varepsilon}{2}$, ε being a constant of expansivity, then (X_B, V_B) is isomorphic to (X_{n_0}, S).

Note that once some (X_{n_0}, S) is not periodic, then (X_n, S) is aperiodic for all $n \geq n_0$.

6.5.4 Sturmian subshifts

We define the morphisms ρ_n and γ_n, $n \in \mathbb{N} \setminus \{0\}$ from $\{0,1\}$ to $\{0,1\}^*$ by

$$\begin{array}{l} \rho_n(0) = 01^{n+1} \\ \rho_n(1) = 01^n \end{array} \quad \text{and} \quad \begin{array}{l} \gamma_n(0) = 10^{n+1} \\ \gamma_n(1) = 10^n \ . \end{array}$$

The next theorem is due to G. A. Hedlund and M. Morse (Morse and Hedlund 1940).

Theorem 6.5.13 *Let $x \in \{0,1\}^{\mathbb{N}}$ be a Sturmian word. Then*

(i) *there is $n \geq 1$ such that x is a concatenation of finite words belonging to the set $\left\{01^{n+1}, 01^n\right\}$ or to the set $\left\{10^{n+1}, 10^n\right\}$,*

(ii) *if $x = \rho_n(z)$ or $x = \gamma_n(z)$, for some $n \geq 1$ and $z \in \{0,1\}^{\mathbb{N}}$, then z is Sturmian.*

Proof Assertion 1 follows from Theorem 7.1 in (Morse and Hedlund 1940) and Assertion 2 is Theorem 8.1 in (Morse and Hedlund 1940). □

Let (X, S) be a Sturmian subshift. For $a \in \{0,1\}$, we recall that $[a] = \left\{ (x_i)_{i\in\mathbb{Z}} \in X \;\middle|\; x_0 = a \right\}$.

Corollary 6.5.14 *Let (X, S) be a Sturmian subshift of $\{0,1\}^{\mathbb{Z}}$,* i.e.*, there exists a Sturmian word x such that all bi-infinite words in X have the same language as x. There exists a sequence $(\zeta_n)_{n\in\mathbb{N}}$ taking values in $\{\rho_1, \gamma_1, \rho_2, \gamma_2, \ldots\}$ such that $(\mathcal{P}(n))_n$ is a sequence of KR-partitions of (X, S) satisfying (KR1), (KR2) and (KR3) where $\mathcal{P}(1) = \{[0],[1]\}$ and, for $n \geq 2$,*

$$\mathcal{P}(n) = \left\{ S^k \zeta_1 \cdots \zeta_n([a]) \mid 0 \leq k < |\zeta_1 \cdots \zeta_{n-1}(a)|,\ a \in \{0,1\} \right\} .$$

It appears that (KR1) is not clearly satisfied. To be convinced, note that, if for example $\zeta_n = \rho_i$, then

$$\zeta_1 \cdots \zeta_n([a]) \subseteq [\zeta_1 \cdots \zeta_{n-1}(1).\zeta_1 \cdots \zeta_n(a)\, \zeta_1 \cdots \zeta_{n-1}(0)] .$$

Let (X, S) be a Sturmian subshift and $(\mathcal{P}(n))_n$ be the sequence of partitions given by Corollary 6.5.14. With such a sequence is associated an ordered Bratteli-Vershik diagram $B = (V, E, \geq)$ which can be described as follows. For all $n \geq 1$, V_n consists of two vertices, the substitution read on E_{n+1} is ζ_n, with $E(1)$ consisting of a simple hat.

It is clear that a Sturmian subshift is linearly recurrent if, and only if, its BV-representation given by Corollary 6.5.14 is such that $\{\zeta_n \mid n \geq 1\}$ is finite. In (Dartnell, Durand, and Maass 2000) it is proven that it is a substitution subshift if, and only if, $(\zeta_n)_n$ is ultimately periodic. This implies the next result (see also (Kůrka 2003)).

Theorem 6.5.15 *Let (X, S) be a Sturmian subshift generated by α. Then, (X, S) is a substitution subshift if, and only if, α is quadratic.*

The proof uses the notion of dimension groups we introduce in Section 6.6. For similar theorems characterising substitutive Sturmian words, see the references in (Lothaire 2002).

6.5.5 Toeplitz subshifts

A word $(x_n)_{n\in\mathbb{K}}$ ($\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$) on the alphabet A satisfying

$$\forall n \in \mathbb{Z},\ \exists p \in \mathbb{N},\ \forall k \in \mathbb{Z},\ x_n = x_{n+kp}$$

is called *Toeplitz word*. The subshifts they generate are called *Toeplitz subshifts*. These words and subshifts have been deeply studied since they were introduced in (Jacobs and Keane 1969). We refer to (Downarowicz 2005) for a nice survey on important dynamical results on these subshifts.

A Bratteli diagram has the *equal path number property* if for all $n \geq 1$ and $u, v \in V(n)$ we have $|r^{-1}(u)| = |r^{-1}(v)|$. This property was defined in (Gjerde and Johansen 2000). Note that the representation of odometers given in Section 6.5.1 shares this property.

Theorem 6.5.16 *A minimal subshift is Toeplitz if, and only if, it has an expansive BV-representation (X_B, V_B) where $B = (V, E, \geq)$ has the equal path number property. Moreover, there exist BV-systems having the equal path number property that are neither expansive nor equicontinuous.*

6.5.6 Interval exchange transformations

Let $\zeta = (\Delta_1, \dots, \Delta_k)$ be a partition of the segment $[0, 1)$ into $k \geq 2$ disjoint intervals of the form $[a, b)$ numbered from left to right. A *(k)-interval exchange transformation* is an onto map $T : [0, 1) \to [0, 1)$ where $T : \Delta_i \to [0, 1)$ is a translation. We remark there exists a permutation $\pi : \{1, \dots, k\} \to \{1, \dots, k\}$ such that the $T\Delta_i$ are positioned on $[0, 1)$, from left to right, as follows: $T\Delta_{\pi^{-1}(1)}$, …, $T\Delta_{\pi^{-1}(k)}$. For more details on interval exchange transformations we refer to (Cornfeld, Fomin, and Sinaĭ 1982). See also Section 7.4.4.

Let us now give what we will later, in the present section, refer to as the *Cantor version of interval exchange transformations*.

Suppose that T is minimal: all its orbits are dense in $[0, 1)$. Let $\mathcal{D}(T) = \{d_1, \dots, d_k\}$ be the set of left extremal points of the intervals Δ_i and set $\mathcal{O}(T) = \{T^j d \mid j \in \mathbb{Z},\ d \in \mathcal{D}(T)\}$. We define

$$X = ([0, 1) \setminus \mathcal{O}(T)) \bigcup \{x^-, x^+ \mid x \in \mathcal{O}(T)\}$$

where $0^- = 1$. Defining $x^- < x^+$ for all $x \in \mathcal{O}(T)$ with the exception of $0^- \geq x$ for all $x \in X$, this extends the natural order on $[0,1)$ to X. Endowed with the topology of intervals, X is a Cantor space because $\mathcal{O}(T)$ is dense in $[0,1)$.

Let $F : X \to X$ defined by $F(y) = T(y)$ if $y \in [0,1) \setminus \mathcal{O}(T)$ and $F(x^\varepsilon) = T(x)^\varepsilon$ if $x \in \mathcal{O}(T)$ where $\varepsilon \in \{+,-\}$. The pair (X, F) is a minimal Cantor dynamical system, we will refer to as the *Cantor version of the interval exchange* T (for more details see (Gjerde and Johansen 2002)).

Let $\varphi : X \to [0,1)$ be defined by $\varphi(x^+) = \varphi(x^-) = x$ and $\varphi(x) = x$ when $x \notin \mathcal{O}(T)$. This is an onto continuous map. It is one-to-one everywhere except on a countable set of points. Moreover, $\varphi \circ F = T \circ \varphi$.

Theorem 6.5.17 *Let (X, F) be the Cantor version of a minimal k-interval exchange. Then, (X, F) has a BV-representation (X_B, V_B) where $B = (V, E, \geq)$ is such that*

(i) $\mathrm{Card}(V(1)) = k$ *and* $\mathrm{Card}(V(i)) - \mathrm{Card}(V(i+1)) \in \{0,1\}$ *for all* $i \geq 1$,

(ii) *for all $i \geq 1$ when $V(i-1) = V(i)$ the incidence matrix $\mathbf{M}(i)$ of $E(i)$ has the following form*

$$\mathbf{M}(i) = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 & s_1 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 & s_2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 & s_l \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 & s_{l+1} \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 & s_{l+2} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 & s_k \end{bmatrix},$$

where $s_i \in \{0, m, m+1\}$, $s_l = m$ and $s_{l+1} = m+1$ for some $m \geq 0$. When $\mathrm{Card}(V(i)) - \mathrm{Card}(V(i+1)) = 1$, the line $l+1$ does not exist. All the entries of $\mathbf{M}(1)$ are equal to 1.

This theorem has been proved in (Gjerde and Johansen 2002). They also showed that there are BV-dynamical systems satisfying the hypothesis of the theorem that are not isomorphic to a Cantor version of an interval exchange transformation.

6.5.7 Representation of non-minimal Cantor dynamical systems

In (Medynets 2006) the author shows that for Cantor dynamical systems without periodic points, but not necessarily minimal, a BV-representation can also be given. It is applied in (Bezuglyi, Kwiatkowski, and Medynets 2008) to subshifts generated by non-primitive substitutions. The authors show that they have stationary BV-representations as in the minimal case.

6.6 Characterisation of Strong Orbit Equivalence

In this section we give the proof of the Strong Orbit Equivalence theorem proved in (Giordano, Putnam, and Skau 1995). The statement is given in terms of Bratteli diagrams and dimension groups.

6.6.1 Dimension groups as ordered groups

An *ordered group* is a pair (G, G^+), where G^+ is a subset of the group G, such that

$$G^+ + G^+ \subseteq G^+, \ G^+ - G^+ = G, \ G^+ \cap (-G^+) = \{0\} \ .$$

For example, $(\mathbb{Z}^d, (\mathbb{Z}^d)^+)$, with $(\mathbb{Z}^d)^+ = \{(e_1, \ldots, e_d) \in \mathbb{Z}^d \mid e_i \geq 0, \ 1 \leq i \leq d\}$, is an ordered group.

The set G^+ is the *positive cone* of G and its elements are called non-negative. We set, for all $h, g \in G$, $g \geq h$ if, and only if, $g - h \in G^+$. An *order unit* of (G, G^+) is a non-negative element u of G such that

$$\forall g \in G^+, \exists n > 0, \ g \leq nu \ .$$

Let (G, G^+, u) and (H, H^+, v) be two ordered groups with order units. When $\varphi : G \to H$ is a homomorphism satisfying $\varphi(G^+) \subseteq H^+$ and $\varphi(u) = v$, we say that φ is a *morphism (of ordered groups with order unit)*. When it is clear from the context we write G instead of (G, G^+, u).

A *dimension group* is an ordered group obtained as a direct limit of a sequence of finitely generated free Abelian groups with standard order and positive group homomorphism as maps. More precisely, let $\left(M(n) : \mathbb{Z}^{d(n-1)} \to \mathbb{Z}^{d(n)}\right)_{n \geq 1}$ be a sequence of homomorphism such that $M(n) \left(\mathbb{Z}^{d(n-1)}\right)^+ \subseteq \left(\mathbb{Z}^{d(n)}\right)^+$ for all $n \geq 1$. Consider the following subgroups of $\prod_{n \geq 0} \mathbb{Z}^{d(n)}$:

$$\Delta = \left\{ (v(n))_{n \geq 0} \in \prod_{n \geq 0} \mathbb{Z}^{d(n)} \mid M(n)v(n) = v(n+1) \text{ for all } n \text{ large enough} \right\}$$

and

$$\Delta^0 = \{(v(n))_n \in \Delta \mid v(n) = 0 \text{ for all } n \text{ large enough}\} \ .$$

Let G be the quotient of Δ by Δ^0 and G^+ be the projection in G of the subset

$$\Delta^+ = \left\{ (v(n))_n \in \Delta \mid v(n) \in \left(\mathbb{Z}^{d(n)}\right)^+ \text{ for all } n \text{ large enough} \right\} \ .$$

We denote by

$$G := \varinjlim_{M(n)} \mathbb{Z}^{d(n)} \ .$$

The pair (G, G^+) is an ordered group called *dimension group.* It can be checked that u is an order unit of (G, G^+) if, and only if, it is the image (by the canonical projection) in G of some $(v(n)) \in \Delta^+ \setminus \Delta^0$. We say that (G, G^+, u) is a *dimension group with order unit.*

6.6.2 Dimension groups and coboundaries

Let (X, T) be a minimal Cantor dynamical system. We denote by $C(X, \mathbb{Z})$ the set of continuous maps from X to $\mathbb{Z}$. Consider the map $\beta : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ defined by $\beta f = f \circ T - f$ for all $f \in C(X, \mathbb{Z})$. The images of β are called *coboundaries.* Let $H(X, T)$ be the quotient group $C(X, \mathbb{Z})/\beta C(X, \mathbb{Z})$ and π be the natural projection from $C(X, \mathbb{Z})$ to $C(X, \mathbb{Z})/\beta C(X, \mathbb{Z})$. We call *order unit* the image by π of the constant function equal to 1, and we denote it by u_X. The positive cone, $H^+(X, T, \mathbb{Z})$, is the image by π of the set of non-negative functions $C(X, \mathbb{Z}_{\geq 0})$. Finally, the triple

$$DG(X, T) = (H(X, T, \mathbb{Z}), H^+(X, T, \mathbb{Z}), u_X)$$

is an ordered group with order unit. In the next section we will see that it is the *dimension group* generated by the incidence matrices of any BV-representation of (X, T).

6.6.3 Dimension groups and KR-partitions

Let us show now that, thanks to sequences of KR-partitions, the two previous definitions coincide in the context of minimal Cantor dynamical systems.

Let (X, T) be a minimal Cantor dynamical system and let $(\mathcal{P}(n))_n$ with

$$\mathcal{P}(n) = \left\{ T^j B_i(n) \mid 0 \leq j < h_i(n), 1 \leq i \leq t(n) \right\}$$

be a sequence of KR-partitions satisfying (KR1), (KR2) and (KR3). We recall that from Theorem 6.4.3, such a sequence always exists.

Let $f \in C(X, \mathbb{Z})$. As $(\mathcal{P}(n))_n$ generates the topology, there exists n_0 such that for all $n \geq n_0$, f is constant on each atom of $\mathcal{P}(n)$. Let $(\mathbf{M}(n))_{n\geq 1}$ be the associated sequence of incidence matrices, set $d(n) = |V(n)|$ for all $n \geq 0$, and consider Δ, Δ^0, Δ^+, G and G^+ as defined in Section 6.6.1. We call χ the canonical projection from Δ to G.

Define the column vector $\tilde{\mathbf{f}}(n) \in \mathbb{Z}^{d(n)}$ by letting $\tilde{\mathbf{f}}_i(n)$ be the sum of the values of f over all the levels of the tower i, $1 \leq i \leq t(n)$. It is clear that

$$\tilde{\mathbf{f}}(n+1) = \mathbf{M}(n+1)\tilde{\mathbf{f}}(n) \ .$$

We set $\tilde{J}(f) = \chi(0, \ldots, 0, \tilde{\mathbf{f}}(n_0), \tilde{\mathbf{f}}(n_0+1), \ldots)$. This defines a homomorphism from $C(X, \mathbb{Z})$ to G.

Let $f \in \beta C(X, \mathbb{Z})$. One has $f = g \circ T - g$ with $g \in C(X, \mathbb{Z})$. Then $\tilde{\mathbf{f}}_i(n) = g \circ T^{h_i(n)}(x) - g(x)$ for all $x \in B_i(n)$. But as the partition satisfies (KR1), $T^{h_i(n)} B_i(n) \subseteq B(n)$ and then $(\tilde{\mathbf{f}}(n))_n$ goes to 0. Moreover the $\tilde{\mathbf{f}}_i(n)$ are integers. Hence there exists n_1 such that for $n \geq n_1$, we have $\tilde{\mathbf{f}}(n) = 0$. Consequently $\beta C(X, \mathbb{Z})$ is included in $\ker \tilde{J}$.

Now, let $f \in C(X, \mathbb{Z})$ and suppose $\tilde{J}(f) = 0$. Then, there exists n_0 such that $\tilde{\mathbf{f}}(n) = 0$ for all $n \geq n_0$. Take $n \geq n_0$. Let us find $g \in C(X, \mathbb{Z})$ satisfying $f = g \circ T - g$. We define g as follows:

$$g(x) = -(f(x) + f(Tx) + \cdots + f(T^{h_i(n)-1-j}x)) \text{ if } x \in T^j B_i(n) \ .$$

Let $x \in T^j B_i(n)$ with $j \neq h_i(n) - 1$. Then, $T(x)$ belongs to the level $j+1$ of the tower i and

$$g(T(x)) = -(f(T(x)) + f(T^2(x)) + \cdots + f(T^{h_i(n)-1-(j+1)}(T(x)))) \ .$$

Hence, $f(x) = g \circ T(x) - g(x)$ for all x that are not in $\cup_{1\leq i\leq t(n)} T^{h_i(n)-1} B_i(n)$. Let $x \in T^{h_i(n)-1} B_i(n)$. Then $T(x)$ belongs to $B_j(n)$ for some j and consequently $g(T(x)) = \tilde{\mathbf{f}}_j(n) = 0$. Finally $g \circ T(x) - g(x) = 0 - (-f(x)) = f(x)$.

This proves that $\ker \tilde{J} = \beta C(X, \mathbb{Z})$ and that $\tilde{J}$ induces a one-to-one homomorphism J from $H(X, T)$ to G. We end this section by showing that J is an isomorphism of dimension groups.

Let $(g(n))_n$ be a representant of an element of G. We define $f \in C(X, \mathbb{Z})$ by $f(x) = g(n)_i$ if $x \in B_i(n)$, for some i, and 0 elsewhere. Then, $\tilde{\mathbf{f}}_i(n) = g(n)_i$ and J is onto. We see that $J(H^+(X, T)) = G^+$ and that $J(u_X) = \chi(\mathbf{M}(n) \cdots \mathbf{M}(2)\mathbf{M}(1))_n) = u_G$. Note that $\mathbf{M}(1)$ is a vector since $V(0)$ is a one-point set.

We thus have proved the following result.

Proposition 6.6.1 *Let (X,T) be a minimal Cantor dynamical system. For every BV-representation $B=(V,E,\geq)$ of (X,T) with incidence matrices $(\mathbf{M}(n))_n$ we have that $DG(X,T)$ is isomorphic, as a dimension group, to (G,G^+,u_G). In particular, the dimension group does not depend on the BV-representation.*

6.6.4 Strong orbit equivalence

In the sequel we present and prove the Strong Orbit Equivalence (SOE) theorem obtained in (Giordano, Putnam, and Skau 1995). We follow the proof proposed in (Glasner and Weiss 1995) giving more details.

We say that two dynamical systems (X,T) and (Y,S) are *orbit equivalent* whenever there exists a homeomorphism $\varphi : X \to Y$ sending orbits to orbits

$$\varphi\left(\{T^n x \mid n\in\mathbb{Z}\}\right) = \{S^n\varphi(x) \mid n\in\mathbb{Z}\} \ ,$$

for all $x\in X$. This induces the existence of maps $\alpha : X\to\mathbb{Z}$ and $\beta : X\to\mathbb{Z}$ satisfying for all $x\in X$

$$\varphi\circ T(x) = S^{\alpha(x)}\circ\varphi(x) \text{ and } \varphi\circ T^{\beta(x)}(x) = S\circ\varphi(x) \ .$$

When α and β have at most one point of discontinuity, we say that (X,T) and (Y,S) are *strongly orbit equivalent* (SOE). It is natural to consider such a definition because M. Boyle proved in (Boyle 1983) that if α is continuous then (X,T) is conjugate to (Y,S) or to (Y,S^{-1}). In (Giordano, Putnam, and Skau 1995) the authors characterised SOE by means of Bratteli diagrams and dimension groups.

Theorem 6.6.2 (Giordano, Putnam, and Skau 1995) *Let $(X^{(1)},T)$ and $(X^{(2)},S)$ be two minimal Cantor dynamical systems. The following are equivalent:*

(i) *There exist two BV-representations, $B^{(1)}=(V^{(1)},E^{(1)},\geq^{(1)})$ of $(X^{(1)},T)$ and $B^{(2)}=(V^{(2)},E^{(2)},\geq^{(2)})$ of $(X^{(2)},S)$, and an unordered Bratteli diagram $B=(V,E)$ of which $B^{(1)}$ and $B^{(2)}$ are contractions.*

(ii) *There exist two BV-representations, $B^{(1)}=(V^{(1)},E^{(1)},\geq^{(1)})$ of $(X^{(1)},T)$ and $B^{(2)}=(V^{(2)},E^{(2)},\geq^{(2)})$ of $(X^{(2)},S)$, and a homeomorphism $F : X_{B^{(1)}}\to X_{B^{(2)}}$ such that $F(x)(n)$ depends only on $x(1)\ldots x(n)$ and $F\left(x_{\mathrm{u}}^{(1)}\right)=x_{\mathrm{u}}^{(2)}$, $u\in\{\min,\max\}$, and having the property that if x and y are cofinal from level n, then $F(x)$ and $F(y)$ are cofinal from level $n+1$.*

(iii) *$(X^{(1)},T)$ and $(X^{(2)},S)$ are SOE.*

(iv) *$DG(X^{(1)},T)$ and $DG(X^{(2)},S)$ are isomorphic as dimension groups with order units.*

Proof Let us show that (i) implies (ii). Note that a contraction of a BV-representation is itself a BV-representation. Hence, by contracting if needed, we can suppose that $B^{(1)}$ is obtained from B by contracting to odd levels while $B^{(2)}$ is obtained by contracting to even levels. Moreover, from Theorem 6.4.6, we can also suppose that all incidence matrices have entries greater than two. This means that every pair of vertices in consecutive levels has at least two connecting edges.

Let $x^{(1)}_{\min}$ and $x^{(1)}_{\max}$ be the minimal and maximal paths of $B^{(1)}$, and $x^{(2)}_{\min}$ and $x^{(2)}_{\max}$ for $B^{(2)}$. There are unique paths $\tilde{x}^{(1)}_{\min}$ and $\tilde{x}^{(2)}_{\min}$ in B that contract respectively to $x^{(1)}_{\min}$ and $x^{(2)}_{\min}$. Choose a path $z_{\min}$ in B passing through the same vertices as $\tilde{x}^{(1)}_{\min}$ does at odd levels and through the same vertices as $\tilde{x}^{(2)}_{\min}$ at even levels. We similarly construct a path $z_{\max}$ by taking care that it does not share any common edge with $z_{\min}$. This is possible because the incidence matrices have entries greater than two.

Let us define two homeomorphisms $F_1 : X_B \to X_{B^{(1)}}$ and $F_2 : X_B \to X_{B^{(2)}}$. In constructing $z_{\min}$, for each even n, we matched a pair of edges in $E_n \circ E_{n+1}$ with an edge in $E^{(1)}_{n/2}$, namely

$$(z_{\min}(n), z_{\min}(n+1)) \to x^{(1)}_{\min}(n/2)\ ,$$

and we match a pair in $E_{n+1} \circ E_{n+2}$ with an edge in $E^{(2)}_{(n+2)/2}$, namely

$$(z_{\min}(n+1), z_{\min}(n+2)) \to x^{(2)}_{\min}((n+2)/2)\ .$$

In the same way $(z_{\max}(n), z_{\max}(n+1))$ is matched with $x^{(1)}_{\max}(n/2)$ and $(z_{\max}(n+1), z_{\max}(n+2))$ with $x^{(2)}_{\max}((n+2)/2)$. Now, for all even n, we extend these matchings in an arbitrary way to bijections respecting the range and source maps from $E(n) \circ E(n+1)$ to $E^{(1)}(n/2)$ and from $E(n+1) \circ E(n+2)$ to $E^{(2)}((n+2)/2)$. This defines two homeomorphisms

$$F_1 : X_B \to X_{B^{(1)}} \text{ and } F_2 : X_B \to X_{B^{(2)}}\ .$$

The homeomorphism $F = F_2 \circ F_1^{-1}$ has the desired properties.

Let us show that (ii) implies (iii). It suffices to show that $(X_{B^{(1)}}, V_{B^{(1)}})$ and $(X_{B^{(2)}}, V_{B^{(2)}})$ are SOE. In a minimal BV-representation, two points belong to the same orbit if, and only if, they are cofinal, except when it is the orbit of the minimal path. Hence F maps orbits to orbits with the possible exception of the orbit of the minimal paths. But as $F\left(x^{(1)}_{\mathrm{u}}\right) = x^{(2)}_{\mathrm{u}}$, $u \in \{\min, \max\}$, this is also true for the orbit of the minimal paths.

Consequently, there are maps $\alpha : X_{B^{(1)}} \to \mathbb{Z}$ and $\beta : X_{B^{(1)}} \to \mathbb{Z}$ uniquely defined by the relations

$$F \circ V_{B^{(1)}}(x) = V_{B^{(2)}}^{\alpha(x)} \circ F(x) \text{ and } F \circ V_{B^{(1)}}^{\beta(x)}(x) = V_{B^{(2)}} \circ F(x)$$

for all $x \in X_{B^{(1)}}$. It remains to prove that α and β are continuous with the possible exception of $x_{\max}^{(1)}$ and $x_{\max}^{(2)}$. We do it for α. It is similar for β.

Let $x = (x_n)_n \in X_{B^{(1)}} \setminus \{x_{\max}^{(1)}\}$ and $k = \alpha(x)$. Let n_0 be such that $(x_1, \ldots, x_{n_0})$ has a non-maximal edge and the minimum number of paths from any vertex in V_{n_0-1} to V_0 is greater than k.

Let y belonging to the cylinder $[x_1, \ldots, x_{n_0+1}]$. It suffices to show that $\alpha(y) = k$. The paths $V_{B^{(1)}}(x)$ and $V_{B^{(1)}}(y)$ start with the same $n_0 + 1$ first edges. Thus, from the property of F, $F \circ V_{B^{(1)}}(x)$ and $F \circ V_{B^{(1)}}(y)$ start with the same $n_0 + 1$ first edges f_1, f_2, …, f_{n_0+1}:

$$F \circ V_{B^{(1)}}(x) = (f_1, f_2, \ldots, f_{n_0+1}, x'_{n_0+2}, \ldots) \text{ and}$$
$$F \circ V_{B^{(1)}}(y) = (f_1, f_2, \ldots, f_{n_0+1}, y'_{n_0+2}, \ldots) \ .$$

For the same reason, and because x and $V_{B^{(1)}}(x)$, and, y and $V_{B^{(1)}}(y)$ are cofinal from $n_0 + 1$, $F(x)$ and $F(y)$ start with the same edges g_1, g_2, …, g_{n_0+1} and

$$F(x) = (g_1, g_2, \ldots, g_{n_0+1}, x'_{n_0+2}, \ldots) \text{ and}$$
$$F(y) = (g_1, g_2, \ldots, g_{n_0+1}, y'_{n_0+2}, \ldots) \ .$$

But as there are at least k paths from any vertex in V_{n_0-1} to V_0, we deduce that $V_{B^{(1)}}^k([g_1, g_2, \ldots, g_{n_0+1}]) = [f_1, f_2, \ldots, f_{n_0+1}]$ because $F \circ V_{B^{(1)}}(x) = V_{B^{(2)}}^k \circ F(x)$. Therefore $F \circ V_{B^{(1)}}(y) = V_{B^{(2)}}^k \circ F(y)$ and $\alpha(y) = k$.

Let us show that (iii) implies (iv). Let $F : (X^{(1)}, T) \to (X^{(2)}, S)$ be a SOE map. Remark that $(X^{(2)}, S)$ is isomorphic to $(X^{(1)}, F^{-1} \circ S \circ F)$. Hence we can suppose $X^{(2)} = X^{(1)} = X$. Then we have

$$T(x) = S^{\alpha(x)}(x) \text{ and } S(x) = T^{\beta(x)}(x)$$

where α and β are continuous everywhere with y as a possible exception.

Let A be a clopen set not containing y. As α is continuous on A the set $\alpha(A)$ is compact and consequently finite: there exist n_1, …, n_k such that $A = \cup_{1 \le i \le k} A \cap \alpha^{-1}(\{n_i\})$. We recall that that the *indicator function* of the set X is denoted by $\mathbb{1}_X$. Hence

$$TA = \bigcup_{1 \le i \le k} S^{n_i}(A \cap \alpha^{-1}(\{n_i\}))$$

and

$$\mathbb{1}_A \circ T^{-1} = \sum_{1 \le i \le k} \mathbb{1}_{A \cap \alpha^{-1}(\{n_i\})} \circ S^{-n_i} \ .$$

But as $f - f \circ S^{-n} = (\sum_{1 \le i \le n} f \circ S^{-i}) \circ S - (\sum_{1 \le i \le n} f \circ S^{-i})$, we deduce that $\mathbb{1}_A \circ T^{-1} - \mathbb{1}_A$ belongs to $\beta_S(C(X, \mathbb{Z}))$.

Now suppose that A contains y. Remark that $\mathbb{1}_A \circ T^{-1} - \mathbb{1}_A = \mathbb{1}_{X \setminus A} - \mathbb{1}_{X \setminus A} \circ T^{-1}$. But as y is not contained in $X \setminus A$ we deduce from the previous case that $\mathbb{1}_A \circ T^{-1} - \mathbb{1}_A$ belongs to $\beta_S(C(X, \mathbb{Z}))$. As T is invertible we proved that for all clopen set E, $\mathbb{1}_E \circ T - \mathbb{1}_E$ belongs to $\beta_S(C(X, \mathbb{Z}))$ and consequently that $\beta_T(C(X, \mathbb{Z})) \subseteq \beta_S(C(X, \mathbb{Z}))$. Proceeding similarly with the equality $S(x) = T^{\beta(x)}(x)$ we obtain $\beta_T(C(X, \mathbb{Z})) = \beta_S(C(X, \mathbb{Z}))$.

Let us show that (iv) implies (i). Let $B^{(1)} = (V^{(1)}, E^{(1)}, \ge^{(1)})$ be a BV-representation of $(X^{(1)}, T)$ and $B^{(2)} = (V^{(2)}, E^{(2)}, \ge^{(2)})$ of $(X^{(2)}, S)$. Let $G^{(1)}$ and $G^{(2)}$ be the dimension groups they induced. We recall that

$$G^{(1)} = \lim_{\overrightarrow{M^{(1)}(n)}} \mathbb{Z}^{|V^{(1)}(n)|} \text{ and } G^{(2)} = \lim_{\overrightarrow{M^{(2)}(n)}} \mathbb{Z}^{|V^{(2)}(n)|} .$$

We know from Proposition 6.6.1 that $G^{(1)}$ is isomorphic to $DG(X^{(1)}, T)$ and $G^{(2)}$ is isomorphic to $DG(X^{(2)}, S)$.

Now we shall construct an unordered Bratteli diagram $B = (V, E)$ that contracts to a contraction of $B^{(1)}$ on odd levels and to a contraction of $B^{(2)}$ on even levels. It suffices to give the sets of vertices $V(n)$ and the incidence matrices $(\mathbf{M}(n))_n$ between consecutive levels.

We set $V(1) = V^{(1)}(1)$ and $\mathbf{M}(1) = M^{(1)}(1)$. Looking at the canonical generators of $\mathbb{Z}^{\mathrm{Card}(V(1))}$ as elements of $G^{(2)}$, we can consider that they are elements of some $\mathbb{Z}^{\mathrm{Card}(V^{(2)}(n_2))}$. We set $V(2) = V^{(2)}(n_2)$, and we call $\mathbf{M}(2)$ the map it defines from $\mathbb{Z}^{\mathrm{Card}(V(1))}$ to $\mathbb{Z}^{\mathrm{Card}(V(2))}$. Again, the elements of $\mathbb{Z}^{\mathrm{Card}(V(2))}$ can be considered as elements of $G^{(1)}$, and thus belong to some $\mathbb{Z}^{\mathrm{Card}(V^{(1)}(n_3))}$. We set $V(3) = V^{(1)}(n_3)$ and we call $\mathbf{M}(3)$ the map that it defines from $\mathbb{Z}^{\mathrm{Card}(V(2))}$ to $\mathbb{Z}^{\mathrm{Card}(V(3))}$. Proceeding like this, we obtain the sequence

$$\mathbb{Z} \xrightarrow{\mathbf{M}(1)} \mathbb{Z}^{\mathrm{Card}(V(1))} \xrightarrow{\mathbf{M}(2)} \mathbb{Z}^{\mathrm{Card}(V(2))} \xrightarrow{\mathbf{M}(3)} \mathbb{Z}^{\mathrm{Card}(V(3))} \cdots$$

that is sufficient to define the Bratteli diagram B we are looking for. □

6.7 Entropy

M. Boyle and D. Handelman have proved in (Boyle and Handelman 1994) that every minimal Cantor dynamical system is SOE to a minimal Cantor dynamical system of entropy zero. In this section we give the details of their proof. It suffices to prove that any Bratteli diagram can be telescoped to a diagram admitting an ordering such that the associated Bratteli-Vershik dynamical system has entropy zero.

We suggest (Walters 1982) as an introduction to topological entropy. Let us recall how it can be defined for subshifts (X, S) and Bratteli-Vershik dynamical systems (X_B, V_B).

The entropy of (X, S), denoted by $h(S)$, is the growth rate of the number of finite words of length n occurring in elements of X:

$$h(S) = \limsup_n \frac{\log |L_n(X)|}{n} ,$$

where $L_n(X) = \{x_i x_{i+1} \cdots x_{i+n-1} \mid i \in \mathbb{Z},\ x \in X\}$ (see also Section 4.4.2).

To give a convenient way to compute the entropy $h(V_B)$ of (X_B, V_B) we need some notations.

For $n \geq 1$, let $P(n)$ be the set of paths from $V(0)$ to $V(n)$. We define π_n on X_B by $\pi_n((e_k)_{k\geq 1}) = (e_1, \ldots, e_n)$. We will consider the set $A_n = \pi_n(X_B)$ as an alphabet. We call S_n the shift on $A_n^{\mathbb{Z}}$. The set

$$X_n = \left\{ \left(\pi_n \left(V_B^k(x)\right)\right)_{k\in\mathbb{Z}} \mid x \in X_B \right\}$$

is included in $A_n^{\mathbb{Z}}$, S_n-invariant and compact. Hence (X_n, S_n) is a subshift. As a consequence of the Theorem 7.6 in (Walters 1982) we have

$$h(V_B) = \lim_{n\to+\infty} h(S_n) .$$

Hence, we need first to compute $h(S_n)$. To this end we will need the following subshifts. When W is a set of finite words, we denote by $\Omega(W)$ the subset of all bi-infinite words formed by concatenation of finite words belonging to W. Let S_W denote the shift map on $\Omega(W)$. It is clear $(\Omega(W), S_W)$ is a subshift.

Lemma 6.7.1 *Let W be a set of m finite words of length at least l. Then*

$$h(S_W) \leq \frac{\log m}{l} .$$

Proof Let k be the greatest length of the finite words in W. Let w be a finite word of length n occurring in some word of $\Omega(W)$. Then there exist r finite words m_1, ..., m_r of W, a prefix s and suffix p of some finite words in W such that $w = s m_1 \cdots m_r p$. Since $r \leq \frac{n}{l}$, we deduce that there are at most $k^2 m^{2+\frac{n}{l}}$ finite words of length n in $L_n(\Omega(W))$, which ends the proof. □

In (Boyle and Handelman 1994) the authors use a different lemma. They show that $h(S_W) = \frac{\log m}{l}$, whenever W is a set of m distinct finite words of length l.

An ordering on a Bratteli diagram is a *consecutive ordering* if whenever edges e, f and g have the same range, e and g have the same source and $e \le f \le g$, then e and f have the same source (see Figure 6.9).

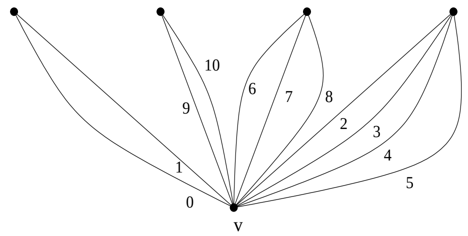

Fig. 6.9. An example of a consecutive ordering viewed from a vertex $v \in V(n+1)$ to $V(n)$.

Proposition 6.7.2 *Let $B = (V, E, \geq)$ be a properly ordered Bratteli diagram where $\geq$ is a consecutive ordering. Suppose that*

$$\lim_{n \to +\infty} \frac{\log(\eta(n+1)\,\mathrm{Card}(V(n)))}{\eta(n+1)} = 0 ,$$

where $\eta(n)$, $n \geq 1$, is the minimum number of edges from a vertex at level $n-1$ to a vertex at level n. Then, $h(V_B) = 0$.

Proof Let A_n and X_n be defined as above when we described $h(V_B)$. Let u be a vertex at level n, and let $p_1, \ldots, p_s$ be the paths from level 0 to u, listed in increasing order. We set $W(u) = p_1 \cdots p_s$. We can consider that it belongs to A_n^s. Now, assume that v is a vertex at level $n+1$, that $a_1, \ldots, a_t$ are the edges from u to v, listed in increasing order, and that y is an infinite path in the Bratteli diagram such that $y_1 \cdots y_{n+1} = p_1 a_1$. Then,

$$\left(\pi_n\left(V_B^k(y)\right)\right)_{0 \leq k \leq st-1} = W(u)^t .$$

Note that t is greater than $\eta(n+1)$. Therefore, X_n is included in $\Omega(\mathcal{W})$ where

$$\mathcal{W} = \left\{W(u)^t \mid u \in V(n),\ \eta(n+1) \leq t < 2\eta(n+1)\right\} ,$$

and consequently, $h(S_n) \leq h(S_{\mathcal{W}})$. As $\mathcal{W}$ consists of at most $\mathrm{Card}(V(n))\eta(n+1)$ finite words of length at least $\eta(n+1)$, from Lemma 6.7.1 we obtain

$$h(S_{\mathcal{W}}) \leq \frac{\log\left(\eta(n+1)\,\mathrm{Card}(V(n))\right)}{\eta(n+1)} .$$

This achieves the proof. □

Theorem 6.7.3 *Any minimal Cantor dynamical system is strongly orbit equivalent to a minimal Cantor dynamical system of entropy zero.*

Proof From Theorem 6.4.6, it suffices to consider a minimal Bratteli-Vershik dynamical system (X_B, V_B). Let $\eta(n)$, $n \geq 1$, be the minimum number of edges from a vertex at level $n-1$ to a vertex at level n.

From Theorem 6.4.6, we know, by contracting if needed, that we can assume the incidence matrices of $B = (V, E, \geq)$ to have strictly positive entries. Hence, contracting again if needed, we can suppose that

$$\lim_{n \to +\infty} \frac{\log(\eta(n+1)\,\mathrm{Card}(V(n)))}{\eta(n+1)} = 0 .$$

Consider $B' = (V, E, \geq_*)$ where $\geq_*$ is a consecutive ordering. Then, from Proposition 6.7.2, $(X_{B'}, V_{B'})$ has zero entropy and, from Theorem 6.6.2, is strongly orbit equivalent to (X_B, V_B). □

In this proof we find all the arguments to prove that all minimal BV-dynamical systems with a consecutive ordering have entropy zero.

This theorem shows that there can be dynamical systems with different entropies in a strong orbit equivalence class. Hence it is natural to ask whether all entropies can be realised inside a given class. M. Boyle and D. Handelman showed in (Boyle and Handelman 1994) that it is true in the class of the odometer $(\mathbb{Z}_2, x \mapsto x+1)$. Later, F. Sugisaki proved in (Sugisaki 2003) that it is true in any strong orbit equivalence class. Finally, he showed that moreover the realisations can be chosen to be subshifts.

Theorem 6.7.4 (Sugisaki 2007) *Let* $\alpha \in [1, +\infty[$ *and* (X, T) *a minimal Cantor dynamical system. There exists a minimal subshift of entropy* $\log \alpha$ *that is strongly orbit equivalent to* (X, T).

6.8 Invariant measures and Bratteli diagrams

In this section we describe the construction of invariant measures through Bratteli diagrams. We know that such measures exists as said in Section 1.6.2 and proven in Proposition 7.2.4.

6.8.1 How to see invariant measures on Bratteli diagrams

The description we give below is very classical and applies to any measure-theoretic dynamical system defined on a compact space. But we present it in the context we work with from the beginning of the present chapter.

Let (X_B, V_B) be the minimal Cantor dynamical system that the properly ordered Bratteli diagram B generates, and let $(\mathcal{P}(n))_n$ be the sequence of KR-partitions that B naturally defines:

$$\mathcal{P}(n) = \left\{ V_B^j B_i(n) \mid 0 \leq j < h_i(n), 1 \leq i \leq \mathrm{Card}(V(n)) \right\} .$$

More precisely:

- $B_i(n)$ is the cylinder set spanned by the unique minimal path from $v_i \in V(n)$ to $V(0)$ where we set $V(n) := \{v_1, v_2, \dots, v_l\}$,
- $(h_i(n)) = \mathbf{M}(n)\mathbf{M}(n-1)\cdots\mathbf{M}(1)$ where the $\mathbf{M}(k)$ are the incidence matrices of the $E(k)$,
- $\cup_{0\leq j<h_i(n)} V_B^j B_i(n)$ is the clopen set spanned by all the finite paths from $v_i \in V(n)$ to $V(0)$,
- the unique minimal path u of B satisfies $\{u\} = \cap_{n\in\mathbb{N}} B(n)$ where $B(n) = \cup_{1\leq i\leq \mathrm{Card}(V(n))} B_i(n)$.

Let μ be a V_B-invariant probability measure. Then $\mu(V_B^j B_i(n))$, $0 \leq j < h_i(n)$, does not depend on j and is equal to $\mu(B_i(n))$. Hence, from standard argument (see for example Theorem 7.1 in (Lang 1993)), the measure μ is completely determined by the values it takes at $B_i(n)$ (for all i and all n). Moreover, a simple computation yields the following fundamental relation:

$$\mu(n) = {}^t\mathbf{M}(n+1)\mu(n+1) \ , \tag{6.7}$$

where $\mu(n)$ is the column vector $(\mu(B_i(n)))_{i\in V(n)}$ for any $n \geq 1$.

Let $\mathcal{M}(X_B, V_B)$ be the of V_B-invariant measures and $\mathcal{E}$ the set of V_B-ergodic measures of (X_B, V_B). Let $\mathcal{C}$ be the set of sequences $(m(n))_{n\geq 0}$ satisfying

$$m(n) \in \mathbb{R}_+^{V(n)}, \ m(n) = {}^t\mathbf{M}(n+1)m(n+1), \ n \geq 0 \text{ and } m(0) = 1 \ .$$

Let us note that it is a convex set. The map F from $\mathcal{M}(X_B, V_B)$ to $\mathcal{C}$ defined by $\mu \mapsto (\mu(n))_n$ is a bijection. Roughly speaking, in order to construct a V_B-invariant measure, it suffices, for all n, to put weights on the minimal paths from every $v \in V(n)$ to $V(0)$ that respect Equality (6.7).

The map F sends extremal points to extremal points. Using this and knowing that the ergodic measures of (X_B, V_B) are the extremal points of the convex set $\mathcal{M}(X_B, V_B)$ (this is proven in Chapter 7, see Proposition 7.2.4), it is not difficult to prove that if (X_B, V_B) has bounded topological rank K, then it has at most K ergodic measures (similar results are proven in Chapter 7).

6.8.2 Invariant measures as homomorphisms of dimension groups

A *state* of the dimension group with order unit (G, G^+, u) is a group homomorphism $w : G \to \mathbb{R}$ such that $w(G^+)$ is included in $[0, +\infty[$ and $w(u) = 1$.

Let (X, T) be a minimal Cantor dynamical system, and denote by $\mathcal{S}(X, T)$ the set of states of its dimension group $DG(X, T)$. We recall that π is the natural projection from $C(X, \mathbb{Z})$ onto the quotient group $C(X, \mathbb{Z})/\beta C(X, \mathbb{Z})$. Let U be a clopen set, and $\mathbb{1}_U$ be its indicator function. Consider the map from $\mathcal{M}(X, T)$ to $\mathcal{S}(X, T)$ defined by

$$\mu \longmapsto \left(\pi(f) \mapsto \int_X f d\mu \right) .$$

The measure μ being T-invariant, this map is well defined. Moreover it is a bijective affine map between $\mathcal{M}(X, T)$ and $\mathcal{S}(X, T)$.

6.8.3 Linearly recurrent dynamical systems are uniquely ergodic

Below we show how to use the Bratteli diagrams, or more precisely their associated KR-partitions, to control the set of invariant measures. We will prove the following result.

Proposition 6.8.1 *Every linearly recurrent Cantor dynamical system is uniquely ergodic.*

In the sequel (X, T) is a linearly recurrent Cantor dynamical system having a BV-representation (X_B, V_B) with B satisfying (i) and (ii) in Theorem 6.5.10. Using the notation in Section 6.8.1, the Bratteli diagram B induces a KR-partition

$$\mathcal{P}(n) = \left\{ V_B^j B_i(n) \mid 0 \le j < h_i(n), 1 \le i \le \mathrm{Card}(V(n)) \right\}$$

such that $\sup_n \mathrm{Card}(V(n)) \le K$ and $\sup_{i,j,n} \frac{h_i(n+1)}{h_j(n)} \le K$ for some K.

Lemma 6.8.2 *Let μ be an invariant measure of (X_B, V_B). Then, for all $n \in \mathbb{N}$ and $k \in V(n)$, we have*

$$h_k(n)\mu(B_k(n)) \ge \frac{1}{K} .$$

Proof Let $k \in V(n)$. By Equation (6.7), since all the entries of $\mathbf{M}(n+1)$ are positive, we get

$$\mu(B_k(n)) \ge \sum_{l \in V(n)} \mu(B_l(n+1)) .$$

But, as for every l we have $h_k(n) \geq h_l(n+1)/K$, thus

$$h_k(n)\mu(B_k(n)) \geq \sum_{l \in V(n)} \frac{h_l(n+1)}{K}\mu(B_l(n+1)) = \frac{1}{K} \ .$$

This completes the proof. ☐

Proof [Proposition 6.8.1] As in Section 6.8.1, given a V_B-invariant probability measure μ, we define the numbers

$$\mu(n)_k = \mu(B_k(n)), \ n \geq 0, \ k \in V(n) \ .$$

From Lemma 6.8, there exists a constant $\delta > 0$ such that

$$\mu(n)_i \geq \delta\mu(n-1)_k$$

for every $n \geq 1$ and $(i,k) \in V(n) \times V(n-1)$, and every invariant measure μ. Without loss of generality we can assume $\delta < 1/2$.

Let μ, μ' be two invariant measures, and $\mu(n)_k, \mu'(n)_k$ be defined as above. We define

$$S_n = \max_k \frac{\mu'(n)_k}{\mu(n)_k} = \frac{\mu'(n)_i}{\mu(n)_i} \ , \ s_n = \min_k \frac{\mu'(n)_k}{\mu(n)_k} = \frac{\mu'(n)_j}{\mu(n)_j} \ , \ \text{and } r_n = \frac{S_n}{s_n}$$

for some i, j. Let $m_{l,k}(n)$ be the entries of the incidence matrix $\mathbf{M}(n)$. For every $k \in V(n-1)$, we have:

$$\begin{aligned}
\mu'(n-1)_k &= \sum_{l \neq j} \mu'(n)_l m_{l,k}(n) + \mu'(n)_j m_{j,k}(n) \\
&\leq S_n \sum_{l \neq j} \mu(n)_l m_{l,k}(n) + s_n \mu(n)_j m_{j,k}(n) \\
&= S_n\mu(n-1)_k - (S_n - s_n)\mu(n)_j m_{j,k}(n) \leq S_n\mu(n-1)_k - (S_n - s_n)\mu(n)_j \\
&\leq \mu(n-1)_k s_n\big(r_n(1-\delta) + \delta\big) \ .
\end{aligned}$$

And in a similar way, for every $k \in V(n-1)$ we have

$$\mu'(n-1)_k \geq \mu(n-1)_k s_n\big(\delta r_n + (1-\delta)\big) \ .$$

We deduce that

$$r_{n-1} \leq \varphi(r_n) \text{ where } \varphi(x) = \frac{(1-\delta)x + \delta}{\delta x + (1-\delta)} \ .$$

The function φ is increasing on $[0, +\infty)$ and tends to $(1-\delta)/\delta$ at $+\infty$. Writing $\varphi^m = \varphi \circ \cdots \circ \varphi$ (m times), for every $n, m \in \mathbb{N}$ we have $1 \leq r_n \leq \varphi^m(r_{n+m}) \leq \varphi^{m-1}((1-\delta)/\delta)$. Taking the limit with $m \to +\infty$, we get $r_n = 1$. ☐

6.9 Eigenvalues of stationary BV-models

6.9.1 Basic knowledge on eigenvalues

Let (X,T) be a topological dynamical system. A complex number λ is a *continuous eigenvalue* of (X,T) if there exists a continuous function $f : X \to \mathbb{C}$, $f \neq 0$, such that $f \circ T = \lambda f$. We say that f is a *continuous eigenfunction* (associated with λ). If (X,T) is minimal, then every continuous eigenvalue is of modulus 1 and every continuous eigenfunction has a constant modulus.

Let μ be a T-invariant probability measure, *i.e.*, $T\mu = \mu$, defined on the Borel σ-algebra $\mathcal{B}_X$ of X. A complex number λ is a $L^2(\mu)$-*eigenvalue* of (X,T) if there exists $f \in L^2(X, \mathcal{B}_X, \mu)$ (shortly denoted $L^2(\mu)$), $f \neq 0$, such that $f \circ T = \lambda f$. We say that f is a $L^2(\mu)$-*eigenfunction* (associated with λ). If the system is ergodic, then every eigenvalue is of modulus 1, and every eigenfunction has a constant modulus. Moreover the eigensubspace of $L^2(\mu)$ associated with λ has dimension 1. To simplify the language we will also say that *an eigenvalue is continuous* when the associated eigenfunction is continuous.

6.9.2 Continuous eigenfunctions versus measurable eigenfunctions

Let (X,T) be a topological dynamical system and μ a T–invariant probability measure. In this section we want to explain how Bratteli diagrams and, or, KR-partitions can be used to study eigenvalues (see Section 6.9.1). We illustrate this by proving a result of B. Host (Host 1986), which firstly gives a characterisation of the eigenvalues having a continuous eigenfunction, and which secondly, deduces from this characterisation that all eigenvalues have one continuous eigenfunction. Since minimal substitution subshifts are uniquely ergodic, let us note that the eigensubspace has dimension 1, and consequently, the result of Host means that in the L^2-equivalence class of each eigenfunction, there is a continuous eigenfunction, *i.e.*, eigenfunctions of minimal substitution subshifts are almost surely equal to continuous eigenfunctions.

Theorem 6.9.1 (Host 1986) *Let $\sigma : A \to A^*$ be a one-to-one primitive substitution. Let (X,S) be the subshift it generates and μ be its unique ergodic measure. Suppose that σ is recognisable. Then,*

(i) *All eigenfunctions of (X,S,μ) are μ-almost everywhere equal to a continuous eigenfunction.*
(ii) *The complex number λ is an eigenvalue of (X,T,μ) if, and only if,*

there exists an integer $p > 0$ such that for all letter $a \in A$ the limit

$$h(a) = \lim_{n \to +\infty} \lambda^{|\sigma^{pn}(a)|}$$

exists and h satisfies: there exists $f : A \to \mathbb{C}$ such that for all finite words ab of two letters, we have $f(b) = f(a)h(a)$.

Let us make some comments about this statement. From Theorem 6.5.5 and Proposition 6.8.1 we know that minimal substitution subshifts are uniquely ergodic. In (Mossé 1992), B. Mossé showed that all primitive substitutions with non-periodic fixed points are recognisable. Consequently the recognisability requirement becomes vacuous. The one-to-one assumption is not necessary because the subshift can be generated (when it is not periodic), up to isomorphism, by an other primitive substitution which is one-to-one (see Exercise 6.25). Moreover, the use of Bratteli diagrams gives a statement without the requirement on the existence of the integer p and on the function h (see Theorem 6.9.2 and 6.9.3).

We split into two parts the result of B. Host and state it by means of Bratteli diagrams. Before we need some notations. In the sequel, $\| \cdot \|$ denotes the distance to the nearest integer. For a vector $\mathbf{v} = {}^t(v_1, \dots, v_m) \in \mathbb{R}^m$, we write

$$\|\mathbf{v}\|_\infty = \max_{1 \le j \le m} |v_j| \text{ and } \|\mathbf{v}\| = \max_{1 \le j \le m} \|v_j\| \ .$$

Theorem 6.9.2 *Let B be a properly ordered stationary Bratteli diagram. Let $\mathbf{M}$ be its $C \times C$ incidence matrix and $\mathbf{H}(n) = \mathbf{M}^{n-1}\mathbf{M}(1) = (h_i(n))_{1 \le i \le C}$. Then, $\lambda \in \mathbb{C}$ is a continuous eigenvalue of (X_B, V_B) if, and only if,*

$$\lim_{n \to +\infty} \lambda^{h_i(n)} = 1 \text{ for all } i \in \{1, \dots, C\} \ .$$

The proof we give later also holds for proper primitive substitutions σ generating a non-periodic subshift, with the necessary and sufficient condition becoming: $\lim_{n \to +\infty} \lambda^{|\sigma^n(a)|} = 1$ for all $a \in A$.

Theorem 6.9.3 *Let (X, S, μ) be a minimal substitution subshift. If $f \in L^2(\mu)$ is an eigenfunction of $\lambda \in \mathbb{C}$, then it is μ-almost surely equal to a continuous eigenfunction (of λ).*

We prove these results in the next section.

In (Ferenczi, Mauduit, and Nogueira 1996) an algebraic characterisation of these eigenvalues is given.

It is in general not true that all eigenvalues of a minimal dynamical system have a continuous eigenfunction as it can be seen for some Toeplitz systems (Iwanik 1996, Downarowicz and Lacroix 1996) and for some interval exchange transformations (Ferenczi, Holton, and Zamboni 2001).

The general question is: When does a measure-theoretic eigenvalue $\lambda \in \mathbb{C}$ of the dynamical system (X, T, μ) have a continuous eigenfunction ?

Such a question also appears in (Nogueira and Rudolph 1997) where the authors show that generically interval exchange transformations are not topologically weakly mixing (*i.e.*, they do not have non-trivial continuous eigenfunctions) and where they "fully expect" that the same holds for (measure-theoretic) weak mixing (*i.e.*, they do not have non-trivial eigenfunctions). This was proven in (Avila and Forni 2007).

6.9.3 Proofs

We first start by two lemmas whose proofs is left to reader.

Lemma 6.9.4 *Let* $\mathbf{M}$ *be a matrix with integer entries. If* $\mathbf{u}$ *is a real vector such that* $\|\mathbf{M}^n\mathbf{u}\|_\infty \to 0$ *when* $n \to \infty$*, then the convergence is exponential,* i.e.*, there exist* $0 \le r < 1$ *and a constant* K *such that* $\|\mathbf{M}^n\mathbf{u}\|_\infty \le Kr^n$ *for all* $n \in \mathbb{N}$*.*

Lemma 6.9.5 *Let* $\mathbf{M}$ *be a matrix with integer entries. Let* u *be a real vector such that* $\|\mathbf{M}^n\mathbf{u}\| \to 0$ *as* $n \to +\infty$*. Then there exist an integer vector* $\mathbf{w}$ *and a real vector* $\mathbf{v}$ *such that*

$$\mathbf{u} = \mathbf{w} + \mathbf{v} \text{ and } \|\mathbf{M}^n\mathbf{v}\|_\infty \to_{n\to+\infty} 0 \ .$$

Proof [Theorem 6.9.2] Let $(\mathcal{P}(n))_n$ be the sequence of KR-partitions that B naturally defines:

$$\mathcal{P}(n) = \left\{ V_B^j B_i(n) \mid 0 \le j < h_i(n),\ 1 \le i \le C \right\} \ .$$

The unique minimal path u of B verifies $\{u\} = \cap_{n\in\mathbb{N}} B(n)$ where $B(n) = \cup_{1\le i\le C} B_i(n)$. Let g be a continuous eigenfunction of λ of modulus equal to 1. Then, it is uniformly continuous. Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $|g(y) - g(u)| < \varepsilon/2$ for all $y \in B(n_0)$.

Let $i \in \{1, \ldots, C\}$ and $n \ge n_0$. Let $x \in B_i(n)$ and set $y = V_B^{h_i(n)}(x)$ which belongs to $B(n)$. We thus have

$$\begin{aligned}\left|\lambda^{h_i(n)} - 1\right| &= \left|g\left(V_B^{h_i(n)}x\right) - g\left(x\right)\right| \\ &\le |g(y) - g(u)| + |g(u) - g(x)| < \varepsilon \ .\end{aligned}$$

This proves the direct implication.

Let $\lambda = \exp(2i\pi\alpha)$. Suppose that $\lim_{n\to+\infty} \max_{1\le i\le C} \left|\lambda^{h_i(n)} - 1\right| =$

0. Recall that $h_i(n)$ is equal to the ith entry of $\mathbf{M}^{n-1}\mathbf{M}(1)$. Then, $\|\alpha\mathbf{M}^{n-1}\mathbf{M}(1)\|$ tends to 0 when n goes to $+\infty$, and from Lemmas 6.9.4 and 6.9.5 (by taking $\mathbf{u} = \alpha\mathbf{M}(1)$), the series $\sum \max_{1\le i\le C} \mid \lambda^{h_i(n)} - 1 \mid$ converges.

For every $n \in \mathbb{N}$, let f_n be the function on X_B defined by

$$f_n(x) = \lambda^j \text{ for } x \in V_B^j B_i(n),\ 0 \le j < h_i(n),\ 1 \le i \le C\ .$$

We compare f_n and f_{n-1}. Let $L = \max_{1\le i\le C} \sum_{1\le j\le C} m_{i,j}$. By construction, for every x, $f_n(x)/f_{n-1}(x) = \lambda^l$ where l is a sum of terms of the form $h_i(n-1)$ and this sum contains at most L terms. We thus obtain

$$\sup_{x\in X_B} |f_n - f_{n-1}| \le L \max_{1\le l\le C} |\lambda^{h_l(n-1)} - 1|\ .$$

Hence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a continuous function f, which is clearly a continuous eigenfunction for λ. □

Lemma 6.9.6 *Let B be a stationary properly ordered Bratteli diagram, with $C \times C$ incidence matrix $\mathbf{M}$, and μ be the unique ergodic measure of (X_B, V_B). Let $(h_i(n))_i = (\mathbf{M}^{n-1}\mathbf{M}(1))_i$. Suppose that $\mathbf{M}$ has positive entries. Then, there exists K such that for all $n \ge 1$ and $(i,l) \in \{1,\dots,C\}^2$ we have*

$$h_l(n+1) \le K h_i(n) \text{ and } h_i(n)\mu(B_i(n)) \ge \frac{1}{K}\ .$$

Proof It is easy to establish that there exists a constant K such that for every $n \ge 1$ and $(i,l) \in \{1,\dots,C\}^2$, we have $h_l(n+1) \le K h_i(n)$. Let $n \ge 1$ and $i \in \{1,\dots,C\}$. We have the relation $\mu(B_i(n-1)) = \sum_{l=1}^{C} m_{l,i}\, \mu(B_l(n))$. Consequently, the entries of $\mathbf{M}$ being positive, we deduce

$$h_i(n)\mu(B_i(n)) \ge \sum_{l=1}^{C} \frac{h_l(n+1)}{K}\mu(B_l(n+1)) = \frac{1}{K}\ .$$

This completes the proof. □

Proof [Theorem 6.9.3] Let $(\mathcal{P}(n))_n$ be a sequence of partitions of X that induced a stationary BV-representation of (X, S): $\mathcal{P}(n) = \{S^j B_i(n) \mid 0 \le j < h_i(n),\ 1 \le i \le C\}$. We call $\mathbf{M}$ the incidence matrix from level 2. We can suppose that it has positive entries. Let K be the constant given in Lemma 6.9.6. As there is a unique minimal path and B is stationary, by contracting if needed, we can suppose that $B(n+1)$ is included in $B_1(n)$ for all $n \ge 1$.

Let $g \in L^2(\mu)$ be an eigenfunction for λ, *i.e.*, $g \circ S = \lambda g$ and let g_n be its conditional expectation with respect to $\mathcal{P}(n)$:

$$g_n = \sum_{A \in \mathcal{P}(n)} \frac{1}{\mu(A)} \int_A g d\mu \ \mathbb{1}_A \ .$$

By ergodicity we can suppose that the modulus of g is equal to 1.

We begin with some remarks. The function g_n is constant in $B_i(n)$ for all $i \in \{1, \ldots, C\}$. We call $d_i(n)$ this constant. For all $y \in S^j B_i(n)$, $0 \leq j < h_i(n)$, $g_n(y) = \lambda^j d_i(n)$. From martingale theory we know that the sequence $(g_n)_n$ converges to g in $L^1(\mu)$. We start by proving that $\lim_{n\to+\infty} \max_{1\leq i\leq C} |d_i(n)| = 1$. We have

$$\begin{aligned}
\int_X |g - g_n| d\mu &= \sum_{i=1}^{C} \sum_{j=0}^{h_i(n)-1} \int_{T^j B_i(n)} |g - g_n| d\mu \\
&= \sum_{i=1}^{C} \sum_{j=0}^{h_i(n)-1} \int_{B_i(n)} |\lambda^j g - \lambda^j d_i(n)| d\mu \\
&= \sum_{i=1}^{C} h_i(n) \int_{B_i(n)} |g - d_i(n)| d\mu \\
&\geq \sum_{i=1}^{C} h_i(n)\mu(B_i(n))(1 - |d_i(n)|) \geq \sum_{i=1}^{C} \frac{1}{K}(1 - |d_i(n)|) \ .
\end{aligned}$$

Since $|d_i(n)| \leq 1$, we obtain that

$$\lim_{n\to+\infty} |d_i(n)| = 1 \ . \tag{6.8}$$

As $B(n+1)$ is included in $B_1(n)$ we have

$$\begin{aligned}
&h_i(n+1)\mu(B_i(n+1))|d_i(n+1) - d_1(n)| \\
\leq & h_i(n+1) \int_{B_i(n+1)} (|g - d_i(n+1)| + |g - d_1(n))|) d\mu \\
\leq & K h_1(n) \int_{B_1(n)} |g - d_1(n)| d\mu + h_i(n+1) \int_{B_i(n+1)} |g - d_i(n+1))| d\mu \\
\leq & (K+1) \int_X |g - g_n| d\mu
\end{aligned}$$

which tends to 0 when n goes to $+\infty$. Hence $\lim_{n\to+\infty} d_i(n+1)/d_1(n) = 1$ and consequently, for all $k \in \{1, \ldots, C\}$

$$\lim_{n\to+\infty} \frac{d_k(n+1)}{d_i(n+1)} = 1 \ . \tag{6.9}$$

Now remark that we have

$$\sum_{k=1}^{C} h_i(n)\mu(B_i(n) \cap S^{-h_i(n)}B_k(n)) = h_i(n)\mu(B_i(n)) \geq \frac{1}{K} \ .$$

Hence, there exists k such that

$$h_i(n)\mu(B_i(n) \cap S^{-h_i(n)}B_k(n)) \geq \frac{1}{KC} \ . \tag{6.10}$$

We set $W(n) = B_i(n) \cap S^{-h_i(n)}B_k(n)$. Thus,

$$\begin{aligned}
&h_i(n)\mu(W(n))|d_i(n) - \lambda^{-h_i(n)}d_k(n)| \\
=&h_i(n)\int_{W(n)} |d_i(n) - \lambda^{-h_i(n)}d_k(n)|d\mu \\
\leq&h_i(n)\int_{B_i(n)} |d_i(n) - g|d\mu + h_i(n)\int_{S^{-h_i(n)}B_k(n)} |g - \lambda^{-h_i(n)}d_k(n)|d\mu \\
\leq&h_i(n)\int_{B_i(n)} |d_i(n) - g|d\mu + Kh_k(n)\int_{B_k(n)} |g - d_k(n)|d\mu \\
\leq&(K+1)\int_X |g - g_n|d\mu \ .
\end{aligned}$$

From (6.8), (6.9) and (6.10) we deduce that

$$\lim_{n\to+\infty} \max_{1\leq i\leq C} |\lambda^{h_i(n)} - 1| = 0 \ .$$

We conclude by invoking Theorem 6.9.2. □

6.9.4 In the context of linearly recurrent dynamical systems

The following theorem applies to substitution subshifts and linearly recurrent subshifts and was proven in (Cortez, Durand, Host, et al. 2003) and (Bressaud, Durand, and Maass 2005). It generalises Theorem 6.9.2.

Theorem 6.9.7 *Let (X,T) be a Cantor dynamical system having a BV-representation (X_B, V_B) with B satisfying* (i) *and* (ii) *in Theorem 6.5.10. Let $(\mathbf{M}(n))_{n\geq 1}$ be the sequence of incidence matrices of B and μ be the unique invariant measure of (X,T). Let $\lambda = \exp(2i\pi\alpha)$. Then,*

(i) *λ is an $L^2(\mu)$-eigenvalue of (X,T) with respect to μ if, and only if,*

$$\sum_{n\geq 2} \|\alpha \mathbf{M}(n)\mathbf{M}(n-1)\cdots\mathbf{M}(1)\|^2 < \infty \ .$$

(ii) λ *is a continuous eigenvalue of the system* (X, T) *if, and only if,*

$$\sum_{n\geq 2} \|\alpha \mathbf{M}(n)\mathbf{M}(n-1)\cdots\mathbf{M}(1)\| < \infty \ .$$

Such a result in the context of bounded topological rank remains open.

We remark that for minimal substitution subshifts the two conditions are equivalent. It is no longer true for linearly recurrent subshifts. In (Bressaud, Durand, and Maass 2005), the authors have constructed an example that has a L^2-eigenvalue which is not continuous.

6.9.5 In the context of tilings, Delone sets, $\mathbb{Z}^d$ and $\mathbb{R}^d$-actions

In (Solomyak 1997) B. Solomyak characterised the eigenvalues of $\mathbb{R}^d$-actions on self-affine tiling spaces. As a consequence he also obtained they are all continuous. In (Cortez, Gambaudo, and Maass 2007) the authors characterised the continuous eigenvalues of the free minimal actions on the Cantor set.

6.10 Exercises

Section 6.3

Exercise 6.1 Show that $X_B^{\max}$ and $X_B^{\min}$ are non-empty sets for every ordered Bratteli diagram B.

Exercise 6.2 Show that the Bratteli compactum X_B of a simple Bratteli diagram B is a Cantor space.

Exercise 6.3 Show that Theorem 6.4.9 does not hold for clopen sets instead of cylinders.
Hint. Eigenvalues could help, see Section 6.9.

Section 6.5

Exercise 6.4 Prove that $\mathbb{Z}_{(p_n)}$ is a compact topological ring.

Exercise 6.5 Give a necessary and sufficient condition for an odometer $(\mathbb{Z}_{(p_n)}, R)$ to have the following property: For all clopen U there exists a clopen $V \subseteq U$ such that (V, R_V) is isomorphic to $(\mathbb{Z}_{(p_n)}, R)$. Give an example having this property which is not an odometer.

Exercise 6.6 Prove Lemma 6.4.5.

Exercise 6.7 Prove the map φ in the proof of Theorem 6.4.6 is a homeomorphism.

Exercise 6.8 Prove that the family of stationary BV-dynamical systems is stable under Kakutani equivalence.

Exercise 6.9 Prove that the sequence of partitions given in Section 6.5.1 satisfies (KR1), (KR2) and (KR3).

Exercise 6.10 Suppose that (X, S) is a minimal substitution subshift on the alphabet A. Let $x \in X$, $\varphi : A^* \to B^*$ be a morphism and $y = \varphi(x)$. Consider (Y, S) the subshift generated by y. Prove that it is isomorphic to a substitution subshift (X_σ, S) where σ is primitive.

Exercise 6.11 Prove Proposition 6.5.6.

Exercise 6.12 Show that every sequence of a linearly recurrent subshift is linearly recurrent with the same constant.

Exercise 6.13 Give examples of dynamical systems that satisfy either (i) or (ii) in Theorem 6.5.10, and that are not linearly recurrent.

Exercise 6.14 Prove that the odometers are minimal dynamical systems.

Exercise 6.15 Show that Theorem 6.4.9 does not hold for clopen sets instead of cylinder sets.

Exercise 6.16 Characterise the odometers having finitely many induced systems on cylinder sets $[a_0, a_1, \ldots, a_n]$ defined in Section 6.5.1 and prove that they have infinitely many induced systems on clopen sets.

Exercise 6.17 Let (X, S) be a minimal substitutive subshift. Prove that for all clopen sets $U \subset X$, there exists a clopen set $V \subset U$ such that $(U, S_{/U})$ is conjugate to $(V, S_{/V})$.

Section 6.7

Exercise 6.18 In Section 6.8.1 show that the map F sends the extremal points of the set $\mathcal{M}(X_B, V_B)$ to the extremal points of $\mathcal{C}$.

Exercise 6.19 Suppose that W is a set of m distinct finite words of length l. Prove that the entropy $h(\Omega(W))$ is equal to $\frac{\log m}{l}$.

Exercise 6.20 Let $B = (V, E, \geq)$ be a properly ordered Bratteli diagram such that $(V(n))_n$ is bounded. Show that the entropy of (X_B, V_B) is zero.

Exercise 6.21 Let B be a properly ordered Bratteli diagram with a consecutive ordering. Show that $h(V_B) = 0$.

Section 6.8

Exercise 6.22 In Section 6.8.1 show that the map F sends the extremal points of $\mathcal{M}(X,T)$ to the extremal points of $\mathcal{C}$.

Exercise 6.23 Prove that if the topological minimal Cantor dynamical system (X,T) has bounded topological rank K then it has at most K ergodic measures.

Exercise 6.24 Show that the map defined in Section 6.8.2 is a bijection between $\mathcal{M}(X,T)$ and $\mathcal{S}(X,T)$.

Section 6.9

Exercise 6.25 Let $\sigma : A \to A^*$ be a primitive substitution which is not one-to-one and which has a non-periodic fixed point. Let (X,S) be the subshift that it generates. Find a one-to-one primitive substitution $\tau : B \to B^*$ and a morphism $\varphi : B \to A^*$ such that $\sigma = \varphi \circ \tau$. Deduce that (X,S) is isomorphic to a subshift generated by a one-to-one primitive substitution.

Exercise 6.26 Construct a linearly recurrent subshift that has a L^2-eigenvalue which is not continuous.

Exercise 6.27 Let $B = (V, E \prec)$ be a properly ordered Bratteli diagram with $\mathrm{Card}(V(n)) = 2$ for infinitely many n. Suppose that it has a continuous eigenvalue $\exp 2i\pi\alpha$ where α is irrational. Prove that it is uniquely ergodic.

Exercise 6.28 Prove that all eigenvalues of a uniquely ergodic Toeplitz subshift with finite topological rank are continuous.

Exercise 6.29 Show that all eigenvalues of a Toeplitz subshift of finite type are rational.

Exercise 6.30 Construct a Toeplitz subshift that has a L^2-eigenvalue which is not continuous.