LINEARLY REPETITIVE DELONE SYSTEMS HAVE A FINITE NUMBER OF NON PERIODIC DELONE SYSTEMS FACTORS.

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Abstract. In this paper we prove linearly repetitive Delone systems have finitely many Delone system factors up to conjugacy. This result is also applicable to linearly repetitive tiling systems.

1. Introduction

The concepts of tiling dynamical system and Delone dynamical system are extensions to $\mathbb{R}^d$-actions of the notion of subshift (see [Ro]). Classical examples are those generated by self-similar tilings, as the Penrose one, which have been extensively studied since the 90's. For details and references see for example [Ro, So1]. Systems arising from self-similar tilings are known to be linearly repetitive (see [So2, Lemma 2.3]), this means there exists a positive constant L, such that every pattern of diameter D appears in every ball of radius LD in any tiling of the system. This concept has been first defined in [LP]. Linearly repetitive tiling and Delone systems can be seen as a generalization to $\mathbb{R}^d$-actions of the notion of linearly recurrent subshift introduced in [DHS].

We study the factor maps between Delone systems. The main result is the following: linearly repetitive Delone systems have finitely many Delone system factors up to conjugacy. As noticed in [So3], tiling systems are topologically conjugate to Delone systems. This conjugacy also preserves linear repetitivity. Consequently, the results that we present can be easily extended to linearly repetitive tiling systems.

The main result of this paper was obtained in the context of subshifts in [Du1]. A key tool used in [Du1], is the existence of sliding-block-codes for factor maps between subshifts (Curtis-Hedlund-Lyndon Theorem). Unlike subshifts, factor maps between two tiling systems are not always sliding-block-codes (see [Pe] and [RS]). The lack of this property appears to be the main difficulty of this work. To surmount this obstacle, we carefully dissect continuity of factor maps, by means of Voronoï cells and return vectors.

This paper is organized as follows: In Section 2 we recall basic concepts and results about Delone systems. In Section 3 we show the factor maps

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from linearly repetitive Delone systems to Delone systems are finite-to-one. Finally, Section 4 is devoted to the proof of the main theorem.

2. Definitions and background

In this section we give the basic definitions and properties concerning Delone sets. For more details we refer to [LP] and [Ro]. Let r and R be two positive real numbers. A (r, R)-*Delone set* X is a discrete subset of $\mathbb{R}^d$ satisfying the following two properties:

(1) *Uniform discreteness*: each open ball of radius $r > 0$ in $\mathbb{R}^d$ contains at most one point of X.
(2) *Relative density*: each closed ball of radius $R > 0$ in $\mathbb{R}^d$ contains at least one point of X.

A (r, R)-Delone set X, in short a *Delone set*, has *finite local complexity* if $X - X$ is *locally finite*, i.e. the intersection of $X - X$ with any bounded set is finite.
The translation by a vector $v \in \mathbb{R}^d$ of a Delone set X, is the Delone set $X - v$ obtained after translating every point of X by $-v$. Observe that $X - v$ has finite local complexity if and only if X has finite local complexity. A Delone set is said to be *non periodic* if $X - v = X$ implies $v = 0$.
Let $R > 0$ and X be a Delone set. We say that $P \subseteq X$ is the *R-patch* of X centered at the point $y \in \mathbb{R}^d$ if

$$P = X \cap B_R(y),$$

where $B_R(y)$ denotes the open ball of a radius R centered at y. If there is no confusion, we refer to a R-patch of X merely as a patch. A *sub-patch* of the patch P is a patch of X included in P. A patch Q is a *translated* of the patch P if there exists $v \in \mathbb{R}^d$ such that $P - v = Q$. The vector $v \in \mathbb{R}^d$ is a *return vector* of the patch P in X if $P - v$ is a patch of X. An *occurrence* of the patch P of X centered at $y \in \mathbb{R}^d$ is a point $w \in \mathbb{R}^d$ such that $y - w$ is a return vector of P. Observe the patch $P - (y - w)$ is the translated of P centered at w.
The *R-atlas* $\mathcal{A}_X(R)$ of X is the collection of all the R-patches centered at a point of X translated to the origin. More precisely:

$$\mathcal{A}_X(R) = \{X \cap B_R(x) - x;\ x \in X\}.$$

The atlas $\mathcal{A}_X$ of X is the union of all the R-atlases, for $R > 0$. Notice that X has finite local complexity if and only if $\mathcal{A}_X(R)$ has finite local complexity for every $R > 0$.
The Delone set X is *repetitive* if for each $R > 0$ there is a finite number $M > 0$, such that for every closed ball B of radius M the set $B \cap X$ contains a translated patch of every R-patch of X. Observe that any repetitive Delone set has necessarily finite local complexity.
The *Voronoï cell* of a point $x \in X$ is the compact subset

$$V_x = \{y \in \mathbb{R}^d;\ ||x - y|| \leq ||x' - y|| \ \text{ for any } x' \in X\}.$$

Notice that if X is a Delone set has finite local complexity, then each Voronoï cell of X is a polyhedra, and there is a finite number of Voronoï cells of X up to translations.

2.1. **Delone systems.** We denote by $\mathcal{D}$ the collection of the Delone sets of $\mathbb{R}^d$. The group $\mathbb{R}^d$ acts on $\mathcal{D}$ by translations:

$$(v, X) \mapsto X - v \text{ for } v \in \mathbb{R}^d \text{ and } X \in \mathcal{D}.$$

Furthermore, this action is continuous with the topology induced by the following distance: take X, X' in $\mathcal{D}$, and define A the set of $\varepsilon \in (0, \frac{1}{\sqrt{2}})$ such that there exist v and v' in $B_\varepsilon(0)$ with

$$(X - v) \cap B_{1/\varepsilon}(0) = (X' - v') \cap B_{1/\varepsilon}(0),$$

we set

$$d(X, X') = \begin{cases} \inf A & \text{if } A \neq \emptyset \\ \frac{1}{\sqrt{2}} & \text{if } A = \emptyset. \end{cases}$$

Roughly speaking, two Delone sets are close if they have the same pattern in a large neighborhood of the origin, up to a small translation.
A *Delone system* is a pair $(\Omega, \mathbb{R}^d)$ such that Ω is a translation invariant closed subset of $\mathcal{D}$. The orbit closure of a Delone set X in $\mathcal{D}$ is the set $\Omega_X = \overline{\{X + v : v \in \mathbb{R}^d\}}$. This is invariant by the $\mathbb{R}^d$-action, and, it is compact if and only if X has finite local complexity (see [Ro] and [Ru]). Every $X' \in \Omega_X$ is a (r, R)-Delone set if X is a (r, R)-Delone set, and for any real $R > 0$, we have $\mathcal{A}_{X'}(R) \subset \mathcal{A}_X(R)$. If all the orbits are dense in Ω_X, the Delone system $(\Omega_X, \mathbb{R}^d)$ is said to be *minimal*. It is shown in [Ro] that the Delone set X is repetitive if and only if the system $(\Omega_X, \mathbb{R}^d)$ is minimal. In that case, for any $X' \in \Omega_X$ and any $R > 0$ the R-atlases $\mathcal{A}_{X'}(R), \mathcal{A}_X(R)$ are the same. If in addition, X is non periodic, then every Delone set in Ω_X is non periodic.
A *factor map* between two Delone systems $(\Omega_1, \mathbb{R}^d)$ and $(\Omega_2, \mathbb{R}^d)$ is a continuous surjective map $\pi : \Omega_1 \to \Omega_2$ such that $\pi(X - v) = \pi(X) - v$, for every $X \in \Omega_1$ and $v \in \mathbb{R}^d$.
In symbolic dynamics it is well-known that topological factor maps between subshifts are always given by sliding-block-codes. There are examples which show that this result can not be extended to Delone systems ([Pe], [RS]). The following lemma shows that factor maps between Delone systems are not far from being sliding-block-codes. A similar result can be found in [HRS].

Lemma 1. *Let X_1 and X_2 be two Delone sets. Suppose X_1 has finite local complexity and $\pi : \Omega_{X_1} \to \Omega_{X_2}$ is a factor map. Then, there exists a constant $s_0 > 0$ such that for every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ satisfying the following: For any $R \geq R_\varepsilon$, if X and X' in Ω_{X_1} verify*

$$X \cap B_{R+s_0}(0) = X' \cap B_{R+s_0}(0),$$

then

$$(\pi(X) - v) \cap B_R(0) = \pi(X') \cap B_R(0)$$

for some $v \in B_\varepsilon(0)$.

Proof. The Delone set X_2 has also finite local complexity because Ω_{X_2} is compact. Let r_0 and R_0 be a positive constant such that X_2 is a (r_0, R_0)-Delone set. Since all the elements of Ω_{X_2} are (r_0, R_0)-Delone sets, if two different points y_1, y_2 of $\mathbb{R}^d$ satisfy $(X - y_1) \cap B_R(a) = (X - y_2) \cap B_R(a)$ for some $X \in \Omega_{X_2}$, $a \in \mathbb{R}^d$ and $R > R_0$, then $\|y_1 - y_2\| \geq \frac{r_0}{2}$ (for the details see [So1]).

Let $0 < \delta_0 < \min\{\frac{r_0}{4}, \frac{1}{R_0}\}$. Since π is uniformly continuous, there exists $s_0 > 1$ such that if X and X' in Ω_{X_1} verify $X \cap B_{s_0}(0) = X' \cap B_{s_0}(0)$ then

$$(\pi(X) - v) \cap B_{\frac{1}{\delta_0}}(0) = \pi(X') \cap B_{\frac{1}{\delta_0}}(0),$$

for some $v \in B_{\delta_0}(0)$. Let $0 < \varepsilon < \delta_0$. By uniform continuity of π, there exists $0 < \delta < \frac{1}{s_0}$ such that if X and X' in Ω_{X_1} verify $X \cap B_{\frac{1}{\delta}}(0) = X' \cap B_{\frac{1}{\delta}}(0)$ then

$$(\pi(X) - v) \cap B_{\frac{1}{\varepsilon}}(0) = \pi(X') \cap B_{\frac{1}{\varepsilon}}(0), \tag{2.1}$$

for some $v \in B_\varepsilon(0)$. Now fix $R \geq R_\varepsilon = \frac{1}{\delta} - s_0$, and let X and X' be two Delone sets in Ω_{X_1} satisfying

$$X \cap B_{R+s_0}(0) = X' \cap B_{R+s_0}(0). \tag{2.2}$$

Observe that X and X' satisfy (2.1), and $(X - a) \cap B_{s_0}(0) = (X' - a) \cap B_{s_0}(0)$, for every a in $B_R(0)$. The choice of s_0 ensures that

$$(\pi(X) - a - t(a)) \cap B_{\frac{1}{\delta_0}}(0) = (\pi(X') - a) \cap B_{\frac{1}{\delta_0}}(0), \tag{2.3}$$

for some $t(a) \in B_{\delta_0}(0)$. Let us prove the map $a \to t(a)$ is locally constant. For $a \in B_R(0)$, let $0 < s_a < \frac{1}{\delta_0} - R_0$ be such that $B_{s_a}(a) \subseteq B_R(0)$. Every $a' \in B_{s_a}(0)$ verifies $B_{\frac{1}{\delta_0} - \|a'\|}(-a') \subset B_{\frac{1}{\delta_0}}(0)$. Let $a' \in B_{s_a}(0)$. This inclusion and (2.3) imply

$$(\pi(X) - a - a' - t(a)) \cap B_{\frac{1}{\delta_0} - \|a'\|}(-a') = (\pi(X') - a - a') \cap B_{\frac{1}{\delta_0} - \|a'\|}(-a'). \tag{2.4}$$

On the other hand, from the definition of the map $a \to t(a)$ we deduce

$$(\pi(X) - a - a' - t(a + a')) \cap B_{\frac{1}{\delta_0}}(0) = (\pi(X') - a - a') \cap B_{\frac{1}{\delta_0}}(0),$$

which implies

$$(\pi(X) - a - a' - t(a + a')) \cap B_{\frac{1}{\delta_0} - \|a'\|}(-a') = (\pi(X') - a - a') \cap B_{\frac{1}{\delta_0} - \|a'\|}(-a'). \tag{2.5}$$

Since $\|t(a) - t(a + a')\| \leq \frac{r_0}{2}$, from equations (2.4), (2.5) and the remark of the beginning of the proof we conclude $t(a) = t(a + a')$ for every $a' \in B_s(0)$. Therefore the map $a \mapsto t(a)$ is constant on $B_{s_a}(a)$.
Furthermore, due to $\delta_0 > \varepsilon$ and (2.2), Equation (2.1) implies there exists $v \in B_\varepsilon(0)$ such that

$$(\pi(X) - v) \cap B_{\frac{1}{\delta_0}}(0) = \pi(X') \cap B_{\frac{1}{\delta_0}}(0). \tag{2.6}$$

For $a = 0$, from (2.3) and (2.6) we have that $t(0) = v$ or $\|v - t(0)\| \geq \frac{r_0}{2}$. Since $\|t(0) - v\| \leq \delta_0 + \varepsilon < 2\delta_0 < \frac{r_0}{2}$, we conclude $t(0) = v$ and then $t(a) = v$ for every $a \in B_R(0)$. This property together with (2.3) and (2.6) imply that

$$(\pi(X) - v) \cap B_R(0) = \pi(X') \cap B_R(0).$$

This conclude the proof. □

3. Preimages of factor maps.

In the rest of this paper we suppose that all the Delone sets have finite local complexity.

A Delone set X is *linearly repetitive* if there exists a constant $L > 0$ such that for every patch P in X, any ball of radius $L\text{diam}(P)$ intersected with X contains a translated patch of P. In this instance we say that X is *linearly repetitive with constant* L. Notice the constant L must be greater or equal than 1, and if X is linearly repetitive with constant L, then it is linearly repetitive with constant L', for every $L' > L$. Every Delone set in the orbit closure of a linearly repetitive Delone set is linearly repetitive with the same constant. When X is linearly repetitive, we call $(\Omega_X, \mathbb{R}^d)$ a *linearly repetitive* Delone system.

The following lemma shows the factors of linearly repetitive systems are also linearly repetitive with a uniform control on the constants. This was already proven for subshifts in [Du1].

Lemma 2. *Let X be a linearly repetitive Delone set with constant L. If X' is a Delone set such that $(\Omega_{X'}, \mathbb{R}^d)$ is a topological factor of $(\Omega_X, \mathbb{R}^d)$, then there exists a constant $\tau_{X'} > 0$ such that if P is a patch of X' with* $\text{diam}(P) \geq \tau_{X'}$, *then for any $y \in \mathbb{R}^d$, the set $X' \cap B_{5L\text{diam}(P)}(y)$ contains a translated patch of P.*

Proof. Let $\pi : \Omega_X \to \Omega_{X'}$ be a topological factor, where X is a (r_X, R_X)-linearly repetitive Delone set with constant L, and X' is a $(r_{X'}, R_{X'})$-Delone set. We can assume that $\pi(X) = X'$. Let $s_0 > 0$ be the constant of Lemma 1. Fix $0 < \varepsilon < Ls_0$ and consider $R_\varepsilon > 0$ as in Lemma 1. We set

$$\tau_{X'} = \max\{s_0, R_\varepsilon, R_X, R_{X'}\}.$$

Let P be a patch in X' with $\text{diam}(P) = D \geq \tau_{X'}$, and let $v \in P \subset X'$. Let $Q = (X - v) \cap B_{D+s_0}(0)$. Since $\text{diam}(Q) \leq 2(D + s_0)$, for every $y \in \mathbb{R}^d$ there exists $w \in B_{2L(D+s_0)}(y)$ such that $(X - w) \cap B_{D+s_0}(0) = Q$. Then, from Lemma 1 there exists $t \in B_\varepsilon(0)$ such that

$$(X' - v) \cap B_D(0) = (X' - w - t) \cap B_D(0).$$

Since $(X' - v) \cap B_D(0)$ contains a translated of P, this shows that every ball of radius $2L(D + s_0) + \varepsilon \leq 5LD$ in X' contains a translated of P as sub-patch. □

The next Lemma follows the same lines of Lemma 2.4 in [So2]. We show the set of occurrences of a R-patch of a linearly repetitive Delone set and its factors is uniformly discrete with a constant depending linearly on R.

Lemma 3. *Let X be a non periodic linearly repetitive Delone set with constant L, and let X' be a non periodic Delone set such that $(\Omega_{X'}, \mathbb{R}^d)$ is a topological factor of $(\Omega_X, \mathbb{R}^d)$. There exists a constant $M_{X'} > 0$ such that for every $R \geq M_{X'}$ and for every R-patch P of X', if $x \in \mathbb{R}^d \setminus \{0\}$ is a return vector of P, then $\|x\| \geq R/(11L)$.*

Proof. Let $R' > 0$ be a real such that any patch of the kind $X' \cap B_{R'}(y)$, with $y \in \mathbb{R}^d$, has diameter greater than $\tau_{X'}$, where $\tau_{X'}$ is the constant given by Lemma 2. Let $M_{X'} = 110LR' + R'$ and P be the R-patch $X' \cap B_R(v)$ with $R > M_{X'}$ and $v \in \mathbb{R}^d$. Suppose there exists $x \in \mathbb{R}^d$, with $0 < \|x\| < R/(11L)$, such that $P + x$ is a patch of X'. For any $y \in \mathbb{R}^d$, consider the patches

$$Q_y = X' \cap B_{R'}(y) \text{ and } S_y = X' \cap B_{R'+\|x\|}(y).$$

Since

$$\tau_{X'} \leq \operatorname{diam}(S_y) \leq 2(R' + \|x\|),$$

from Lemma 2, every ball of radius $10L(R' + \|x\|)$ intersected with X' contains a translated of S_y. By the very hypothesis, we have

$$10L(R' + \|x\|) < 10LR' + \frac{10R}{11} \leq \frac{R}{11} + \frac{10R}{11} = R.$$

This implies there exists $w \in \mathbb{R}^d$ such that $S_y + w$ is a sub-patch of $X' \cap B_R(v) = P$. Because $P + x$ is also a patch of X', we have $Q_y + w + x$ is also a patch of X' and a sub-patch of $S_y + w$. Hence $Q_y + w + x = Q_{y+x} + w$ and

$$Q_y + x = Q_{y+x}.$$

Since y is arbitrary, we conclude that $X' + x = X'$, which contradicts the non periodicity of X' if $x \neq 0$. □

We recall the following definition: A factor map $\pi : (\Omega, \mathbb{R}^d) \to (\Omega', \mathbb{R}^d)$ is said to be *finite-to-one* (with constant D) if for all $y \in Y$ we have $|\pi^{-1}(\{y\})| \leq D$. The next result is a technical lemma we use in Proposition 5 to show that factor maps between linearly repetitive Delone systems are finite-to-one.

Lemma 4. *Let $\pi : (\Omega_X, \mathbb{R}^d) \to (\Omega_{X'}, \mathbb{R}^d)$ be a factor map, where X is a linearly repetitive Delone set with constant L, and X' is a non periodic Delone set. We denote by s_0 the constant given by Lemma 1.*
For every $0 < \varepsilon < \frac{s_0}{2}$, there exists a constant R_π such that for any $R > R_\pi$ there are at most $n \leq (55L^2)^d$ patches $P_1, \ldots, P_n$ satisfying for every $1 \leq i \leq n$ the following conditions:

i) $P_i = (X - w_i) \cap B_{R+s_0}(0)$, *for some* $w_i \in \mathbb{R}^d$, *ii) If X" belongs to Ω_X and $X" \cap B_{R+s_0}(0) = P_i$, then there exists $v \in B_\epsilon(0)$ such that*

$$(\pi(X") - v) \cap B_R(0) = \pi(X) \cap B_R(0),$$

iii) The patch $(X - w_i) \cap B_{R+s_0-2\epsilon}(0)$ is not a sub-patch of P_j, for every $1 \le j \le n$, $j \ne i$.

Proof. Let $0 < \varepsilon < \frac{s_0}{2}$, $R_\pi = \max\{s_0, M_{X'}, R_\epsilon\}$ and $R > R_\pi$, where $M_{X'}$ is the constant given by Lemma 3 and R_ϵ by Lemma 1. Let $P_1, \ldots, P_n$ be n patches of X satisfying the conditions $i)$, $ii)$, $iii)$.
Let $1 \le i \le n$. We have

$$\mathrm{diam}(P_i) \le 2(R + s_0) \le 4R.$$

Linear repetitivity implies there exists $v_i \in B_{4LR}(0)$ such that

$$(X - v_i) \bigcap B_{R+s_0}(0) = P_i.$$

Then by $ii)$, there is $u_i \in B_\epsilon(0)$ satisfying

$$Q = (\pi(X - v_i) + u_i) \cap B_R(0) = (\pi(X) - v_i + u_i) \cap B_R(0),$$

where $Q = \pi(X) \cap B_R(0)$ (observe that Q does not depend on i). This means the set $Q + v_i - u_i$ is a patch of $\pi(X)$. As $\{v_i - u_i,\ 1 \le i \le n\}$ is included in $B_{4LR+\epsilon}(0)$ and $R > M_{X'}$, Lemma 3 implies the number of elements in $\{v_i - u_i,\ 1 \le i \le n\}$ is bounded by

$$\frac{\mathrm{vol}(B_{4LR+\epsilon}(0))}{\mathrm{vol}\left(B_{\frac{R}{11L}}(0)\right)} \le (55L^2)^d.$$

If n is greater than $(55L^2)^d$, then there exist $i \ne j$ such that $v_i - u_i = v_j - u_j$, and $||v_i - v_j|| < 2\epsilon$. This implies the patch $(X - v_i) \cap B_{R+s_0-2\epsilon}(0)$ is included in the patch $(X - v_j) \cap B_{R+s_0}(0) = P_j$, which contradicts the condition $iii)$. □

The next result was proven in [Du1] for subshifts. We use it with Proposition 6 to conclude the proof of the main theorem.

Proposition 5. *Let X be a linearly repetitive Delone set with constant L. If $\pi : (\Omega_X, \mathbb{R}^d) \to (\Omega_{X'}, \mathbb{R}^d)$ is a factor map such that X' is a non periodic Delone set, then π is finite-to-one with constant $(55L^2)^d$.*

Proof. Let $X'_0 \in \Omega_{X'}$. Suppose there exist $n > (55L^2)^d$ elements $X_1, \ldots, X_n$ of Ω_X, such that $\pi(X_i) = X'_0$, for each $1 \le i \le n$. Since they are all different, there exists $R_0 > 0$ such that for any $R \ge R_0$, the patches $X_i \cap B_R(0)$ are pairwise distinct.
Let $0 < \varepsilon < \frac{s_0}{2}$ and R_π be the constant given by Lemma 4. Lemma 1 ensures that for any $Y \in \Omega_X$ satisfying $Y \cap B_R(0) = X_i \cap B_R(0)$, with $1 \le i \le n$ and $R > \max\{R_0, R_\epsilon + s_0, R_\pi + s_0\}$, there exists $v \in B_\epsilon(0)$ such that $(\pi(Y) - v) \cap B_{R-s_0}(0) = X'_0 \cap B_{R-s_0}(0)$. This means the patches $X_1 \cap B_R(0), \cdots, X_n \cap B_R(0)$ satisfy conditions $i)$ and $ii)$ of Lemma 4. Then

we deduce there exist different $i(R)$ and $j(R)$ in $\{1,\dots,n\}$ such that the patch $X_{i(R)} \cap B_{R-2\epsilon}(0)$ is a sub-patch of $X_{j(R)} \cap B_R(0)$. In other words, there exists $v_R \in B_{2\epsilon}(0)$ such that $X_{i(R)} \cap B_{R-2\epsilon}(0) = (X_{j(R)} + v_R) \cap B_{R-2\epsilon}(0)$. By the pigeonhole principle, there exist different i_0 and j_0 in $\{1,\dots,n\}$, and an increasing sequence $(R_p)_{p\geq 0}$, tending to ∞ with p, such that $i(R_p) = i_0$ and $j(R_p) = j_0$, for every $p \geq 0$. By compactness, we can also assume that $(v_{R_p})_{p\geq 0}$ converges to a vector v. Thus, for every $p \geq 0$ we get

$$X_{i_0} \cap B_{R_p-2\epsilon}(0) = (X_{j_0} + v_{R_p}) \cap B_{R_p-2\epsilon}(0),$$

which implies that $X_{i_0} = X_{j_0} + v$ and $X'_0 = \pi(X_{i_0}) = \pi(X_{j_0} + v) = X'_0 + v$. Since $X_{i_0} \neq X_{j_0}$, the vector v is different from zero, but this contradicts the non periodicity of X'_0. □

The following proposition is a straightforward generalization of Lemma 21 in [Du1].

Proposition 6. *Let $(\Omega, \mathbb{R}^d)$ be a minimal Delone system and $\phi_1 : (\Omega, \mathbb{R}^d) \to (\Omega_1, \mathbb{R}^d)$, $\phi_2 : (\Omega, \mathbb{R}^d) \to (\Omega_2, \mathbb{R}^d)$ be two factor maps. Suppose that $(\Omega_2, \mathbb{R}^d)$ is non periodic and ϕ_1 is finite-to-one. If there exist $X, Y \in \Omega$ and $v \in \mathbb{R}^d$ such that $\phi_1(X) = \phi_1(Y)$ and $\phi_2(X) = \phi_2(Y - v)$, then $v = 0$.*

Proof. There exists a sequence $(v_i)_{i\in\mathbb{N}} \subset \mathbb{R}^d$ such that $\lim_{i\to+\infty} X - v_i = Y$. By compactness, we can suppose that the sequence $(Y - v_i)_{i\in\mathbb{N}}$ converges to a point $Y_2 \in \Omega$. By continuity, we have $\phi_1(Y) = \phi_1(Y_2)$, and $\phi_2(Y) = \phi_2(Y_2) - v$. By compactness, we can suppose that the sequence of points $(Y_2 - v_i)_{i\in\mathbb{N}} \subset \Omega$ converges to a point Y_3. So we have $\phi_1(Y_2) = \phi_1(Y_3)$ and $\phi_2(Y_2) = \phi_2(Y_3) - v$. Hence we construct by induction a sequence $(Y_n)_{n\in\mathbb{N}} \subset \Omega$ such that $\phi_1(Y_n) = \phi_1(Y_{n+1})$ and $\phi_2(Y_n) = \phi_2(Y_{n+1}) - v$ for all $n \geq 1$. Since the map ϕ_1 is finite-to-one, there exist $i < j$ such that $Y_i = Y_j$. Then, we have

$$\begin{aligned}\phi_2(Y_i) = \phi_2(Y_{i+1}) - v = \phi_2(Y_{i+2}) - 2v = \ldots &= \phi_2(Y_j) - (j-i)v \\ &= \phi_2(Y_i) - (j-i)v.\end{aligned}$$

Since $(\Omega_2, \mathbb{R}^d)$ is non periodic, we conclude $v = 0$. □

Remark. Following the lines of the proof of Proposition 6, this result can be generalized to $\mathbb{Z}^d$ or $\mathbb{R}^d$ actions, more precisely: Let G be $\mathbb{R}^d$ or $\mathbb{Z}^d$. Let (X, G) be a minimal dynamical system and $\phi_1 : (X, G) \to (X_1, G)$, $\phi_2 : (X, G) \to (X_2, G)$ be two factor maps. Suppose that (X_2, G) is free and ϕ_1 is finite-to-one. If there exist $x, y \in X$ and $g \in G$ such that $\phi_1(x) = \phi_1(y)$ and $\phi_2(x) = \phi_2(g.y)$, then g is the identity in G.

4. Number of factors of linearly repetitive Delone systems.

Let X be a Delone set having finite local complexity, and $P = X \cap B_R(x)$ be a patch of X. We define

$$X_P = \{v \in \mathbb{R}^d : P + v \text{ is a patch of } X\}.$$

Observe that 0 always belongs to X_P. It is straightforward to check that X_P is a Delone set when X is repetitive. Furthermore, X_P is a Delone set having finite local complexity because of $X_P - X_P \subset X - X$. Then we define the *Voronoï cell of* P associated to $v \in X_P$ as the Voronoï cell of $v + x \in X_P + x$. That is,

$$V_{P,v} = \{y \in \mathbb{R}^d : \|y - (x+v)\| \leq \|y - (x+u)\|, \forall u \in X_P\}.$$

Notice the Voronoï cell of P associated to $v \in X_P$ is the Voronoï cell of $v \in X_P$ translated by the vector x.

Remark 7. It follows from the definition that a (r, R)-Delone set X satisfies the following: for any $x \in X$, the diameter of the Voronoï cell V_x is smaller or equal to $2R$ and $B_{\frac{r}{2}}(x)$ is contained in V_x. If X is linearly recurrent with constant L, Lemma 3 implies for every sufficiently large R and every patch $P = X \cap B_R(x)$ of X, the collection X_P is a $(\frac{R}{11L}, 2LR)$-Delone set. Therefore, in this instance we have $\mathrm{diam}(V_{P,v}) \leq 4LR$ and $B_{\frac{R}{11L}}(x+v) \subseteq V_{P,v}$, for every $v \in X_P$.

In the next lemma, we bound the number of ways we can extend a given patch P to a bigger one. More precisely, this gives an upper bound of the number (up to translation) of R'-patches $X \cap B_{R'}(x)$, such that $X \cap B_R(x)$ is a translated of P.

Lemma 8. *Let X be a linearly repetitive Delone set with constant L, and consider $0 < R_1 < R_2$, with R_1 sufficiently large. Then there are at most $n \leq (44L^2)^d \left(\frac{R_2}{R_1}\right)^d$ patches $P_1, \cdots, P_n$ of X, up to translation, satisfying for every $1 \leq i \leq n$ the following two conditions:*

i) there exists $v_i \in \mathbb{R}^d$ such that $P_i = X \cap B_{R_2}(v_i)$.

ii) $(X - v_i) \cap B_{R_1}(0) = (X - v_j) \cap B_{R_1}(0)$, for every $1 \leq j \leq n$.

Proof. Applying Lemma 3 to the identity factor map on $(\Omega_X, \mathbb{R}^d)$, we deduce there exists $M_X > 0$, such that for every $R \geq M_X$ and $x \in \mathbb{R}^d$, the distance between two different occurrences of $P = X \cap B_R(x)$ is greater or equal to $R/(11L)$.

Let $M_X \leq R_1 < R_2$ and $n \in \mathbb{N}$. Suppose $P_1, \cdots, P_n$ are patches of X satisfying conditions *i)* and *ii)*, and such that for every $1 \leq i \leq n$,

iii) P_i is not a translated of P_j, for every $j \in \{1, \cdots, n\} \setminus \{i\}$.

Condition *i)* and linear repetitivity of X imply for every $1 \leq i \leq n$, there exists $w_i \in \mathbb{R}^d$ such that $P_i + w_i$ is a sub-patch of $X \cap B_{2LR_2}(0)$. From condition *ii)* it follows that for every $1 \leq i \leq n$, the point $v_i + w_i$ is an occurrence of the patch $X \cap B_{R_1}(v_1)$ in the ball $B_{2LR_2}(0)$. Finally, by the choice of R_1, conditions *ii)*, *iii)* and Lemma 3, for every i and j in $\{1, \cdots, n\}$ such that $i \neq j$, we get $\|v_i + w_i - (v_j + w_j)\| \geq \frac{R_1}{11L}$, which implies

$$n \leq \frac{\mathrm{vol}(B_{2LR_2}(0))}{\mathrm{vol}(B_{\frac{R_1}{22L}}(0))} = (44L^2)^d \left(\frac{R_2}{R_1}\right)^d,$$

and achieves the proof. □

The following lemma is certainly well-known, but we did not find any reference. This shows that a Voronoi cell of a point x in a (r, R)-Delone set X is completely determined by the points in $X \cap B_{4R}(x)$.

Lemma 9. *Let X be a (r, R)-Delone set. Then for every $x \in X$ one has*

$$V_x = \{y \in \mathbb{R}^d : \|x - y\| \leq \|x' - y\|, \text{ for every } x' \in X \cap B_{4R}(x)\}.$$

Proof. Let $C_x = \{y \in \mathbb{R}^d : \|x - y\| \leq \|x' - y\|, \text{ for every } x' \in X \cap B_{4R}(x)\}$. By definition of Voronoï cell, the inclusion $V_x \subseteq C_x$ is direct.
Observe the set C_x is convex because is obtained as intersection of convex sets. Now, suppose there exists $y \in C_x \setminus V_x$. Then there exist $x' \in X$, satisfying $V_x \cap V_{x'} \neq \emptyset$, and $z \in ([x, y] \cap V_{x'}) \setminus V_x$, where $[x, y]$ is the segment with extreme points x and y. Since $\|x - x'\| \leq 4R$ and $\|z - x'\| < \|z - x\|$, definition of C_x implies $z \notin C_x$, which contradicts the convexity of C_x. □

Lemma 10. *Let X be a non periodic linearly repetitive Delone set with constant L. There exists a positive constant $c(L)$ such that for every sufficiently large R and every patch $P = X \cap B_R(x)$, the collection $\{X \cap V_{P,v} : v \in X_P\}$ contains at most $c(L)$ elements up to translation.*

Proof. Let R be a big enough positive number, in order to apply Lemma 8 to $R_1 = R$ and $R_2 = 8LR$.
Let $x \in \mathbb{R}^d$, $P = X \cap B_R(x)$ and $v \in X_P$. Since $X_P + x$ is a Delone set with constant of uniform density equal to $2LR$ (see Remark 7), Lemma 9 implies $V_{P,v}$ is completely determined by the patch $X \cap B_{8RL}(v + x)$. Furthermore, the Voronoï cell $V_{P,v}$ is contained in the ball $B_{4RL}(v + x)$ (see Remark 7). Then it follows there are at most as many Voronoï cells of P and patches of the kind $X \cap V_{P,v}$, up to translation, as patches Q satisfying the following two conditions: *i*) there exists $w \in \mathbb{R}^d$ such that $Q = X \cap B_{8RL}(w)$ and *ii*) w is an occurrence of a translated of P. These two conditions and Lemma 8 imply there are at most

$$c(L) \leq (44L^2)^d \left(\frac{8LR}{R}\right)^d = (352L^3)^d$$

patches of the kind $X \cap V_{P,v}$ up to translation. □

We have already defined the notion of return vector of a patch, now let us define the notion of return vector of a Voronoï cell of a patch. For a patch $P = X \cap B_R(x)$ of X and $v \in X_P$, we say that $w \in \mathbb{R}^d$ is a *return vector of* $V_{P,v} \cap X$ if $(X - w) \cap V_{P,v} = X \cap V_{P,v}$. We set

$$P_{n,w,v} \text{ the patch } (X - w - x - v) \cap B_{L^n R}(0).$$

Notice that $P_{n,w,v} + v + w + x$ is a patch of X. When there is no confusion about n and v, we write P_w instead of $P_{n,w,v}$.

Lemma 11. *Let $n \in \mathbb{N}$ and X be a non periodic linearly repetitive Delone set with constant L. For every sufficiently large $R > 0$ and every R-patch P, the collection $\{P_w : w$ is a return vector of $V_{P,v} \cap X\}$ has at most $c(n, L)$ elements, for every $v \in X_P$.*

Proof. Let $R_1 = R$ and $R_2 = L^n R$ be sufficiently large positive numbers in order to apply Lemma 8. Let $P = X \cap B_R(x)$ be a patch of X and $v \in X_P$. Since $X_P + x$ is a Delone set with constant of uniform discreteness equal to $\frac{R}{11L}$, the Voronoï cell $V_{P,v}$ contains the ball $B_{\frac{R}{22}}(v+x)$. This implies for every pair of return vectors u and w of $V_{P,v}$ it holds that $P_w \cap B_{\frac{R}{22}}(0) = P_u \cap B_{\frac{R}{22}}(0)$. Thus, from Lemma 8 it follows there are at most

$$c(n, L) \leq (44L^2)^d \left(\frac{L^n R}{\frac{R}{22L}} \right)^d = (968L^{n+3})^d$$

patches of the kind P_w. □

Let $n \in \mathbb{N}$. We call $M(n, L)$ the number of coverings of a set with $c(L)c(n, L)$ elements, where $c(L)$ and $c(n, L)$ are the constants of Lemma 10 and Lemma 11 respectively.

Theorem 12. *Let X be a linearly repetitive Delone set. There are finitely many Delone system factors of $(\Omega_X, \mathbb{R}^d)$ up to conjugacy. Moreover, the number of factors only depends on the linearly recurrence constant of X.*

Proof. Let X be a non periodic linearly repetitive Delone set with constant $L > 1$. Let $n \in \mathbb{N}$ be such that

$$L^n - 1 - 12L - 176L^2 > 1, \tag{4.1}$$

and let $R_1 > 1$ be a constant such that for every $R \geq R_1$, Lemma 10 and Lemma 11 are applicable to R-patches of X.
For every $1 \leq i \leq M(n, L) + 1$, let X_i be a non periodic Delone set such that there exists a topological factor map $\pi_i : \Omega_X \to \Omega_{X_i}$, and let $X_0 = X$. Let M_{X_i} be the constant of Lemma 3 associated to X_i.
Fix $0 < \varepsilon < 1$. For every $1 \leq i \leq M(n, L) + 1$, consider $R_\varepsilon^{(i)}$ and $s_0^{(i)}$ the constants of Lemma 1 associated to π_i. We define

$$R_\varepsilon = \max_i \{R_\varepsilon^{(i)}\},\ s_0 = \max_i \{s_0^{(i)}\} \text{ and } M = \max_i \{M_{X_i}\}.$$

Observe in an open ball of radius $r/22L$, there is at most one return vector of a r-patch of X_i, with $r \geq M$, for every $1 \leq i \leq M(n, L) + 1$.
We take

$$R > \max\{R_\varepsilon, s_0, M + \varepsilon, R_1, 45L\},$$

Consider the patch $P = B_R(0) \cap X$, and $v_1, \cdots, v_N \in X_P$ such that for every $v \in X_P$, there exist $1 \leq i \leq N$ and $u \in \mathbb{R}^d$ satisfying $V_{P,v} \cap X = (V_{P,v_i} \cap X) + u$. Roughly speaking, every set of the kind $V_{P,v} \cap X$ is a translated of some set $V_{P,v_i} \cap X$. Since $R > R_1$, Lemma 10 ensures $N \leq c(L)$.

For every $1 \leq j \leq N$, let $w_{j,1}, \cdots, w_{j,m_j}$ be return vectors of $V_{P,v_j} \cap X$, chosen in order that for every return vector w of $V_{P,v_j} \cap X$, there exists $1 \leq i \leq m_j$ such that P_w is equal to $P_{w_{j,i}}$. Since $R > R_1$, Lemma 11 implies that $m_j \leq c(n, L)$, for every $1 \leq j \leq N$. Therefore, the collection

$$\mathcal{F} = \{P_{w_{j,l}} : 1 \leq l \leq m_j,\ 1 \leq j \leq N\}$$

contains at most $c(L)c(n, L)$ elements.
Let R' be the constant given by

$$R' = (L^n - 1)R - \varepsilon - 4LR.$$

The choice of n ensures that $R' > 0$.
For every $1 \leq i \leq M(n, L) + 1$, we define the following relation on $\mathcal{F}$:

$P_{w_{j,l}} \mathcal{R}_i P_{w_{k,m}}$ if and only if for every $X', X'' \in \Omega_X$ such that $X' \cap B_{L^n R}(0) = P_{w_{j,l}}$ and $X'' \cap B_{L^n R}(0) = P_{w_{k,m}}$, there exist $v \in B_{2\varepsilon}(0)$ and $w \in B_{4LR}(0)$ such that $\pi_i(X') \cap B_{R'}(0) = (\pi_i(X'') + v + w) \cap B_{R'}(0)$.

Since $L^n R - s_0 \geq (L^n - 1)R \geq R > R_\varepsilon$, from Lemma 1 it follows this relation is reflexive, so non empty. Since the cardinal of $\mathcal{F}$ is bounded by $c(L)c(n, L)$, there are at most $M(n, L)$ different relations of this kind. So, there exist $1 \leq i < j < M(n, L) + 1$ such that $\mathcal{R}_i = \mathcal{R}_j$.
In the sequel, we will prove that $(\Omega_{X_i}, \mathbb{R}^d)$ and $(\Omega_{X_j}, \mathbb{R}^d)$ are conjugate. For that, it is sufficient to show that if $Y, Z \in \Omega_X$ are such that $\pi_i(Y) = \pi_i(Z)$ then $\pi_j(Y) = \pi_j(Z)$.
Let Y and Z be two Delone sets in Ω_X such that $\pi_i(Y) = \pi_i(Z)$. Without loss of generality, we can suppose that 0 is an occurrence of P in Y and in $Z - u_0$, where u_0 is some point in $B_{4LR}(0)$. The patches of Y and Z are translated of the patches of X. This implies there exist $1 \leq q_0, r_0 \leq N$ such that

$$Y \cap B_{L^n R}(0) = P_{w_{q_0,l_0}} \text{ and } (Z - u_0) \cap B_{L^n R}(0) = P_{w_{r_0,k_0}},$$

for some $1 \leq l_0 \leq m_{q_0}$ and $1 \leq k_0 \leq m_{r_0}$

Claim 1: $P_{w_{q_0,l_0}} \mathcal{R}_i P_{w_{r_0,k_0}}$.

Proof of Claim 1: Let X' and X'' be two Delone sets in Ω_X such that $X' \cap B_{L^n R}(0) = P_{w_{q_0,l_0}}$ and $X'' \cap B_{L^n R}(0) = P_{w_{r_0,l_0}}$. Since $R \geq s_0$, $R \geq R_\varepsilon$ and

$$X' \cap B_{L^n R}(0) = Y \cap B_{L^n R}(0),\ X'' \cap B_{L^n R}(0) = (Z - u_0) \cap B_{L^n R}(0),$$

By the choice of n and R, Lemma 1 implies there exits z_1 and z_2 in $B_\varepsilon(0)$ such that

$$(\pi_i(X') + z_1) \cap B_{(L^n - 1)R}(0) = \pi_i(Y) \cap B_{(L^n - 1)R}(0), \text{ and}$$
$$(\pi_i(X'') + z_2) \cap B_{(L^n - 1)R}(0) = \pi_i(Z - u_0) \cap B_{(L^n - 1)R}(0).$$

Then we get

$$
\begin{aligned}
&(\pi_i(X'') + z_2 + u_0) \cap B_{(L^n-1)R-4LR}(0) \\
=&\pi_i(Z) \cap B_{(L^n-1)R-4LR}(0) \\
=&\pi_i(Y) \cap B_{(L^n-1)R-4LR}(0) \\
=&(\pi_i(X') + z_1) \cap B_{(L^n-1)R-4LR}(0).
\end{aligned}
$$

Therefore

$$
(\pi_i(X'') + z_2 + u_0 - z_1) \cap B_{(L^n-1)R-4LR-\varepsilon}(0) = \pi_i(X') \cap B_{(L^n-1)R-4LR-\varepsilon}(0),
$$

which implies that $P_{w_{q_0,l_0}} \mathcal{R}_i P_{w_{r_0,k_0}}$.

Since $\mathcal{R}_i = \mathcal{R}_j$, from Claim 1 we get $P_{w_{q_0,l_0}} \mathcal{R}_j P_{w_{r_0,k_0}}$.
Let s be any other occurrence of P in Y. Repeating the same argument for $Y + s$ and $Z + s$, we deduce there exist $u_s \in B_{4LR}(0)$ and $1 \leq q_s, r_s \leq N$ such that

$$
(Y + s) \cap B_{L^n R}(0) = P_{w_{q_s,l_s}} \text{ and } (Z - u_s) \cap B_{L^n R}(0) = P_{w_{r_s,k_s}},
$$

for some $1 \leq l_s \leq m_{q_s}$ and $1 \leq k_s \leq m_{r_s}$. Then from Claim 1 we get $P_{w_{q_s,l_s}} \mathcal{R}_j P_{w_{r_s,k_s}}$. This implies there exist $t_s \in B_{2\varepsilon}(0)$ and $w_s \in B_{4LR}(0)$ such that

$$
\pi_j(Y + s) \cap B_{R'}(0) = (\pi_j(Z + s - u_s) + t_s + w_s) \cap B_{R'}(0).
$$

Claim 2: The vector $w_s - u_s + t_s$ does not depend on s, i.e, there exists $y \in \mathbb{R}^d$ such that $w_s - u_s + t_s = y$ for every occurrence s of P in Y.

Proof of Claim 2: Let s_1 and s_2 be two occurrences of P in Y such that the Voronoï cells of s_1 and s_2, with respect to set of occurrences of P in Y, have common points in their borders. Since the diameter of these Voronoï cells is smaller or equal to $4RL$ (see remark 7), we get $\|s_1 - s_2\| \leq 8LR$. Then Then

$$
\begin{aligned}
& (\pi_j(Z) + s_1 + (s_2 - s_1) - u_{s_1} + t_{s_1} + w_{s_1}) \cap B_{R'-8LR}(0) \\
= \;& (\pi_j(Y) + s_1 + (s_2 - s_1)) \cap B_{R'-8LR}(0) \\
= \;& (\pi_j(Z) + s_2 - u_{s_2} + t_{s_2} + w_{s_2}) \cap B_{R'-8LR}(0).
\end{aligned}
$$

This implies $(-u_{s_1} + t_{s_1} + w_{s_1}) - (-u_{s_2} + t_{s_2} + w_{s_2})$ is a return vector of a $(R' - 8LR)$-patch of $\pi_j(Z) + s_2$. Since

$$
R' - 8LR = R(L^n - 1 - 12L) - \varepsilon \geq R - \varepsilon > M,
$$

Lemma 3 implies the non zero vectors of the $(R' - 8LR)$-patches of $\pi_j(Z) + s_2$ have norm greater or equal to $(R' - 8LR)/11L$. Thus, due to

$$
\| -u_{s_1} + t_{s_1} + w_{s_1} - (-u_{s_2} + t_{s_2} + w_{s_2})\| \leq 16LR + 4\varepsilon,
$$

and

$$\begin{aligned} 11(16LR+4\varepsilon) &= 176L^2R+44L\varepsilon \\ &< (L^n-1-12L-1)R+44L\varepsilon \\ &= R'-8LR+\varepsilon-R+44L\varepsilon \\ &< R'-8LR+L-R+44L < R'-8LR, \end{aligned}$$

we deduce $-u_{s_1}+t_{s_1}+w_{s_1} = -u_{s_2}+t_{s_2}+w_{s_2}$, which shows Claim 2.

¿From Claim 2 we get there exists $y \in \mathbb{R}^d$ such that for every occurrence s of P in Y,

$$\pi_j(Y+s)\cap B_{R'}(0) = (\pi_j(Z+s)+y)\cap B_{R'}(0), \text{ and then}$$
$$\pi_j(Y)\cap B_{R'}(s) = (\pi_j(Z)+y)\cap B_{R'}(s).$$

¿From Remark 7, the diameter of the Voronoï cells of P is less than $4LR$, which is less than R'. Hence,

$$\pi_j(Y) = \pi_j(Z)+y.$$

We conclude with Lemma 5 and Proposition 6. □

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