ON THE EIGENVALUES OF FINITE RANK BRATTELI-VERSHIK DYNAMICAL SYSTEMS

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Abstract. In this article we study conditions to be a continuous or a measurable eigenvalue of finite rank minimal Cantor systems, that is, systems given by an ordered Bratteli diagram with a bounded number of vertices per level. We prove that continuous eigenvalues always come from the stable subspace associated to the incidence matrices of the Bratteli diagram and we study rationally independent generators of the additive group of continuous eigenvalues. Given an ergodic probability measure, we provide a general necessary condition to be a measurable eigenvalue. Then we consider two families of examples. A first one to illustrate that measurable eigenvalues do not need to come from the stable space. Finally we study Toeplitz type Cantor minimal systems of finite rank. We recover classical results in the continuous case and we prove that measurable eigenvalues are always rational but not necessarily continuous.

1. Introduction

The study of eigenvalues of dynamical systems has been extensively considered in ergodic theory to understand and build the Kronecker factor and also to study the weak mixing property. In topological dynamics one also consider continuous eigenvalues, that is, eigenvalues associated to continuous eigenfunctions, to study topological weak mixing (at least in the minimal case). Since continuous eigenvalues are also eigenvalues, a recurrent question is to know whether they coincide. In general the answer is negative, since there are minimal topologically weakly mixing systems that are not weakly mixing for some invariant measures. A positive answer to that question has been given for the class of primitive substitution systems in [Ho]. The same question has been considered for linearly recurrent minimal Cantor systems in [CDHM] and [BDM] concluding that in general not all eigenvalues are continuous. Nevertheless, in those articles explicit necessary and sufficient conditions are given to check whether a complex number is an eigenvalue, continuous or not, allowing to recover the result in [Ho]. Those conditions only depend on the incidence matrices associated to a Bratteli-Vershik representation of the linearly recurrent minimal system and not on the partial order of the diagram. The independence of the order seems to be characteristic of linearly recurrent systems.

In this article we consider the same question for minimal Cantor systems that admit a proper Bratteli-Vershik representation with the same number of vertices per level. We call them (topologically) finite rank minimal Cantor systems. The

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motivations are different. First, this class contains linearly recurrent systems but it is much larger and natural (some systems in this class appear as the symbolic representation of well studied classes of dynamical systems like interval exchange transformations [GJ]). Second, the knowledge about this class of systems is very small, one of its main (and recent) properties is that they are either expansive or equicontinuous [DM]. Thus, up to odometers, that are well known, this is a huge class of symbolic minimal systems.

In Section 3 we study continuous eigenvalues. The main result states that continuous eigenvalues of minimal Cantor systems always come from the stable space associated to the sequence of matrices of the Bratteli-Vershik representation of the system and we provide a general necessary condition to be a continuous eigenvalue. In Section 4 these results are used to get a bound for the maximal number of rationally independent continuous eigenvalues of a finite rank system. In Section 5 we give a necessary condition to be an eigenvalue of a finite rank minimal system with respect to some invariant probability measure. Next two sections are devoted to examples. In Section 6 we construct a uniquely ergodic non weakly mixing finite rank minimal Cantor system to illustrate that eigenvalues do not always come from the stable space associated to the sequence of matrices given by the Bratteli-Vershik representation of the system. In the last section we study Toeplitz type minimal Cantor systems of finite rank. The main property we deduce is that eigenvalues are always rational but not always continuous.

2. Basic definitions

2.1. **Dynamical systems and eigenvalues.** A *topological dynamical system*, or just dynamical system, is a compact Hausdorff space X together with a homeomorphism $T : X \to X$. One uses the notation (X,T). If X is a Cantor set one says that the system is Cantor. That is, X has a countable basis of closed and open sets and it has no isolated points. A dynamical system is *minimal* if all orbits are dense in X, or equivalently the only non trivial closed invariant set is X.
A complex number λ is a *continuous eigenvalue* of (X,T) if there exists a continuous function $f : X \to \mathbb{C}$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called a *continuous eigenfunction* (associated to λ). Let μ be a T-invariant probability measure, i.e., $T\mu = \mu$, defined on the Borel σ-algebra $\mathcal{B}(X)$ of X. A complex number λ is an *eigenvalue* of the dynamical system (X,T) with respect to μ if there exists $f \in L^2(X,\mathcal{B}(X),\mu)$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called an *eigenfunction* (associated to λ). If the system is ergodic, then every eigenvalue is of modulus 1 and every eigenfunction has a constant modulus μ-almost surely. Of course continuous eigenvalues are eigenvalues.

2.2. **Bratteli-Vershik representations.** Let (X,T) be a minimal Cantor system. It can be represented by an *ordered Bratteli-Vershik diagram*. For details on this theory see [HPS]. We give a brief outline of such constructions emphasizing the notations used in this paper.

2.2.1. *Bratteli-Vershik diagrams.* A Bratteli-Vershik diagram is an infinite graph (V,E) which consists of a vertex set V and an edge set E, both of which are divided into levels $V = V_0 \cup V_1 \cup \ldots$, $E = E_1 \cup E_2 \cup \ldots$ and all levels are pairwise disjoint. The set V_0 is a singleton $\{v_0\}$, and for $k \geq 1$, E_k is the set of edges joining vertices in V_{k-1} to vertices in V_k. It is also required that every vertex in V_k is the "end-point" of some edge in E_k for $k \geq 1$ and an "initial-point" of some edge in E_{k+1} for $k \geq 0$. One puts $V_k = \{1, \ldots, C(k)\}$. The *level* k is the subgraph consisting of the vertices in $V_k \cup V_{k+1}$ and the edges E_{k+1} between these vertices. Level 0 is called the *hat* of the Bratteli-Vershik diagram and it is uniquely determined by an integer vector $H(1) = \left(h_1(1), \ldots, h_{C(1)}(1)\right)^T \in \mathbb{N}^{C(1)}$, where each component represents the number of edges joining v_0 and a vertex of V_1.

We describe the edge set E_k using a $V_k \times V_{k-1}$ incidence matrix $M(k)$ for which its (i,j)–entry is the number of edges in E_k joining vertex $j \in V_{k-1}$ with vertex $i \in V_k$. For $1 \leq k \leq l$ one defines $P(k) = M(k) \cdots M(2)$ and $P(l,k) = M(l) \cdots M(k+1)$ with $P(1) = I$ and $P(k,k) = I$; where I is the identity map. Also, put $H(k) = P(k)H(1) = (h_1(k), \ldots, h_{C(k)}(k))^T$.

2.2.2. *Ordered Bratteli-Vershik diagrams.* An *ordered* Bratteli-Vershik diagram is a triple $B = (V, E, \preceq)$ where (V, E) is a Bratteli-Vershik diagram and $\preceq$ a partial ordering on E such that : Edges e and e' are comparable if and only if they have the same end-point.

Let $k \leq l$ in $\mathbb{N} \setminus \{0\}$ and let $E_{k,l}$ be the set of all paths in the graph joining vertices of V_{k-1} with vertices of V_l. The partial ordering of E induces another in $E_{k,l}$ given by $(e_k, \ldots, e_l) \preceq (f_k, \ldots, f_l)$ if and only if there is $k \leq i \leq l$ such that $e_j = f_j$ for $i < j \leq l$ and $e_i \preceq f_i$.

Given a strictly increasing sequence of integers $(m_n)_{n \geq 0}$ with $m_0 = 0$ one defines the *contraction* of $B = (V, E, \preceq)$ (with respect to $(m_n)_{n \geq 0}$) as

$$\left((V_{m_n})_{n \geq 0}, \left(E_{m_n+1, m_{n+1}}\right)_{n \geq 0}, \preceq \right),$$

where $\preceq$ is the order induced in each set of edges $E_{m_n+1, m_{n+1}}$. The converse operation is called *miscroscoping* (see [HPS] for more details).

Given an ordered Bratteli-Vershik diagram $B = (V, E, \preceq)$ one defines X_B as the set of infinite paths $(e_1, e_2, \cdots)$ starting in v_0 such that for all $i \geq 1$ the end-point of $e_i \in E_i$ is the initial-point of $e_{i+1} \in E_{i+1}$. We topologize X_B by postulating a basis of open sets, namely the family of *cylinder sets*

$$[e_1, e_2, \ldots, e_k] = \{ (x_1, x_2, \ldots) \in X_B \mid x_i = e_i, \text{ for } 1 \leq i \leq k \}.$$

Each $[e_1, e_2, \ldots, e_k]$ is also closed, as is easily seen, and so we observe that X_B is a compact, totally disconnected metrizable space.

When there is a unique $x = (x_1, x_2, \ldots) \in X_B$ such that x_i is maximal for any $i \geq 1$ and a unique $y = (y_1, y_2, \ldots) \in X_B$ such that y_i is minimal for any $i \geq 1$, one says that $B = (V, E, \preceq)$ is a *properly ordered* Bratteli diagram. Call these particular points $x_{\max}$ and $x_{\min}$ respectively. In this case one defines a dynamic V_B over X_B called the *Vershik map.* The map V_B is defined as follows: let $x = (x_1, x_2, \ldots) \in X_B \setminus \{x_{\max}\}$ and let $k \geq 1$ be the smallest integer so that x_k is not a maximal edge. Let y_k be the successor of x_k and $(y_1, \ldots, y_{k-1})$ be the unique minimal path in $E_{1,k-1}$ connecting v_0 with the initial point of y_k. One sets $V_B(x) = (y_1, \ldots, y_{k-1}, y_k, x_{k+1}, \ldots)$ and $V_B(x_{\max}) = x_{\min}$.

The dynamical system (X_B, V_B) is called the *Bratteli-Vershik system* generated by $B = (V, E, \preceq)$. The dynamical system induced by any contraction of B is topologically conjugate to (X_B, V_B). In [HPS] it is proved that any minimal Cantor system (X, T) is topologically conjugate to a Bratteli-Vershik system (X_B, V_B).

One says that (X_B, V_B) is a *Bratteli-Vershik representation* of (X,T). In what follows, we identify (X,T) with any of its Bratteli-Vershik representations.
In the sequel we will need to assume some extra technical conditions on the Bratteli-Vershik representations that make this work consistent with previous ones of the authors ([CDHM],[BDM]), that we think do not represent a "real" restriction for our purposes:
(H1) $H(1) = (1, \dots, 1)^T$;
(H2) For every $k \geq 2$, $M(k) > 0$;
(H3) For every level $k \geq 2$, all maximal edges of E_k start in the same vertex of V_{k-1}. We can assume it is $C(k-1)$.
Abusing of the language, a Bratteli-Vershik representation of a system (X,T) verifying (H1), (H2) and (H3) will be called proper.
A minimal Cantor system is of (topological) finite rank $d \geq 1$ if it admits a proper Bratteli-Vershik representation such that the number of vertices per level verify $C(k) \leq d$ for any $k \geq 1$. Contracting and microscoping the diagram if needed one can assume (this is done in the sequel) that $C(k) = d$ for any $k \geq 2$.
A minimal Cantor system is linearly recurrent if it admits a proper Bratteli-Vershik representation such that the set $\{M(n); n \geq 1\}$ is finite. Clearly, linearly recurrent minimal Cantor systems are of finite rank (for details about these systems see [Du1]).

2.2.3. *Associated Kakutani-Rohlin partitions and invariant measures.* Let (X,T) be the minimal Cantor system defined by the ordered Bratteli diagram $B = (V, E, \preceq)$. This diagram defines for each $n \geq 0$ a clopen *Kakutani-Rohlin* (KR) partition of X

$$\mathcal{P}(n) = \{T^{-j} B_k(n); k \in V_n,\ 0 \leq j < h_k(n)\}\ ,$$

with $B_k(n) = [e_1, \dots e_n]$, where $(e_1, \dots, e_n)$ is the unique path from v_0 to vertex k such that each e_i is maximal for the ordering of B. For each $k \in V_n$ the set $\{T^{-j} B_k(n); 0 \leq j < h_k(n)\}$ is the *k-th tower* of $\mathcal{P}(n)$. This corresponds to the set of all the paths from v_0 to $k \in V_n$ (there are exactly $h_k(n)$ of such paths). The map $\tau_n : X \to V_n$ is given by $\tau_n(x) = k$ if x belongs to the k-th tower of $\mathcal{P}(n)$. Denote by $\mathcal{T}_n$ the σ-algebra generated by the partition $\mathcal{P}(n)$; that is, the finite paths joining v_0 with any vertex of V_n.
Let μ be a T-invariant measure. It is determined by its value in $B_k(n)$ for each $n \geq 0$ and $k \in V_n$. Define $\mu(n) = (\mu_1(n), \dots, \mu_{C(n)})^T$ with $\mu_k(n) = \mu(B_k(n))$. A simple computation yields to the following fundamental relation:

$$\mu(n) = M^T(n+1)\mu(n+1) \tag{2.1}$$

for any $n \geq 0$.
Fix $n \in \mathbb{N}$. The return time of x to $B_{\tau_n(x)}(n)$ is given by $r_n(x) = \min\{j \geq 0; T^j x \in B_{\tau_n(x)}(n)\}$. Define the map $s_n : X \to \mathbb{N}^{C(n)}$, called *suffix map of order* n, by

$$(s_n(x))_k = |\{e \in E_{n+1}; x_{n+1} \preceq e, x_{n+1} \neq e, k \text{ is the initial vertex of } e\}|$$

for all $x = (x_n; n \geq 1) \in X$ and $k \in V_n$. A classical computation gives (see for example [BDM])

$$r_n(x) = s_0(x) + \sum_{k=1}^{n-1} \langle s_k(x), P(k)H(1) \rangle \ . \tag{2.2}$$

Observe that if $H(1) = (1, \ldots, 1)^T$ then $s_0(x) = 0$.

3. Continuous eigenvalues : general necessary and sufficient conditions

Let (X, T) be a minimal Cantor system given by a proper Bratteli-Vershik representation $B = (V, E, \preceq)$. Recall the associated sequence of matrices is $(M(n); n \geq 1)$ with $M(n) > 0$, $P(n) = M(n) \cdots M(2)$ for $n \geq 2$ and $P(n, k) = M(n) \cdots M(k+1)$ for $1 \leq k < n$.
First we recall a general necessary and sufficient condition to be a continuous eigenvalue of (X, T) proved in [BDM].

Proposition 1. *Let $\lambda = \exp(2i\pi\alpha)$. The following conditions are equivalent,*

(1) *λ is a continuous eigenvalue of the minimal Cantor system (X, T);*
(2) *$(\lambda^{r_n(x)}; n \geq 1)$ converges uniformly in x, i.e., the sequence $(\alpha r_n(x); n \geq 1)$ converges $(\mathrm{mod}\ \mathbb{Z})$ uniformly in x.*

It follows that,

Corollary 2. *Let $\lambda = \exp(2i\pi\alpha)$. If λ is a continuous eigenvalue of (X, T) then*

$$\lim_{n \to \infty} \lambda^{h_{j_n}(n)} = 1$$

uniformly in $(j_n; n \in \mathbb{N}) \in \prod_{n \in \mathbb{N}} \{1, \ldots, C(n)\}$.

The following theorem states that necessary condition in Theorem 1 part (2) in in [BDM] is true in general for proper Bratteli-Vershik representations. In fact, from Corollary 2 we know that, contracting if needed, we have the convergence of the sum. But we do not know a priori how much we should contract. The next theorem says that it suffices to contract in order to be proper.
Denote by $|\!|\!| \cdot |\!|\!|$ the distance to the nearest integer vector and $\| \cdot \|$ the norm defined by $\|v\| = \max_i |v_i|$ for all $v \in \mathbb{R}^d$.

Theorem 3. *Let (X, T) be a minimal Cantor system given by a proper Bratteli-Vershik representation $B = (V, E, \preceq)$. If $\lambda = \exp(2i\pi\alpha)$ is a continuous eigenvalue of (X, T) then*

$$\sum_{n \geq 1} |\!|\!| \alpha P(n)H(1) |\!|\!| < \infty$$

To prove Theorem 3 we need an intermediate statement.

Lemma 4. *Assume coefficients of matrices $(M(n); n \geq 1)$ are strictly bigger than 1. Let $(j(n); n \in \mathbb{N})$ and $(i(n); n \in \mathbb{N})$ be sequences of positive integers such that $j(n+1) - j(n) \geq 2$ and $i(n) \in \{1, \ldots, C(j(n))\}$ for all $n \in \mathbb{N}$. Then there exist different points $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ in X such that:*

(1) *$\forall\ n \in \mathbb{N}$, $s_{j(n)}(x) - s_{j(n)}(y) = e_{i(n)}$ (the $i(n)$-th canonical vector);*
(2) *$s_j(x) - s_j(y) = (0, \ldots, 0)^T$ whenever $j \notin \{j(n); n \in \mathbb{N}\}$.*

Proof. Recall (X,T) is given by a proper Bratteli-Vershik representation $B = (V,E,\preceq)$. Let $n \geq 1$. Since matrices $(M(n); n \geq 1)$ have coefficients larger than two, then there are successive edges $e_n, f_n \in E_{j(n)+1}$ for the local order, with $e_n \preceq f_n$, such that the starting vertex of f_n is $i(n)$ and the end vertex of the edges is $C(j(n)+1)$. Recall we are assuming that by hypothesis (H3) all maximal edges of level $j(n)+2$ start at this vertex. Define $x_{j(n)+1} = e_n$ and $y_{j(n)+1} = f_n$. The edges between $x_{j(n)+1}$ and $x_{j(n+1)+1}$ are defined as the maximal path joining $x_{j(n)+1}$ with $x_{j(n+1)+1}$ and similarly we complete edges in y. This is possible since $j(n+1) - j(n) \geq 2$.
Since f_n is the successor of e_n, property (1) holds at each $n \geq 1$. Property (2) follows from hypothesis (H3): for each $j \notin \{j(n); n \in \mathbb{N}\}$ all maximal edges start at $C(j)$. □

Proof of Theorem 3. We can assume without loss of generality that the coefficients of the matrices $(M(n); n \geq 1)$ are strictly bigger than 1. Indeed, by hypothesis H(2) one has that $M(n+1)M(n)$ has coefficients strictly larger than 1 for any $n \geq 2$. Then contracting at even and odd levels produces two different representations verifying the desired property of the corresponding Bratteli-Vershik diagram. Each case serves to prove separately that $\sum_{n\geq 0} |||\alpha P(2n+1)H(1)||| < \infty$ and that $\sum_{n\geq 1} |||\alpha P(2n)H(1)||| < \infty$, which together proves the theorem.
Let λ be a continuous eigenvalue of (X,T). We deduce from Corollary 2 that $|||\alpha P(n)H(1)|||$ converges to 0 as $n \to \infty$. Then for all $n \geq 1$ there exist a real vector $v(n)$ and an integer vector $w(n)$ such that

$$\alpha P(n)H(1) = v(n) + w(n) \text{ and } \lim_{n\to\infty} v(n) = 0 \ .$$

Thus it is enough to prove that $\sum_{n\geq 1} \|v(n)\| < \infty$.
For $n \geq 1$ let $l(n) \in \{1,...,C(n)\}$ be such that

$$|\langle e_{l(n)}, v(n)\rangle| = \max_{l\in\{1,\ldots,C(n)\}} |\langle e_l, v(n)\rangle|$$

where e_l is the l-th canonical vector of $\mathbb{R}^{C(n)}$. Let

$$I^+ = \{n \geq 1; \langle e_{l(n)}, v(n)\rangle \geq 0\},\ I^- = \{n \geq 1; \langle e_{l(n)}, v(n)\rangle < 0\}.$$

To prove $\sum_{n\geq 1} \|v(n)\| < \infty$ one only needs to show that

$$\sum_{n\in I^+} \langle e_{l(n)}, v(n)\rangle < \infty \text{ and } -\sum_{n\in I^-} \langle e_{l(n)}, v(n)\rangle < \infty.$$

Since the arguments we will use are similar in both cases we only prove the first one. Moreover, to prove $\sum_{n\in I^+} \langle e_{l(n)}, v(n)\rangle < \infty$ we only show $\sum_{n\in I^+\cap 2\mathbb{N}} \langle e_{l(n)}, v(n)\rangle < \infty$. Analogously, one can prove that $\sum_{n\in I^+\cap(2\mathbb{N}+1)} \langle e_{l(n)}, v(n)\rangle < \infty$.
Assume $I^+ \cap 2\mathbb{N}$ is infinite, if not the result follows directly. Order its elements $j(0) < j(1) < \ldots < j(n) < \ldots$ and define $i(n) = l(j(n))$ for $n \in \mathbb{N}$. Let $x, y \in X$ be the points given by Lemma 4 using these sequences.
Now, from equality (2.2) and Proposition 1, one deduces that

$$\begin{aligned}
&\alpha(r_m(x)-r_m(y))\\
&=\alpha \sum_{n\in\{1,\dots,m-1\}\cap I^+\cap 2\mathbb{N}} \langle (s_n(x)-s_n(y)), P(n)H(1)\rangle + (s_0(x)-s_0(y))\\
&=\alpha \sum_{n\in\{1,\dots,m-1\}\cap I^+\cap 2\mathbb{N}} \langle e_{l(n)}, P(n)H(1)\rangle\\
&= \sum_{n\in\{1,\dots,m-1\}\cap I^+\cap 2\mathbb{N}} \langle e_{l(n)}, v(n)+w(n)\rangle
\end{aligned}$$

converges $\mod \mathbb{Z}$ when $m \to \infty$. Then $\sum_{n\in\{1,\dots,m-1\}\cap I^+\cap 2\mathbb{N}}\langle e_{l(n)}, v(n)\rangle$ converges $\mod \mathbb{Z}$ when $m \to \infty$. But $\langle e_{l(n)}, v(n)\rangle$ tends to 0, hence the series $\sum_{n\in I^+\cap 2\mathbb{N}}\langle e_{l(n)}, v(n)\rangle$ converges. □

The following theorem states that continuous eigenvalues are always constructed from the subspaces,

$$\left\{v \in \mathbb{R}^{C(m)}; P(n,m)v \to_{n\to\infty} 0\right\},\ m \geq 2,$$

defined by the sequence of incidence matrices $(M(n); n \geq 1)$.

Theorem 5. *Let (X,T) be a minimal Cantor system given by a proper Bratteli-Vershik representation $B = (V,E,\preceq)$. Let $\lambda = \exp(2i\pi\alpha)$ be a continuous eigenvalue of (X,T). Then, there exist $m \in \mathbb{N}$, $v \in \mathbb{R}^{C(m)}$ and $w \in \mathbb{Z}^{C(m)}$ such that*

$$\alpha P(m)H(1) = v + w \text{ and } P(n,m)v \to_{n\to\infty} 0.$$

Proof. We deduce from Corollary 2 that $|||\alpha P(n)H(1)|||$ converges to 0 as n tends to ∞. Hence, for every $n \geq 2$ one can write $\alpha P(n)H(1) = v(n) + w(n)$, where $w(n)$ is an integer vector and $v(n)$ is a real vector with $\|v(n)\| \to 0$ as $n \to \infty$. Clearly

$$\alpha P(n+1)H(1) = M(n+1)v(n) + M(n+1)w(n) = v(n+1) + w(n+1). \tag{3.1}$$

We start proving that there exists $m \geq 1$ such that for all $n \geq m$ one has $M(n+1)v(n) = v(n+1)$. From Proposition 1 one knows there exists $m \geq 1$ such that for all $n \geq m$ and all $x \in X$ it holds

$$|||\langle s_n(x), \alpha P(n)H(1)\rangle||| < \frac{1}{4} \text{ and } \|v(n)\| < \frac{1}{4}. \tag{3.2}$$

Hence, for all $n \geq m$ and all $x \in X$, one gets

$$|||\langle s_n(x), v(n)\rangle||| < \frac{1}{4}. \tag{3.3}$$

Fix $n \geq m$. Consider $x \in B_k(n+1)$ for some $1 \leq k \leq C(n+1)$ and let $0 = j_1 < j_2 < \cdots < j_l$ be the collection of all the integers $0 \leq j < h_k(n+1)$ such that $T^{-j}x \in \cup_{i\in\{1,\dots,C(n)\}} B_i(n)$. Remark that

$$\|s_n^T(T^{-j_l}x) - e_k^T M(n+1)\| = 1 \tag{3.4}$$

$$\|s_n(T^{-j_{m+1}}x) - s_n(T^{-j_m}x)\| = 1 \tag{3.5}$$

for all $1 \leq m \leq l-1$. Let $1 \leq m \leq l-1$ and suppose $\left|\langle s_n(T^{-j_m}x), v(n)\rangle\right| < 1/4$. Then, from (3.2) and (3.5),

$$
\begin{aligned}
&|\langle s_n(T^{-j_{m+1}}x), v(n)\rangle| \\
&= |\langle s_n(T^{-j_m}x), v(n)\rangle + \langle s_n(T^{-j_{m+1}}x), v(n)\rangle - \langle s_n(T^{-j_m}x), v(n)\rangle| \\
&< \frac{1}{2}.
\end{aligned}
$$

From (3.3) one gets that $|\langle s_n(T^{-j_{m+1}}x), v(n)\rangle| < \frac{1}{4}$. Thus, as $|\langle s_n(x), v(n)\rangle| = 0$, it follows by induction that $\langle s_n(T^{-j_l}x), v(n)\rangle < \frac{1}{4}$. Therefore, from (3.4) one deduces that $|\langle e_k, M(n+1)v(n)\rangle| < 1/2$. This is true for all $1 \leq k \leq C(n+1)$, then $\|M(n+1)v(n)\| < 1/2$.
Finally, from (3.1) one deduces that for all $n \geq m$,

$$M(n+1)v(n) = v(n+1) \text{ and } M(n+1)w(n) = w(n+1). \tag{3.6}$$

To conclude it is enough to set $v = v(m)$ and $w = w(m)$. □

Lemma 6. *Let (X,T) be a minimal Cantor system given by a Bratteli-Vershik representation $B = (V, E, \preceq)$. Consider $m \in \mathbb{N}$ and $v \in \mathbb{R}^{C(m)}$ such that $P(n,m)v \to 0$ as $n \to \infty$. Then $\langle v, \mu(m)\rangle = 0$ for any T-invariant probability measure μ.*

Proof. From definition one has

$$
\begin{aligned}
\langle v, \mu(m)\rangle &= \langle v, P^T(n,m)\mu(n)\rangle = \langle P(n,m)v, \mu(n)\rangle \\
&\leq \|P(n,m)v\| \cdot \|\mu(n)\|
\end{aligned}
$$

and the last term converges to 0 as $n \to \infty$. Thus $\langle v, \mu(m)\rangle = 0$. □

Corollary 7. *Let (X,T) be a minimal Cantor system given by a proper Bratteli-Vershik representation $B = (V, E, \preceq)$ and let μ be a T-invariant probability measure. Let $\lambda = \exp(2i\pi\alpha)$ be a continuous eigenvalue of (X,T). Then one of the following conditions holds:*

1. *α is rational with a denominator dividing $\gcd(h_i(m); 1 \leq i \leq C(m))$ for some $m \in \mathbb{N}$.*
2. *There exist $m \in \mathbb{N}$ and an integer vector $w \in \mathbb{Z}^{C(m)}$ such that $\alpha = \langle w, \mu(m)\rangle$.*

Proof. Let m, v and w be as in Theorem 5. We recall that $P(m)H(1) = H(m)$. Assume $v = 0$. Then $\alpha H(m) = w$ and thus α is rational with a denominator dividing $\gcd(h_i(m); 1 \leq i \leq C(m))$. Now suppose $v \neq 0$. From Lemma 6, $\langle v, \mu(m)\rangle = 0$. Thus, from $w = \alpha H(m) - v$ and $\langle \mu(m), H(m)\rangle = 1$ one gets $\alpha = \langle w, \mu(m)\rangle$. □

Part (2) of previous lemma left open the question whether any integer vector $w \in \mathbb{Z}^{C(m)}$, for some $m \in \mathbb{N}$, can produce a continuous eigenvalue of the system by taking $\alpha = \langle w, \mu(m)\rangle$. It is enough to consider topological weakly mixing minimal Cantor systems to see that in some cases not all integer vectors can produce a continuous eigenvalue. In general, the set of integer vectors from which one can define continuous eigenvalues of the system is a discrete group. But most of the time it is very difficult to describe it explicitly. In the next section we give a slightly more precise description of such group in the finite rank case.

4. Special properties of continuous eigenvalues of finite rank systems

Let (X,T) be a minimal Cantor system of finite rank d. Fix a proper Bratteli-Vershik representation of (X,T) with exactly d vertices per level which sequence of incidence matrices is $(M(n); n \geq 1)$.

4.1. Rationally independent continuous eigenvalues.

Let $E(X,T)$ be the additive group of continuous eigenvalues of (X,T), that is,

$$E(X,T) = \{\alpha \in \mathbb{R}; \exp(2i\pi\alpha) \text{ is a continuous eigenvalue of } (X,T)\} \ .$$

In this section we study the maximal number $\eta(X,T)$ of rationally independent elements in $E(X,T)$ and its relation with the number of ergodic measures. Remark that 1 is always an eigenvalue of (X,T) so $\mathbb{Z} \subseteq E(X,T)$. We need the following simple lemma whose proof is left to the reader.

Lemma 8. *Let (X,T) be a minimal Cantor system of finite rank d. Then, there are at most d ergodic measures $\mu_1,\ldots,\mu_l$ ($l \leq d$). Moreover, there exists $m \in \mathbb{N}$ such that for all $n \geq m$ vectors $\mu_1(n),\ldots,\mu_l(n)$ are linearly independent.*

Theorem 9. *Let (X,T) be a minimal Cantor system of finite rank d. Let $\mu_1,\ldots,\mu_l$, $l \leq d$, be all its ergodic measures. Then, $\eta(X,T) \leq d-l+1$.*

Proof. Put $\eta = \eta(X,T)$ and assume $\eta > d-l+1$. Let $\{\alpha_1,\ldots,\alpha_\eta\}$ be a set of rationally independent elements in $E(X,T)$. From Theorem 5, there exist $m \in \mathbb{N}$ and vectors $v_i \in \mathbb{R}^d$, $w_i \in \mathbb{Z}^d$, for $i \in \{1,\ldots,\eta\}$, such that $\alpha_i H(m) - v_i = w_i$ and $P(n,m)v_i \to_{n\to\infty} 0$. Consider m so large that Lemma 8 is also verified from such an integer.
From Lemma 6 one has that for all $1 \leq i \leq \eta$ and all $1 \leq j \leq l$, $\langle w_i, \mu_j(m)\rangle = \alpha_i$. Thus, $\langle w_i, \mu_1(m) - \mu_j(m)\rangle = 0$ for $2 \leq j \leq l$. Now, from Lemma 8 one deduces that $\{\mu_1(m)-\mu_2(m),\ldots,\mu_1(m)-\mu_l(m)\}$ generates a $(l-1)$-dimensional vector space. We conclude that the linear space generated by $w_1,\ldots,w_\eta$ is of dimension at most $d-l+1$. Consequently, there exist integers $\epsilon_1,\ldots,\epsilon_{d-l+2}$ with $|\epsilon_1|+\ldots+|\epsilon_{d-l+2}| \neq 0$ and

$$\epsilon_1 w_1 + \ldots + \epsilon_{d-l+2} w_{d-l+2} = 0 \ .$$

Thus,

$$\epsilon_1 \alpha_1 + \ldots + \epsilon_{d-l+2}\alpha_{d-l+2} = 0$$

which contradicts the fact that $\{\alpha_1,\ldots,\alpha_\eta\}$ is a set of rationally independent elements in $E(X,T)$. □

Remark that from the proof of the theorem, a set of rationally independent generators of $E(X,T)$ can be determined from a single level m. Of course, this level m can be very large and difficult to get.

Put $\eta = \eta(X,T)$. If $\eta = d$ we say that (X,T) is of maximal type. From Theorem 9 one has that maximal type systems are uniquely ergodic. Conversely, if (X,T) is uniquely ergodic, then it has at most d rationally independent continuous eigenvalues but it is not necessarily of maximal type. To illustrate this it suffices to consider the Chacon subshift which is uniquely ergodic and has no eigenvalue.
Since $1 \in E(X,T)$, one can always produce rationally independent generators of $E(X,T)$ containing 1. Observe that rational eigenvalues are associated to 1. Fix

$\{1, \alpha_1, \dots, \alpha_{\eta-1}\}$ a set of rationally independent generators of $E(X,T)$. Let μ be an ergodic measure of (X,T).
From Theorem 5 there is $m \in \mathbb{N}$ such that for all $1 \le i \le \eta - 1$ there exist a real vector $v_i \in \mathbb{R}^d$ and an integer vector $w_i \in \mathbb{Z}^d$ satisfying $\alpha_i H(m) = v_i + w_i$ and $P(n,m)v_i \to 0$ as $n \to \infty$. From Lemma 6, each v_i is orthogonal to the linear space spanned by $\mu(m)$, thus

$$\alpha_i = \langle w_i, \mu(m) \rangle. \tag{4.1}$$

Proposition 10. *Both families of vectors* $\{v_1, \dots, v_{\eta-1}\}$ *and* $\{w_1, \dots, w_{\eta-1}, H(m)\}$ *are linearly independent.*

Proof. Suppose $\sum_{i=1}^{\eta-1} \delta_i w_i = 0$ with some $\delta_i \neq 0$. Since $w_1, \dots, w_{\eta-1}$ are integer vectors we can assume the δ_i's are integer numbers. From $\alpha_i = \langle w_i, \mu(m) \rangle$ for all $1 \le i \le \eta - 1$ one gets $\sum_{i=1}^{\eta-1} \delta_i \alpha_i = 0$ with some $\delta_i \neq 0$. But $\alpha_1, \dots, \alpha_{\eta-1}$ are rationally independent, then coefficients δ_i's must be 0, a contradiction. Then $w_1, \dots, w_{\eta-1}$ are linearly independent.
Now, it holds that $H(m) \notin \langle \{w_1, \dots, w_{\eta-1}\} \rangle$. Indeed, if $H(m) = \sum_{j=1}^{\eta-1} q_j w_j$, with rational coefficients, then by taking the inner product with $\mu(m)$ one gets that $1 = \sum_{j=1}^{\eta-1} q_j \alpha_j$. This contradicts the fact that $1, \alpha_1, \dots, \alpha_{\eta-1}$ are rationally independent. One concludes $w_1, \dots, w_{\eta-1}, H(m)$ are linearly independent.
Therefore, from $\sum_{j=1}^{\eta-1} \lambda_j v_j = 0$ one deduces $(\sum_{j=1}^{\eta-1} \lambda_j \alpha_j) H(m) - \sum_{j=1}^{\eta-1} \lambda_j w_j = 0$ and thus $\lambda_1 = \dots = \lambda_{\eta-1} = 0$. □

For each $n \ge 1$, define $\zeta(\mu, n)$ to be the maximal number of rationally independent components of $\mu(n)$.

Proposition 11. *For all* $n \ge m$, $\zeta(\mu, n) \ge \eta$. *In particular, if the system is of maximal type, then* $\zeta(\mu, n) = d$ *for* $n \ge m$.

Proof. We give a proof for $n = m$, for a general n it is analogous. Let $q = (q_1, \dots, q_d)^T \in \mathbb{Q}^d$ be such that $\langle q, \mu(m) \rangle = 0$. Thus q is not contained in the linear space $\langle \{w_1, \dots, w_{\eta-1}, H(m)\} \rangle$. Indeed, if $q = \sum_{i=1}^{\eta-1} r_i w_i + r H(m)$ with r and the r_i's rational numbers, then, taking the inner-product with $\mu(m)$, one obtains $0 = \sum_{i=1}^{\eta-1} r_i \alpha_i + r$, which implies $r_1 = \dots = r_{\eta-1} = r = 0$ (recall $1, \alpha_1, \dots, \alpha_{\eta-1}$ are rationally independent).
Assume for all subsets J of $\{1, \dots, d\}$ with cardinality η there is a non zero rational vector $q^J \in \mathbb{Q}^d$ with $q^J_j = 0$ for $j \in \{1, \dots, d\} \setminus J$ such that $\langle q^J, \mu(m) \rangle = 0$. At least $d - \eta + 1$ of such vectors must be linearly independent. To prove this fact consider the family $J_i = \{i, \dots, i + \eta - 1\}$ for $i \in \{1, \dots, d - \eta + 1\}$ and the corresponding vectors $q^{J_1}, \dots, q^{J_{d-\eta+1}}$. From the first part of the proof one concludes that $H(m), w_1, \dots, w_{\eta-1}, q^{J_1}, \dots, q^{J_{d-\eta+1}}$ are $d+1$ independent vectors in $\mathbb{R}^d$, which is a contradiction. Therefore, there is $J \subseteq \{1, \dots, d\}$ with cardinality η such that $\mu_j(m)$, $j \in J$, are rationally independent components of $\mu(m)$. This gives $\zeta(\mu, n) \ge \eta$. The maximal type case follows directly. □

Observe that the inequality in the proposition can be strict.

4.2. **Dimension group and geometric interpretation of eigenvalues.** Observe that it is not enough to have $v \in \mathbb{R}^d$ with $P(n,m)v \to 0$ as $n \to \infty$ and

$w \in \mathbb{Z}^d$ such that $v + w = \alpha H(m)$ for some $m \geq 1$ to ensure $\exp(2i\pi\alpha)$ is a continuous eigenvalue of (X, T). In addition, from Proposition 1, it is also necessary that the series $\sum_{n \geq m} \langle s_n(x), P(n,m)v \rangle$ converges modulo $\mathbb{Z}$. In the next two propositions we try to give a more precise statement involving the so called dimension group associated to the sequence of matrices $(M(n); n \geq 1)$.
Fix $m \geq 1$ and an invariant measure μ. Put

$$\mathcal{V}(m) = \langle \{\mu(m)\} \rangle^{\perp}.$$

Let $\mathcal{V}^s(m)$ be the subspace of $\mathbb{R}^d$ that is asymptotically contracted by $(M(n); n \geq m)$:

$$\mathcal{V}^s(m) = \{v \in \mathbb{R}^d; P(n,m)v \to 0 \text{ as } n \to \infty\}.$$

Also distinguish the subspaces of $\mathcal{V}^s(m)$,

$$\mathcal{V}_0(m) = \left\{v \in \mathbb{R}^d; \exists n \geq m, P(n,m)v = 0\right\} = \bigcup_{n \geq m} \mathrm{Ker}(P(n,m))$$

and

$$\mathcal{V}_1(m) = \left\{ v \in \mathbb{R}^d; \sum_{n \geq m} \|P(n,m)v\| < \infty \right\} .$$

Obviously,

$$\mathcal{V}_0(m) \subseteq \mathcal{V}_1(m) \subseteq \mathcal{V}^s(m) \subseteq \mathcal{V}(m) .$$

One has $P(m)\mathcal{V}(1) \subseteq \mathcal{V}(m)$. Equality holds if the matrices $(M(n); n \geq 1)$ are invertible.

Proposition 12. *There exist $m \geq 1$ and a linear space $\mathcal{V}^{(m)} \subseteq \mathcal{V}(m)$ such that $P(n,m) : \mathcal{V}^{(m)} \to \mathcal{V}^{(n)}$ is onto and one to one for any $n > m$.*

Proof. Let us choose a subspace $\mathcal{V}^{(1)}$ of $\mathbb{R}^d$ such that $\mathcal{V}^s(1) = \mathcal{V}^{(1)} \oplus \mathcal{V}_0(1)$. Let $m \geq 1$ and assume subspaces $\mathcal{V}^{(n)}$ are defined for all $1 \leq n \leq m$ such that: $\mathcal{V}^s(n) = \mathcal{V}_0(n) \oplus \mathcal{V}^{(n)}$ and $P(n,k)\mathcal{V}^{(k)} \subseteq \mathcal{V}^{(n)}$ for all $1 \leq k \leq n \leq m$.
Choose a subspace $\mathcal{W}^{(m+1)}$ of $\mathcal{V}^s(m+1)$ such that $\mathcal{V}^s(m+1) = \mathcal{V}_0(m+1) \oplus M(m+1)\mathcal{V}^{(m)} \oplus \mathcal{W}^{(m+1)}$ and set $\mathcal{V}^{(m+1)} = M(n+1)\mathcal{V}^{(m)} \oplus \mathcal{W}^{(m+1)}$. This procedure defines recursively a sequence of subspaces $(\mathcal{V}^{(m)}; m \geq 1)$ verifying for all $m \geq 2$ and all $1 \leq k \leq m$, $\mathcal{V}^s(m) = \mathcal{V}_0(m) \oplus \mathcal{V}^{(m)}$ and $P(m+1,k)\mathcal{V}^{(k)} \subseteq \mathcal{V}^{(m+1)}$.
Since $P(n,m)\mathcal{V}^{(m)} \subseteq \mathcal{V}^{(n)}$ and in view of the definition of $\mathcal{V}_0(m)$, $P(n,m)$ is injective on $\mathcal{V}^{(m)}$, then the sequence $(\dim(\mathcal{V}^{(m)}); m \geq 1)$ is non decreasing. Since it is bounded by d, there is $m \in \mathbb{N}$ such that for all $n \geq m$, $P(n,m)\mathcal{V}^{(m)} = \mathcal{V}^{(n)}$. This concludes the proof. □

Fix the integer m found in the previous proposition. Notice that if the matrices $(M(n); n \geq 1)$ are invertible, then one can take $m = 1$.
Consider the discrete subgroup of $\mathbb{R}^d$

$$\mathcal{G}(m) = \bigcup_{n \geq m} P(n,m)^{-1}\mathbb{Z}^d = \left\{z \in \mathbb{Q}^d; \exists n \geq m, P(n,m)z \in \mathbb{Z}^d\right\}$$

and the one dimensional subspace $\Delta(m) = \{tH(m); t \in \mathbb{R}\} \subseteq \mathbb{R}^d$.

Proposition 13. *Let $\lambda = \exp(2i\pi\alpha)$. If λ is a continuous eigenvalue of (X, T) then $\alpha H(m) \in (\mathcal{G}(m) + \mathcal{V}_1(m)) \cap \Delta(m)$.*

Proof. According to Theorem 3,

$$\sum_{n\geq 1} |\!|\!| \alpha P(n)H(1) |\!|\!| < \infty \ .$$

From Lemma 6, there exist $m' \geq 1$, an integer vector $w' \in \mathbb{Z}^d$ and a real vector $v' \in \mathbb{R}^d$ with

$$\alpha H(m') = v' + w' \text{ and } \|P(n,m')v'\| \to 0 \text{ as } n \to \infty \ .$$

One can assume $m' \geq m$.
Since $v' \in \mathcal{V}^s(m')$, it splits into $v' = v'_0 + v'_s$, with $v'_0 \in \mathcal{V}_0(m')$ and $v'_s \in \mathcal{V}^{(m')}$. There is $v \in \mathcal{V}^{(m)}$ such that $v'_s = P(m',m)v$. Hence,

$$\alpha P(m',m)H(m) = w' + v'_0 + P(m',m)v.$$

Let n be such that $P(n,m')v'_0 = 0$. One has,

$$\alpha P(n,m)H(m) = P(n,m')w' + 0 + P(n,m)v.$$

One deduces that

$$P(n,m)\left(\alpha H(m) - v\right) \in \mathbb{Z}^d,$$

which means that $\alpha H(m) - v \in (\mathcal{G}(m) + \mathrm{Ker}(P(n,m)))$. Since for n large enough $|\!|\!| \alpha P(n)H(1) |\!|\!| = |\!|\!| P(n,m)v |\!|\!| = \|P(n,m)v\|$, then v must belong to $\mathcal{V}_1(m)$. This fact and $\mathrm{Ker}(P(n,m)) \subseteq \mathcal{V}_1(m)$ imply that

$$\alpha H(m) \in (\mathcal{V}_1(m) + \mathcal{G}(m)) \cap \Delta(m).$$

□

Remark 14. $\mathcal{G}(m)$ is associated to the so called dimension group. It is classically presented as a quotient $\mathcal{G}' = \mathcal{H}/\sim$, where

$$\mathcal{H} = \left\{(z,p) \in \mathbb{Q}^d \times \mathbb{N}; \exists n \geq p, P(n,p)z \in \mathbb{Z}^d\right\}$$

and

$$(z,p) \sim (y,q) \Leftrightarrow \exists n \geq p, n \geq q, P(n,p)z = P(n,q)y.$$

If the matrices $(M(n); n \geq 1)$ are invertible, each $g \in \mathcal{G}'$ can be represented by the unique element $(z,1) \in \mathcal{H}$ in class g. In the general case, some elements of $\mathcal{G}'$ do not have a representative of this type. Nevertheless, by previous propositions one can choose an appropriate $m \geq 1$ and identify each $g \in \mathcal{G}'$ with a representative of the form $(z,m) \in \mathcal{H}$ if it is not asymptotically null, and with $(0,m)$ if it is asymptotically null. The difference here is that two elements of $\mathcal{G}(m)$ may correspond to the same element of $\mathcal{G}'$ if their images coincide after a while. One has that $\mathcal{G}'$ is isomorphic to $\mathcal{G}(m)/\approx$ with $z \approx y \Leftrightarrow \exists n \geq m, P(n,m)z = P(n,m)y$.

Fix $m \geq 1$ as before and such that $1, \alpha_1, \ldots, \alpha_{\eta-1}$ is a base of rationally independent continuous eigenvalues of (X,T) with $\alpha_i H(m) = v_i + w_i$, $v_i \in \mathcal{V}^s(m)$ and $w_i \in \mathbb{Z}^d$. When $\zeta(\mu,m) = d$, the eigenvalues can be described from $\langle\{w_1, \ldots, w_{\eta-1}, H(m)\}\rangle$.

Proposition 15. *Assume $\zeta(\mu,m) = d$ (in particular if (X,T) is of maximal type). Consider $\alpha = q + \sum_{i=1}^{\eta-1} q_i\alpha_i \in E(X,T)$ with $q, q_1, \ldots, q_{\eta-1} \in \mathbb{Q}$. Then $qH(m) + \sum_{i=1}^{\eta-1} q_i w_i$ belongs to $\mathcal{G}(m)$. Moreover, if α is a rational continuous eigenvalue then $\alpha H(m)$ belongs to $\mathcal{G}(m)$. Conversely, if $q, q_1, \ldots, q_{\eta-1} \in \mathbb{Q}$ are such that $qH(m) + \sum_{i=1}^{\eta-1} q_i w_i \in \mathcal{G}(m)$ then $\alpha = q + \sum_{i=1}^{\eta-1} q_i\alpha_i \in E(X,T)$.*

Proof. Take $\alpha \in E(X,T)$ as in the statement of the proposition. By Proposition 13, there are $v' \in \mathcal{V}^s(m)$ and $w' \in \mathcal{G}(m)$ such that $\alpha H(m) = v' + w'$. Thus, $v - v' = w' - w$, where $v = \sum_{i=1}^{\eta-1} q_i v_i$ and $w = qH(m) + \sum_{i=1}^{\eta-1} q_i w_i$. From $\langle v - v', \mu(m)\rangle = 0$ one deduces that $\langle w - w', \mu(m)\rangle = 0$. But $\zeta(\mu, m) = d$, thus $w = w'$ and consequently $v = v'$. This proves the first result. If $\alpha \in \mathbb{Q}$ then $v = 0$, which proves the second result.
Now consider $q, q_1, \ldots, q_{\eta-1} \in \mathbb{Q}$ such that $qH(m) + \sum_{i=1}^{\eta-1} q_i w_i \in \mathcal{G}(m)$. Put $v = \sum_{i=1}^{\eta-1} q_i v_i$. The series $\sum_{n\geq m} \langle s_n(x), P(n,m)v\rangle$ converges uniformly in x modulo $\mathbb{Z}$, because the corresponding series with v_i instead of v does. This proves $\alpha = q + \sum_{i=1}^{\eta-1} q_i \alpha_i$ belongs to $E(X,T)$. □

Define the matrix $W = [w_1, \ldots, w_{\eta-1}, H(m)]$. From (4.1), it follows that

$$W^T \mu(m) = (\alpha_1, \ldots, \alpha_{\eta-1}, 1)^T .$$

Corollary 16. *If $\zeta(\mu, m) = d$ (in particular if (X,T) is of maximal type) then $E(X,T)$ is isomorphic (as a group) with the discrete subgroup of $\mathbb{Q}^d$, $\mathbb{Q}(X,T) = \{z \in \mathbb{Q}^d; W^T z \in \mathcal{G}(m)\}$.*

5. Measurable eigenvalues of finite rank systems: a general necessary condition

Let (X,T) be a minimal Cantor system of finite rank d and μ be an ergodic probability measure. By contracting the associated Bratteli-Vershik diagram one can always assume there exists $I \subseteq \{1, \ldots, d\}$ such that:
(1) For all $k \in I$, $\liminf_{n\to\infty} \mu\{\tau_n = k\} > 0$;
(2) For all $k \in I^c$, $\sum_{n\geq 1} \mu\{\tau_n = k\} < \infty$.
A diagram verifying these properties will be called *clean*. From conditions (1) and (2) one deduces that $\tau_n(x)$ belongs to I from some n for almost all $x \in X$.
Consider a measurable eigenfunction $f : X \to \mathbb{C}$ of (X,T) with respect to μ associated to the eigenvalue $\lambda = \exp(2i\pi\alpha)$ with $|f| = 1$ μ-almost surely. For $n \geq 1$ put

$$f_n = \mathbb{E}(f|\mathcal{T}_n) = \sum_{k=1}^{d} \sum_{j=0}^{h_k(n)-1} \mathbf{1}_{T^{-j}B_k(n)} \frac{1}{\mu_k(n)} \int_{B_k(n)} \lambda^{-j} f d\mu$$

and set

$$\frac{1}{\mu_k(n)} \int_{B_k(n)} f d\mu = c_k(n) \lambda^{\rho_k(n)},$$

with $c_k(n) \geq 0$. Clearly $f_n(x) = \lambda^{-r_n(x) + \rho_{\tau_n(x)}(n)} c_{\tau_n(x)}(n)$.

Lemma 17. *For any $k \in I$ one has $c_k(n) \to 1$ as $n \to \infty$.*

Proof. By construction, $||f_n||_2^2 = \sum_{k=1}^{d} \mu\{\tau_n = k\} c_k(n)^2 \to 1$ as $n \to \infty$. Since the $c_k(n)$ are bounded by one, then $c_k(n) \to 1$ as $n \to \infty$ for each $1 \leq k \leq d$ such that $\liminf_{n\to\infty} \mu\{\tau_n = k\} > 0$. □

For $n \geq 1$ and $k, l \in \{1, \ldots, d\}$ define $S_n(l,k) = \{s_n(x); x \in X, \tau_n(x) = l, \tau_{n+1}(x) = k\}$.

Proposition 18. *Let (X,T) be a minimal Cantor system of finite rank d and μ be an ergodic measure. Assume (X,T) is given by a proper clean Bratteli-Vershik representation. If $\lambda = \exp(2i\pi\alpha)$ is an eigenvalue of (X,T) with respect to μ, then* *for $n \geq 1$ there exist real numbers $\rho_1(n), \dots, \rho_d(n)$ such that the following series converges,*

$$\sum_{n\geq 1} \max_{(l,k)\in J} \frac{1}{M_{k,l}(n+1)} \sum_{s\in S_n(l,k)} |1-\lambda^{\langle s,H(n)\rangle-\rho_k(n+1)+\rho_l(n)}|^2 \tag{5.1}$$

where $J = \{(l,k)\in\{1,\dots,d\}^2; \liminf_{n\to\infty} \mu\{\tau_n = l, \tau_{n+1}=k\} > 0\}$.

Proof. Let I be the subset of $\{1,\dots,d\}$ verifying (1) and (2) in the definition of a clean Bratteli-Vershik representation.
Let $f : X \to S^1$ be an eigenfunction for the eigenvalue $\lambda = \exp(2i\pi\alpha)$. As above, for $n \geq 1$ and $k \in \{1,\dots,d\}$, we set $f_n = \mathbb{E}(f|\mathcal{T}_n)$ and $\frac{1}{\mu_k(n)}\int_{B_k(n)} f d\mu = c_k(n)\lambda^{\rho_k(n)}$ with $c_k(n) \geq 0$. From Lemma 17, $c_k(n) \to 1$ as $n \to \infty$ if $k \in I$. Let $l, k \in J$. Observe that $l, k \in I$ too. One has

$$\frac{b}{M_{k,l}(n+1)} \leq \frac{\mu\{\tau_n = l, \tau_{n+1} = k\}}{M_{k,l}(n+1)} = \frac{M_{k,l}(n+1)h_l(n)\mu_k(n+1)}{M_{k,l}(n+1)} \tag{5.2}$$

$$= h_l(n)\mu_k(n+1)\ , \tag{5.3}$$

for some $b > 0$ and n large enough. On the other hand, since the sequence $(f_n; n \geq 1)$ is a martingale, then $\sum_{n\geq 1} ||f_{n+1}-f_n||_2^2$ converges and

$$\begin{aligned}
&||f_{n+1}-f_n||_2^2\\
&= \int_X |f_{n+1}-f_n|^2 d\mu\\
&= \int_X \left| c_{\tau_{n+1}(x)}(n+1)\lambda^{-r_{n+1}(x)+\rho_{\tau_{n+1}(x)}(n+1)} - c_{\tau_n(x)}(n)\lambda^{-r_n(x)+\rho_{\tau_n(x)}(n)}\right|^2 d\mu\\
&= \int_X c_{\tau_n(x)}(n)\cdot \left| \frac{c_{\tau_{n+1}(x)}(n+1)}{c_{\tau_n(x)}(n)} - \lambda^{r_{n+1}(x)-r_n(x)-\rho_{\tau_{n+1}(x)}(n+1)+\rho_{\tau_n(x)}(n)}\right|^2 d\mu\\
&= \sum_{k=1}^d\sum_{l=1}^d h_l(n)\mu_k(n+1)\sum_{s\in S_n(l,k)} c_l(n)\cdot\left|\frac{c_k(n+1)}{c_l(n)} - \lambda^{\langle s,H(n)\rangle-\rho_k(n+1)+\rho_l(n)}\right|^2 .
\end{aligned}$$

Consequently, from the convergence of $c_l(n)$ to 1 as $n \to \infty$ for $l \in I$ and (5.2) one deduces,

$$\sum_{n\geq 1}\sum_{(l,k)\in J} \frac{1}{M_{k,l}(n+1)} \sum_{s\in S_n(l,k)} \left|\frac{c_k(n+1)}{c_l(n)} - \lambda^{\langle s,H(n)\rangle-\rho_k(n+1)+\rho_l(n)}\right|^2$$

converges. But $|\frac{c_k(n+1)}{c_l(n)} - \lambda^{\langle s,H(n)\rangle-\rho_k(n+1)+\rho_l(n)}| \geq |\frac{c_k(n+1)}{c_l(n)} - 1|$, then one also gets that

$$\sum_{n\geq 1}\sum_{(l,k)\in J}\left|\frac{c_k(n+1)}{c_l(n)} - 1\right|^2$$

converges. One concludes that

$$\sum_{n\geq 1}\sum_{(l,k)\in J} \frac{1}{M_{k,l}(n+1)} \sum_{s\in S_n(l,k)} |1-\lambda^{\langle s,H(n)\rangle-\rho_k(n+1)+\rho_l(n)}|^2.$$

converges, which gives the result. □

Remark from last theorem that $\frac{1}{M_{k,l}(n+1)}\sum_{s\in S_n(l,k)}\lambda^{\langle s,H(n)\rangle-\rho_k(n+1)+\rho_l(n)}$ converges to 1 as $n\to\infty$ for any $(l,k)\in J$. This suggests a strong condition on the distribution of powers of λ in S^1 in relation to the local ordering of the Bratteli-Vershik representation.

6. Example 1: measurable eigenvalues do not always come from the stable subspace

In Theorem 5 we proved that if $\lambda=\exp(2i\pi\alpha)$ is a continuous eigenvalue of a minimal Cantor system (X,T) given by a proper Bratteli-Vershik representation, then for some $m\geq 1$ there exist $v\in\mathbb{R}^{C(m)}$ with $P(n,m)v\to 0$ as $n\to\infty$ and $w\in\mathbb{Z}^{C(m)}$ such that $\alpha H(m)=v+w$. In this section we construct a uniquely ergodic minimal Cantor system of finite rank 2 and a measurable eigenvalue λ for which this property is not verified. In particular λ will not be a continuous eigenvalue.

6.1. **Matrices for the Bratteli-Vershik diagram.** We start by constructing a suitable sequence of matrices $(M(n);n\geq 2)$. Let $A=\begin{pmatrix}2&1\\1&1\end{pmatrix}$. Its eigenvalues are $\varphi=\frac{3+\sqrt{5}}{2}$ and $\varphi^{-1}=\frac{3-\sqrt{5}}{2}$. Observe that $\varphi=\theta^2$ where $\theta=\frac{1+\sqrt{5}}{2}$. Consider a orthonormal basis of $\mathbb{R}^2$ made of eigenvectors $e_u=\frac{1}{\sqrt{\theta^2+1}}(\theta,1)^T$ and $e_s=\frac{1}{\sqrt{\theta^2+1}}(-1,\theta)^T$ associated to φ and φ^{-1} respectively. They generate the unstable subspace $E^u=\{v\in\mathbb{R}^2;||A^nv||\to_{n\to\infty}\infty\}$ and the stable subspace $E^s=\{v\in\mathbb{R}^2;A^nv\to_{n\to\infty}0\}$.
We put $H(1)=(1\ 1)^T$ and for $n\geq 2$ the matrix $M(n)=A^{k_n}$ for some integer $k_n\geq 2$ to be defined. Recall $P(1)=I$ and $P(n)=M(n)\cdots M(2)$ for $n\geq 2$ and $P(n,m)=M(n)\cdots M(m+1)$ for $1\leq m\leq n$. We set $K_n=\sum_{i=2}^n k_i$, so $P(n)=A^{K_n}$. For convenience we set $k_1=K_1=0$.

Lemma 19. *Let $(\epsilon_n;n\geq 1)$ and $(\delta_n;n\geq 1)$ be sequences of real numbers in $]0,\varphi^{-1}]$ and $v_1\in]0,\epsilon_1[$. There exist a real number $0<\beta<1$ and a sequence $(k_n;n\geq 2)$ of integers larger than 2 such that for all $v>v_1$ and all $n\geq 1$*
$$A^{K_n}(\beta e_u+ve_s)=z_n+u_ne_u+(v_n+\varphi^{-K_n}(v-v_1))e_s$$
with $0<v_n<\epsilon_n$, $0<u_n\leq\delta_nv_n$ and $z_n\in\mathbb{Z}^2$.

Proof. First we construct recursively the sequence $(k_n;n\geq 2)$ and a sequence $(\alpha_n;n\geq 1)$ such that for all $n\geq 1$
$$z_n=A^{K_n}(\alpha_ne_u+v_1e_s)-v_ne_s\in\mathbb{Z}^2$$
for some $0<v_n<\epsilon_n$. We start the recursion with $\alpha_1=0$, $v_1>0$ and $z_1=0$ and we put $t_1=1$.
Assume the construction is achieved up to $n\geq 1$. Let $k_{min}\geq 2$ be large enough so that $\varphi^{-k_{min}}v_n<\epsilon_{n+1}$ and $\varphi^{-k_{min}}<\epsilon_{n+1}$. The direction given by e_u has irrational slope. Thus there exist $t_{n+1}>0$, $0<s_{n+1}<\epsilon_{n+1}-\varphi^{-k_{min}}v_n$ and $\bar{z}_{n+1}\in\mathbb{Z}^2$ such that $t_{n+1}e_u=\bar{z}_{n+1}+s_{n+1}e_s$. Choose $k_{n+1}>k_{min}$ so that $\varphi^{-k_{n+1}}t_{n+1}<\min(\frac{\varphi-1}{\varphi}\delta_nv_n,\varphi^{-2}t_n)$, where t_n comes from previous step. Let
$$v_{n+1}=\varphi^{-k_{n+1}}v_n+s_{n+1}\ \text{ and }\ \alpha_{n+1}=\alpha_n+\varphi^{-K_{n+1}}t_{n+1}\ .$$

One has

$$\begin{aligned}
&A^{K_{n+1}}(\alpha_{n+1}e_u + v_1 e_s) - v_{n+1}e_s \\
&= (\varphi^{K_{n+1}}\alpha_n + t_{n+1})e_u + (\varphi^{-K_{n+1}}v_1 - \varphi^{-k_{n+1}}v_n - s_{n+1})e_s \\
&= \varphi^{k_{n+1}}(\varphi^{K_n}\alpha_n)e_u + \varphi^{-k_{n+1}}(\varphi^{-K_n}v_1 - v_n)e_s + (t_{n+1}e_u - s_{n+1}e_s) \\
&= A^{k_{n+1}}[\varphi^{K_n}\alpha_n e_u + (\varphi^{-K_n}v_1 - v_n)e_s] + (t_{n+1}e_u - s_{n+1}e_s) \\
&= A^{k_{n+1}}z_n + \bar{z}_{n+1} = z_{n+1} \in \mathbb{Z}^2,
\end{aligned}$$

and $0 < v_{n+1} = \varphi^{-k_{n+1}}v_n + s_{n+1} < \epsilon_{n+1}$ by construction.
Let $m \geq 0$. It holds,

$$\begin{aligned}
\alpha_{n+m} - \alpha_n = \sum_{j=n}^{n+m-1}(\alpha_{j+1} - \alpha_j) &= \sum_{j=n}^{n+m-1}\varphi^{-K_{j+1}}t_{j+1} \\
&< \varphi^{-K_{n+1}}t_{n+1}\sum_{j=0}^{m-1}\varphi^{-j} < \varphi^{-K_n}\delta_n v_n < \varphi^{-K_n}\delta_n\epsilon_n \ ,
\end{aligned}$$

where we used that $\varphi^{-k_{n+1}}t_{n+1} < \min(\frac{\varphi-1}{\varphi}\delta_n v_n, \varphi^{-2}t_n)$.
Then the sequence $(\alpha_n; n \geq 1)$ converges. Put $\beta = \lim_{n\to\infty}\alpha_n$. It is clear that $0 < \beta < 1$ and $\varphi^{K_n}(\beta - \alpha_n) < \delta_n v_n$ for $n \geq 1$. To conclude define $u_n = \varphi^{K_n}(\beta - \alpha_n)$ and observe that for all $v > v_1$

$$A^{K_n}(\beta e_u + v e_s) = z_n + \varphi^{K_n}(\beta - \alpha_n)e_u + (v_n + \varphi^{-K_n}(v - v_1))e_s$$

where by construction $v_n < \epsilon_n$. □

In previous lemma we gave a procedure to construct one value of β. In fact it is possible to construct a whole Cantor set of such numbers associated to the same sequence $(k_n; n \geq 1)$.

Proposition 20. *Let $(\epsilon_n; n \geq 1)$ and $(\delta_n; n \geq 1)$ be sequences of real numbers in $]0, \varphi^{-1}]$ and $0 < v_1 < \epsilon_1$. There exist a real number $0 < \beta < 1$ such that $\beta H(1) \notin \mathbb{Z}^2 + E^s$ and a sequence $(k_n; n \geq 2)$ of integers larger than 2 such that for all $v > v_1$ and all $n \geq 2$*

$$A^{K_n}(\beta e_u + v e_s) = z_n + u_n e_u + (v_n + \varphi^{-K_n}(v - v_1))e_s$$

with $0 < v_n < \epsilon_n$, $0 < u_n \leq \delta_n v_n$ and $z_n \in \mathbb{Z}^2$.

Proof. The construction in previous lemma can be modified in order to produce a complete Cantor set of β's. We give the main ingredients of the new recursive procedure. In the first step, choose two different values $t_2^{(0)} > 0$ and $t_2^{(1)} > 0$ and then k_2 large enough so that conditions in the construction are satisfied for both values. As a result we get also $v_2^{(0)}, v_2^{(1)}, \alpha_2^{(0)}, \alpha_2^{(1)}, z_2^{(0)}$ and $z_2^{(1)}$.
At the inductive step in the proof of the lemma, assume that for any $n \geq 2$ and $s \in \{0,1\}^{n-1}$ we have constructed $v_n^{(s)}, \alpha_n^{(s)}, t_n^{(s)}, z_n^{(s)}$ and k_n (independent of s) verifying the conditions of the construction. For each $s \in \{0,1\}^{n-1}$ proceed with the recursive procedure from data given by $v_n^{(s)}, \alpha_n^{(s)}, t_n^{(s)}, z_n^{(s)}$ and k_n in two different ways to construct $t_{n+1}^{(s0)}$ and $t_{n+1}^{(s1)}$. Moreover, we can proceed in order that all values in $(t_{n+1}^{(s)}; s \in \{0,1\}^n)$ are different. Since we manage a finite number of situations the integer k_{n+1} can be choosen independent of the previous choice. Then complete the auxiliary data to finish the recursion.

The rule "$\varphi^{-k_{n+1}}t_{n+1} < \min(\frac{\varphi-1}{\varphi}\delta_n v_n, \varphi^{-2}t_n)$" implies that each choice of $s \in \{0,1\}^{\mathbb{N}}$ to follow the algorithm, produces a different value $\beta^{(s)}$. Therefore, not all the possible values $\beta^{(s)}$ comes from the so called regular weak stable space, that is $\beta^{(s)}H(1) \notin \mathbb{Z}^2 + E^s$, because the intersection of $\mathbb{Z}^2+E^s$ with $\langle\{H(1)\}\rangle$ is countable. This achieves the proof. □

Corollary 21. *Let $(\epsilon_n; n \geq 1)$ and $(\delta_n; n \geq 1)$ be sequences of real numbers in $]0, \varphi^{-1}]$. There is a sequence $(k_n; n \geq 2)$ of integers larger than 2 and a real number $\alpha > 0$ such that for all $n \geq 2$*

$$\alpha P(n)H(1) = w_n + z_n \ ,$$

where $z_n \in \mathbb{Z}^2$ and $w_n = u_n e_u + \bar{v}_n e_s \in \mathbb{R}^2$ with $||w_n|| \leq C\epsilon_n$ for some positive constant C and $0 < u_n \leq \delta_n \epsilon_n$. Moreover, $(w_n)_1 < 0$ and $(w_n)_2 > 0$.

Proof. Let $v_1 < \min\left(\epsilon_1, ||\frac{1}{\langle e_u, H(1)\rangle}H(1) - e_u||\right)$. Let $0 < \beta < 1$ be given by Proposition 20 with v_1 and the sequences of *epsilon*'s and *delta*'s given here. We consider the intersection of $\{tH(1); t \in \mathbb{R}\}$ with $\{e_1 + \beta e_u + t e_s; t \in \mathbb{R}\}$ where $e_1 = (1\ 0)^T$. Call it $\alpha H(1) = e_1 + \beta e_u + v e_s$. By the choice of v_1 one has $v > v_1$. Then by Proposition 20 for $n \geq 2$, $\alpha P(n)H(1) = P(n)e_1 + z_n + u_n e_u + (v_n + \varphi^{-K_n}(v - v_1))e_s$. Thus as $\varphi^{-K_n} \leq \epsilon_n$, $v_n \leq \epsilon_n$ and $u_n \leq \delta_n\epsilon_n$ one concludes $||w_n|| \leq C\epsilon_n$ for some positive constant C. Also, since $\theta u_n - v_n \leq (\theta\delta_n - 1)v_n \leq (\theta\varphi^{-1} - 1)v_n < 0$ one deduces that $(w_n)_1 < 0$. □

6.2. **Order for the Bratteli-Vershik diagram.** Now the matrices $(M(n); n \geq 2)$ have been constructed we will proceed to give an ordering to the Bratteli diagram induced by them. We introduce the notion of *best ordering* associated to (w, h) where $w = (w_1, w_2)^T \in \mathbb{R}^2$ with $w_1 < 0$, $w_2 > 0$, $h = (h_1, h_2)^T \in \mathbb{N}^2$ has strictly positive coordinates and $\langle h, w\rangle > 0$. This ordering is described by a word $p = p_1 \ldots p_l 1$ in $\{1,2\}^*$ of length $l+1 = h_1 + h_2$ defined recursively by: for $0 \leq n \leq l-1$,

$$p_{n+1} = \begin{cases} 1 & \text{if } \langle\sum_{i=1}^n e_{p_i} - h, w\rangle > 0 \\ 2 & \text{otherwise} \end{cases}$$

where e_1, e_2 are the canonical vectors of $\mathbb{R}^2$. Since $\langle h, w\rangle > 0$ then $p_1 = 2$. This motivates the following definition: $K(p) = \inf\{i > 1 : p_i = 1\} - 2$.
Let $w^\perp$ be a vector orthogonal to w and consider the line $L = \{h + tw^\perp; t \in \mathbb{R}\}$. Notice that, a point $y \in \mathbb{R}^2$ is above this line (in the direction of w) if and only if $\langle y - h, w\rangle > 0$. Thus $p_{n+1} = 1$ if the integer vector $\sum_{i=1}^n e_{p_i}$ is above the line L and is equal to 2 otherwise.

Lemma 22. *It holds,*

- $K(p) \leq \frac{\langle h, w\rangle}{w_2}$.
- *For all $K(p) < j \leq l+1$, $\left|\langle\sum_{i=j}^{l+1} e_{p_i}, w\rangle\right| \leq 3||w||$.*

Proof. The intersection point of L with the vertical axis is $\frac{\langle h, w\rangle}{w_2}e_2$. This gives the first inequality.

Let $K(p) < j \leq l+1$. Since $w_1 < 0$ and $w_2 > 0$, $\mathrm{dist}(\sum_{i=1}^{j-1} e_{p_i}, L) \leq 1$ and $\sum_{i=1}^{l+1} e_{p_i} = h$ or $(h + e_1 - e_2)$. Therefore,

$$\left|\langle \sum_{i=j}^{l+1} e_{p_i}, w\rangle\right| = \left|\langle \sum_{i=1}^{l+1} e_{p_i} - \sum_{i=1}^{j-1} e_{p_i}, w\rangle\right| \leq \left|\langle h - \sum_{i=1}^{j-1} e_{p_i}, w\rangle\right| + \left|\langle \tilde{h}, w\rangle\right|$$

where $\tilde{h}$ is either h or $(h + e_1 - e_2)$. One concludes that,

$$\left|\langle \sum_{i=j}^{l+1} e_{p_i}, w\rangle\right| \leq \left(\mathrm{dist}(\sum_{i=1}^{j-1} e_{p_i}, L) + \sqrt{2}\right) ||w|| \leq 3||w|| \ .$$

□

Fix two decreasing sequences of real numbers $(\epsilon_n; n \geq 1)$ and $(\delta_n; n \geq 1)$ with values in $]0, \varphi^{-1}]$ such that $\delta_n \leq \epsilon_n$ for $n \geq 1$. Let $0 < v_1 < \epsilon_1$. Let α and $(k_n; n \geq 1)$ be as in Corollary 21. Then $\alpha P(n) H(1) = w_n + z_n$ where $z_n \in \mathbb{Z}^2$ and $w_n \in \mathbb{R}^2$ for $n \geq 2$. From construction it follows that $(w_n)_1 < 0$ and $(w_n)_2 > 0$.
Let (X, T) be the minimal Cantor system defined from the proper Bratteli-Vershik representation described as follows: (i) the vertex at each level are labelled by $\{1, 2\}$, (ii) the incidence matrices are given by $M(n) = A^{k_n}$ for $n \geq 2$, and (iii) for $j \in \{1, 2\}$ and $n \geq 2$ the order of the edges arriving at vertex j in level n is given by the best order associated to $(w_n, m_n(j))$ described by the word $p^{(n,j)}$, where $m_j(n) = (M_{j,1}(n), M_{j,2}(n))^T$. Since $p_1^{(n,j)} = 2$ and $p_{l+1}^{(n,j)} = 1$ this diagram has unique minimal and maximal points and the diagram is proper.

6.3. **Construction of measurable eigenvalue.** For any $x = (x_1, x_2, \ldots) \in X$ and $n \geq 1$ its suffix $s_n(x)$ is given by,

$$s_n(x) = \sum_{k=o_{n+1}(x)+1}^{\langle m_j(n+1), H(1)\rangle} e_{p_k^{(n+1,j)}},$$

where $\tau_{n+1}(x) = j$ and $o_{n+1}(x) \in \{1, \ldots, \langle m_j(n+1), H(1)\rangle\}$ is the local order of coordinate x_{n+1}.
Let $i, j \in \{1, 2\}$ and $1 \leq o \leq \langle m_j(n+1), H(1)\rangle$ such that $p_o^{(n+1,j)} = i$. Put $\gamma_n = \sum_{k=o+1}^{\langle m_j(n+1), H(1)\rangle} e_{p_k^{(n+1,j)}}$. It is clear that γ_n is the suffix $s_n(x)$ of some $x \in X$ with $\tau_n(x) = i$ and $\tau_{n+1}(x) = j$. Let μ be the unique invariant measure of (X, T) (it is unique since $\langle \mu(n), e_s\rangle = 0$ for all $n \geq 1$). A direct computation yields to

$$\mu\{s_n = \gamma_n \,|\, \tau_n = i, \tau_{n+1} = j\} = \frac{1}{M_{j,i}(n+1)}.$$

Lemma 23. *There is a positive constant C such that for all $n \geq 1$*

$$\mu\{|\langle s_n, w_n\rangle| > 3||w_n||\} \leq C\epsilon_n \ .$$

Proof. Let $i, j \in \{1, 2\}$. Set $K_j(n+1) = K(p^{(n+1,j)})$. From the second statement of Lemma 22 one gets

$$\begin{aligned}\mu\{|\langle s_n, w_n\rangle| > 3||w_n||| \,|\, \tau_n = i, \tau_{n+1} = j\} &\leq \mu\{1 \leq o_{n+1} \leq K_j(n+1) - 1 \,|\, \tau_n = i, \tau_{n+1} = j\} \\ &\leq \frac{|\{1 \leq o \leq K_j(n+1) - 1 \ ; \ p_o^{(n+1,j)} = i\}|}{M_{j,i}(n+1)}\end{aligned}$$

If $\tau_n(x) = 1$ then necessarily $o_{n+1}(x) > K_j(n+1)$, while if $\tau_n(x) = 2$ then $|\{1 \leq o \leq K_j(n+1) - 1 \ ; \ p_o^{(n+1,j)} = 2\}| \leq K_j(n+1)$. So,

$$\mu\{|\langle s_n, w_n\rangle| > 3||w_n|| \,|\, \tau_n = i, \tau_{n+1} = j\} \leq \frac{K_j(n+1)}{M_{j,2}(n+1)} \ .$$

Let $f_n = \frac{|(w_n)_1|}{(w_n)_2}$ (recall that $(w_n)_1 < 0$). From Lemma 22 one gets

$$\begin{aligned}\frac{K_j(n+1)}{M_{j,2}(n+1)} &\leq \frac{M_{j,1}(n+1)}{M_{j,2}(n+1)}\left(\frac{M_{j,2}(n+1)}{M_{j,1}(n+1)} - f_n\right) \\ &= \frac{M_{j,1}(n+1)}{M_{j,2}(n+1)}\left(\left(\frac{M_{j,2}(n+1)}{M_{j,1}(n+1)} - \theta^{-1}\right) + \left(\theta^{-1} - f_n\right)\right) \ .\end{aligned}$$

Let $w_n = \bar{v}_n e_s + u_n e_u$. Recall from construction that $\bar{v}_n = v_n + \varphi^{-K_n}(v - v_1)$, $v_n \leq \epsilon_n$ and $u_n \leq \delta_n v_n$. Also $\varphi^{-k_n} \leq \epsilon_n$. Therefore,

$$f_n = \frac{\theta^{-1}\bar{v}_n - u_n}{\bar{v}_n + \theta^{-1}u_n} = \frac{\theta^{-1} - \frac{u_n}{\bar{v}_n}}{1 + \theta^{-1}\frac{u_n}{\bar{v}_n}}$$

and

$$|f_n - \theta^{-1}| = \left|\frac{u_n}{\bar{v}_n}\frac{1+\theta^{-2}}{1 + \theta^{-1}\frac{u_n}{\bar{v}_n}}\right| \leq \frac{u_n}{v_n}(1+\theta^{-2}) \leq (1+\theta^{-2})\delta_n \leq (1+\theta^{-2})\epsilon_n \ .$$

On the other hand, $\frac{M_{j,2}(n+1)}{M_{j,1}(n+1)}$ approaches θ^{-1} at speed $\varphi^{-k_{n+1}} \leq \epsilon_{n+1} \leq \epsilon_n$. Thus

$$\frac{K_j(n+1)}{M_{j,2}(n+1)} \leq C\epsilon_n$$

for some positive constant C.
To conclude, one integrates this uniform bound with respect to i and j. □

Now, we assume $(\epsilon_n; n \geq 1)$ is summable, that is, $\sum_{n\geq 1}\epsilon_n < \infty$.

Theorem 24. *The complex number* $\exp(2i\pi\alpha)$ *is an eigenvalue of* (X, T) *with respect to* μ *that is not continuous.*

Proof. The fact that it is not continuous follows directly from construction and Theorem 5.
First we prove the series $\sum_{n\geq 1} |||\langle s_n(x), \alpha P(n)H(1)\rangle|||$ converges μ-almost surely. Since $\sum_{n\geq 1}\mu\{|\langle s_n, w_n\rangle| > 3||w_n||\} \leq C\sum_{n\geq 1}\epsilon_n < \infty$, then by Borel-Cantelli Lemma one has for μ-almost every $x \in X$

$$\sum_{n\geq 1} 1_{\{|\langle s_n(x), w_n\rangle| > 3||w_n||\}} < \infty.$$

Denote by $N_0(x)$ the first integer such that for all $N > N_0(x)$, $|\langle s_n(x), w_n\rangle| \leq 3||w_n||$. Since N_0 is almost surely finite, one has for μ-almost all $x \in X$ and $N > N_0(x)$,

$$\sum_{n>N} |||\langle s_n(x), \alpha P(n)H(1)\rangle||| \leq \sum_{n>N} |\langle s_n(x), w_n\rangle| \leq 3\sum_{n>N} ||w_n|| \leq \sum_{n>N} \epsilon_n \ .$$

Hence the series $\sum_{n\geq 1} |||\langle s_n(x), \alpha P(n)H(1)\rangle|||$ converges almost surely.
To conclude observe that $f(x) = \exp(-2i\pi \sum_{n\geq 1}\langle s_n(x), \alpha P(n)H(1)\rangle)$ is an eigenfunction of (X,T) associated to $\exp(2i\pi\alpha)$. Indeed, a simple computation yields to $\sum_{n\geq 1}\langle s_n(Tx) - s_n(x), \alpha P(n)H(1)\rangle = -\alpha$. □

7. Example 2: continuous and measurable eigenvalues of Toeplitz type systems of finite rank

It is known that any subgroup of $S^1 = \{z \in \mathbb{C}\,; |z| = 1\}$ can be the set of measurable eigenvalues of a Toeplitz system (see [DL] or [Dow]). The main motivation of this section is to show a class of examples of Toeplitz minimal Cantor systems where the finite rank assumption restricts the possibilities of measurable eigenvalues.
A properly ordered Bratteli-Vershik diagram $B = (V, E, \preceq)$ is of Toeplitz type if for all $n \geq 1$ and for all $u, v \in V_n$ the number of edges in E_n finishing at u coincides with the number of edges in E_n finishing at v. Denote this number by q_n and set $p_n = q_n q_{n-1}\cdots q_1$. We say $(q_n; n \geq 1)$ is the characteristic sequence of the diagram. A minimal Cantor system is said to be of Toeplitz type if it is given by a Bratteli-Vershik diagram of this type. This definition is motivated by the characterization of Toeplitz subshifts in [GJ]. That is, a properly ordered Bratteli-Vershik diagram of Toeplitz type is a Toeplitz subshift whenever it is expansive. First we prove a known result for Toeplitz subshifts.

Theorem 25. *Let (X,T) be a minimal Cantor system of Toeplitz type given by a Bratteli-Vershik diagram with characteristic sequence $(q_n; n \geq 1)$. Then, $\exp(2i\pi\alpha)$ is a continuous eigenvalue of (X,T) if and only if $\alpha = \frac{a}{p_n}$ for some $a \in \mathbb{Z}$ and $n \geq 1$.*

Proof. Assume without loss of generality that $H(1) = (1,\ldots,1)^T$.
Let $\exp(2i\pi\alpha)$ be a continuous eigenvalue of (X,T) with $\alpha \in]0,1[$. Let $\alpha = \sum_{i\geq 1}\frac{a_i}{p_i}$, with $a_i \in \{0,\ldots,q_i - 1\}$ for all $i \geq 1$, be the expansion of α in base $(p_n; n \geq 1)$. By Corollary 2 one has that $\alpha p_n \to 0 \mod \mathbb{Z}$ as $n \to \infty$, which implies that $\sum_{i\geq n+1}\frac{a_i}{q_{n+1}\cdots q_i} \to_{n\to\infty} 0$.
Assume $(a_n; n \geq 1)$ is not ultimately equal to 0. Let $x_n \in X$ be such that $\langle s_n(x_n), H(1)\rangle = \lfloor \frac{q_{n+1}}{2a_{n+1}}\rfloor$. It exists since $\langle s_n(x), H(1)\rangle$ can take any value between $\{0,\ldots,q_{n+1} - 1\}$. A simple computation yields to: $\alpha p_n\langle s_n(x_n), H(1)\rangle$ converges to $\frac{1}{2} \mod \mathbb{Z}$ as $n \to \infty$. But, from Proposition 1 one knows that $\alpha p_n\langle s_n(x), H(1)\rangle$ converges to 0 modulo $\mathbb{Z}$ and uniformly in x, then last statement is a contradiction. One concludes that $(a_n; n \geq 1)$ is ultimately equal to 0 and that $\alpha = \frac{a}{p_m}$ for some $a \in \mathbb{N}$ and $m \in \mathbb{N}$.
Conversely, assume $\alpha = \frac{a}{p_m}$ for some $a \in \mathbb{N}$ and $m \in \mathbb{N}$. Then for all $x \in X$ and $n \geq m$,

$$\langle s_n(x), \alpha P(n)H(1)\rangle = a\frac{p_n}{p_m}\langle s_n(x), H(1)\rangle = aq_{m+1}\cdots q_n\langle s_n(x), H(1)\rangle,$$

which belongs to $\mathbb{Z}$. Then $\sum_{n\geq 1}\langle s_n(x), \alpha P(n)H(1)\rangle$ converges uniformly modulo $\mathbb{Z}$. One concludes by using Proposition 1. $\square$

The next proposition shows that in the class of linearly recurrent systems of Toeplitz type, continuous and measurable eigenvalues coincide.

Theorem 26. *Let (X,T) be a minimal Cantor system of Toeplitz type and finite rank given by a Bratteli-Vershik diagram with bounded characteristic sequence $(q_n; n\geq 1)$. Let μ be its unique invariant probability measure. Then $\exp(2i\pi\alpha)$ is an eigenvalue of (X,T) associated to μ if and only if $\alpha = \frac{a}{p_m}$ for some $a\in\mathbb{Z}$ and $m\in\mathbb{N}$. In particular, they are all continuous eigenvalues.*

Proof. Let $\exp(2i\pi\alpha)$ be a measurable eigenvalue with $\alpha\in]0,1[$ and $\alpha=\sum_{i\geq 1}\frac{a_i}{p_i}$ be its expansion in base $(p_n; n\geq 1)$. From Theorem 1 (1) in [BDM] one deduces that $\langle \alpha P(n)H(1), e_1\rangle = p_n\alpha \to_{n\to\infty} 0 \mod \mathbb{Z}$. This implies that $\frac{a_n}{q_n}$ goes to zero when n goes to infinity. Since, the characteristic sequence is bounded then $(a_n; n\geq 1)$ is ultimately equal to 0 and $\alpha=\frac{a}{p_m}$ for some $a\in\mathbb{N}$ and $m\in\mathbb{N}$. We conclude using Theorem 25. $\square$

Let (X,T) be a minimal Cantor system of Toeplitz type and finite rank d given by a Bratteli-Vershik diagram with characteristic sequence $(q_n; n\geq 1)$. Let μ be an ergodic probability measure. Consider a measurable eigenfunction $f: X\to\mathbb{C}$ of (X,T) with respect to μ associated to the eigenvalue $\lambda=\exp(2i\pi\alpha)$ with $|f|=1$, μ-almost surely. One has from Martingale theorem that $f_n=\mathbb{E}_\mu(f|\mathcal{T}_n)$ converges μ-almost surely and in $L^2(X,\mathcal{B}(X),\mu)$ to f. Following the notations of Section 5, we recall

$$f_n(x) = \frac{\int_{B_k(n)} f d\mu}{\mu_k(n)}\lambda^{-j} = c_k(n)\lambda^{\rho_k(n)-j}$$

whenever $x\in T^{-j}B_k(n)$ for some $1\leq k\leq d$ and $0\leq j< h_k(n)$. We set $c'_k(n) = c_k(n)\lambda^{\rho_k(n)}$. Remark that,

$$j=\sum_{i=1}^{n-1}\langle s_i(x), P(i)H(1)\rangle = \sum_{i=1}^{n-1} p_i\langle s_i(x), H(1)\rangle = \sum_{i=1}^{n-1} p_i\bar{s}_i(x),$$

where $\bar{s}_i(x)=\langle s_i(x), H(1)\rangle$. Since the system is of Toeplitz type one knows that $0\leq \bar{s}_i(x) < q_{i+1}$. Given $1\leq i,k\leq d$ and $n\geq 2$ define $S_{k,i}(n)=\{\bar{s}_n(x); x\in X, \tau_n(x)=k, \tau_{n+1}(x)=i\}$.

Theorem 27. *Let (X,T) be a minimal Cantor system of Toeplitz type and finite rank d. Let μ be an ergodic probability measure. Then all measurable eigenvalues of (X,T) with respect to μ are rational.*

Proof. Let $(q_n; n\geq 1)$ be the characteristic sequence of the Bratteli-Vershik diagram given the system and $\lambda=\exp(2i\pi\alpha)$ be a measurable eigenvalue of (X,T) with respect to μ. Since $f_n\to f$ as $n\to\infty$ μ-almost surely, then $\lim_{n\to\infty}|{c'}_{\tau_n(x)}(n)|=1$ almost surely too. Hence, by Egoroff theorem, for $\rho<\frac{1}{8d^2}$ there is a measurable set A such that $\mu(A)>1-\rho$ and $(f_n; n\geq 1)$ converges to f and $(|{c'}_{\tau_n(x)}(n)|; n\geq 1)$ converges to 1 uniformly on A.
Let $\epsilon=\frac{1}{8d^4}$. Then for all $n<N$ large enough and $x\in A$ one has $|f_n(x)-f_N(x)|\leq\epsilon$ and $|{c'}_{\tau_n(x)}(n)|>2/3$. By using the expression of f_n one deduces that,

$$\left| \frac{c'_{\tau_N(x)}(N)}{c'_{\tau_n(x)}(n)} - (\lambda^{p_n})^{\bar{s}_{n,N}(x)} \right| \leq \frac{\epsilon}{|c'_{\tau_n(x)}(n)|}$$

for every $x \in A$, where $\bar{s}_{n,N}(x) = \sum_{i=n}^{N-1} q_{n+1} \cdots q_i \bar{s}_i(x)$. Put $Q_{n,N} = q_{n+1} \cdots q_N$. Clearly $0 \leq \bar{s}_{n,N} < Q_{n,N}$.
Assume α is irrational. For any interval $L \subseteq S^1$, by the unique ergodicity of the rotation by λ^{p_n}, one has

$$d_{n,N}(L) = \frac{1}{Q_{n,N}} |\{0 \leq s < Q_{n,N}; \lambda^{p_n s} \in L\}| \to_{N\to\infty} |L|$$

uniformly in L of the same length (here $|L|$ denotes the normalized length in S^1). Let L be an interval in S^1 such that $|L| = \frac{1}{4d^2}$ and fix N such that $d_{n,N}(L) > |L|/2$. In addition, we can assume the interval L is disjoint from the set $\left\{ \frac{c'_i(N)}{c'_j(n)}; 1 \leq i,j \leq d \right\}$ and that the distance between L and $\left\{ \frac{c'_i(N)}{c'_j(n)}; 1 \leq i,j \leq d \right\}$ is bigger than 2ϵ. Therefore,

$$\begin{aligned} \mu\{x \in X; \lambda^{p_n \bar{s}_{n,N}(x)} \in L\} &= \sum_{i=1}^{d} \mu_i(N) p_n Q_{n,N} d_{n,N}(L) \\ &= d_{n,N}(L) > \frac{|L|}{2} > \rho \ . \end{aligned}$$

This implies that $\mu\{x \in A; \lambda^{p_n \bar{s}_{n,N}(x)} \in L\} > 0$ and thus there is $x \in A$ such that $\lambda^{p_n \bar{s}_{n,N}(x)} \in L$ while

$$2\epsilon \leq \left| \frac{c'_{\tau_N(x)}(N)}{c'_{\tau_n(x)}(n)} - (\lambda^{p_n})^{\bar{s}_{n,N}(x)} \right| \leq \frac{\epsilon}{|c'_{\tau_n(x)}(n)|} \ .$$

Thus, $|c'_{\tau_n(x)}(n)| \leq 1/2$, which is a contradiction. One concludes α is rational. □

Let us now show that the measurable eigenvalues cannot be any rational number.

Theorem 28. *Let (X,T) be a minimal Cantor system of Toeplitz type and finite rank d with characteristic sequence $(q_n; n \geq 1)$. Let μ be an ergodic probability measure. If $\exp(2i\pi p/q)$, with $(p,q) = 1$, is a non continuous rational eigenvalue of (X,T) with respect to μ then for all n large enough $\frac{q}{(q,p_n)} \leq d$.*

Proof. Let $\lambda = \exp(2i\pi p/q)$ with $p, q \in \mathbb{N}$ and $(p,q) = 1$ be a non continuous eigenvalue of (X,T) with respect to μ. Then $\exp(2i\pi/q)$ is also a non continuous eigenvalue with respect to μ. From Theorem 25 we deduce that for all n large enough $\exp(2i\pi(q,p_n)/q)$ is a non continuous eigenvalue. Hence we can assume $(q,p_n) = 1$ for all n large enough (consider $\frac{q}{(q,p_n)}$ instead of q).
Let δ be the minimum distance to the integers of the non zero elements in $\{c(a-b)/q; 0 \leq a,b,c < q\}$. Take $0 < \epsilon < \min(1/2qd, \delta/2)$. Contracting the diagram if needed, one can suppose $q/q_n < \epsilon$ for all $n \geq 1$.
From Theorem 7 in [BDM], for each $n \geq 1$ there exists a real function $\rho_n : X \to \mathbb{R}$ such that $\tau_n(x) = \tau_n(y)$ implies $\rho_n(x) = \rho_n(y)$ and $((1/q)r_n - \rho_n; n \in \mathbb{N})$ converges μ-almost everywhere modulo $\mathbb{Z}$. Thus, from Egoroff theorem there exists $A \in \mathcal{B}(X)$ with $\mu(A) \geq 1-\epsilon$ such that $((1/q)r_n - \rho_n; n \in \mathbb{N})$ converges uniformly on A modulo

$\mathbb{Z}$. Thus, since $r_{n+1} - r_n = \bar{s}_n p_n$ with $0 \le \bar{s_n} < q_{n+1}$, $((1/q)\bar{s}_n - \rho_{n+1} + \rho_n; n \in \mathbb{N})$ converges uniformly to 0 on A modulo $\mathbb{Z}$.
Let n be large enough such that $(q, p_n) = 1$ and the distance to the integers of $(1/q)\bar{s}_n - \rho_{n+1} + \rho_n$ is uniformly lower than $\epsilon/2q$ in A.
For all $0 \le a < q$ set $S_n(a) = \{x \in X; \bar{s}_n = a \mod q\}$. Let $q_{n+1} = kq + r$ with $0 \le r < q$. For all $t \in \{1, \dots, d\}$ one has that

$$\frac{1}{q} - \epsilon \le \frac{1}{q} - \frac{1}{q_{n+1}} \le \frac{1}{q} - \frac{r}{qq_{n+1}} \le \frac{k}{q_{n+1}} \le \mu\{S_n(a)|\tau_{n+1} = t\}. \tag{7.1}$$

There exists $t \in \{1, \dots, d\}$ such that $\mu\{\tau_{n+1} = t\} \ge 1/d$. Hence, using (7.1), one obtains $\mu(S_n(a) \cap \{\tau_{n+1} = t\} \cap A) \ge \left((\frac{1}{q} - \epsilon)/d\right) - \epsilon \ge \frac{1}{qd} - 2\epsilon > 0$ for any $0 \le a < q$. Suppose $q > d$. Then, by the pigeonhole principle, there exist $0 \le a, b < q$ with $a \ne b$, $x \in S_n(a) \cap \{\tau_{n+1} = t\} \cap A$ and $y \in S_n(b) \cap \{\tau_{n+1} = t\} \cap A$ with $\tau_n(x) = \tau_n(y)$. Since $\rho_n(x) = \rho_n(y)$ and $\rho_{n+1}(x) = \rho_{n+1}(y)$, then the following equality holds modulo $\mathbb{Z}$,

$$\frac{p_n}{q}(a - b) = p'_n \left(\left(\frac{\bar{s_n}(x)}{q} - \rho_{n+1}(x) - \rho_n(x) \right) - \left(\frac{\bar{s_n}(y)}{q} - \rho_{n+1}(y) - \rho_n(y) \right) \right)$$

where $1 \le p'_n < q$ and $(p'_n, q) = 1$. Thus,

$$\text{dist}(\frac{p_n}{q}(a - b), \mathbb{Z}) \le q(\epsilon/2q + \epsilon/2q) \le \epsilon \le \delta/2$$

This contradicts the definition of δ and $(q, p_n) = 1$. Hence $q \le d$. □

The last two theorems implies that an arbitrary subgroup of S^1 cannot be the set of eigenvalues of a Toeplitz minimal Cantor system of finite rank. Also, it is not difficult to deduce from last theorem that there is a unique $q \le d$ with $(q, p_n) = 1$ for all enough large n such that all non continuous eigenvalues of the same type are in the subgroup generated by $1/q$. Finally observe that from [DM] it follows that Toeplitz type minimal Cantor systems of finite rank that have non continuous eigenvalues are expansive (thus subshifts).
In the following example we provide a Toeplitz system of finite rank 3 where $\lambda = -1$ is a non continuous eigenvalue.

Example. Let $(l_n; n \ge 1)$ be a strictly increasing sequence of integers with $l_1 = 0$. Put $q_n = 3^{l_n}$ for $n \ge 1$. Consider the Toeplitz system (X, T) of finite rank 3 given by the properly ordered Bratteli-Vershik diagram $B = (V, E, \preceq)$ with characteristic sequence $(q_n; n \ge 1)$ and such that for all $n \ge 1$: (1) $V_n = \{1, 2, 3\}$, (2) the local order of the q_{n+1} arrows arriving at a vertex of level $n + 1$ is given by the following associated sequences of starting vertices in level n,

$$1 \to (12)^{t_{n+1}-3}131, \; 2 \to 1(12)^{t_{n+1}-3}31, \; 3 \to (12)^{t_{n+1}-3}131,$$

where $q_{n+1} = 2t_{n+1} - 3$.
Let $n \ge 1$. Define $\rho_1(n) = -\rho_2(n) = -\rho_3(n) = 1$ and $f_n(x) = (-1)^j \rho_k(n)$ if $x \in T^{-j} B_k(n)$ for $k \in \{1, 2, 3\}$ and $0 \le j < h_k(n)$. We set $A_n = \{x \in X; f_n(x) \ne f_{n+1}(x)\}$. Let $1 \le k, l \le 3$ and $x \in X$ with $\tau_n(x) = k$ and $\tau_{n+1}(x) = l$. It is direct that, if $f_n(x) = (-1)^j \rho_k(n)$ then

$$f_{n+1}(x) = (-1)^{j + N_n(x)p_n} \rho_l(n + 1) = f_n(x)(-1)^{N_n(x)p_n} \frac{\rho_l(n + 1)}{\rho_k(n)}$$

for some $N_n(x) \in \mathbb{N}$. Recall $p_n = q_1 \cdots q_n = 3^{l_1+\ldots+l_n}$, therefore the difference between $f_{n+1}(x)$ and $f_n(x)$ depends on the parity of $N_n(x)$, k and l. For instance if $k = l = 1$ then $N_n(x)$ is always even and consequently $f_{n+1}(x) = f_n(x)$; if $l = 3$ and $k = 2$ then $N_n(x)$ is odd, which implies that $f_{n+1}(x) \neq f_n(x)$. The study of all possible cases yields to,

$$A_n = \{\tau_{n+1} = 3\} \cup \{\tau_{n+1} = 2, \tau_n = 3\} \cup \left(\bigcup_{\substack{0 \leq j < p_n \\ p_{n+1} - p_n \leq j < p_{n+1}}} T^{-j} B_2(n+1) \right)$$

Let μ be an ergodic measure. From construction $M_{i3}(n+2) = 1$ for any $i \in V_{n+2}$, then $\mu_3(n+1) = \mu_1(n+2) + \mu_2(n+2) + \mu_3(n+2)$ and thus

$$\begin{aligned} \mu\{\tau_{n+1} = 3\} &= p_{n+1}\mu_3(n+1) \\ &= \tfrac{1}{q_{n+2}} \left(\mu\{\tau_{n+2} = 1\} + \mu\{\tau_{n+2} = 2\} + \mu\{\tau_{n+2} = 3\}\right) = \tfrac{1}{q_{n+2}} \end{aligned}$$

Finally,

$$\begin{aligned} \mu(A_n) &= \mu\{\tau_{n+1} = 3\} + M_{23}(n+1)p_n\mu_2(n+1) + 2p_n\mu_2(n+1) \\ &= \mu\{\tau_{n+1} = 3\} + \tfrac{3}{q_{n+1}}\mu\{\tau_{n+1} = 2\} \\ &= \tfrac{1}{q_{n+2}} + \tfrac{3}{q_{n+1}}\mu\{\tau_{n+1} = 2\} \leq \tfrac{4}{q_{n+1}} \end{aligned}$$

Since $\sum_{n\geq 1} \frac{1}{q_{n+1}}$ converges, one deduces that $\mu(\limsup_{n\to\infty} A_n) = 0$. Then, $(f_n; n \geq 1)$ converges μ-almost everywhere to some function f. One easily checks that $f \circ T = -f$ μ-almost everywhere, hence -1 is a measurable eigenvalue of the system. We use Theorem 25 to conclude that -1 is not a continuous eigenvalue.

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