LOCAL RATES OF POINCARÉ RECURRENCE FOR ROTATIONS AND WEAK MIXING

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Abstract. We study the lower and upper local rates of Poincaré recurrence of rotations on the circle by means of symbolic dynamics. As a consequence, we show that if the lower rate of Poincaré recurrence of an ergodic dynamical system $(X, \mathcal{F}, \mu, T)$ is greater or equal to 1 μ-almost everywhere, then it is weakly mixing.

1. Introduction and main result

Let T, acting on the Lebesgue probability space $(X, \mathcal{F}, \mu)$, be a (non necessarily invertible) measure-preserving ergodic map. Let U be a measurable subset of X and $x \in U$. Define the first return time of x to U by $\tau_U(x) \stackrel{\text{def}}{=} \inf\{k \geq 1 : T^k x \in U\}$. If $\mu(U) > 0$ then $\tau_U(x)$ is finite for μ-almost every x by Poincaré recurrence theorem. Now define the Poincaré recurrence of the set U by

$$\tau(U) \stackrel{\text{def}}{=} \inf\{\tau_U(x) : x \in U\}\,.$$

It is easy to check that

$$\tau(U) = \inf\{k > 0 : T^k U \cap U \neq \emptyset\} = \inf\{k > 0 : T^{-k} U \cap U \neq \emptyset\}\,.$$

The notion of Poincaré recurrence of a set is used in [1] to define a notion of dimension similar to Hausdorff dimension where diameters of sets are replaced by their Poincaré recurrence (see also [3, 4, 5, 11, 17, 16, 18]).
Suppose now that ζ is a finite measurable partition of X and denote, as usually, by ζ_n the partition $\zeta \vee T^{-1}\zeta \vee \cdots \vee T^{-n+1}\zeta$ ($n \geq 1$, $\zeta_1 \stackrel{\text{def}}{=} \zeta$) and by $\zeta_n(x)$ the atom of this partition containing point x. We define the (lower and upper) local rate of Poincaré recurrence for the partition ζ respectively as follows:

$$\underline{\mathcal{R}}_\zeta(x) \stackrel{\text{def}}{=} \liminf_{n\to\infty} \frac{\tau(\zeta_n(x))}{n}\,, \quad \overline{\mathcal{R}}_\zeta(x) \stackrel{\text{def}}{=} \limsup_{n\to\infty} \frac{\tau(\zeta_n(x))}{n}\,.$$

(Of course, these quantities depend on the map T but we omit this dependence in the notation.) If $x \in X$ is a periodic point, then obviously $\underline{\mathcal{R}}_\zeta(x) = \overline{\mathcal{R}}_\zeta(x) = 0$. A useful fact to be used later is that both $\underline{\mathcal{R}}_\zeta$ and $\overline{\mathcal{R}}_\zeta$ are sub-invariant functions, namely $\underline{\mathcal{R}}_\zeta \circ T \leq \underline{\mathcal{R}}_\zeta$ and $\overline{\mathcal{R}}_\zeta \circ T \leq \overline{\mathcal{R}}_\zeta$. This is because for any $n \geq 1$ and any $x \in X$, $\tau(\xi_{n-1}(Tx)) \leq \tau(\xi_n(x))$. If we assume μ is an ergodic probability measure, then it follows by basic arguments that $\underline{\mathcal{R}}_\zeta$ and $\overline{\mathcal{R}}_\zeta$ are μ-a.e. constant functions (see [3] for details).

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The definition of the (lower and upper) local rate of Poincaré recurrence first appeared in [13] in connection with the error term in the approximation by the exponential law of the distribution of rescaled return times to cylinder sets. In [2] the authors prove that the following result, where $h_\mu(T,\zeta)$ denotes the measure-theoretic entropy of $(X,\mathcal{F},\mu,T)$ with respect to the partition ζ.

Theorem 1. *Let $(X,\mathcal{F},\mu,T)$ be as above and ζ a finite partition of X. If $h_\mu(T,\zeta)>0$ then $\underline{\mathcal{R}}_\zeta(x)\geq 1$ μ-almost everywhere.*

It is equivalent to state that if $\underline{\mathcal{R}}_\zeta(x)<1$ μ-almost everywhere, then $h_\mu(T,\zeta)=0$. In the case when (X,T) has the specification property (think of a topologically mixing subshift of finite type over a finite alphabet as a typical example), one has $\overline{\mathcal{R}}_\zeta(x)\leq 1$ for all $x\in X$, where ζ is the canonical partition labelled by the alphabet (see [3]). Therefore, in this case $\underline{\mathcal{R}}_\zeta(x)=\overline{\mathcal{R}}_\zeta(x)=1$ μ-almost everywhere where μ is any ergodic probability measure such that $h_\mu(T,\zeta)>0$. Outside such sets of full measure, the specification property allows one to construct some points x such that $0<\underline{\mathcal{R}}_\zeta(x)=\overline{\mathcal{R}}_\zeta(x)<1$. On another hand, the positiveness of entropy is an unavoidable assumption in Theorem 1. Indeed, it is shown in [3] that $\underline{\mathcal{R}}_\zeta(x)=0$ for Lebesgue almost every x for a special class of rotations where ζ is the canonical partition into two atoms given by the rotation angle. A natural question is thus to know whether or not the converse to Theorem 1 holds. In the course of the present note, we will provide an example, namely the Morse system, showing that this is not the case.
The motivation of the present work was to figure out which property of an ergodic dynamical system $(X,\mathcal{F},\mu,T)$, equipped with a partition ζ, the property "$\underline{\mathcal{R}}_\zeta(x)\geq 1$ μ-almost surely" is related to. In this direction, we have the following result.

Theorem 2. *Let T, acting on the Lebesgue probability space $(X,\mathcal{F},\mu,T)$, be a measure-preserving ergodic map. If $\underline{\mathcal{R}}_\zeta(x)\geq 1$ for μ-almost every $x\in X$ and every non-trivial measurable partition ζ then $(X,\mathcal{F},\mu,T)$ is weakly mixing.*

(By 'non-trivial' we mean that no atom of ζ has measure 0 or 1.) For definitions and properties of the classical notions of mixing in ergodic theory, we refer the reader to e.g. [19].
Our approach to prove this theorem is as follows. The point is that a non weakly mixing system has a non trivial eigenvalue, hence a measure-theoretical factor which is a rotation. Moreover, every measurable partition of the factor induces a measurable partition of the original system. Therefore we are led to show that the lower local rate of Poincaré recurrence for rotations is strictly less than 1 for some partition. But the rotations are measure theoretically isomorphic to Sturmian subshifts. Hence, it suffices to prove that the lower rate of Poincaré recurrence for Sturmian subshifts is strictly less than 1 for some partitions. In fact we prove:

Theorem 3. *Let (Ω_α,S) be the Sturmian subshift generated by $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ and μ be its unique ergodic measure. Let ζ be the partition $\{[0],[1]\}$.*

(1) *The following statement are equivalent.*
   (a) *The coefficients of the continued fraction of α are bounded;*

   (b) $\underline{\mathcal{R}}_\zeta(x) > 0$ *for μ-almost every x;*
   (c) $\overline{\mathcal{R}}_\zeta(x) < \infty$ *for μ-almost every x.*
(2) *Moreover, if the coefficients of the continued fraction of α are bounded, then $\underline{\mathcal{R}}_\zeta(x) < 1$ and $\overline{\mathcal{R}}_\zeta(x) > 1$ for μ-almost every x.*
(3) *The same results hold for $([0,1[, x \mapsto x+\alpha \mod 1)$ and $\zeta = \{[0,1-\alpha[, [1-\alpha,1[\}$.*

After completing our work, we discovered the preprint [14] in which the author computes the precise values of $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ for rotations. See also [15] for substitutive subshifts.

2. Local rates of recurrence for some zero entropy systems

In this section we establish some results about the local rate of recurrence for some zero entropy systems that will be useful in the proof of Theorem 2.
Let A be a finite alphabet. We endow $A^{\mathbb{Z}}$ and $A^{\mathbb{N}}$ with the product topology. Let $S : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be the shift map: $(Sx)_n = x_{n+1}$ where $x = (x_n) \in A^{\mathbb{Z}}$. If X is a closed and S-invariant set of $A^{\mathbb{Z}}$ then (X,S) is called a subshift. It is a continuous map. For all word u on the alphabet A, we call cylinder generated by u the set $[u] = \{(x_n) \in X : x_0x_1 \cdots x_{|u|-1} = u\}$.

2.1. Linearly recurrent subshifts.

Let $x = (x_n)$ be a sequence in $A^{\mathbb{Z}}$ where A is a finite alphabet. We call $L(x)$ the set of all finite words appearing in x. The length $|u|$ of a word $u \in L(x)$ is the number of symbols in u. Let $u, w \in L(x)$. For a subshift (X,S) we set $L(X) = \bigcup_{x\in X} L(x)$. We say an element x of $A^{\mathbb{N}}$ or $A^{\mathbb{Z}}$ generates (X,S) if $L(x) = L(X)$. We say that w is a return word to u of x if wu belongs to $L(x)$, u is a prefix of wu and u has exactly two occurrences in wu. (There is a more general definition of return words in [10].) We say that x is uniformly recurrent if all $u \in L(x)$ appears infinitely many times in x, and, for all $u \in L(x)$ there exists K_u such that for all return word w to u, $|w| \leq K_u|u|$. Let us recall that if the subshift (X,S) is minimal then all its points are uniformly recurrent.
We say that x is linearly recurrent (LR) (with constant $K \in \mathbb{R}\backslash\{0\}$) if it is uniformly recurrent and if for all $u \in L(x)$ and all return word w to u, we have $|w| \leq K|u|$. We say that a subshift (X,S) is LR (with constant K) if it is minimal and contains a LR sequence (with constant K). Notice that a minimal subshift is LR if and only if all its elements are linearly recurrent.
Let u be a word and $\alpha \in \mathbb{R}_+$. The prefix of length $\lfloor |u|\alpha \rfloor$ of the sequence $uuu\ldots$ is denoted by u^α, where $\lfloor . \rfloor$ is the integer part map.

Proposition 4 ([10])**.** *Let (X,S) be an aperiodic LR subshift with constant K. Then X is $(K+1)$-power free (i.e. $u^{K+1} \in L(X)$ if and only if $u = \emptyset$) and for all $u \in L(X)$ and for all $w \in \mathcal{R}_u$ we have $(1/K)|u| < |w|$.*

Let (X,S) be a subshift on the alphabet A. Let ζ be the partition into 1-cylinders, i.e. $\zeta = \{[a]; a \in A\}$. There is a bijective correspondence between the partition ζ_n and the set of the words of length n of X. Moreover, for all $x = (x_n) \in X$ and $n \in \mathbb{N}$ we have

$$\tau(\zeta_n(x)) = \min\{|w| : w \,\text{is a return word to}\, x_0x_1\ldots x_{n-1}\}\,.$$

Hence by Proposition 4 we have the following result.

Proposition 5. *Let (X,S) be an aperiodic LR subshift with constant K and ζ the partition into 1-cylinders. Then for all $x \in X$:*

$$\frac{1}{K} \leq \underline{\mathcal{R}}_\zeta(x) \leq \overline{\mathcal{R}}_\zeta(x) \leq K.$$

Let (X,S) be a subshift and ζ be the partition into 1-cylinders. If a subshift is $(K+1)$-power free then $1/K \leq \underline{\mathcal{R}}_\zeta(x)$ for all $x \in X$. Hence if we define

$$\delta = \inf\{K \in \mathbb{R}\backslash\{0\} : (X,S) \text{ is } (K+1) - \text{power free}\}$$

then it comes that $1/\delta \leq \underline{\mathcal{R}}_\zeta(x)$ for all $x \in X$. It is known [21] that $\delta = 1$ for the Morse sequence (this is the fixed point of the substitution σ defined by $\sigma(0) = 01$ and $\sigma(1) = 10$). For the subshift (X,S) of $\{0,1\}^{\mathbb{N}}$ generated by the S-orbit closure of the Morse sequence, one has $\underline{\mathcal{R}}_\zeta(x) \geq 1$ for all $x \in X$, where $\zeta = \{[0],[1]\}$. This gives an example of a zero-entropy dynamical system having a local rate of Poincaré recurrence greater or equal to one. Therefore the converse to Theorem 1 is false.

2.2. **Sturmian subshifts.** Let $0 < \alpha < 1$ be an irrational number. We define the map $R_\alpha : [0,1[\to [0,1[$ by $R_\alpha(t) = t + \alpha \pmod 1$ and the map $I_\alpha : [0,1[\to \{0,1\}$ by $I_\alpha(t) = 0$ if $t \in [0, 1-\alpha[$ and $I_\alpha(t) = 1$ otherwise. Let Ω_α be the closure of the set $\left\{ (I_\alpha(R_\alpha^n(t)))_{n\in\mathbb{Z}} \;\middle|\; t \in [0,1[\right\}$. The subshift (Ω_α, S) is the *Sturmian subshift* generated by α and its elements are called *Sturmian sequences*. There exists a factor map (see [12]) $\phi : (\Omega_\alpha, S) \to ([0,1[, R_\alpha)$ such that $\left|\phi^{-1}(\{\beta\})\right| = 2$ if $\beta \in \{n\alpha \mid n \in \mathbb{Z}\}$ and $\left|\phi^{-1}(\{\beta\})\right| = 1$ otherwise. Consequently ϕ is a measure-theoretic isomorphism. It is well-known that (Ω_α, S) is a non-periodic uniquely ergodic minimal subshift.

Proposition 6 ([8, 9])**.** *A Sturmian subshift (Ω_α, S) is LR if and only if the coefficients of the continued fraction expansion of α are bounded.*

In the sequel we will make use of the following morphisms ρ_n and γ_n, $n \in \mathbb{N}\backslash\{0\}$, from $\{0,1\}$ to $\{0,1\}^*$ defined by

$$\begin{matrix} \rho_n(0) = 01^{n+1} \\ \rho_n(1) = 01^n \end{matrix} \quad \text{and} \quad \begin{matrix} \gamma_n(0) = 10^{n+1} \\ \gamma_n(1) = 10^n \end{matrix} .$$

Proposition 7. *Let (Ω_α, S) be a Sturmian subshift. There exists a sequence $(\kappa_n)_{n\in\mathbb{N}}$ taking values in $\{\rho_1, \gamma_1, \rho_2, \gamma_2, \ldots\}$ such that*

(1) $y = \lim_{n\to+\infty} \kappa_1 \cdots \kappa_n(00\cdots)$ *exists and generates (Ω_α, S);*
(2) (Ω_α, S) *is uniquely ergodic;*
(3) $1 \leq \frac{|\kappa_1\cdots\kappa_n(0)|}{|\kappa_1\cdots\kappa_n(1)|} \leq \frac{3}{2}$;
(4) *Let $P_0 = \{[0],[1]\}$, and for $n \geq 1$, let*

$$P_n = \left\{ S^k \kappa_1 \cdots \kappa_n([a]) \;\middle|\; 0 \leq k < |\kappa_1 \cdots \kappa_n(a)|,\ a \in \{0,1\} \right\}$$

is a partition of Ω_α with the following properties:
(a) $\kappa_1 \cdots \kappa_{n+1}([0]) \cup \kappa_1 \cdots \kappa_{n+1}([1]) \subseteq \kappa_1 \cdots \kappa_n([0]) \cup \kappa_1 \cdots \kappa_n([1])$,
(b) $P_n \prec P_{n+1}$ *as partitions,*

Proof. The statements 1., 2. and 4. are mainly due to Hedlund and Morse [12] (see [7]). Point 3. follows from the definition of ρ_n and γ_n given above. □

3. Proof of Theorem 2 and Theorem 3

3.1. **Proof of Theorem 3.** In this subsection $\alpha \in [0,1[$ is an irrational number, (Ω_α, S) the Sturmian subshift it defines, μ its unique ergodic measure and $(\kappa_n)_{n\in\mathbb{N}}$ the sequence given by Proposition 7.
For all $n \in \mathbb{N}$ and all $a \in \{0,1\}$ we set
(1)

$$\mu_n(a) = \sum_{0\le k<|\kappa_1\cdots\kappa_n(a)|} \mu\left(S^k\kappa_1\cdots\kappa_n([a])\right) = |\kappa_1\cdots\kappa_n(a)|\,\mu\left(\kappa_1\cdots\kappa_n([a])\right).$$

We remark that $\mu_n(0)+\mu_n(1)=1$.

Lemma 8. *Let $n \in \mathbb{N}$ and $m \in \mathbb{R}$ with $1 \le m \le |\kappa_n(0)|-2$. We set $\kappa_n(0) = ab^{|\kappa_n(0)|-1}$. Let U_n be the union of the sets $S^k\kappa_1\cdots\kappa_n([0])$, where*

$$|\kappa_1\cdots\kappa_{n-1}(a)| \le k \le |\kappa_1\cdots\kappa_n(0)| - \lfloor|\kappa_1\cdots\kappa_{n-1}(b)|(m+1)\rfloor, \tag{2}$$

and V_n be the union of the sets $S^l\kappa_1\cdots\kappa_n([1])$, where

$$|\kappa_1\cdots\kappa_{n-1}(a)| \le l \le |\kappa_1\cdots\kappa_n(1)| - \lfloor|\kappa_1\cdots\kappa_{n-1}(b)|m\rfloor. \tag{3}$$

If $x \in U_n$ then there exists a word v such that $x \in [v^{m+1}]$. If $x \in V_n$ then there exists a word v such that $x \in [v^m]$. In both case $|v| = |\kappa_1\cdots\kappa_{n-1}(b)|$. Moreover for all $n \in \mathbb{N}$

$$\mu(U_n) \ge \mu_n(0)\left(\frac{|\kappa_n(0)|-m-2}{3/2+|\kappa_n(0)|-1}\right) \text{ and } \mu(V_n) \ge \mu_n(1)\left(\frac{|\kappa_n(0)|-m-2}{3/2+|\kappa_n(0)|-2}\right).$$

Proof. Let $n \in \mathbb{N}$ and $m \in [1, |\kappa_n(0)|-2]$. Let $x \in S^k\kappa_1\cdots\kappa_n[0]$ where $k \in \mathbb{N}$ satisfies (2). Let w be the prefix of length $\lfloor|\kappa_0\cdots\kappa_{n-1}(b)|(m+1)\rfloor$ of x. We have $x \in [w]$ and

$$\lfloor|\kappa_0\cdots\kappa_{n-1}(b)|(m+1)\rfloor = |\kappa_0\cdots\kappa_{n-1}(b)|(\lfloor m\rfloor+1)+\lfloor|\kappa_0\cdots\kappa_{n-1}(b)|(m-\lfloor m\rfloor)\rfloor.$$

The sequence x belongs to $S^k[\kappa_1\cdots\kappa_{n-1}(ab^{|\kappa_n(0)|-1})]$. Consequently, the hypotheses on k imply that there exist two words s and p such that

$$w = s(\kappa_0\cdots\kappa_{n-1}(b))^{\lfloor m\rfloor}p(sp)^{m-\lfloor m\rfloor} = (sp)^{m+1},$$

where $ps = \kappa_0\cdots\kappa_{n-1}(b)$. The other case can be treated in the same way.
From Proposition 7 we deduce that

$$\begin{aligned}\mu(U_n) &= \mu_n(0)\frac{|\kappa_1\cdots\kappa_n(0)| - |\kappa_1\cdots\kappa_{n-1}(a)| - \lfloor|\kappa_1\cdots\kappa_{n-1}(b)|(m+1)\rfloor + 1}{|\kappa_1\cdots\kappa_n(0)|}\\ &\ge \mu_n(0)\left(\frac{|\kappa_1\cdots\kappa_{n-1}(b)|(|\kappa_n(0)|-m-2)}{|\kappa_1\cdots\kappa_{n-1}(a)|+|\kappa_1\cdots\kappa_{n-1}(b)|\left(|\kappa_n(0)|-1\right)}\right)\\ &\ge \mu_n(0)\left(\frac{|\kappa_n(0)|-m-2}{3/2+|\kappa_n(0)|-1}\right).\end{aligned}$$

The same computations can be done for V_n. □

Proof of the statement (1) *of Theorem 3.* To prove that (a) is equivalent to (b) it suffices to prove that if the coefficients of the continued fraction of α are not bounded then $\underline{\mathcal{R}}_\zeta(x) = 0$ for μ-almost every x. The other part of the proof follows from Proposition 5 and 6. Hence, $\underline{\mathcal{R}}_\zeta$ being S-invariant and μ ergodic, it is enough to prove that $\mu\left(\{x; \underline{\mathcal{R}}_\zeta(x) = 0\}\right) > 0$.
Let (U_n) and (V_n) be the sequences of open sets given by Lemma 8. From Proposition 1.1 in [9] and Proposition 6 there exists a strictly increasing sequence (n_i) such that $\lim_{i\to+\infty} |\kappa_{n_i}(0)| = +\infty$. For all $i \in \mathbb{N}$ we set $m_i = \lfloor\sqrt{|\kappa_{n_i}(0)| - 2}\rfloor$. For all $i \in \mathbb{N}$ and all $x \in U_{n_i} \cup V_{n_i}$, by Lemma 8, there exists v_i such that x belongs to the cylinder $[v_i^{m_i}]$ and consequently

$$\frac{\tau\left(\zeta_{(m_i-1)|v_i|}(x)\right)}{(m_i-1)|v_i|} = \frac{1}{m_i-1}.$$

Hence, if $x \in \cap_{j\in\mathbb{N}} \cup_{i\geq j} (U_{n_i} \cup V_{n_i})$ then $\underline{\mathcal{R}}_\zeta(x) = 0$. But, from Lemma 8, we also have $\mu(\cap_{j\in\mathbb{N}} \cup_{i\geq j} (U_{n_i} \cup V_{n_i})) \geq 2/3$. Thus, (a) is equivalent to (b).

To prove that (a) is equivalent to (c) it suffices to prove that if the coefficients of the continued fraction of α are not bounded then $\overline{\mathcal{R}}_\zeta(x) = \infty$ for μ-almost every x. The other part of the proof follows from Proposition 5 and 6. Hence, $\underline{\mathcal{R}}_\zeta$ being S-invariant and μ ergodic, it is enough to prove that $\mu\left(\{x; \overline{\mathcal{R}}_\zeta(x) \geq h\}\right) > 0$ for all $h \geq 0$.
Let $h \geq 2$. Let $n \in \mathbb{N}$ be such that $\kappa_{n+1}(0) = ab^{i+1}$ with $i \geq 2$. We set $l_n = |\kappa_1 \cdots \kappa_n(a)|$, $k_n = |\kappa_1 \cdots \kappa_n(b^i)|/h$ and $W_n = \cup_{1\leq k\leq k_n} S^{-k}\kappa_1 \cdots \kappa_n([abb])$. Take $x \in S^{-k}\kappa_1 \cdots \kappa_n([abb])$ with $1 \leq k \leq k_n$. We remark $\kappa_1 \cdots \kappa_n([abb])$ is contained in $\kappa_1 \cdots \kappa_{n+1}([a]) \cup \kappa_1 \cdots \kappa_{n+1}([b])$ (the proof is left to the reader). The words $\kappa_{n+1}(a)$ and $\kappa_{n+1}(b)$ end with the word b^i. Thus, we can write $\kappa_1 \cdots \kappa_n(b) = uv$ in such a way that $v(\kappa_1 \cdots \kappa_n(b))^p \kappa_1 \cdots \kappa_n(ab^j)$ is a prefix of x for some $p \leq i$, where $j = \max(i, 2)$. It can be seen that $\tau\left(\zeta_{k_n+l_n-1}(x)\right)$ is greater than $|\kappa_1 \cdots \kappa_n(ab^j)|$. Consequently

$$\frac{\zeta_{k_n+l_n-1}(x)}{k_n+l_n} \geq \frac{j|\kappa_1 \cdots \kappa_n(b)| + |\kappa_1 \cdots \kappa_n(a)|}{|\kappa_1 \cdots \kappa_n(b^i)|/h + |\kappa_1 \cdots \kappa_n(a)|} \geq \frac{j+3/2}{i/h+3/2} = f(h,i,j). \tag{4}$$

Hence if $x \in \mathcal{W}(h) = \cap_{j\in\mathbb{N}} \cup_{i\geq j} W_i$ then $\overline{\mathcal{R}}(x) \geq f(h,i,j)$. But for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
\mu(W_n) &= k_n\mu(\kappa_1 \cdots \kappa_n([abb])) \geq k_n\mu\left(\kappa_1 \cdots \kappa_{n+1}([0]) \cup \kappa_1 \cdots \kappa_{n+1}([1])\right) \\
&= \frac{i|\kappa_1 \cdots \kappa_n(b)|}{h}\left(\frac{\mu_{n+1}(0)}{|\kappa_1 \cdots \kappa_n(ab^{i+1})|} + \frac{\mu_{n+1}(1)}{|\kappa_1 \cdots \kappa_n(ab^i)|}\right) \\
&\geq \frac{i}{h}\frac{|\kappa_1 \cdots \kappa_n(b)|}{|\kappa_1 \cdots \kappa_n(ab^{i+1})|} = \frac{i}{h}\frac{|\kappa_1 \cdots \kappa_n(b)|}{|\kappa_1 \cdots \kappa_n(a)| + (i+1)|\kappa_1 \cdots \kappa_n(b)|} \\
&\geq \frac{i}{h}\frac{1}{3/2+(i+1)} \geq \frac{2}{h}\frac{1}{3/2+(2+1)} \geq \frac{4}{9h}.
\end{aligned}$$

Hence $\mu(\mathcal{W}(h)) \geq 4/9h1$. From Proposition 1.1 in [9] and Proposition 6 there exists a strictly increasing sequence (n_i) such that $\lim_{i\to+\infty} |\kappa_{n_i}(0)| = +\infty$. Thus, using (4) it comes that $\overline{\mathcal{R}}(x) \geq h$ for all $x \in \mathcal{W}(h)$. This proves that (a) is equivalent (c). □

Proof of the statement (2) *of Theorem 3.* Using (4) we conclude that $\overline{\mathcal{R}}(x) > \frac{7}{5}$ for μ-almost every $x \in X$.
Now we prove the other part of the statement. It suffices to prove that for some $\theta < 1$ we have $\mu(\{x : \underline{\mathcal{R}}_\zeta(x) \le \theta\}) > 0$.
From the hypotheses the sequence $(|\kappa_n(0)|; n \in \mathbb{N})$ is bounded by some constant K. We will need the following lemma.

Lemma 9. *For all $a \in \{0,1\}$ and all $n \in \mathbb{N}$ we have $\mu_n(a) \ge 2/(3K+1)$.*

Proof. Suppose $\kappa_{n+1}(0) = ab^{i+1}$ then one can check we have

$$\mu(\kappa_1 \dots \kappa_n([a])) = \mu(\kappa_1 \dots \kappa_{n+1}([0])) + \mu(\kappa_1 \dots \kappa_{n+1}([1])) \text{ and}$$
$$\mu(\kappa_1 \dots \kappa_n([b])) = (i+1)\mu(\kappa_1 \dots \kappa_{n+1}([0])) + i\mu(\kappa_1 \dots \kappa_{n+1}([1])).$$

Consequently using (1) we obtain

$$\frac{3}{2i} \ge \frac{\mu_n(a)}{\mu_n(b)} \ge \frac{2}{3}\frac{\mu(\kappa_1 \dots \kappa_n([a]))}{\mu(\kappa_1 \dots \kappa_n([b]))} \ge \frac{2}{3(i+1)}.$$

We conclude using the facts that $i \ge K$ and $\mu_n(a) + \mu_n(b) = 1$. □

We consider several cases. Suppose there exists a strictly increasing sequence (n_i) such that $|\kappa_{n_i}(0)| \ge 4$ for all $i \in \mathbb{N}$. We set $m_i = 3$, $i \in \mathbb{N}$. Let $(U_{n_i})_{i\in\mathbb{N}}$ and $(V_{n_i})_{i\in\mathbb{N}}$ be the sequences of open sets given by Lemma 8 and associated to the sequence $(m_i)_{i\in\mathbb{N}}$. For all $i \in \mathbb{N}$ and all $x \in U_{n_i}$, by Lemma 8, there exists v_i such that x belongs to the cylinder $[v_i^{5/2}]$ and $\lim_{i\to\infty} |v_i| = +\infty$. Consequently

$$\frac{\tau\left(\zeta_{(3/2)|v_i|-1}(x)\right)}{\lfloor (3/2)|v_i| \rfloor} \le \frac{|v_i|}{(3/2)|v_i|} = \frac{2}{3},$$

and, if $x \in \cap_{j\in\mathbb{N}} \cup_{i\ge j} U_{n_i}$ then $\underline{\mathcal{R}}_\zeta(x) \le 2/3$. Lemma 9 and Lemma 8 imply

$$\mu(U_{n_i}) \ge \mu_n(0)\frac{|\kappa_{n_i}(0)| - 7/2}{1/2 + |\kappa_{n_i}(0)|} \ge \frac{2(K-7/2)}{(3K+1)(1/2+K)} > 0.$$

Hence $\mu(\cap_{j\in\mathbb{N}} \cup_{i\ge j} U_{n_i}) > 0$ and $\mu(\{x : \underline{\mathcal{R}}_\zeta(x) \le 2/3\}) > 0$.
It remains to treat the following case: There exists i_0 such that for all $i \ge i_0$, $|\kappa_i(0)| = 3$. For all $n \in \mathbb{N}$ we define W_n to be the union of the sets $S^k \kappa_1 \cdots \kappa_{2n}[0]$, where

$$|\kappa_1 \cdots \kappa_{2n-2}(01)| \le k \le |\kappa_1 \cdots \kappa_{2n-2}(01)| + \lfloor |\kappa_1 \cdots \kappa_{2n-2}(1)|/2 \rfloor. \tag{5}$$

Let $x \in W_n$. We consider four cases.
First case: Suppose $\kappa_{2n-1}\kappa_{2n} = \rho_1^2$. We have $\kappa_{2n-1}\kappa_{2n}(0) = 0110101$. Hence x belongs to $T^k[\kappa_1\kappa_2 \cdots \kappa_{2n-2}(0110101)]$ for some k satisfying (5). We can write $\kappa_1\kappa_2 \cdots \kappa_{2n-2}(1) = uv$, with $|v| \ge \lfloor |\kappa_1 \cdots \kappa_{2n-2}(1)|/2 \rfloor$, in such a way that the word

$$v\ \kappa_1\kappa_2 \cdots \kappa_{2n-2}(0)\ uv\ \kappa_1\kappa_2 \cdots \kappa_{2n-2}(0)\ uv$$

is a prefix of x. We set $k_n = |\kappa_1 \cdots \kappa_{2n-2}(01)| + |v|$. We have

$$\begin{aligned}\frac{\tau\left(\zeta_{k_n-1}(x)\right)}{k_n} &= \frac{|\kappa_1\cdots\kappa_{2n-2}(01)|}{k_n} \leq \frac{|\kappa_1\cdots\kappa_{2n-2}(01)|}{|\kappa_1\cdots\kappa_{2n-2}(01)|+|\kappa_1\cdots\kappa_{2n-2}(1)|/2-1}\\ &\leq \theta_1 = \frac{2}{2+1/5} < 1,\end{aligned}$$

Second case: $\kappa_{2n-1}\kappa_{2n} = \gamma_1\rho_1$. We have $\kappa_{2n-1}\kappa_{2n}(0) = 1001010$. As previously we obtain that there exists $\theta_2 < 1$ such that

$$\frac{\tau\left(\zeta_{k_n-1}(x)\right)}{k_n} \leq \theta_2 < 1.$$

Third case: $\kappa_{2n-1}\kappa_{2n} = \gamma_1^2$. We have $\kappa_{2n-1}\kappa_{2n}(0) = 10100100$. We write $\kappa_1\kappa_2\cdots\kappa_{2n-2}(1) = uv$ such that $v\kappa_1\kappa_2\cdots\kappa_{2n-2}(00)uv\kappa_1\kappa_2\cdots\kappa_{2n-2}(00)$ is a prefix of x and $|v| \geq \lfloor|\kappa_1\cdots\kappa_{2n-2}(1)|/2\rfloor$. But the images of 0 and 1 by $\kappa_{2n-1}\kappa_{2n}$ begin with the letter 1. Furthermore the word

$$v\ \kappa_1\kappa_2\cdots\kappa_{2n-2}(00)\ uv\ \kappa_1\kappa_2\cdots\kappa_{2n-2}(00)\ uv$$

is a prefix of x. We set $l_n = |\kappa_1\cdots\kappa_{2n-2}(001)| + |v|$. We have

$$\begin{aligned}\frac{\tau\left(\zeta_{l_n-1}(x)\right)}{l_n} &= \frac{|\kappa_1\cdots\kappa_{2n-2}(001)|}{l_n}\\ &\leq \frac{|\kappa_1\cdots\kappa_{2n-2}(001)|}{|\kappa_1\cdots\kappa_{2n-2}(001)|+|\kappa_1\cdots\kappa_{2n-2}(1)|/2-1} \leq \theta_3 = \frac{2}{2+1/8} < 1.\end{aligned}$$

Fourth case: $\kappa_{2n-1}\kappa_{2n} = \rho_1\gamma_1$. We have $\kappa_{2n-1}\kappa_{2n}(0) = 01011011$. Proceeding as in the third case we obtain that there exists $\theta_4 < 1$ such that

$$\frac{\tau\left(\zeta_{m_n-1}(x)\right)}{m_n} \leq \theta_4 < 1,$$

where $m_n = |\kappa_1\cdots\kappa_{2n-2}(011)| + |v|$.

To conclude we set $\theta = \max\{\theta_1,\theta_2,\theta_3,\theta_4\}$ and we remark we have

$$\begin{aligned}\mu(W_n) &\geq \mu_n(0)\frac{|\kappa_1\cdots\kappa_{2n-2}(1)|}{2|\kappa_1\cdots\kappa_{2n}(0)|}\\ &\geq \mu_n(0)\frac{|\kappa_1\cdots\kappa_{2n-2}(1)|}{5|\kappa_1\cdots\kappa_{2n-2}(0)|+5|\kappa_1\cdots\kappa_{2n-2}(1)|} \geq \frac{4}{25(3K+1)}.\end{aligned}$$

Hence $\mu(\cap_{j\in\mathbb{N}}\cup_{i\geq j} W_i) > 0$ and $\mu(\{x : \underline{\mathcal{R}}_\zeta(x) \leq \theta\}) > 0$. □

Proof of the statement (3). It suffices to remark that $[0] = \phi^{-1}([0, 1-\alpha[)$ and $[1] = \phi^{-1}([1-\alpha, 1[)$ (where ϕ is the measure-theoretical isomorphism given in Subsection 2.2). □

3.2. **Proof of Theorem 2.** If the system is not weakly mixing then it has a nontrivial eigenvalue $\exp(2i\pi\alpha)$, therefore a measure-theoretical factor which is a rotation: $([0,1[, R_\alpha)$. Furthermore, every measurable partition of the factor induces a measurable partition of the original system. If α is a rational number then clearly there exists a non-trivial partition ζ of $[0,1[$ so that $\underline{\mathcal{R}}_\zeta(x) = 0$ for μ-almost every x. If α is irrational then by Theorem 3 there is a non-trivial partition ζ of $[0,1[$ such that $\underline{\mathcal{R}}_\zeta(x) = 0$ for μ-almost every x if the coefficients of the continued fration of α are not bounded and $0 < \underline{\mathcal{R}}_\zeta(x) < 1$ for μ-almost every x otherwise. This ends the proof. □

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