WORDS AND MORPHISMS WITH STURMIAN ERASURES

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Abstract. We say $x \in \{0,1,2\}^{\mathbb{N}}$ is a word with Sturmian erasures if for any $a \in \{0,1,2\}$ the word obtained erasing all a in x is a Sturmian word. A large family of such words is given coding trajectories of balls in the game of billiards in the cube. We prove that the monoid of morphisms mapping all words with Sturmian erasures to words with Sturmian erasures is not finitely generated.

1. Introduction

In this paper we are interested in words x defined on the alphabet $A_3 = \{0,1,2\}$ having the following property: For any letter $a \in A_3$, the word obtained erasing all a in x is a Sturmian word. We say x is a *word with Sturmian erasures.*

Sturmian words are well-known objects that can be defined in many ways. For example, a word is Sturmian if and only if for all $n \in \mathbb{N}$ the number of distinct finite words of length n appearing in x is $n+1$ (see [Lo] for complete references about Sturmian words). Sturmian words can also be viewed as trajectories of balls in the game of billiards in the square. We will see that a large family of words with Sturmian erasures is the family of trajectories of balls in the game of billiards in the cube.

Here we are interested in the morphisms $f : A_3 \to A_3^*$ (the free monoid generated by A_3) that send all words with Sturmian erasures to words with Sturmian erasures. We call such f the *morphisms with Sturmian erasures* and we denote by MSE the set of all these morphisms. Our main result is the following:

Theorem 1. *We have:*

1. *The monoid* MSE *is not finitely generated;*
2. MSE *is the union of* $\mathrm{MSE}^{\varepsilon}$ *and the set of permutation of* A_3.
3. *If* $f : A_3 \to A_3^*$ *is locally with Sturmian erasures such that* $f(i)$ *is the empty word for some* $i \in A_3$ *then it is a morphism with Sturmian erasures;*

Where $\mathrm{MSE}^{\varepsilon}$ is the set of morphisms with Sturmian erasures having the empty word as an image of a letter and locally with Sturmian erasures means that there exists a word with Sturmian erasures such that $f(x)$ is a word with Sturmian erasures.

We recall that F. Mignosi and P. Séébold proved in [MS] that the monoid of the morphisms sending all Sturmian words to Sturmian words is finitely generated.

In the last section we give some other informations about the words with Sturmian erasures: symbolic complexity, link with the game of billiards in the cube, balanced property and palyndroms.

1991 *Mathematics Subject Classification.* Primary: 68R15.

Key words and phrases. Words with Sturmian erasures, Sturmian words, Sturmian morphisms.

2. Definitions, notations and background

2.1. **Words, morphisms and matrices.** We call *alphabet* a finite set of elements called *letters*. Let A be an alphabet and A^* be the free monoid generated by A. The elements of A^* are called *words*. The neutral element of A^*, also called the *empty word*, is denoted by ε. We set $A^+ = A^* \setminus \{\varepsilon\}$. Let $u = u_0 u_1 \cdots u_{n-1}$ be a word of A^*, $u_i \in A$, $0 \leq i \leq n-1$. Its *length* is n and is denoted by $|u|$. In particular, $|\varepsilon| = 0$. If $a \in A$ then $|u|_a$ denotes the number of occurrences of the letter a in the word u. We call *infinite words* the elements of $A^{\mathbb{N}}$ and we set $A^\infty = A^{\mathbb{N}} \cup A^*$. Let $x \in A^\infty$ and $y \in A^*$. We say that y is a *factor* of x if there exist $u \in A^*$ and $v \in A^\infty$ such that $x = uyv$. In particular if $u = \varepsilon$ then y is a *prefix* of x and if $v = \varepsilon$ then y is a *suffix* of x. An infinite word $x = (x_n; n \in \mathbb{N})$ of $A^{\mathbb{N}}$ is called *eventually periodic* if there exist two words $u \in A^*$ and $v \in A^+$ such that $x = uvvv\ldots$.
The *complexity* function of an infinite word x is the function $P_x : \mathbb{N} \to \mathbb{N}$ where $P_x(n)$ is the number of factors of length n of x.
Let A, B and C be three alphabets. A *morphism* f is a map from A to B^*. It induces by concatenation a map from A^* to B^*. If $f(A)$ is included in B^+, it induces a map from $A^{\mathbb{N}}$ to $B^{\mathbb{N}}$. All these maps are also written f.
To a morphism $f : A \to B^*$ is associated the matrix $M_f = (m_{i,j})_{i \in B, j \in A}$ where $m_{i,j}$ is the number of occurrences of i in the word $f(j)$. If g is a morphism from B to C^* then we can check we have $M_{g \circ f} = M_g M_f$.

2.2. **Sturmian words and Sturmian morphisms.** Let A be a finite alphabet. An infinite word $x \in A^{\mathbb{N}}$ is *Sturmian* if for all $n \in \mathbb{N}, P_x(n) = n + 1$. Since $P_x(1) = 2$, we can suppose $A = \{0, 1\}$ (see [Lo] for more informations about these words).
A morphism f from A to A^* is *Sturmian* if for all Sturmian word x the word $f(x)$ is Sturmian. A morphism f is *locally Sturmian* if there exists at least a Sturmian word x such that $f(x)$ is Sturmian. We call St the semigroup generated by the morphisms E, φ, and $\widetilde{\varphi}$ defined by

$$\begin{array}{rcccrcccrccc} E: & A^* & \longrightarrow & A^* & \varphi: & A^* & \longrightarrow & A^* & \widetilde{\varphi}: & A^* & \longrightarrow & A^* \\ & 0 & \longmapsto & 1 & & 0 & \longmapsto & 01 & & 0 & \longmapsto & 10 \\ & 1 & \longmapsto & 0 & & 1 & \longmapsto & 0 & & 1 & \longmapsto & 0 \end{array}$$

Theorem 2. [BS, MS] *The following three conditions are equivalent*

1. $f \in \mathrm{St}$*;*
2. f *locally Sturmian;*
3. f *Sturmian.*

2.3. **Words with Sturmian erasures.** Let $A_3 = \{0, 1, 2\}$ and let x be a infinite word of $A_3^{\mathbb{N}}$. For $i \in A_3$ we denote $\pi_i : A_3 \to A_3^*$ the morphism defined by $\pi_i(j) = j$ if $j \in A_3$ with $j \neq i$ and $\pi_i(i) = \varepsilon$.

Definition 3. *An infinite word* $x \in A_3^{\mathbb{N}}$ *is called* word with Sturmian erasures *if and only if the word* $\pi_i(x)$ *is a Sturmian word for all* $i \in A_3$*. We say* $f : A_3 \longrightarrow A_3^*$ *is a* morphism with Sturmian erasures *if* $f(x)$ *is a word with Sturmian erasures for all words* $x \in A_3^{\mathbb{N}}$ *with Sturmian erasures.*

We call WSE the set of words with Sturmian erasures and MSE the set of morphisms with Sturmian erasures. We remark MSE is a monoid for the composition law of morphism. The image of a Sturmian word by a morphism with Sturmian erasures is a word with Sturmian erasures. Hence WSE is not empty.

Example 1. Let $g : A_3 \to A_3^*$ be the morphism defined by : $g(0) = 02$, $g(1) = 10$ and $g(2) = \varepsilon$. Let $F_0 = 0$ and for $n \geq 0$ $F_{n+1} = \varphi(F_n)$. Let $F \in \{0,1\}^{\mathbb{N}}$ be the unique fixed point of φ in $\{0,1\}^{\mathbb{N}}$ (see [Qu]). Then for each $n \geq 0$ F_n is a prefix of F, and we have $F = 0100101001001....$ This word is called the *Fibonacci word* (remark that $|F_{n+2}| = |F_{n+1}| + |F_n|$, $n \geq 0$). It is a Sturmian word. From Theorem 2 we deduce that

$$g(F) = 021002021002100202100202100210020210 0210\dots$$

is a word with Sturmian erasures. Hence WSE is not empty.
Let $x \in A_3^{\mathbb{N}}$ be a word with Sturmian erasures. The word $\pi_2(x)$ is a Sturmian word and $g \circ \pi_2 = g$. Moreover $\pi_2 \circ g_{|\{0,1\}}$ is a Sturmian morphism. Hence

$$\pi_2 \circ g(x) = \pi_2 \circ g \circ \pi_2(x) = \pi_2 \circ g_{|\{0,1\}}(\pi_2(x)) = g_{|\{0,1\}}(\pi_2(x))$$

is a Sturmian word. We can also show that $\pi_0 \circ g(x)$ and $\pi_1 \circ g(x)$ are words with Sturmian erasures. Hence, g is a morphism with Sturmian erasures and MSE is not empty.

3. Proofs of points (2) et (3) of Theorem 1

We denote by MSE$^\varepsilon$ the set of morphisms of MSE such that there exists $l \in A_3$ with $f(l) = \varepsilon$. We will prove that MSE is the union of MSE$^\varepsilon$ with the set of permutations on A_3. This last set is generated by

$$\begin{array}{llll} E_0 : & A_3^* & \longrightarrow & A_3^* \\ & 0 & \longmapsto & 0 \\ & 1 & \longmapsto & 2 \\ & 2 & \longmapsto & 1 \end{array} \quad \begin{array}{llll} E_1 : & A_3^* & \longrightarrow & A_3^* \\ & 0 & \longmapsto & 2 \\ & 1 & \longmapsto & 1 \\ & 2 & \longmapsto & 0 \end{array} \quad \begin{array}{llll} E_2 : & A_3^* & \longrightarrow & A_3^* \\ & 0 & \longmapsto & 1 \\ & 1 & \longmapsto & 0 \\ & 2 & \longmapsto & 2 \end{array}$$

We denote by MSE$_i$, $i \in A_3$, the set of morphisms with Sturmian erasures such that $f(i) = \varepsilon$. We have MSE$^\varepsilon$ = MSE$_0 \cup$ MSE$_1 \cup$ MSE$_2$.

3.1. **Proof of the point (3) of Theorem 1.** We start with the following proposition.

Proposition 4. *If a morphism $f : \{0,1\} \to A_3^*$ maps a Sturmian word defined on $\{0,1\}^{\mathbb{N}}$ into a word with Sturmian erasures then it maps any Sturmian word into a word with Sturmian erasures.*

Proof. Let $x \in \{0,1\}^{\mathbb{N}}$ be a Sturmian word and $f : \{0,1\} \to A_3^*$ be a morphism such that $f(x)$ is a word with Sturmian erasures.
Let i be a letter of A_3 and $\overline{E} : A_3 \to \{0,1\}^*$ be a morphism such that $\{0,1\} = \{\overline{E}(a); i \neq a,\ a \in A_3\}$. Then $\overline{E} \circ \pi_i \circ f(x)$ is Sturmian and $\overline{E} \circ \pi_i \circ f : \{0,1\} \to \{0,1\}^*$ is a locally Sturmian morphism. Hence, from Theorem 2 it is Sturmian. It follows that for every Sturmian word y, $f(y)$ is a word with Sturmian erasures. This ends the proof. □

Now we prove the point (3) of Theorem 1. Let $f : A_3 \to A_3^*$, such that $f(i) = \varepsilon$ for some $i \in A_3$, and $x \in A_3^{\mathbb{N}}$ be a word with Sturmian erasures such that $f(x)$ is a word with Sturmian erasures. We remark that we have $f \circ \pi_i(x) = f(x)$ and that $\pi_i(x)$ is a Sturmian word.

We can suppose $A_3 \setminus \{i\} = \{0, 1\}$. Hence the morphism $f \circ \pi_{i|\{0,1\}}$ satisfy the hypothesis of Proposition 4. Consequently if y is a word with Sturmian erasures then $f(y) = f \circ \pi_i(y) = f \circ \pi_{i|\{0,1\}}(\pi_i(y))$ is a word with Sturmian erasures. □

Example 2. We can remark that there exist morphisms $f : A_3 \to A_3^*$ such that for some word $x \in$ WSE we have $f(x) \in$ WSE but f is not a morphism with Sturmian erasures.
For example let F be the Fibonacci word, f be defined by $f(0) = 0$, $f(1) = 1$ and $f(2) = 012$, $g : A_3 \to A_3^*$ be defined by $g(0) = 01$, $g(1) = 02$ and $g(2) = \varepsilon$, and, $h : A_3 \to A_3^*$ be defined by $h(0) = 02$, $h(1) = 10$ and $h(2) = \varepsilon$.
As in Example 1 we can prove that g, h and $f \circ g$ are morphisms with Sturmian erasures and consequently $x = g(F)$ and $f(x) = f \circ g(F)$ are words with Sturmian erasures.
But we remark that $f \circ h(F)$ is not a word with Sturmian erasures. Indeed 001210 is a prefix of $f(y)$ and 00110 is a prefix of $w = \pi_2 f(y)$. Consequently $P_w(2) = 4$ and w is not a Sturmian word.

3.2. **Proof of the point (2) of Theorem 1.** We need the following lemma that follows from Theorem 2 and the fact that the determinant of the matrices associated to φ, $\widetilde{\varphi}$ and E belong to $\{-1, 1\}$.

Lemma 5. *Let M be the matrix associated to the Sturmian morphism f. Then* $\det M = \pm 1$.

Let us prove the point (2) of Theorem 1. This proof is due to D. Bernardi.
Let f be a morphism of MSE. Let $i \in A_3$ and $\overline{E} : A_3 \setminus \{i\} \to \{0, 1\}^*$ be a morphism such that $\{0, 1\} = \{\overline{E}(a); a \neq i,\ a \in A_3\}$. We set $h = \overline{E} \circ \pi_i \circ f \circ g_{|\{0,1\}}$ where $g : A_3 \longrightarrow A_3^*$ is the morphism defined by: $g(0) = 02$, $g(1) = 12$ and $g(2) = \varepsilon$.
As the letter i does not appear in the images of $\pi_i \circ f$, we consider $\pi_i \circ f$ as a morphism from A_3 to $(A_3 \setminus \{i\})^*$. We set $M_{\pi_i \circ f} = (u, v, w)$ where u, v and w are column vectors belonging to $\mathbb{R}^2$. We recall g is a morphism with Sturmian erasures (see Example 1 of the subsection 2.3). Hence the morphism h is Sturmian and we have $M_h = (u + w, v + w)$. From Lemma 5

$$\det(u, v) + \det(u, w) + \det(w, v) = \det(u + w, v + w) = \pm 1.$$

We do the same with g being one of the two following morphisms : $(0 \longmapsto 01,$ $1 \longmapsto 12,\ 2 \mapsto \varepsilon)$ and $(0 \longmapsto 02,\ 1 \longmapsto 01,\ 2 \mapsto \varepsilon)$. We obtain finally

$$\begin{aligned} &(1) \quad \det(u, v) + \det(u, w) + \det(v, w) = \pm 1. \\ &(2) \quad \det(u, v) + \det(w, u) + \det(w, v) = \pm 1. \\ &(3) \quad \det(u, v) + \det(u, w) + \det(w, v) = \pm 1. \end{aligned}$$

The combinations of the equations (1) and (2), (2) and (3), and, (3) and (1) imply respectively that $\det(u, v)$, $\det(w, u)$ and $\det(w, v)$ belong to $\{-1, 0, 1\}$. From (1), one of three determinants $\det(u, v)$, $\det(u, w)$ or $\det(v, w)$ is different from 0.
We suppose $\det(u, v) \neq 0$ (the other cases can be treated in the same way). The set $\{u, v\}$ is a base of $\mathbb{R}^2$, hence there exist two real numbers a and b such that $w = au + bv$. We have $a = \det(w, v)/\det(u, v)$ and $b = \det(w, u)/\det(v, u)$. Moreover from (1) and (2) we see that $\det(u, w) + \det(v, w)$ and $-(\det(u, w) +$

$\det(v,w)) = \det(w,u)+\det(w,v)$ belong to $\{\det(v,u)-1, \det(v,u)+1\}$ which is equal to $\{0,2\}$ or $\{-2,0\}$. Consequently $\det(u,w)+\det(v,w) = 0$. Hence $a = b$ and $w = a(u+v)$. The vector w is the column of the matrix of a morphism therefore it has non-negative coordinates which implies that a is non-negative.
Suppose $a > 0$. Then $\det(w,v)$ and $\det(u,v)$ are positive and have the same sign. Hence one of the equations (2) or (3) is equal to -3 or 3 which is not possible. Consequently $a = 0$ and $w = 0$.
Therefore for all $i \in A_3$ the matrix $M_{\pi_i \circ f} = (m_i(c,d))_{c\in A_3\setminus\{i\}, d\in A_3}$ has a column $(m_i(c,d_i))_{c\in A_3\setminus\{i\}}$ with entries equal to 0.
Two cases occurs.
1- There exists $i,j \in A_3$ $(i \neq j)$ such that $d_i = d_j$. In this case we easily check that $f(d_i) = \varepsilon$. Consequently f belongs to $\mathrm{MSE}^{\varepsilon}$.
2- The sets $\{d_0,d_1,d_2\}$ and A_3 are equal. In this case we can check that f is a permutation. □

4. Prime morphisms

4.1. **Some technical definitions.** Let A be an alphabet and $f : A \to A^*$ be a morphism. A letter a is called *f-nilpotent* if there exists an integer n such that $f^n(a) = \varepsilon$ (if it is not ambiguous we will say it is nilpotent). The set of f-nilpotent letters is denoted by $\mathcal{N}_f$. We call $\mathcal{P}'_f$ the set of letters a such that there exists an integer n satisfying $\pi_{\mathcal{N}_f}(f^n(a)) = a$ where $\pi_{\mathcal{N}_f}(b) = \varepsilon$ if $b \in \mathcal{N}_f$ and b otherwise. The set of such letters is denoted by $\mathcal{P}'_f$.
We say the letter a is *f-permuting* if there exists an integer n such that $f^n(a) \in (\mathcal{N}_f \cup \mathcal{P}'_f)^* \setminus \mathcal{N}_f{}^*$. We denote by $\mathcal{P}_f$ the set of such letters. We remark that $\mathcal{P}'_f$ is included in $\mathcal{P}_f$.
A letter a is called *f-expansive*, or expansive when the context is clear, if it is neither nilpotent nor permuting. We remark the letter $a \in A$ is f-expansive if and only if $\lim_{n\to+\infty} |f^n(a)| = +\infty$ and it is f-permuting if and only if the sequence $(|f^n(a)|; n \in \mathbb{N})$ is bounded and is never equal to 0.
The morphism f is *nilpotent* if $f(A)$ is included in $\mathcal{N}_f^*$, i.e., if there exists an integer n such that $f^n(a) = \varepsilon$ for all $a \in A$. A morphism f is called *expansive* if there exists a f-expansive letter. A morphism f is a *unit* if it is neither nilpotent nor expansive. In others words if $f(A)$ is included in $(\mathcal{N}_f \cup \mathcal{P}_f)^*$.
Let $\mathcal{M}$ be a monoid of morphisms. A morphism $f \in \mathcal{M}$ is said to be *prime* in $\mathcal{M}$ if for any morphisms g and h in $\mathcal{M}$ such that $f = g \circ h$, then g or h is a unit of $\mathcal{M}$. We say that f is of *degree* n in $\mathcal{M}$, $n \in \mathbb{N}$, if any decomposition of f into a product of prime and unit morphisms of $\mathcal{M}$ contains at least n prime morphisms and there exists at least one decomposition of f into prime and unit morphisms of $\mathcal{M}$ containing exactly n prime morphisms.
The set of prime morphisms in St is $\{f \circ g \circ h; f,h \in \{Id, E\},\ g \in \{\varphi, \widetilde{\varphi}\}\}$.

4.2. **Some conditions to be a prime morphism.** In the sequel we need the following morphisms which are extensions to the alphabet A_3 of the morphisms φ and $\widetilde{\varphi}$:

$$\begin{array}{rccc} \varphi_1 : & A_3^* & \longrightarrow & A_3^* \ , \\ & 0 & \longmapsto & 01 \\ & 1 & \longmapsto & 0 \\ & 2 & \longmapsto & \varepsilon \end{array} \qquad \begin{array}{rccc} \widetilde{\varphi_1} : & A_3^* & \longrightarrow & A_3^* \\ & 0 & \longmapsto & 10 \\ & 1 & \longmapsto & 0 \\ & 2 & \longmapsto & \varepsilon \ . \end{array}$$

In the next proposition we need the following lemma.

Lemma 6. *Let $g \in \mathrm{MSE}_2$. Then, $|g(a)| \geq 2$, $|g(a)|_0 + |g(a)|_1 \geq 1$ and $|g(01)|_a \geq 1$ for all $a \in \{0,1\}$.*

Proof. Suppose for $a \in \{0,1\}$ we have $|g(a)| = 1$, for example $g(a) = b$. Then $\pi_b \circ g(x)$ is periodic for all $x \in A_3^{\mathbb{N}}$. This contradicts the fact that g belongs to MSE_2. If $|g(a)| = 0$ we have the same conclusion. This proves the first part of the lemma.
Suppose $|g(a)|_0 + |g(a)|_1 = 0$. Then $\pi_2 \circ g(x)$ is periodic for all $x \in A_3^{\mathbb{N}}$. This proves the second inequality.
Suppose $|g(01)|_a = 0$, then a does not appear in $g(x)$. This contradicts the fact that g belongs to MSE_2. □

Proposition 7. *Let $i \in A_3$ and $f \in \mathrm{MSE}_i$. We set $A_3 = \{i,j,k\}$.*
1) If $f(j)$ is neither a prefix nor a suffix of $f(k)$ and that $f(k)$ is neither a prefix nor a suffix of $f(j)$, then f is prime in MSE_i.
2) Moreover, if f is prime in MSE_i and if we have $|f(012)|_j > |f(012)|_k \geq |f(012)|_i$, then $f(j)$ is neither a prefix nor a suffix of $f(k)$ and $f(k)$ is neither a prefix nor a suffix of $f(j)$.

Proof. We only make the proof in the case $i = 2$.
1) We suppose $f(0)$ is neither prefix nor suffix of $f(1)$ and that $f(1)$ is neither prefix nor suffix of $f(0)$. We proceed by contradiction, i.e., we suppose there exist $g, h \in \mathrm{MSE}_2$ which are not units such that $f = g \circ h$.
Let $h_1 = \pi_2 \circ h$. We have $g \circ h = g \circ h_1$. We define $\overline{h_1} : \{0,1\} \to \{0,1\}^*$ by $\overline{h_1}(i) = h_1(i)$ for all $i \in \{0,1\}$. We remark that $\overline{h_1}$ is a Sturmian morphism hence it is a product of $\varphi, \widetilde{\varphi}$ and E (Theorem 2). Therefore h_1 is a product of $\varphi_1, \widetilde{\varphi}_1$ and $\pi_2 \circ E_2$.
We consider two cases.
Suppose $h_1 \notin \{\pi_2 \circ E_2, \pi_2 \circ E_2 \circ E_2\}$. Then, for example, h_1 is equal to $h_2 \circ \varphi_1$ where h_2 is a product of $\varphi_1, \widetilde{\varphi}_1$ and $\pi_2 \circ E_2$. The other cases ($h_1 = h_2 \circ \widetilde{\varphi_1}$ or $h_1 = h_2 \circ \varphi_1 \circ \pi_2 \circ E_2$ or $h_1 = h_2 \circ \widetilde{\varphi_1} \circ \pi_2 \circ E_2$) can be treated in the same way. We have $f(0) = g \circ h_2 \circ \varphi_1(0) = g \circ h_2(0) g \circ h_2(1)$ and $f(1) = g \circ h_2(0)$. which contradicts the hypothesis.
Suppose $h_1 \in \{\pi_2 \circ E_2, \pi_2 \circ E_2 \circ E_2\}$, then $h_1(0) = 1$, $h_1(1) = 0$ and $h_1(2) = \varepsilon$ or $h_1(0) = 0$, $h_1(1) = 1$ and $h_1(2) = \varepsilon$. In both case we easily check that h is a unit of MSE_2. This ends the first part of the proof.

2) We now suppose f is a prime morphism in MSE_2 such that $|f(01)|_j > |f(01)|_k \geq |f(01)|_2$, where $A_3 = \{j,k,2\}$.
We proceed by contradiction. We suppose $f(1)$ is a prefix of $f(0)$. The other case can be treated in the same way. There exist u and v in A_3^* such that $f(0) = uv$ and $f(1) = u$. We define $g, h : A_3 \to A_3^*$ by $g(0) = u$, $g(1) = v$, $g(2) = \varepsilon$, $h(0) = 01$, $h(1) = 02$ and $h(2) = \varepsilon$. We remark that h is not a unit and $f = g \circ h$. To end the proof it suffices to show that g is not a unit of MSE_2. We start proving g belongs to MSE_2.
Let $x \in \mathrm{WSE}$. As in Example 1 we can prove that h belongs to MSE_2. Consequently $h(x)$ belongs to WSE. Moreover $f(x) = g(h(x))$ belongs to WSE. From the point (3) of Theorem 1 it comes that g belongs to MSE_2. From Lemma 6 we have $|f(01)| = |g(010)| \geq 6$. Consequently $|f(01)|_0 + |f(01)|_1 +$

$|f(01)|_2 \geq 6$. From the hypothesis it comes that $|g(010)|_j = |f(01)|_j \geq 3$. Hence $2|g(01)|_j - |g(1)|_j \geq 3$ and $|g(01)|_j \geq 2$.
Now we prove by induction that for all $n \in \mathbb{N}$ we have

$$|\pi_2 \circ g^n(01)| \geq n+2, \ |\pi_2 \circ g^n(0)| \geq 1, \text{ and } |\pi_2 \circ g^n(1)| \geq 1.$$

This is true for $n = 0$. We suppose it is true for $n \in \mathbb{N}$. From Lemma 6 we have

$$|\pi_2 \circ g^{n+1}(01)| = |\pi_2 \circ g^n(g(01))| \geq |\pi_2 \circ g^n(jjk)| = |\pi_2 \circ g^n(01)| + |\pi_2 \circ g^n(j)| \geq n+3.$$

Moreover, from Lemma 6 in $g(j)$ occurs a letter $a \in \{0, 1\}$. Consequently,

$$|\pi_2 \circ g^{n+1}(j)| = |\pi_2 \circ g^n(g(j))| \geq |\pi_2 \circ g^n(a)| \geq 1.$$

We proceed in the same way for the letter k. This concludes the induction. Therefore, it is clear g is expansive. This concludes the proof. □

5. The monoid MSE is not finitely generated

5.1. **Some preliminary results.** To prove the point (1) of Theorem 1 we need the following subset of MSE. Let MSE' be the set of morphisms $f \in \mathrm{MSE}_2$ such that for some $n \in \mathbb{N}$

$$\pi_2 \circ f \in F_n, \pi_1 \circ f \in G_n \text{ and } \pi_0 \circ f \in H_n \text{ where } F_n = \{\varphi_1, \widetilde{\varphi_1}\}^n,$$
$$G_n = E_0 \circ \{\varphi_1, \widetilde{\varphi_1}\} \circ E_2 \circ \{\varphi_1, \widetilde{\varphi_1}\}^{n-1} \text{ and } H_n = E_2 \circ E_0 \circ \{\varphi_1, \widetilde{\varphi_1}\}^{n-1}.$$

With the two following lemmata we prove that MSE' is not empty. Before we need a new definition and we make some remarks.
Let $u \in \{0, 1\}^*$, $v \in \{0, 2\}^*$ and $w \in \{1, 2\}^*$ be three words. We say that u, v and w *intercalate between them* if and only if there exists $x \in A_3^*$ such that $\pi_2(x) = u$, $\pi_1(x) = v$ and $\pi_0(x) = w$.
Let $(u_n)_{n\in\mathbb{N}}$ be the Fibonacci word : $u_{n+1} = u_n + u_{n-1}$ for all $n \geq 1$, $u_0 = 0$ and $u_1 = 1$. We can remark that for all $n \geq 1$ we have

$$M_{\varphi_1^n} = M_{\widetilde{\varphi}_1^n} = M_{\varphi_1}^n = \begin{bmatrix} u_{n+1} & u_n & 0 \\ u_n & u_{n-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Lemma 8. *Let $n \geq 2$, $f \in F_n$, $g \in G_n$ and $h \in H_n$. Then, for all $a \in \{0, 1\}$ we have $|f(a)|_0 = |g(a)|_0$, $|f(a)|_1 = |h(a)|_1$ and $|g(a)|_2 = |h(a)|_2$.*

Proof. It suffices to remark that $M_f = M_{\varphi_1^n}$,

$$M_g = \begin{bmatrix} u_{n+1} & u_n & 0 \\ 0 & 0 & 0 \\ u_{n-1} & u_{n-2} & 0 \end{bmatrix} \text{ and } M_h = \begin{bmatrix} 0 & 0 & 0 \\ u_n & u_{n-1} & 0 \\ u_{n-1} & u_{n-2} & 0 \end{bmatrix}.$$

□

Lemma 9. *Let f, g and h be three morphisms from A_3 to A_3^* such that $f(a), g(a)$ and $h(a)$ are respectively words on the alphabets $\{0, 1\}$, $\{0, 2\}$ and $\{1, 2\}$ for all $a \in A_3$. Then, $f(a)$, $g(a)$ and $h(a)$ intercalate between them for all $a \in A_3$ if and only if there exists a morphism $\psi : A_3 \to A_3^*$ such that $\pi_2 \circ \psi = f$, $\pi_1 \circ \psi = g$ and $\pi_0 \circ \psi = h$.*

Proof. For all $a \in A_3$ let $\psi(a)$ be the word obtained intercalating $f(a)$, $g(a)$ and $h(a)$. This defines a morphism $\psi : A_3 \to A_3^*$. We can check it satisfies $\pi_2 \circ \psi = f$, $\pi_1 \circ \psi = g$ and $\pi_0 \circ \psi = h$. The reciprocal is left to the reader. □

Lemma 10. *For all $n \in \mathbb{N}^*$, $\varphi_1^n(1)$ is a prefix but not a suffix of $\varphi_1^n(0)$. And for all $n \in \mathbb{N}^*\backslash\{1\}$ if $g = E_0 \circ \widetilde{\varphi_1} \circ E_2 \circ \widetilde{\varphi_1}^{n-1}$ then $g(1)$ is a suffix but not a prefix of $g(0)$.*

Proof. Let $n \in \mathbb{N}^*$. We have $\varphi_1^n(0) = \varphi_1^{n-1}(01) = \varphi_1^n(1)\varphi_1^{n-1}(1)$. Hence $\varphi_1^n(1)$ is a prefix of $\varphi_1^n(0)$.
We proceed by induction to prove that $\varphi_1^n(1)$ is not suffix of $\varphi_1^n(0)$. For $n = 1$ is it clear. Suppose it is true for $n \in \mathbb{N}^*$. We prove it is also true for $n + 1$.
We have $\varphi_1^{n+1}(0) = \varphi_1^{n+1}(1)\varphi_1^n(1)$ and $\varphi_1^{n+1}(1) = \varphi_1^n(0)$. Suppose $\varphi_1^{n+1}(1)$ is a suffix of $\varphi_1^{n+1}(0)$. Looking at $M_{\varphi_1^n}$ we remark that $|\varphi_1^n(1)| < |\varphi_1^n(0)|$, therefore $\varphi_1^n(1)$ is a suffix of $\varphi_1^n(0)$ which contradicts the hypothesis. This concludes the first part of the proof. The other part can be achieved in the same way. □

Lemma 11. *Let $n \in \mathbb{N}^*$, $f_n = \varphi_1^n$, $g_n = E_0 \circ \widetilde{\varphi_1} \circ E_2 \circ \widetilde{\varphi_1}^{n-1}$ and $h_n = E_2 \circ E_0 \circ \widetilde{\varphi_1}^{n-1}$. Then there exists a morphism $\psi_n \in \mathrm{MSE}_2$ such that $\pi_2 \circ \psi_n = f_n$, $\pi_1 \circ \psi_n = g_n$ and $\pi_0 \circ \psi_n = h_n$.*

Proof. We easily check that if ψ is a morphism such that $\pi_2 \circ \psi = f_n$, $\pi_1 \circ \psi = g_n$ and $\pi_0 \circ \psi = h_n$, for some $n \in \mathbb{N}$, then ψ belongs to MSE_2.
We proceed by induction on n to prove what remains. For $n = 1$, we have

$$\begin{array}{llll} f_1 : & A_3^* & \longrightarrow A_3^*, & g_1 : & A_3^* & \longrightarrow A_3^*, & h_1 : & A_3^* & \longrightarrow A_3^*, & \psi_1 : & A_3^* & \longrightarrow A_3^* \\ & 0 & \longmapsto 01 & & 0 & \longmapsto 0 & & 0 & \longmapsto 1 & & 0 & \longmapsto 01 \\ & 1 & \longmapsto 0 & & 1 & \longmapsto 20 & & 1 & \longmapsto 2 & & 1 & \longmapsto 20 \\ & 2 & \longmapsto \varepsilon & & 2 & \longmapsto \varepsilon & & 2 & \longmapsto \varepsilon & & 2 & \longmapsto \varepsilon. \end{array}$$

The morphism ψ_1 is such that $\pi_2 \circ \psi_1 = f_1$, $\pi_1 \circ \psi_1 = g_1$ and $\pi_0 \circ \psi_1 = h_1$, and consequently ψ_1 belongs to MSE_2. For $n = 2$, we have

$$\begin{array}{llll} f_2 : & A_3^* & \longrightarrow A_3^*, & g_2 : & A_3^* & \longrightarrow A_3^*, & h_2 : & A_3^* & \longrightarrow A_3^*, & \psi_2 : & A_3^* & \longrightarrow A_3^* \\ & 0 & \longmapsto 010 & & 0 & \longmapsto 200 & & 0 & \longmapsto 21 & & 0 & \longmapsto 2010 \\ & 1 & \longmapsto 01 & & 1 & \longmapsto 0 & & 1 & \longmapsto 1 & & 1 & \longmapsto 01 \\ & 2 & \longmapsto \varepsilon & & 2 & \longmapsto \varepsilon & & 2 & \longmapsto \varepsilon & & 2 & \longmapsto \varepsilon. \end{array}$$

The morphism ψ_2 is such that $\pi_2 \circ \psi_2 = f_2$, $\pi_1 \circ \psi_2 = g_2$ and $\pi_0 \circ \psi_2 = h_2$, and consequently ψ_2 belongs to MSE_2. Now we suppose the result is true for $n - 1$ and $n \geq 2$. We prove it is also true for $n + 1$. We have

$$\begin{array}{lllllllll} f_{n+1}(0) & = & f_{n-1}(0)f_{n-1}(1)f_{n-1}(0), & f_{n+1}(1) & = & f_n(0), & f_{n+1}(2) & = & \varepsilon, \\ g_{n+1}(0) & = & g_{n-1}(0)g_{n-1}(1)g_{n-1}(0), & g_{n+1}(1) & = & g_n(0), & g_{n+1}(2) & = & \varepsilon, \\ h_{n+1}(0) & = & h_{n-1}(0)h_{n-1}(1)h_{n-1}(0), & h_{n+1}(1) & = & h_n(0) \text{ and} & h_{n+1}(2) & = & \varepsilon. \end{array}$$

From the induction hypothesis and Lemma 9 we know $f_i(a)$, $g_i(a)$, $h_i(a)$ intercalate between them for all $a \in A_3$ and all $i \in \{n-1, n\}$. Consequently, using Lemma 9, there is a morphism $\psi : A_3 \to A_3^*$ such that $\pi_2 \circ \psi_{n+1} = f_{n+1}$, $\pi_1 \circ \psi_{n+1} = g_{n+1}$ and $\pi_0 \circ \psi_{n+1} = h_{n+1}$. □

Proposition 12. *For all $n \in \mathbb{N}^*$, the morphism ψ_n defined in Lemma 11 is prime in MSE_2.*

Proof. We keep the notations of Lemma 11. Let $n \in \mathbb{N}^*$. From Proposition 7 it suffices to prove that $\psi_n(1)$ is neither a prefix nor a suffix of $\psi_n(0)$. We proceed by contradiction: We suppose $\psi_n(1)$ is a prefix or a suffix of $\psi_n(0)$.

Suppose that $\psi_n(1)$ is a prefix of $\psi_n(0)$. Then $\pi_1 \circ \psi_n(1)$ is a prefix of $\pi_1 \circ \psi_n(0)$ and consequently $g_n(1)$ is a prefix of $g_n(0)$. This contradicts Lemma 10.
Suppose that $\psi_n(1)$ is a suffix of $\psi_n(0)$. Then $\pi_2 \circ \psi_n(1)$ is a suffix of $\pi_2 \circ \psi_n(0)$ and consequently $f_n(1)$ is a suffix of $f_n(0)$. This contradicts Lemma 10 and proves the lemma. □

Corollary 13. *The set* MSE_2 *contains infinitely many primes.*

Proof. We left as an exercise to prove that for all $n \in \mathbb{N}^*$ we have $\psi_n \neq \psi_{n+1}$, where ψ_n is defined in Lemma 11. Proposition 12 ends the proof. □

5.2. **Proof of the point (1) of Theorem 1.** We proceed by contradiction: We suppose there exists $F = \{f_1, \ldots, f_l\} \subset \mathrm{MSE}$ generating MSE, i.e., all $g \in \mathrm{MSE}$ is a composition of elements belonging to F.

Let $N = \sup_{a \in A_3, 1 \leq i \leq l} |f_i(a)|$, $(\psi_n)_{n \in \mathbb{N}}$ be the morphisms defined in Lemma 11 and $(u_n)_{n \in \mathbb{N}}$ be the Fibonacci word defined in the previous section. We remark

$$\lim_{n \to +\infty} \max_{a \in A_3} |\psi_n(a)| \geq \lim_{n \to +\infty} u_{n+1} = +\infty.$$

We fix $n \in \mathbb{N}$ such that $\max_{a \in A_3} |\psi_n(a)| > N$. By hypothesis there exist $g_1, \ldots, g_k$ in F such that $\psi_n = g_1 \circ \cdots \circ g_k$. We set $h = g_2 \circ \cdots \circ g_k$. The morphism ψ_n belongs to MSE_a for some $a \in A_3$. It implies $\psi_n(2) = \psi_n(a) = \varepsilon$ and consequently $a = 2$. There exists $b \in A_3$ such that $g_1 \in \mathrm{MSE}_b$. We remark $\psi_n = g_1 \circ h = g_1 \circ \pi_b \circ h$. Two cases occurs.

First case: For all $a \in \{0, 1\}$ we have $|\pi_b \circ h(a)| = 1$.
The morphism h being a morphism with Sturmian erasures we cannot have $\pi_b \circ h(0) = \pi_b \circ h(1)$. Consequently $\psi_n = g_1 \circ E_2$ or $\psi_n = g_1$. This implies there exists $a \in A_3$ such that $|g_1(a)| > N$ which is not possible.

Second case: There exists $a \in \{0, 1\}$ such that $|\pi_b \circ h(a)| > 1$.
We remark $\pi_b \circ h = \pi_b \circ h \circ \pi_2$
If $b = 2$ then $\pi_b \circ h_{|\{0,1\}} : \{0, 1\} \to \{0, 1\}^* \subset A_3^*$ is a Sturmian morphism different from E and $Id_{\{0,1\}}$. Hence from a remark we make in Subsection 3.2 there exist i and j in $\{0, 1\}$, $i \neq j$, such that the word $\pi_b \circ h_{|\{0,1\}}(i)$ is a prefix or a suffix of $\pi_b \circ h_{|\{0,1\}}(j)$. Hence $\psi_n(i)$ is a prefix or a suffix of $\psi_n(j)$. Proposition 7 implies ψ_n is not prime in MSE_2 which contradicts Proposition 12.
Let $b \neq 2$. We set $\{b, c\} = \{0, 1\}$. Then, $E_c \circ \pi_b \circ h_{|\{0,1\}} : \{0, 1\} \to \{0, 1\}^* \subset A_3^*$ is a Sturmian morphism different from E and $Id_{\{0,1\}}$. Hence from a remark we make in Subsection 3.2 there exist i and j in $\{0, 1\}$, $i \neq j$, such that the word $E_c \circ \pi_b \circ h_{|\{0,1\}}(i)$ is a prefix or a suffix of $E_c \circ \pi_b \circ h_{|\{0,1\}}(j)$. Hence $\psi_n(i)$ is a prefix or a suffix of $\psi_n(j)$. Proposition 7 implies ψ_n is not prime in MSE_2 which contradicts Proposition 12.
This concludes the proof.

6. Some further facts about words with Sturmian erasures

6.1. **Geometrical remarks.** We recall that a Sturmian word can be viewed as a coding of a straight half line in $\mathbb{R}^2$ with direction $(1, \alpha)$ where α is a positive irrational number, or in other terms as a trajectory of a ball in the game of billiards in the square with elastic reflexion on the boundary. We do not give the details here, we refer the reader to [Lo].

Let us extend the construction given in [Lo] to obtain what is usually called *billiard words in the unit cube* $[0,1]^3$. Let $d = (d_0, d_1, d_2) \in [0,+\infty[^3$ and $\rho = (\rho_0, \rho_1, \rho_2) \in [0,1[^3$. Let D be the half line with direction d and intercept ρ that is to say $D = \{td + \rho; t \geq 0\}$. Consider the intersections of D with the planes $x = a$, $y = a$, $z = a$, $a \in \mathbb{Z}$: We denote by $I_0, I_1, \dots$ these consecutive intersection points . We say I_n crosses the face F_i, $i \in \{0,1,2\}$, if the $i+1$-th coordinate of I_n is an integer and the $i+1$-th coordinate of $I_{n+1} - I_n$ is not equal to 0.
We set $\Omega_n = \{i \in \{0,1,2\}; I_n \text{ crosses } F_i\}$. Let $x = u_0u_1\dots$ be a word such that

$$u_n = \begin{cases} i & \text{if } \Omega_n = \{i\}, \\ ij & \text{if } \Omega_n = \{i,j\} \text{ where } i \neq j, \\ ijk & \text{if } \Omega_n = \{i,j,k\} = \{0,1,2\}. \end{cases}$$

We say x is a *billiard word in the unit cube* $[0,1]^3$ (with direction d and intercept ρ). We can also say that x is a coding of D. Of course a half line can have several codings. One of the codings of a half line is periodic if and only if $d \in \gamma\mathbb{Z}^3$ for some $\gamma \in \mathbb{R}_+$.
When one of the coordinates of the direction is equal to zero and the two others are rationally independent we can easily deduce from [Lo] (Chapter 2) that x is a Sturmian word. The reciprocal is also true: All Sturmian words can be obtained in this way (see [Lo]).
We remark that if x is a non-periodic cubic billiard word then $\pi_0(x)$ is a Sturmian word with direction $d = (0, d_1, d_2)$ and intercept $\rho = (0, \rho_1, \rho_2)$ (i.e. the orthogonal projection of D onto $\{0\} \times [0,+\infty[\times [0,+\infty[$). We have the analogous remark for $\pi_1(x)$ and $\pi_2(x)$. It is easy to conclude that a cubic billiard word is a word with Sturmian erasures if and only if it is non periodic and $d \in]0,+\infty[^3$.
There exist words with Sturmian erasures that are not cubic billiard words. For example, take the Fibonacci word x (Example 1) and the morphism ψ defined by $\psi(0) = 0012$ and $\psi(1) = 01$. It is easy to see that $y = \psi(x)$ is a word with Sturmian erasures. Let us show that if y was a cubic billiard word then the word 102 should appear in x, which is not the case.
We briefly sketch the proof. Suppose y is a cubic billiard word with direction $d = (1, \alpha, \beta)$ and intercept ρ, then the words $\pi_0(y)$ and $\pi_2(y)$ are Sturmian words with respective directions $(0, \alpha, \beta)$ and $(1, \alpha, 0)$. It can be shown that $\alpha = \theta - 1$ and $\beta = (\theta - 1)^2$ where $\theta = (\sqrt{5}+1)/2$. But with such a direction $d = (1, \theta - 1, (\theta - 1)^2)$ easy calculus show that the word 102 should appear in y.

6.2. **Balanced words.** Let us recall a characterization of Sturmian words due to Hedlund and Morse [HM2]. Let A be a finite alphabet. We say a word $x \in A^{\mathbb{N}}$ is balanced if for all factors u and v of x having the same length we have $||u|_a - |v|_a| \leq 1$ for all $a \in A$. Suppose $\text{Card}A = 2$. A word $x \in A^{\mathbb{N}}$ is Sturmian if and only if x is non eventually periodic and balanced. P. Hubert characterizes in [Hu] the words on a three letters alphabet that are balanced. This characterization shows that such words are not words with Sturmian erasures.

Definition 14. *Let A be a finite alphabet. We say $x \in A^{\mathbb{N}}$ is n-balanced if n is the least integer such that: For all words u and v appearing in x and having the same length we have $\| u |_a - | v |_a| \leq n$ for all $a \in A$.*

Clearly, Sturmian words are 1-balanced.

Proposition 15. *If $x \in A_3^{\mathbb{N}}$ is a word with Sturmian erasures then x is non eventually periodic and 2-balanced.*

Proof. It is clear x is non eventually periodic. From a previous remark we know that x is not 1-balanced.
Suppose x is n-balanced with $n \geq 3$: There exist $e \in A_3$ and two words u and v appearing in x and having the same length such that $||u|_e - |v|_e| \geq 3$.
For all $a \in A_3$ we set $n(a) = ||u|_a - |v|_a|$. Then we can set $A_3 = \{a, b, c\}$ where $n(a) \geq 3$ and $n(a) \geq n(b) \geq n(c)$. Without loss of generality we suppose $|u|_a - |v|_a = n(a)$. As $|u| = |v|$ we have $n(a) = (|v|_b - |u|_b) + (|v|_c - |u|_c)$. Consequently we necessarily have $|v|_b - |u|_b \geq 0$ and $|v|_c - |u|_c \geq 0$ because $n(a) \geq n(b) \geq n(c)$. Thus $n(b) = |v|_b - |u|_b$, $n(c) = |v|_c - |u|_c$ and $n(a) = n(b) + n(c)$. We also see that $n(b) \geq 2$ and $n(c) \geq 0$.
Suppose there exists a factor u' of the word u verifying $|\pi_c(u')| = |\pi_c(v)|$ and $|\pi_c(u')|_a - |\pi_c(v)|_a \geq 2$. Then this would say that $\pi_c(x)$ is not balanced and a fortiori not Sturmian which would end the proof.
Let us find such a u'. We have $|\pi_c(u)| \geq |\pi_c(v)| \geq |v|_b \geq 2$. Hence there exists a non-empty word u' satisfying $|\pi_c(u')| = |\pi_c(v)|$ and having an occurrence in u. Moreover

$$|\pi_c(u')|_a + |\pi_c(u')|_b = |\pi_c(u')| = |\pi_c(v)| = |v| - |v|_c$$
$$= |u| - |v|_c = |u|_a + |u|_b + |u|_c - |v|_c = |\pi_c(u)|_a + |\pi_c(u)|_b - n(c).$$

Hence

$$|\pi_c(u)|_a - n(c) = |\pi_c(u')|_a + |\pi_c(u')|_b - |\pi_c(u)|_b \leq |\pi_c(u')|_a$$

and then

$$2 \leq n(b) = n(a) - n(c) = |\pi_c(u)|_a - |\pi_c(v)|_a - n(c) \leq |\pi_c(u')|_a - |\pi_c(v)|_a,$$

which ends the proof. □

6.3. **Complexity.** Let x be a word with Sturmian erasures and f be a morphism belonging to MSE_i for some $i \in A_3$. Then $\pi_i(x)$ is a Sturmian word and $f(x) = f(\pi_i(x))$. Consequently from a result of Coven and Hedlund [CH] we deduce there exist two integers n_0 and k such that $P_x(n) = n + k$ for all $n \geq n_0$. For example, let F be the Fibonacci word and $f : \{0, 1, 2\} \to \{0, 1, 2\}^*$ be the morphism defined by $f(0) = 0102$, $f(1) = 01$ and $f(2) = \varepsilon$. It is a morphism with Sturmian erasures and $y = f(F)$ is a word with Sturmian erasures. In fact it is a cubic billiard word with direction $d = (1, \theta - 1, (\theta - 1)^2)$ and intercept $\rho = (0, \theta - 1, (\theta - 1)^2)$, where θ is the golden mean $(\sqrt{5} + 1)/2$.
This does not contradict the result in [AMST] saying that if $1, \alpha$ and β are rationally independent then the complexity of the cubic billiard word with direction $(1, \alpha, \beta)$ and intercept $\rho \in]0, 1[^3$ is $n^2 + n + 1$, because $-1 + (\alpha - 1) + (\alpha - 1)^2 = 0$.

6.4. **Conclusion.** Many generalizations of the Sturmian words were tried (more letters, applications of $\mathbb{Z}^2$ to $\{0, 1\}$, ...) but none appeared to be entirely suitable in the sense that it seems impossible to extend these properties to a more general domain astonishing varieties of the properties characterizing these words. The example which we chose for this paper, does not derogate from this rule. Nevertheless, the fact that MSE is not given by a finite generator shows a fundamental difference between the Sturmian words and any generalization with more than two letters because the definition adopted here was less “compromising” possible.

Furthermore, this definition gives a words of a complexity structurally similar to the one of the Sturmian words.

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