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Corrigendum and addendum to 'Linearly recurrent subshifts have a finite number of non-periodic factors'

Fabien Durand

Laboratoire Amiénois de Mathématiques Fondamentales et Appliquées, CNRS-UMR 6140, Université de Picardie Jules Verne, 33 rue Saint Leu, 80039 Amiens Cedex 01, France
(e-mail: fabien.durand@u-picardie.fr)

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Abstract. We prove that a subshift (X, T) is linearly recurrent if and only if it is a primitive and proper S-adic subshift. This corrects Proposition 6 in F. Durand (*Ergod. Th. & Dynam. Sys.* **20** (2000), 1061–1078).

1. *Introduction and definitions*

In this paper we freely use the definitions and the notations of [**Du**]. Proposition 6 in [**Du**] is false: There exist primitive S-adic subshifts that are not linearly recurrent (LR). We will give an example. Nevertheless the other part of this proposition is true: If a subshift is LR then it is primitive S-adic.

The author apologizes for the mistake. We correct this proposition with the next one, but before we have to give and to recall some definitions. Let A and B be two finite alphabets.

Let x be an element of $A^{\mathbb{N}}$ or $A^{\mathbb{Z}}$. We call *occurrence* of $u \in A^*$ in x every integer i such that $x_{[i,i+|u|-1]} = x_i x_{i+1} \ldots x_{i+|u|-1} = u$. A *return word* to $u \in A^*$ in x is a word w such that wu has an occurrence in x, u is a prefix of wu and u has exactly 2 occurrences in wu. We say that x is *linearly recurrent* (LR) (with constant $K \in \mathbb{N}$) if it is uniformly recurrent and if for all u having an occurrence in x and all return words, w, to u in x we have $|w| \leq K|u|$.

Let T be the shift transformation defined on $A^{\mathbb{Z}}$. We say the subshift (X, T) is generated by x if X is the set of the sequences z such that $z_{[i,j]}$ has an occurrence in x for all intervals $[i, j] \subset \mathbb{Z}$. The subshift (X, T) is linearly recurrent if it is minimal and contains a LR sequence. We remark that if $x \in A^{\mathbb{Z}}$ is linearly recurrent then x and $x_{[0,+\infty)}$ generate the same LR subshift.

Let a be a letter of A, S a finite set of morphisms σ from $A(\sigma) \subset A$ to A^* and $(\sigma_n : A_{n+1} \to A_n^*; n \in \mathbb{N})$ be a sequence of $S^{\mathbb{N}}$ such that $(\sigma_0\sigma_1 \cdots \sigma_n(aa\cdots); n \in \mathbb{N})$ converges in $A^{\mathbb{N}}$ to x. We will say that x is a *S-adic sequence on A* (generated by $(\sigma_i; i \in \mathbb{N}) \in S^{\mathbb{N}}$ and a). If there exists an integer s_0 such that for all non-negative integers r and all $b \in A_r$ and $c \in A_{r+s_0+1}$, the letter b has an occurrence in $\sigma_{r+1}\sigma_{r+2}\cdots\sigma_{r+s_0}(c)$, then we say that x is a *primitive* S-adic sequence (with constant s_0).

Let $\sigma : A \to B^*$ be a morphism. We say σ is proper if there exist two letters $r, l \in B$ such that for all $a \in A$ the first letter of $\sigma(a)$ is l and the last letter of $\sigma(a)$ is r. We say the sequence $x \in A^{\mathbb{N}}$ is a *proper S-adic sequence* if it is a S-adic sequence and the morphisms in S are proper. The subshift generated by a proper S-adic sequence is called *proper S-adic subshift*.

PROPOSITION 1.1. *The subshift (X,T) is LR if and only if it is a primitive and proper S-adic subshift.*

2. *Counterexample to the Proposition 6 of [**Du**]*

In this section we give a counterexample to Proposition 6 in [**Du**], i.e., a primitive S-adic subshifts that is not LR.

Let $A = \{a, b, c\}$ be an alphabet, and, $\sigma : A \to A^*$ and $\tau : A \to A^*$ be two morphisms defined by

$$\begin{aligned} \sigma(a) &= acb\ , & \tau(a) &= abc\ , \\ \sigma(b) &= bab\ , & \tau(b) &= acb\ , \\ \sigma(c) &= cbc\ , & \tau(c) &= aac\ . \end{aligned}$$

We call $\mathcal{M}$ the set of all finite composition of elements of $S = \{\sigma, \tau\}$. For each element ρ of $\mathcal{M}$ there exists a unique $n \in \mathbb{N}$ such that $|\rho(a)| = |\rho(b)| = |\rho(c)| = 3^n$. We set $|\rho| = 3^n$. We first give a lemma which proof is left to the reader.

LEMMA 2.1. *Let $z \in A^{\mathbb{N}}$ and $n \in \mathbb{N}$. The difference between two successive occurrences of the word ca in $\sigma^n\tau(z)$ is greater than 3^{n+1}.*

Let $x \in A^{\mathbb{N}}$ be the primitive S-adic sequence defined by

$$x = \lim_{n \to +\infty} \sigma\tau\sigma^2\tau \cdots \sigma^n\tau(aaa\dots).$$

We show x is not LR. Let n be an integer, we set $\rho_n = \sigma\tau\sigma^2\tau\cdots\sigma^n\tau$ and $y = \lim_{l\to+\infty} \sigma^{n+2}\tau\sigma^{n+3}\tau\cdots\sigma^{n+l}\tau(aaa\dots)$. We have $x = \rho_n\sigma^{n+1}\tau(y)$.

Since $\sigma(a)$ is a prefix of y, $\tau\sigma(a) = abcaacacb$, $\sigma^{n+1}(a) = au$ and $\sigma^{n+1}(c) = vc$, for some $u, v \in A^*$, then ca appears in $\sigma^{n+1}\tau(y)$. Let w be a return word to ca in $\sigma^{n+1}\tau(y)$. Hence wca ($= caw'ca$ for some w') appears in x. The word $\rho_n(ca)$ appears exactly twice in $\rho_n(caw'ca)$ (the proof of this fact is left to the reader) hence $\rho_n(w)$ is a return word to $\rho_n(ca)$ in x.

Moreover from Lemma 2.1 we have $|w| \geq 3^{n+2}$. It implies x is not LR because

$$\frac{|\rho_n(w)|}{|\rho_n(ca)|} = \frac{|w||\rho_n|}{2|\rho_n|} \geq \frac{3^{n+2}}{2}.$$

Let $z \in B^{\mathbb{N}}$ where B is a finite alphabet. We denote by $L(z)$ the set of all words having an occurrence in z. For all $n \in \mathbb{N}$ we define $p_z(n)$ to be the number of distinct words of length n in $L(z)$. In [**Du**] it is proved that if z is LR with constant K then $p_z(n) \leq Kn$ for all $n \in \mathbb{N}$. Even if x is not LR, there exists a constant C such that $p_n(x) \leq Cn$ for all $n \in \mathbb{N}$. This is a consequence of the following proposition.

PROPOSITION 2.1. *Let A be a finite alphabet, a be a letter of A and $(\sigma_n : A_{n+1} \to A_n^+ ; n \in \mathbb{N})$ be a sequence of morphisms such that $A_n \subset A$ for all $n \in \mathbb{N}$, $a \in \cap_{n\in\mathbb{N}} A_n$ and*

$$y = \lim_{n\to+\infty} \sigma_0\sigma_1 \cdots \sigma_n(aaa\dots).$$

Suppose moreover $\inf_{c\in A_{n+1}} |\sigma_0\sigma_1\cdots\sigma_n(c)|$ tends to $+\infty$ and there exists a constant D such that

$$|\sigma_0\sigma_1\cdots\sigma_{n+1}(b)| \leq D|\sigma_0\sigma_1\cdots\sigma_n(c)|$$

for all $b \in A_{n+2}$ and $c \in A_{n+1}$, and all $n \in \mathbb{N}$. Then $p_y(n) \leq D(\mathrm{Card}(A))^2 n$.

Proof. This proof follows the lines of the proof of Proposition V.19 in [**Qu**].

Let $n \geq 1$. The sequence $(\inf_{c\in A_{k+1}} |\sigma_0\sigma_1\cdots\sigma_k(c)|)_{k\in\mathbb{N}}$ is non-decreasing and tends to $+\infty$, hence there exists $p \in \mathbb{N}$ such that

$$\inf_{c\in A_p} |\sigma_0\sigma_1\cdots\sigma_{p-1}(c)| \leq n \leq \inf_{c\in A_{p+1}} |\sigma_0\sigma_1\cdots\sigma_p(c)|.$$

From that, every word $w \in L(y)$ of length n has an occurrence i in some $\sigma_0\sigma_1\cdots\sigma_p(bc)$, where b and c are two letters of A, with $i \leq |\sigma_0\sigma_1\cdots\sigma_p(b)| - 1$. Consequently

$$\begin{aligned} p_y(n) &\leq (\mathrm{Card}(A))^2 \sup_{c\in A_{p+1}} |\sigma_0\sigma_1\cdots\sigma_p(c)| \\ &\leq (\mathrm{Card}(A))^2 D \inf_{c\in A_p} |\sigma_0\sigma_1\cdots\sigma_{p-1}(c)| \leq D(\mathrm{Card}(A))^2 n. \end{aligned}$$

This ends the proof. □

COROLLARY 2.1. *Let A be a finite alphabet, a be a letter of A, l be a positive integer and $(\sigma_n : A \to A^*; n \in \mathbb{N})$ be a sequence of morphisms of constant length l and*

$$y = \lim_{n\to+\infty} \sigma_0\sigma_1\cdots\sigma_n(aaa\dots).$$

Then $p_y(n) \leq l(\mathrm{Card}(A))^2 n$.

3. *A sufficient condition for a primitive S-adic sequence to be LR*

This sufficient condition is given in the following lemma and will be used in the sequel.

LEMMA 3.1. *Let S be a finite set of morphisms. Let x be a primitive S-adic sequence generated by $(\sigma_i : A_{i+1} \longrightarrow A_i^*; i \in \mathbb{N})$ and a (with constant s_0). For all $n \in \mathbb{N}$ suppose $\lim_{l\to+\infty} \sigma_n\sigma_{n+1}\cdots\sigma_l(aaa\dots)$ exists and call it $x^{(n)}$. Let D_n be* *the largest difference between two consecutive occurrences of a word of length 2 in* $x^{(n)}$.

If $(D_n; n \in \mathbb{N})$ *is bounded then* x *is LR.*

Proof. Let $x = \lim_{n \to +\infty} \sigma_0 \sigma_1 \cdots \sigma_n(aaa\dots)$. It follows from Lemma 7 of [**Du**] that x is uniformly recurrent. We set $S_k = \sigma_0 \cdots \sigma_k$ for all $k \in \mathbb{N}$. Let u be a non-empty word of $L(x)$ such that $|u| \geq \max\{|S_{s_0}(b)|; b \in A_{s_0+1}\}$, and v be a return word to u. We denote by k_0 the smallest positive integer k such that $|u| < \min\{|S_k(b)|; b \in A_{k+1}\}$; we remark that $k_0 \geq s_0 + 1$. There exists a word of length 2, bc, of $L(x^{(k_0+1)})$ such that u has an occurrence in $S_{k_0}(bc)$. The largest difference between two successive occurrences of bc in $x^{(k_0+1)}$ is bounded by $D = \max_{n \in \mathbb{N}} D_n$ (which does not depend on k_0), hence we have

$$\begin{aligned} |v| &\leq D \max\{|S_{k_0}(d)|; d \in A_{k_0+1}\} \leq DK \min\{|S_{k_0}(d)|; d \in A_{k_0+1}\} \\ &\leq DK \max\{|S_{k_0-1}(d)|; d \in A_{k_0}\} \min\{|\sigma_{k_0}(d)|; d \in A_{k_0+1}\} \\ &\leq DK^2 \min\{|S_{k_0-1}(d)|; d \in A_{k_0}\} \min\{|\sigma_{k_0}(d)|; d \in A_{k_0+1}\} \\ &\leq DK^2 \min\{|\sigma_{k_0}(d)|; d \in A_{k_0+1}\} |u|, \end{aligned}$$

where K is the constant given by Lemma 8 of [**Du**], i.e., K is such that for all integers r, s with $s - r \geq s_0 + 1$ and all b, c of A_{s+1} we have $|\sigma_r \cdots \sigma_s(b)| \leq K|\sigma_r \cdots \sigma_s(c)|$. We set $M = DK^2 \max\{|\sigma_i(d)|; i \in \mathbb{N}, d \in A_{i+1}\}$. For all u of $L(x)$ greater than $\max\{|S_{s_0}(b)|; b \in A_{s_0+1}\}$ and all v in $\mathcal{R}_u$ we have $|v| \leq M|u|$. Hence x is LR. □

4. *A necessary and sufficient condition to be LR*

In the original proof of Proposition 6 in [**Du**] we use the notion of return word. In the proof of Proposition 1.1 we will do the same but we will use an extension of this notion which was defined in [**DHS**]. We will take a sequence x belonging to X and, using these "new" return words, we will show that $x^+ = x_{[0,+\infty)}$ is a primitive and proper S-adic sequence. The subshift (X, T) being minimal we see that it is generated by x^+ and, consequently, (X, T) is a primitive and proper S-adic subshift.

Let A be a finite alphabet, $x \in A^{\mathbb{Z}}$, and, u and v two words of A^*. We say $w \in A^*$ is a *return word to* $u.v$ in x if there exist two consecutive occurrences j, k of uv in x such that $w = x_{[j+|u|, k+|u|)}$. It is immediate to check that a word $w \in A^+$ is a return word to $u.v$ in x if and only if:

1) uwv has an occurrence in x, and
2) v is a prefix of wv and u is a suffix of uw, and
3) the word uwv contains exactly two occurrences of the word uv.

We denote by $\mathcal{R}_{x,u.v}$ the set of return words to $u.v$ in x. If u is the empty word ϵ then the return words to $u.v$ are the return words to v defined in [**Du**] and we set $\mathcal{R}_{x,u.v} = \mathcal{R}_{x,v}$. The return words to $u.v$ are different from the return words to $\epsilon.uv$ but we have $\#\mathcal{R}_{x,u.v} = \#\mathcal{R}_{x,\epsilon.uv} = \#\mathcal{R}_{x,uv}$.

We suppose now that x is a uniformly recurrent sequence. It is easy to see that for all $u, v \in L(x)$ the set $\mathcal{R}_{x,u.v}$ is finite. It will be convenient to label the return words. We enumerate the elements w of $\mathcal{R}_{x,u.v}$ in the order of the first appearance of uwv in $x_{[-|u|,+\infty)}$. This defines a bijective map $\Theta_{x,u.v} : R_{x,u.v} \to \mathcal{R}_{x,u.v} \subset A^+$ where $R_{x,u.v} = \{1, \dots, \#\mathcal{R}_{x,u.v}\}$: $u\Theta_{x,u.v}(k)v$ is the k-th word of the type uwv, $w \in \mathcal{R}_{x,u.v}$, appearing in $x_{[-|u|,+\infty)}$.

We consider $R_{x,u.v}$ as an alphabet. The map $\Theta_{x,u.v}$ defines a morphism from $R_{x,u.v}$ to A^* and the set $\Theta_{x,u.v}(R^*_{x,u.v})$ consists of all concatenations of return words to $u.v$.

The following proposition is important in the proof of Proposition 1.1.

Proposition 4.1 ([**DHS**]) *The map $\Theta_{x,u.v} : R^+_{x,u.v} \to A^+$ is one to one.*

Proof of Proposition 1.1. Let S be a finite set of proper morphisms and suppose (X, T) is a primitive S-adic subshift generated by $(\sigma_i : A_{i+1} \to A_i^*; i \in \mathbb{N}) \in S^{\mathbb{N}}$ and a (with constant s_0). Let

$$x = \lim_{n \to +\infty} \sigma_0 \sigma_1 \cdots \sigma_n(aaa...).$$

We prove that x is LR and consequently that the subshift it generates is LR.

As the morphisms are proper the limit $\lim_{l \to +\infty} \sigma_n \sigma_{n+1} \cdots \sigma_l(aaa \dots)$ exists for all $n \in \mathbb{N}$. We call it $x^{(n)}$ and we define D_n as in Lemma 3.1.

The composition of two proper morphism is again proper. Consequently, from the primitivity, we can suppose that $s_0 = 0$ and that for all $n \in \mathbb{N}$, all $a \in A_{n+1}$ and all $b \in A_n$ the letter b appears in $\sigma_n(a)$.

Let $n \in \mathbb{N}$ and set $\tau = \sigma_n \sigma_{n+1}$. It is a proper substitution. Let l and r be respectively the first and the last letter of the images of σ_n. Let y be a one-sided sequence of $A^{\mathbb{N}}_{n+2}$ and $z = \sigma_n \sigma_{n+1}(y)$. The words of length 2 having an occurrence in z are exactly the words of length 2 having an occurrence in some $\sigma_n(e)$, $e \in A_{n+1}$, and the word rl. On the other hand the letters of $\sigma_{n+1}(y)$ appear with gaps bounded by $K_n = 2 \max\{|\sigma_{n+1}(e)|; e \in A_{n+2}\}$. Consequently the words of length 2 of z appear with gaps bounded by $K_n \max\{|\sigma_n(e)|; e \in A_{n+1}\}$ and, a fortiori, $D_n \leq K_n \max\{|\sigma_n(e)|; e \in A_{n+1}\}$ for all $n \in \mathbb{N}$. Moreover S being finite the sequence $(D_n; n \in \mathbb{N})$ is bounded. Lemma 3.1 implies (X, T) is LR.

We suppose now that (X, T) is LR. The periodic case is trivial hence we suppose that (X, T) is not periodic. From Proposition 5 in [**Du**] there exists $K \geq 2$ such that

$$(\forall u \in L(X))(\forall w \in \mathcal{R}_u) \left(|u|/K \leq |w| \leq K|u| \right),$$

We set $\alpha = K^2(K + 1)$. Let $x = (x_n; n \in \mathbb{Z})$ be an element of X. It suffices to prove that $x_{[0,+\infty)}$ is a primitive and proper S-adic sequence.

For each non-negative integer n we set $u_n = x_{-\alpha^n} \cdots x_{-2} x_{-1}$, $v_n = x_0 x_1 \cdots x_{\alpha^n - 1}$, $\mathcal{R}_n = \mathcal{R}_{x,u_n.v_n}$, $R_n = R_{x,u_n.v_n}$ and $\Theta_n = \Theta_{x,u_n.v_n}$. Let n be a positive integer and w be a return word to $u_n.v_n$. The word w is a concatenation of return words to $u_{n-1}.v_{n-1}$. The map Θ_{n-1} being one to one (Proposition 4.1), this induces a map λ_n from R_n to R^*_{n-1} defined by $\Theta_{n-1} \lambda_n = \Theta_n$. We set $\lambda_0 = \Theta_0$.

For each letter b of R_n we have $|\Theta_{n-1}\lambda_n(b)| \leq K|u_n v_n| = 2K\alpha^n$. Moreover each element of $\mathcal{R}_{n-1}$ is greater than $(2\alpha^{n-1})/K$ hence

$$|\lambda_n(b)| \leq \frac{K^2\alpha^n}{\alpha^{n-1}} = \alpha K^2.$$

By Proposition 5 of [**Du**] we have $\# R_n = \#\mathcal{R}_{x,u_n v_n} \leq K(K+1)^2$, consequently the set $M = \{\lambda_n; n \in \mathbb{N}\}$ is finite. The definition of R_n implies that $\Theta_n(1)x_0x_1\cdots x_{\alpha^n-1}$ is a prefix of $x_{[0,+\infty)}$ for all $n \in \mathbb{N}$ and $\lambda_0\lambda_1\cdots\lambda_n(1) = \Theta_n(1)$. Proposition 4 of [**Du**] implies that the length of $\Theta_n(1)$ tends to infinity with n and

$$x = \lim_{n\to+\infty} \lambda_0\lambda_1\cdots\lambda_n(11\cdots).$$

Let n be an integer greater than 1. Each word of length $2K\alpha^n$ has an occurrence in each word of length $2K(K+1)\alpha^n$ (Proposition 5 of [**Du**]). Hence each element of $\mathcal{R}_n$ has an occurrence in each word of length $2K(K+1)\alpha^n$. Let w be an element of $\mathcal{R}_{n+1}$, we have $|w| \geq 2\alpha^{n+1}/K = 2K(K+1)\alpha^n$. Therefore each element of $\mathcal{R}_n$ has an occurrence in each element of $\mathcal{R}_{n+1}$. It means that if b belongs to R_{n+1} then each letter of R_n has an occurrence in $\lambda_{n+1}(b)$. Hence x is a primitive S-adic sequence.

It remains to show each λ_n is proper. Let w be a return word of $\mathcal{R}_n$. The word wv_{n-1} is a concatenation of return words to v_{n-1}. Let $p \in \mathcal{R}_{n-1}$ be such that pv_{n-1} is a prefix of wv_{n-1} and consequently of wv_n. We know v_n is also a prefix of wv_n and

$$|v_n| = \alpha^n = \alpha^{n-1}(K^3+K^2) \geq (2K+1)\alpha^{n-1} \geq |p| + |v_{n-1}|.$$

Consequently pv_{n-1} is a prefix of v_n. Let $l \in R_{n-1}$ be such that $\Theta_{n-1}(l) = p$. Then l is the first letter of $\lambda_n(c)$ for all $c \in R_n$.

In the same way there exists $s \in \mathcal{R}_{n-1}$ and $r \in R_{n-1}$ such that $\Theta_{n-1}(r) = s$ and $u_{n-1}s$ is a suffix of u_n. Hence r is the last letter of $\lambda_n(c)$ for all $c \in R_n$ and λ_n is proper. □

5. *LR sturmian sequences*

We give a correct proof of the next proposition (which is stated in [**Du**]) because the original proof used Proposition 6 in [**Du**].

Proposition 5.1. *A sturmian subshift (Ω_α, T) is LR if and only if the coefficients of the continued fraction of α are bounded.*

Proof. Let $0 < \alpha < 1$ be an irrational real number and $[0 : i_1+1, i_2, i_3, \dots]$ be its continued fraction. From Proposition 9 in [**Du**] we know that $\Omega_\alpha = \Omega(x)$ where

$$x = \lim_{k\to+\infty} \tau^{i_1}\sigma^{i_2}\tau^{i_3}\sigma^{i_4}\cdots\tau^{i_{2k-1}}\sigma^{i_{2k}}(00\cdots),$$

$\tau(0) = 0$, $\tau(1) = 10$, $\sigma(0) = 01$ and $\sigma(1) = 1$. We just have to prove that if the coefficients of the continued fraction of α are bounded then the sequence x is LR. The other part of the proof is in [**Du**] and do not use Proposition 6 in [**Du**].

Let i, j and k be in $\mathbb{N}^*$. We have

$$\tau^i \circ \sigma^j \circ \tau^k(0) = 0(10^i)^j \text{ and } \tau^i \circ \sigma^j \circ \tau^k(1) = 10^i(0(10^i)^j)^k.$$

Consequently if x belongs to $\{0,1\}^{\mathbb{N}}$ then the set of the words of length 2 having an occurrence in $y = \tau^i \circ \sigma^j \circ \tau^k(x)$ is $\{00, 01, 10\}$. Moreover the difference of two successive occurrences of 01 (resp. 10) in y is less than $i+2$ (resp. $i+2$), and, the difference between two successive occurrences of 00 in y is less than $2j+3$ if $i=1$ and less than 3 if $i \geq 2$. Consequently the difference between two successive occurrences of a word of length 2 in y is bounded by $2\max\{i,j,k\}+3$.

The same bound can be found for $\sigma^i \circ \tau^j \circ \sigma^k$.

For all $n \in \mathbb{N}$, let $x^{(n)}$ and D_n be defined as in Lemma 3.1. Hence, if the sequence $(i_n; n \in \mathbb{N})$ is bounded by K, then D_n is bounded by $2K+3$ for all $n \in \mathbb{N}$. Lemma 3.1 ends the proof. □

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