A NOTE ON LIMIT LAWS FOR MINIMAL CANTOR SYSTEMS WITH INFINITE PERIODIC SPECTRUM

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Abstract. Recently in [L] the author proves that any distribution function can be obtained as a limit law of return time for any ergodic aperiodic system. In this note we provide an alternative construction, based on Bratteli-Vershik representations of systems, which works for any minimal Cantor system having an infinite periodic spectrum. In particular, it provides a very simple construction for odometers.

1. Preliminaries

The study of limit laws of entrance and return times for different dynamical systems has been undertaken in several works in the last decade, see for example [C], [HSV] and the references therein. In most of these works exponential limit laws or piecewise linear functions are obtained. Recently Y. Lacroix in [L] proved that for any ergodic aperiodic system, every distribution function is a limit law of return times. The purpose of this paper is to show the same kind of results but based on the representation of Cantor minimal systems by Bratteli-Vershik systems following the same lines as [DM]. We would like to understand how "natural" limit laws depend on the particular representation of the system. For minimal Cantor systems with an infinite periodic spectrum we obtain arbitrary limit laws of return time. Our proof is based on a simple and explicit representation of the system from which we can see limit laws. In some cases it is very simple and explicit like odometers.

We recall (X, T) is a Cantor dynamical system if X is a Cantor set, that is, it has a countable basis of open and closed sets (clopen sets) and no isolated points, and $T : X \to X$ is an homeomorphism.

The representation of Cantor minimal systems by means of ordered Bratteli diagrams has been introduced in [HPS]. It is a very nice way to describe sequences of nested Kakutani-Rokhlin partitions which provides a very efficient study of return time in terms of matrices and combinatorics of graphs. A Bratteli diagram is an infinite graph (V, E) which consists of a vertex set V and an edge set E, both of which are divided into levels $V = V_0 \cup V_1 \cup \cdots$, $E = E_1 \cup E_2 \cup \cdots$ and all levels are pairwise disjoint. The set V_0 is a singleton $\{v_0\}$, and for $k \geq 1$, E_k is the set of edges joining vertices in V_{k-1} to vertices in V_k. It is also required that every vertex in V_k is the "end-point" of some edge in E_k for $k \geq 1$, and an "initial-point" of some edge in E_{k+1} for $k \geq 0$. By level k we will mean the subgraph consisting of the vertices in $V_k \cup V_{k+1}$ and the edges E_{k+1} between these vertices. For every $e \in E_k$, $\mathbf{s}(e) \in V_{k-1}$ and $\mathbf{t}(e) \in V_k$ are the starting and terminal vertices of e respectively.

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An ordered Bratteli diagram $B = (V, E, \preceq)$ is a Bratteli diagram (V, E) together with a partial ordering $\preceq$ on E. Edges e and e' are comparable if and only if they have the same end-point. We call $succ(e)$ the successor of e with respect to this partial order when e is not a maximal edge.

Let $k < l$ in $\mathbb{N} \setminus \{0\}$ and let $E(k,l)$ be the set of all paths in the graph joining vertices of V_{k-1} with vertices of V_l. The partial ordering of E induces another in $E(k,l)$ given by $(e_k, \ldots, e_l) \prec (f_k, \ldots, f_l)$ if and only if there is $k \leq i \leq l$ such that $e_j = f_j$ for $i < j \leq l$ and $e_i \prec f_i$ (this is the lexicographical order).

Given a strictly increasing sequence of integers $(m_n)_{n\geq 0}$ with $m_0 = 0$ we define the contraction of $B = (V, E, \preceq)$ (with respect to $(m_n)_{n\geq 0}$) as

$$\left((V_{m_n})_{n\geq 0}, (E(m_n + 1, m_{n+1}))_{n\geq 0}, \preceq \right),$$

where $\preceq$ is the order induced in each set of edges $E(m_n + 1, m_{n+1})$. The inverse operation of contracting is microscoping (see [HPS]).

Given an ordered Bratteli diagram $B = (V, E, \preceq)$ we define X_B as the set of infinite paths $(e_1, e_2, \cdots)$ starting in v_0 such that for all $i \geq 1$ the end-point of $e_i \in E_i$ is the initial-point of $e_{i+1} \in E_{i+1}$. We topologize X_B by postulating a basis of open sets, namely the family of cylinder sets $[e_1, e_2, \ldots, e_k] = \{(f_1, f_2, \ldots) \in X_B \ : \ f_i = e_i, \text{ for } 1 \leq i \leq k \}$. Each $[e_1, e_2, \ldots, e_k]$ is also closed, as is easily seen, and so we observe that X_B is a compact, totally disconnected metrizable space.

When there is a unique $x = (x_1, x_2, \ldots) \in X_B$ such that x_i is maximal for any $i \geq 1$ and a unique $y = (y_1, y_2, \ldots) \in X_B$ such that y_i is minimal for any $i \geq 1$, we say that $B = (V, E, \preceq)$ is a properly ordered Bratteli diagram. Call these particular points $x_{\max}$ and $x_{\min}$ respectively. In this case we can define a dynamic V_B over X_B called Vershik map. The map V_B is defined as follows: let $(e_1, e_2, \ldots) \in X_B \setminus \{x_{\max}\}$ and let $k \geq 1$ be the smallest integer so that e_k is not a maximal edge. Let f_k be the successor of e_k and $(f_1, \ldots, f_{k-1})$ be the unique minimal path in $E_{1,k-1}$ connecting v_0 with the initial point of f_k. We set $V_B(x) = (f_1, \ldots, f_{k-1}, f_k, e_{k+1}, \ldots)$ and $V_B(x_{\max}) = x_{\min}$. The dynamical system (X_B, V_B) is called Bratteli-Vershik system generated by $B = (V, E, \preceq)$. The dynamical system induced by any contraction of B is topologically conjugate to (X_B, V_B). In [HPS] it is proved that any minimal Cantor system (X, T) is topologically conjugate to a Bratteli-Vershik system (X_B, V_B). We say that (X_B, V_B) is a Bratteli-Vershik representation of (X, T).

Let $(p_k : k \in \mathbb{N})$ be a sequence of positive integers. The inverse limit of the sequence of groups $(\mathbb{Z}/p_1 \cdots p_k\mathbb{Z} : k \in \mathbb{N})$ endowed with the addition of 1 is called odometer with base $(p_k : k \in \mathbb{N})$. These systems are minimal and uniquely ergodic. We say it is of constant base if the sequence $(p_k : k \in \mathbb{N})$ is ultimately constant. The classical representation by Bratteli–Vershik systems is given in left part of figure 2.

In this paper we will consider the following conditions over the order of a Bratteli–Vershik system which will be appropriate for our purpose,

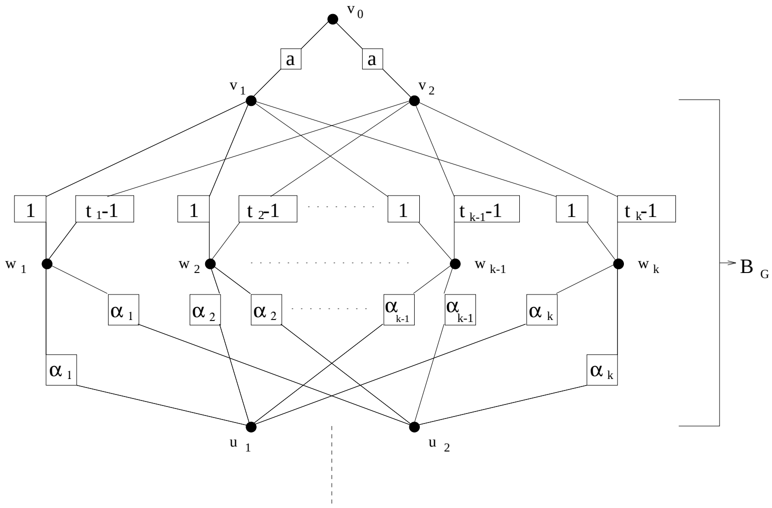

FIGURE 1. $a \geq 1$, $t_1, ..., t_k \geq 2$, $\alpha_1, ..., \alpha_k \geq 1$.

(H1) for every vertex $i \in V_1$ there is a unique edge from v_0 to i;

(H2) $\forall n \in \mathbb{N} \setminus \{0\}, \forall i \in V_n = \{1, ..., |V_n|\}, e = min\{f \in E_n : \mathbf{t}(f) = i\} \Rightarrow \mathbf{s}(e) = 1$.

Given a Bratteli–Vershik system, there is a always a sequence of contactions such that the resulting (conjugate system) verifies these conditions. So we will always assume that these conditions are verified.

For a topological Cantor dynamical system (X, T) the periodic spectrum of T is the set of integers p for which there is a clopen set $A \subseteq X$ such that $X = \cup_{i=0}^{p-1} T^i A$ and the sets $T^i A, i \in \{0, ..., p-1\}$ are pairwise disjoints.

2. RESULTS

Given a Cantor minimal system (X, T), μ a T-invariant probability measure and U a clopen (closed and open) subset of X, we define for $t \geq 0$, $G_U(t) = \mu(\{x \in U : \mu(U)\tau_U(x) \leq t\})/\mu(U)$, where $\tau_U(x) = \inf\{t > 0 : T^t(x) \in U\}$. Since the system is minimal and U is clopen there exists a finite number of return times to U, $\hat{t}_1, ..., \hat{t}_k$, and

$$G_U(t) = \sum_{i=1}^{k} \frac{\mu(U \cap \{\tau_U = \hat{t}_i\})}{\mu(U)} 1_{\{t \geq \mu(U)\hat{t}_i\}}.$$

In the figure, the numbers inside squares mean the number of paths joining the corresponding vertices. The following lemma shows our fundamental construction.

Lemma 2.1. *Let (X_B, V_B) be a Bratteli-Vershik system where the diagram is given by figure 1 and the order satisfies conditions (H1-H2), and let μ be a V_B-invariant probability measure on X_B. Consider $G : [0, +\infty) \to [0, 1]$ the function defined by $G(t) = \sum_{i=1}^{k} \beta_i 1_{\{t \geq d_i\}}$, where*

(1) $\beta_1, ..., \beta_k \in \mathcal{Q} \cap [0, 1]$ *and* $\sum_{i=1}^{k} \beta_i = 1$,

(2) $d_1 < ... < d_k$ *in* $\mathcal{Q}^+$ *and* $\sum_{i=1}^{k} d_i \beta_i = 1$,

(3) *for* $i \in \{1, ..., k-1\}$

$$\frac{t_i}{t_{i+1}} = \frac{d_i}{d_{i+1}}, \ \frac{\alpha_i}{\alpha_{i+1}} = \frac{\beta_i}{\beta_{i+1}}. \tag{1}$$

If U is any cylinder set from v_0 to v_1, or from v_0 to v_2, then $G_U = G$.

Proof. In this proof we use the notation in figure 1. Let U be a cylinder set from v_0 to v_1, the other case is analogous. Notice the return times to U are $\hat{t}_i = a\ t_i$ for $i \in \{1, ..., k\}$. Since $G_U(t) = (1/\mu(U)) \sum_{i=1}^{k} \mu(U \cap \{\tau_U = \hat{t}_i\})\ 1_{\{t \geq \mu(U)\hat{t}_i\}}$, then it is enough to show $d_i = \mu(U) \cdot \hat{t}_i$ and $\beta_i = \mu(U \cap \{\tau_U = \hat{t}_i\})/\mu(U)$ for $i \in \{1, ..., k\}$.

We recall that, being μ V_B-invariant, the measure of any cylinder set generated by a path starting in v_0 only depends on the last vertex of such a path. The number of paths from v_1 to u_1 and from v_2 to u_2 coincides, then

$$\nu_1 + \nu_2 = \frac{1}{a \cdot \sum_{j=1}^{k} \alpha_j t_j} \text{ and } \mu(U) = (\sum_{j=1}^{k} \alpha_j)(\nu_1 + \nu_2) = \frac{\sum_{j=1}^{k} \alpha_j}{a \sum_{j=1}^{k} \alpha_j t_j},$$

where ν_j is the measure of any path from v_0 to u_j, $j = 1, 2$.

The points $x \in U$ such that $\tau_U(x) = \hat{t}_i$ are those passing by vertex w_i and u_1 or u_2, therefore,

$$\frac{\mu(U \cap \{\tau_U = \hat{t}_i\})}{\mu(U)} = \frac{(\nu_1 + \nu_2)\alpha_i}{\mu(U)} = \frac{\alpha_i}{\sum_{j=1}^{k} \alpha_j}.$$

On the other hand

$$\sum_{j=1}^{k} \alpha_j = \sum_{j=1}^{k} \frac{\beta_j}{\beta_i} \alpha_i = \frac{\alpha_i}{\beta_i} \text{ and } \sum_{j=1}^{k} \alpha_j t_j = \sum_{j=1}^{k} \frac{\beta_j}{\beta_i} \alpha_i \frac{d_j}{d_i} t_i = \frac{\alpha_i}{\beta_i} \cdot \frac{t_i}{d_i}, \tag{2}$$

$i \in \{1, ..., k\}$. Hence $\mu(U \cap \{\tau_U = \hat{t}_i\})/\mu(U) = \beta_i$ and $\mu(U)\hat{t}_i = d_i$. This proves the lemma. □

Let $\mathcal{F}$ be the set of non-decreasing, right continuous functions $G : [0, +\infty) \to [0, 1]$ such that $\int_0^{+\infty} (1 - G(x))dx = 1$. Let $\mathcal{D} \subseteq \mathcal{F}$ be any dense set (for pointwise convergence) of simple functions with rational coefficients $G : [0, +\infty) \to [0, 1]$ such that $G(t) = \sum_{j=1}^{k} \beta_j 1_{\{t \geq d_j\}}$, where $d_1 < ... < d_k$, $\sum_{i=1}^{k} \beta_i = 1$ and $\sum_{i=1}^{k} d_i \beta_i = 1$. For instance, we can consider $\mathcal{D} = \mathcal{D}(p)$ to be the set of functions such that $\beta_1, ..., \beta_k$ are in $\mathcal{Q}^+(p) = \{\frac{k}{p^m} : k \in \mathbb{N}, m \in \mathbb{N}\}$, $p \geq 2$. For each $G \in \mathcal{D}$, $G(t) = \sum_{j=1}^{k} \beta_j 1_{\{t \geq d_j\}}$, we choose $\alpha_1, ..., \alpha_k \in \mathbb{N} \setminus \{0\}$, $t_1, ..., t_k \in \mathbb{N} \setminus \{0, 1\}$ such that $\frac{\alpha_i}{\alpha_{i+1}} = \frac{\beta_i}{\beta_{i+1}}$, $\frac{d_i}{d_{i+1}} = \frac{t_i}{t_{i+1}}$ for $i \in \{1, ..., k\}$. It is straightforward that this election can be done. Now we associate to G the block B_G of figure 1.

Since $\mathcal{D}$ is countable, we can write $\mathcal{D} = \{G_n : n \in \mathbb{N}\}$. We denote by $B(\mathcal{D}) = (V, E, \preceq)$ the ordered Bratteli diagram constructed as the concatenation of the blocks $B_n = B_{G_n}$ (defined as B_G but with the parameters of G_n), where the order verifies conditions (H1-H2). Let us notice that the system $(X_{B(\mathcal{D})}, V_{B(\mathcal{D})})$ is uniquely ergodic.

Corollary 2.2. *If $\mathcal{D} = \mathcal{D}(\mathcal{Q}^+(p))$, then $B(\mathcal{D})$ can be chosen to be an odometer in base p.*

Proof. We only need to prove that for each block B_i associated to $G_i \in \mathcal{D}$ we can choose parameters $(\alpha_j)_{j=1}^{k}$ and $(t_j)_{j=1}^{k}$ satisfying conditions (1) for G_i and such that $q_i = \sum_{j=1}^{k} \alpha_j t_j$ is a power of p.

Since $G_i = \sum_{j=1}^{k} \frac{\hat{\beta}_j}{p^m} 1_{\{t \geq \hat{d}_j / p^n\}}$, where $\hat{\beta}_j, \hat{d}_j \in \mathbb{N}$, then if we take $\alpha_j = p^l \hat{\beta}_j$ and $t_j = p^l \hat{d}_j$ for some $l \in \mathbb{N}$ we conclude from equalities (2) that q_i is a power of p. Moreover, B is an odometer

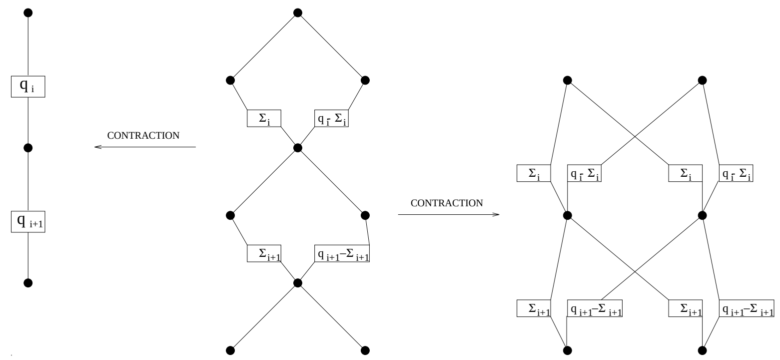

FIGURE 2.

in base p. This fact can be obtained by the contraction of Bratteli diagrams shown in figure 2, where $\Sigma_i = \sum_{i=1}^{k} \alpha_i$. □

Let $(U_n : n \in \mathbb{N})$ be any decreasing family of cylinder sets in $X_{B(\mathcal{D})}$ such that $G_{U_n} = G_n$ given inductively by Lemma 2.1. Let x be the intersection point. We notice tha it can be chosen to be $x_{\min}$.

Remark 2.3. *Since $\mathcal{D}$ is dense, given $G \in \mathcal{F}$ there exist $(G_{n_i} : i \in \mathbb{N})$ in $\mathcal{D}$ converging to G pointwise, and then uniformly on each closed interval where G is continuous. We conclude $\lim_{i\to\infty} G_{n_i} = \lim_{i\to\infty} G_{U_{n_i}} = G$ uniformly.*

The order used in the construction of block B_G in figure 1 has not played any role up to now. In fact, it only satisfies the general conditions (H1-H2). When the order is chosen to be from left to right we obtain an odometer, where the base $(q_i)_{i\in\mathbb{N}}$ is given by $q_i = \sum_{j=1}^{k^{(i)}} \alpha_j^{(i)} t_j^{(i)}$ such that the $\alpha_j^{(i)}, t_j^{(i)}, k^{(i)}$ are taken from B_{G_i}. Other orders could provide a Toeplitz subshift if $V_{B(\mathcal{D})}$ is expansive since $B(\mathcal{D})$ is of Toeplitz type as can be seen from the characterization in [GJ].

Theorem 2.4. *Let (X, T, μ) be a minimal Cantor system with infinite periodic spectrum and $x \in X$. Then there exist a decreasing sequence of clopen sets $(U_i : i \in \mathbb{N})$ in X containing x, such that $\mu(\cap_{i\in\mathbb{N}} U_i) = 0$, and for every $G \in \mathcal{F}$ there is a subsequence $(V_i : i \in \mathbb{N}) \subseteq (U_i : i \in \mathbb{N})$ for which $G_{V_i} \to G$ pointwise (and uniformly on each closed interval of continuity of G) as $i \to \infty$. If (X, T, μ) is an odometer then $x = \cap_{i\in\mathbb{N}} U_i$.*

Proof. From the condition on the periodic spectrum, we deduce that (X, T) has an odometer (Y, S) as a factor, which base can be assumed to be $(p_i^{(1)} p_i^{(2)} p_i^{(3)} : i \in \mathbb{N})$, $p_i^{(1)} p_i^{(2)} p_i^{(3)} < p_{i+1}^{(1)} p_{i+1}^{(2)} p_{i+1}^{(3)}$ for $i \in \mathbb{N}$, and $p_i^{(j)}$ goes to infinity with i for each, $j = 1, 2, 3$. First we construct a Bratteli–Vershik representation of this odometer. Let $\mathcal{R}$ be the set of simple functions with rational coefficients in $\mathcal{F}$. Define

$$\mathcal{D}R = \left\{ G = \sum_{j=1}^{k} \frac{\hat{\beta}_j}{p_i^{(1)}} 1_{\{t \geq \hat{d}_j / p_i^{(2)}\}} \in \mathcal{F} : i, k, \hat{\beta}_j, \hat{d}_j \in \mathbb{N} \right\}.$$

Standard approximation arguments show that $\mathcal{D}R$ is dense in $\mathcal{R}$ with respect to pointwise convergence. For each $G \in \mathcal{R}$ choose a sequence $(G_k(G) : k \in \mathbb{N}) \subseteq \mathcal{D}R$ converging to G. To each $G_k(G)$ we associate in the natural way a triple $(p_i^{(1)}, p_i^{(2)}, p_i^{(3)})$. Then we have produced a set $\mathcal{D} = \{G_k(G) : k \in \mathbb{N}, G \in \mathcal{R}\}$ which is dense in $\mathcal{F}$ with respect to pointwise convergence. Furthermore, we can assume that each element of $\mathcal{D}$ is associated to a different triple $(p_i^{(1)}, p_i^{(2)}, p_i^{(3)})$ for some $i \in \mathbb{N}$, and that all triples are used. Let G_i be the element of $\mathcal{D}$ associated to this triple. If $G_i = \sum_{j=1}^k \frac{\hat{\beta}_j}{p_i^{(1)}} 1_{\{t \geq \hat{d}_j / p_i^{(2)}\}}$, then we construct the block B_{G_i} by setting $\alpha_j = \hat{\beta}_j$ and $t_j = p_i^{(3)} \hat{d}_j$ for each $j \in \{1, ..., k\}$. Therefore, $\sum_{j=1}^k \alpha_j t_j = p_i^{(1)} p_i^{(2)} p_i^{(3)}$. This fact proves, that the Bratteli–Vershik system obtained from $\mathcal{D}$ using the blocks B_{G_i}, $(X_{B(\mathcal{D})}, V_{B(\mathcal{D})})$, is topologically conjugated to the odometer (Y, S).
Let $(U_i : i \in \mathbb{N})$ be the decreasing sequence of cylinder sets containing x. We conclude, using Remark 2.3 and Lemma 2.1, that for every $G \in \mathcal{F}$ there is a decreasing sequence of cylinder sets $(V_i : i \in \mathbb{N}) \subseteq (U_i : i \in \mathbb{N})$ in Y such that G_{V_i} converges to G pointwise as $i \to \infty$. To finish we take the preimage of the sequence $(V_i : i \in \mathbb{N})$ with respect to the factor map. It is clear that return laws coincide. □

3. Conclusions

The result presented in this paper and the article [L] shows that a work must done to enlight the notion of characteristic sequence of decreasing sequences of sets to be considered in order to obtain the good limit laws for the system. Most of the time sets are chosen to be balls in the respective metric. A natural question is how the limit laws depend or not of the representation of the system.

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