

A characterization of substitutive sequences using return words

F. Durand

*Institut de Mathématiques de Luminy, UPR 9016 CNRS, 163 Avenue de Luminy,
Case 930 13288 Marseille Cedex 9, France*

Abstract

We prove that a sequence is primitive substitutive if and only if the set of its derived sequences is finite; we defined these sequences here.

1 Introduction

The purpose of this paper is to characterize primitive substitutive sequences, i.e. sequences defined as the image, under a morphism, of a fixed point of a primitive substitution. No general characterizations of these sequences are known. In this paper we give one, it uses a notion we introduce here, the *return word*. We don't know reference or study about it, except in the area of symbolic dynamical systems, where it is closely related to *induced transformations* ([7,11]). But no combinatoric results are obtained.

For some classes of sequences arising from substitutions characterizations exist. The first class appearing in the literature is the class of the sequences generated by q -substitutions, also called *uniform tag sequences* ([6]). These are images, under a "letter to letter" morphism, of a fixed point of a substitution of constant length q . A. Cobham [6] proved that a sequence is generated by a q -substitution if and only if it is q -automatic, i.e. if it is generated by a q -automaton (see also [8]). When q is a prime power, an algebraic characterization is given by G. Christol, T. Kamae, M. Mendès France and G. Rauzy [5]. And S. Fabre [9] generalized the theorem of A. Cobham to a small class of substitutions of non-constant length, by introducing the concept of Θ -automaton and Θ -substitution.

It turns out that our characterization is similar to the following known one: a sequence $X = (X_n)_{n \in \mathbb{N}}$ is generated by a q -substitution if and only if the set of the sequences $(X_{q^k n + a})_{n \in \mathbb{N}}$, where $0 \leq a \leq q^k - 1$ and $k \geq 0$, is finite ([8]).

Our main tool of this paper is the *return word over u* , where u is a non-empty prefix of a minimal sequence X . It is defined as a word separating two successive factors of u . We introduce a "coding" of X with the return words over u and obtain an other minimal sequence $\mathcal{D}_u(X)$ we call *derived sequence* of X . The main result can be formulated as follows: *a sequence is primitive substitutive if and only if the set of derived sequences is finite.*

The paper is divided into six sections. The next section contains the definitions, the notation and the basic results needed to state the main theorem. The third section shows that the condition of the characterization is sufficient. In the following one, we extend a result, useful for the proof of the main theorem, due to B. Mossé [10], concerning the "power of words", to primitive substitutive sequences and we deduce some return word properties of these sequences. In the fifth section we prove the main theorem, considering first the case of the fixed point, and then the case of its image by a morphism. The last one deals with perspectives and open questions.

2 Definitions, notations

2.1 Words and sequences

Let us recall some standard notations we can find in [12]. We call *alphabet* a finite set of elements called *letters*; symbols \mathcal{A} and \mathcal{B} will always denote alphabets. A *word* on \mathcal{A} of *length* n is an element $x = x_0x_1 \cdots x_{n-1}$ of \mathcal{A}^n , $|x| = n$ is the length of x and \emptyset denotes the *empty-word* of length 0. The set of non-empty words on \mathcal{A} is denoted by \mathcal{A}^+ , and $\mathcal{A}^* = \mathcal{A}^+ \cup \{\emptyset\}$. If \mathcal{S} is a subset of \mathcal{A}^+ , we denote \mathcal{S}^+ the set of words which can be written as a concatenation of elements of \mathcal{S} , and $\mathcal{S}^* = \mathcal{S}^+ \cup \{\emptyset\}$. A *sequence* on \mathcal{A} is an element of $\mathcal{A}^{\mathbb{N}}$. If $X = X_0X_1 \cdots$ is a sequence, and, l and k are two non-negative integers, with $l \geq k$, we write $X_{[k,l]}$ the word $X_kX_{k+1} \cdots X_l$; a word u is a *factor* of X if there exists $k \leq l$ with $X_{[k,l]} = u$; if $u = X_{[0,l]}$ for some l , we say that u is a *prefix* of X and we write $u \prec X$. The empty-word is a prefix of X . The set of factors of length n is written $L_n(X)$, and the set of factors of X by $L(X)$. If u is a factor of X , we will call *occurrence* of u in X every integer i such that $X_{[i,i+|u|-1]} = u$, i.e. such that u is a prefix of $X_{[i,+\infty]}$. The sequence X is *ultimately periodic* if there exist a word u and a word v such that $X = uv^\omega$, where v^ω is the infinite concatenation of the word v . It is *periodic* if u is the empty-word. We will call $\mathcal{A}(X)$ the set of letters occurring in X .

If X is a word, we use the same terminology with the similar definitions. A word u is a *suffix* of the word X if $|u| \leq |X|$ and $u = X_{[|X|-|u|,|X|-1]}$. The empty-word is a suffix of X .

Every map $\phi : \mathcal{A} \rightarrow \mathcal{B}^+$ induces by concatenation a map from \mathcal{A}^* to \mathcal{B}^* , and a map from $\mathcal{A}^{\mathbb{N}}$ to $\mathcal{B}^{\mathbb{N}}$. All these maps are written ϕ also.

2.2 Substitutions and substitutive sequences

In this part we give a definition of what we call *substitution*. In the literature we can find the notions of *iterated morphism* ([3]), *tag system* ([6]) and *CDOL system* ([13]). They are similar to the notion of substitution.

Definition 1 A substitution is a triple $\zeta = (\zeta, \mathcal{A}, \alpha)$, where \mathcal{A} is an alphabet, ζ a map from \mathcal{A} to \mathcal{A}^+ and α an element of \mathcal{A} such that:

- (i) the first letter of $\zeta(\alpha)$ is α ,
- (ii) $|\zeta^n(\alpha)| \rightarrow +\infty$, when $n \rightarrow +\infty$.

This definition could be more general but it is not very restrictive. Indeed, as we study sequences generated by morphisms we need at least one letter satisfying the condition ii. Among these letters there is always one, β , such that β and a power of ζ satisfy the condition i. Details can be found in [12].

The sequence $\lim_{i \rightarrow +\infty} \zeta^i(\alpha)$ exists (see [12]) and is called *fixed point* of $\zeta = (\zeta, \mathcal{A}, \alpha)$, we will denote it X_ζ ; it is characterized by $\zeta(X_\zeta) = X_\zeta$ and α is the first letter of X_ζ .

Definition 2 We say that a substitution $\zeta = (\zeta, \mathcal{A}, \alpha)$ is primitive if there exists an integer k such that for all elements β and γ in \mathcal{A} , γ is a factor of $\zeta^k(\beta)$.

If $\zeta = (\zeta, \mathcal{A}, \alpha)$ is primitive then for all β belonging to \mathcal{A} we have $|\zeta^n(\beta)| \rightarrow +\infty$, when n tends to infinity.

Definition 3 Let X be a sequence on \mathcal{A} . The sequence X is minimal (or uniformly recurrent) if for every integer l there exists an integer k such that each word of $L_l(X)$ occurs in every word of length k .

Equivalently, a sequence X is minimal if for every non-empty factor u of X , the maximal difference between two successive occurrences of u is bounded. If ζ is primitive then the fixed point X_ζ is minimal (see [12]). But there exist non-primitive substitutions with minimal fixed points. Let $\zeta = (\zeta, \{1, 2, 3\}, 1)$ be the substitution defined by $\zeta(1) = 123$, $\zeta(2) = 2$ and $\zeta(3) = 13$. It is not primitive but its fixed point is minimal.

Definition 4 A sequence Y over the alphabet \mathcal{A} is called substitutive if there exist a substitution $\zeta = (\zeta, \mathcal{B}, \alpha)$ and a map $\varphi : \mathcal{B} \rightarrow \mathcal{A}$ such that $\varphi(X_\zeta) = Y$,

and substitutive primitive when ζ is primitive.

It is easy to check that a substitutive primitive sequence is minimal.

2.3 Return words

Let X be a minimal sequence over the alphabet \mathcal{A} and u a non-empty prefix of X . We call *return word over u* every factor $X_{[i,j-1]}$, where i and j are two successive occurrences of u in X , and $\mathcal{H}_{X,u}$ will denote the set of return words over u . Later we will see that $\mathcal{H}_{X,u}$ is a code; every element of $\mathcal{H}_{X,u}^*$ has an unique decomposition in a concatenation of element of $\mathcal{H}_{X,u}$ ([2,4]). The sequence X can be written in an unique way as a concatenation

$$X = m_0 m_1 m_2 \cdots$$

of sequence of elements of $\mathcal{H}_{X,u}$. Let us give to $\mathcal{H}_{X,u}$ the linear order defined by the rank of first appearance in this sequence. This defines a one to one and onto map

$$\Lambda_{X,u} : \mathcal{H}_{X,u} \rightarrow \{1, \dots, \text{Card}(\mathcal{H}_{X,u})\} = \mathcal{N}_{X,u},$$

and the sequence

$$\mathcal{D}_u(X) = \Lambda_{X,u}(m_0) \Lambda_{X,u}(m_1) \Lambda_{X,u}(m_2) \cdots;$$

this sequence of alphabet $\mathcal{N}_{X,u}$ is called a *derived sequence of X* ; it is easy to check that $\mathcal{D}_u(X)$ is minimal. Now we are able to state the main result of this paper:

Theorem 5 *A sequence X is substitutive primitive if and only if the number of its different derived sequences is finite.*

We will denote the reciprocal map of $\Lambda_{X,u}$ by $\Theta_{X,u} : \mathcal{N}_{X,u} \rightarrow \mathcal{H}_{X,u}$. The minimal sequence $\mathcal{D}_u(X)$ belonging to $\mathcal{N}_{X,u}^{\mathbb{N}}$ is characterized by

$$\Theta_{X,u}(\mathcal{D}_u(X)) = X.$$

The next proposition states some elementary facts about return words, we will use them very often.

Proposition 6 *Let u be a non-empty prefix of X .*

(i) *Let w be an element of \mathcal{A}^+ . Then w belongs to $\mathcal{H}_{X,u}$ if and only if wu*

is an element of $L(X)$, u is a prefix of wu and there are exactly two occurrences of u in wu .

- (ii) Let v_1, v_2, \dots, v_k be elements of $\mathcal{H}_{X,u}$. The occurrences of the word u in $v_1v_2 \dots v_k u$ are $0, |v_1|, |v_1| + |v_2|, \dots, \sum_{i=1}^k |v_i|$.
- (iii) The set $\mathcal{H}_{X,u}$ is a code and the map $\Theta_{X,u} : \mathcal{N}_{X,u}^* \rightarrow \mathcal{H}_{X,u}^*$ is one to one and onto.
- (iv) If u and v are two prefixes of X such that u is a prefix of v then $\mathcal{H}_{X,v}$ is included in $\mathcal{H}_{X,u}^+$.
- (v) Let v be a non-empty prefix of $\mathcal{D}_u(X)$ and $w = \Theta_{X,u}(v)u$. Then w is a prefix of X and $\mathcal{D}_v(\mathcal{D}_u(X)) = \mathcal{D}_w(X)$.

Proof. Claim 1. comes from the definition of $\mathcal{H}_{X,u}$. An induction on k prove the claim 2., and 3. follow from 2..

To prove the claim 4., let w be an element of $\mathcal{H}_{X,v}$ and $v = ux$. The word wv is a factor of X , hence the word wu belongs to $L(X)$. The word v is a prefix of wv , consequently the word u is a prefix of wv and the word w belongs to $\mathcal{H}_{X,u}^+$. It remains to prove the claim 5..

The word w is a prefix of X because v is a prefix of $\mathcal{D}_u(X)$. Let $Y = \mathcal{D}_u(X) = q_0q_1q_2 \dots$ and $X = p_0p_1p_2 \dots$ where q_i belongs to $\mathcal{H}_{Y,v}$, p_i belongs to $\mathcal{H}_{X,w}$ for all integers i . Let i be an element of \mathbb{N} . The word p_iw is a factor of X . Moreover u is a prefix of w and w is a prefix of p_iw then p_i belongs to $\mathcal{H}_{X,u}^+$. We write $p_i = \Theta_{X,u}(j_0) \dots \Theta_{X,u}(j_l)$ where j_k belongs to $\mathcal{N}_{X,u}$, $0 \leq k \leq l$.

In addition v is a prefix of $j_0 \dots j_lv$ because $\Theta_{X,u}(v)$ is a prefix of the word $p_i\Theta_{X,u}(v)$. And the word $j_0 \dots j_l$ belongs to $\mathcal{H}_{Y,v}^+$. Let us suppose there are at least three occurrences of v in $j_0 \dots j_lv$, then there are at least three occurrences of w in $p_i\Theta_{X,u}(v)u$. It is impossible because p_i is an element of $\mathcal{H}_{X,w}$. Then there are exactly two occurrences of v in $j_0 \dots j_lv$; consequently $\Lambda_{X,u}(p_i) = j_0 \dots j_l$ belongs to $\mathcal{H}_{Y,v}$. The sequence Y has an unique concatenation decomposition in elements of $\mathcal{H}_{Y,v}$. Hence $\Lambda_{X,u}(p_i) = q_i$, and the map $\Lambda_{X,u}$ from $\mathcal{H}_{X,u}$ to $\mathcal{N}_{X,u}$ is one to one. Then $\Lambda_{Y,v}(q_l) = \Lambda_{X,w}(p_l)$ for all integers l , and finally $\mathcal{D}_v(\mathcal{D}_u(X)) = \mathcal{D}_w(X)$. \square

The set $\mathcal{H}_{X,u}$ is a code, this allows us to extend by concatenation the map $\Lambda_{X,u}$ to $\mathcal{H}_{X,u}^*$.

Lemma 7 *If the minimal sequence X is ultimately periodic, then it is periodic.*

Proof. There exist two non-empty words, u and v , such that $X = uv^\omega$. By minimality, for every integer $k \geq 1$ there exists an integer l such that the word

$X_{[0,k|v|-1]}$ is a factor of v^l . Hence, there exist a suffix, v_1 , and a prefix, v_2 , of v such that the set $\{k \in \mathbb{N}; X_{[0,k|v|-1]} = v_1 v^{k-1} v_2\}$ is infinite. We conclude $X = m^\omega$ where $m = v_1 v_2$. \square

In this paper we only study sequences which are minimal. According to the Lemma 7 we will not make any difference between "periodic" and "ultimately periodic". We will use "periodic" for both notions.

Proposition 8 *If the sequence X is periodic and minimal, then there exists a prefix u of X satisfying: for all words v such that u is a prefix of v and v is a prefix of Y , we have $\text{Card}(\mathcal{H}_{Y,u}) = 1$ and $\mathcal{H}_{Y,u} = \mathcal{H}_{Y,v}$.*

Proof. Let u be a word such that $Y = u^\omega$. The word u belongs to $\mathcal{H}_{Y,u}^+$ and $\Theta_{Y,u}(1)$ is a prefix of Y , consequently $u = \Theta_{Y,u}(1)x$. The word uu is a prefix of Y , hence $\Theta_{Y,u}(1)x\Theta_{Y,u}(1)$ is a prefix of Y . Moreover $\Theta_{Y,u}(1)u$ is a prefix of Y , thus $u = x\Theta_{Y,u}(1)$. It follows:

$$Y = u^\omega = \Theta_{Y,u}(1)x(\Theta_{Y,u}(1)x)^\omega = \Theta_{Y,u}(1)u^\omega \text{ and } Y = (\Theta_{Y,u}(1))^\omega.$$

The decomposition of Y in elements of $\mathcal{H}_{Y,u}$ is unique, consequently $\mathcal{H}_{Y,u} = \{\Theta_{Y,u}(1)\}$.

We know that the word $\Theta_{Y,v}(1)$ belongs to $\mathcal{H}_{Y,u}^+$. There exists an integer k such that $\Theta_{Y,v}(1) = (\Theta_{Y,u}(1))^k$. Then $Y = (\Theta_{Y,v}(1))^\omega$ and $\mathcal{H}_{Y,v} = \{(\Theta_{Y,u}(1))^k\}$. Hence the word $\Theta_{Y,u}(1)$ belongs to $\mathcal{H}_{Y,v}^+$. Finally $k = 1$, this completes the proof. \square

3 The condition is sufficient

To prove that "if the number of the different derived sequences of a minimal sequence X is finite then X is substitutive primitive". We need the following proposition. It states that the image of a minimal fixed point by a morphism from \mathcal{A} to \mathcal{B}^+ is a substitutive primitive sequence.

Proposition 9 *Let $\zeta = (\zeta, \mathcal{A}, \alpha)$ be a primitive substitution, \mathcal{B} an alphabet and ϕ a map from \mathcal{A} to \mathcal{B}^+ . Then the sequence $\phi(X_\zeta)$ is substitutive primitive.*

Proof. Let $\mathcal{C} = \{(a, k); a \in \mathcal{A} \text{ and } 1 \leq k \leq |\phi(a)|\}$ and $\psi : \mathcal{A} \rightarrow \mathcal{C}^+$ be the map defined by:

$$\psi(a) = (a, 1) \dots (a, |\phi(a)|).$$

As ζ is primitive, substituting ζ^n for ζ if needed, we can assume $|\zeta(a)| \geq |\phi(a)|$, because this does not change the fixed point.

Let τ be the map from \mathcal{C} to \mathcal{C}^+ defined by:

$$\begin{aligned} \tau((a, k)) &= \psi(\zeta(a)_{[k,k]}) && \text{if } k < |\phi(a)|, \\ \text{and } \tau((a, |\phi(a)|)) &= \psi(\zeta(a)_{[|\phi(a)|, |\zeta(a)|]}) && \text{otherwise.} \end{aligned}$$

For a in \mathcal{A}

$$\begin{aligned} \tau(\psi(a)) &= \tau((a, 1) \cdots (a, |\phi(a)|)) = \psi(\zeta(a)_{[1,1]}) \cdots \psi(\zeta(a)_{[|\zeta(a)|, |\zeta(a)|]}) \\ &= \psi(\zeta(a)), \end{aligned}$$

thus $\tau(\psi(X_\zeta)) = \psi(\zeta(X_\zeta)) = \psi(X_\zeta)$.

In this way $\psi(X_\zeta)$ is the fixed point of $\tau = (\tau, \mathcal{C}, (\alpha, 1))$ and $\psi(X_\zeta) = X_\tau$. The substitution ζ is primitive and $\tau^n \psi = \psi \zeta^n$, hence τ is primitive. If χ is the map from \mathcal{C} to \mathcal{B} which sends the k -th letter of $\phi(a)$ to (a, k) , we obtain :

$$\chi(\psi(a)) = \chi((a, 1) \cdots (a, |\phi(a)|)) = \phi(a),$$

and $\chi(X_\tau) = \chi(\psi(X_\zeta)) = \phi(X_\zeta)$.

Therefore $\phi(X_\zeta)$ is primitive substitutive. \square

Lemma 10 *Let X be a minimal sequence, on the alphabet \mathcal{A} , which is not periodic. Then*

$$m_n = \inf\{|v|; v \in \mathcal{H}_{X, X[0,n]}\} \rightarrow +\infty \text{ when } n \rightarrow +\infty.$$

Proof. We have seen, it is the Proposition 6, that $\mathcal{H}_{X, X[0, n+1]}$ is included in $\mathcal{H}_{X, X[0, n]}^+$, then $m_n \leq m_{n+1}$. Let suppose $(m_n)_{n \in \mathbb{N}}$ stationary at the rank n_0 : there exist an integer k and, for every $n > n_0$, a word v_n where $|v_n| = k$ and v_n is an element of $\mathcal{H}_{X, X[0, n]}$. If $n \geq k$ then $X[0, n]$ is a prefix of $v_n X[0, n]$. Therefore, for all integers j such that $0 \leq j \leq n - k$, we deduce $X[j] = X[k+j]$. It follows that X is periodic with period k . This completes the proof. \square

In the following proposition we choose two prefixes u and v of a non-periodic minimal sequence X satisfying: u is a prefix of v , such that each word tu , where t belongs to $\mathcal{H}_{X, u}$, is a factor of every w belonging to $\mathcal{H}_{X, v}$. The minimality of X and the Lemma 10 allows us to set such a hypothesis.

Proposition 11 *Let X be a non-periodic minimal sequence. Let u and v be two prefixes of X , where u is a prefix of v , such that each word tu , where t belongs to $\mathcal{H}_{X,u}$, is a factor of every w belonging to $\mathcal{H}_{X,v}$. If $\mathcal{D}_u(X) = \mathcal{D}_v(X)$, then $\mathcal{D}_u(X)$ is the fixed point of a primitive substitution and X is substitutive primitive.*

Proof. Suppose $\mathcal{D}_u(X) = \mathcal{D}_v(X)$. Then we have

$$\mathcal{N}_{X,u} = \mathcal{A}(\mathcal{D}_u(X)) = \mathcal{A}(\mathcal{D}_v(X)) = \mathcal{N}_{X,v}.$$

And $\mathcal{H}_{X,v}$ is included in $\mathcal{H}_{X,u}^+$ because u is a prefix of v . Therefore we obtain $\text{Card}(\mathcal{N}_{X,u}) = |\{\Lambda_{X,u}\Theta_{X,v}(a); a \in \mathcal{N}_{X,v}\}|$. Let us consider the map ζ from $\mathcal{N}_{X,u}$ to $\mathcal{N}_{X,u}^+$ defined by $\zeta = \Lambda_{X,u}\Theta_{X,v}$. It is easy to see that the first letter of $\zeta(1)$ is 1. Moreover, for every i, j belonging to $\mathcal{N}_{X,u}$, $\Theta_{X,u}(j)u$ is a factor of $\Theta_{X,v}(i)$. Therefore j appears in $\Lambda_{X,u}\Theta_{X,v}(i) = \zeta(i)$ and $\zeta = (\zeta, \mathcal{N}_{X,u}, 1)$ is a primitive substitution. And $\mathcal{D}_u(X)$ is the fixed point of ζ because

$$\zeta(\mathcal{D}_u(X)) = \Lambda_{X,u}\Theta_{X,v}(\mathcal{D}_u(X)) = \Lambda_{X,u}\Theta_{X,v}(\mathcal{D}_v(X)) = \Lambda_{X,u}(X) = \mathcal{D}_u(X).$$

With the Proposition 9, we conclude that X is substitutive primitive because $\Theta_{X,u}(\mathcal{D}_u(X)) = X$. \square

Now we are able to show that the condition of the characterization is sufficient.

Theorem 12 *If the minimal sequence X has a finite number of derived sequences, then it is substitutive primitive.*

Proof. If X is periodic, then it is easy to check that X is the fixed point of a primitive substitution of constant length.

Let X be non-periodic. There exists a prefix u of X such that the set $K = \{v \prec X; \mathcal{D}_v(X) = \mathcal{D}_u(X)\}$ is infinite. By minimality, we can choose n so large that every factor of length n of X contains each element of $\mathcal{H}_{X,u}$ as a factor. By Lemma 10 there exists a word v belonging to K such that $|w| \geq n$ for all w belonging to $\mathcal{H}_{X,v}$. With u and v the hypothesis of Proposition 11 are fulfilled, this completes the proof. \square

4 Return words of a substitutive primitive sequence

In this section $\zeta = (\zeta, \mathcal{A}, \alpha)$ is a primitive substitution, $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is a map, we call it *projection*, and $Y = \phi(X_\zeta)$. We have to introduce some notations.

We define

$$S(\zeta) = \sup\{|\zeta(a)|; a \in \mathcal{A}\} \text{ and } I(\zeta) = \inf\{|\zeta(a)|; a \in \mathcal{A}\}.$$

4.1 Power of words in a substitutive primitive sequence

The following result comes from [12].

Lemma 13 *There exists a constant Q such that for all integers n*

$$S(\zeta^n) \leq QI(\zeta^n).$$

Definition 14 *A word v is primitive if it does not exist an integer n and a word $w \neq v$ such that $v = w^n$.*

A word is always a power of a primitive word. The proof of the next result can be found in [10], so we omit the proof.

Proposition 15 *Let $w = v^n$, where v is a primitive word and $n \geq 2$. If vu is a factor of w , then u is a power of v .*

The two following results have been proved by B. Mossé in [10] for fixed points of primitive substitutions, we adapted her proof to substitutive primitive sequences.

Lemma 16 *If there exist a primitive word v and two integers N and p such that:*

- (i) *for all words ab of $L_2(X_\zeta)$, $\phi(\zeta^p(ab))$ is a factor of v^N ,*
- (ii) $2|v| \leq I(\zeta^p)$,

then Y is periodic.

Proof. According to the conditions 1. and 2., for all letters a of \mathcal{A} , there exist an integer $n_a \geq 1$, a prefix w_a of v and a suffix v_a de v such that $|w_a|, |v_a| < |v|$ and $\phi(\zeta^p(a)) = v_a v^{n_a} w_a$. If ab belongs to $L_2(X_\zeta)$ then $v^{n_a} w_a v_b v^{n_b}$ belongs to $L(Y)$. The word v is primitive then $w_a v_b = v$ or $w_a v_b = \emptyset$. Hence $Y = \phi(X_\zeta) = \phi(\zeta^p(X_\zeta))$ is periodic. \square

Theorem 17 *If Y is not periodic, there exists an integer N such that w^N is a factor of Y if and only if $w = \emptyset$.*

Proof. By minimality, there exists an integer r such that each word of length 2 of Y is a factor of each factor of length r of Y . Let v be a non-empty

primitive word of \mathcal{B}^+ , $M > 0$ be an integer such that v^M belongs to $L(Y)$ and p be the integer defined by $I(\zeta^{p-1}) \leq 2|v| < I(\zeta^p)$. There is an occurrence of each word $\phi(\zeta^p(ab))$, where ab belongs to $L(X_\zeta)$, in each word of length $2 \sup\{|\zeta^p(w)|; w \in L(X_\zeta), |w| = r\}$. Hence, according to the Lemma 16, we deduce

$$|v^M| = M|v| < 2 \sup\{|\phi(\zeta^p(w))|; w \in L(X_\zeta), |w| = r\}.$$

Then, using the Lemma 13, and its constant Q ,

$$\begin{aligned} M &< \frac{2 \sup\{|\phi(\zeta^p(w))|; w \in L(X_\zeta), |w| = r\}}{\frac{1}{2}I(\zeta^{p-1})} \\ &\leq 4r \frac{S(\zeta^p)}{I(\zeta^{p-1})} \leq 4rS(\zeta) \frac{S(\zeta^{p-1})}{I(\zeta^{p-1})} \leq 4rS(\zeta)Q. \end{aligned}$$

With $N = 4rS(\zeta)Q$ we complete the proof. \square

4.2 Return words

Here we prove that, for a substitutive primitive sequence, the length of the return words over u is proportional to the length of u , and that the number of return words are bounded independently of u .

Theorem 18 *If Y is not periodic, there exist three positive constants K , L and M such that: for all non-empty prefixes u of Y ,*

- (i) *for all words v of $\mathcal{H}_{Y,u}$, $L|u| \leq |v| \leq M|u|$,*
- (ii) *$\text{Card}(\mathcal{H}_{Y,u}) \leq K$.*

Proof. Let u be a non-empty prefix of Y .

1. Let v be an element of $\mathcal{H}_{Y,u}$ and k be the smallest integer such that $|u| \leq I(\zeta^k)$. The choice of k entails that there exists an element ab of $L_2(X_\zeta)$ such that u is a factor of $\phi(\zeta^k(ab))$. Let R be the largest difference between two successive occurrences of an element of $L_2(X_\zeta)$ in X_ζ . We have

$$|v| \leq RS(\zeta^k) \leq RQI(\zeta^k) \leq RQS(\zeta)I(\zeta^{k-1}) \leq RQS(\zeta)|u|,$$

and we put $M = RQS(\zeta)$.

Theorem 17 gives us $|v| \geq |u|/N$, where N is the constant of this theorem.

2. Let n be the smallest integer such that $I(\zeta^n) \geq (M + 1)|u|$. For all v

belonging to $\mathcal{H}_{Y,u}$ we have $|vu| \leq (M+1)|u|$. Hence there exists a word ab of $L_2(X_\zeta)$ such that vu is a factor of $\phi(\zeta^n(ab))$. The word $\phi(\zeta^n(ab))$ contains at the most $q = |\phi(\zeta^n(ab))|L^{-1}|u|^{-1}$ occurrences of u . Consequently at the most q factors vu where v belongs to $\mathcal{H}_{Y,u}$. On the other hand

$$\begin{aligned} |\phi(\zeta^n(ab))| &\leq 2S(\zeta^n) \leq 2S(\zeta)S(\zeta^{n-1}) \\ &\leq 2S(\zeta)QI(\zeta^{n-1}) \leq 2S(\zeta)Q(M+1)|u|, \end{aligned}$$

hence $q \leq 2S(\zeta)Q(M+1)L^{-1}$ and

$$\text{Card}(\mathcal{H}_{Y,u}) \leq 2S(\zeta)Q(M+1)(\text{Card}(\mathcal{A}))^2L^{-1},$$

this completes the proof. \square

5 The condition is necessary

To prove "if X is a primitive substitutive sequence then the number of its different derived sequences is finite", this section is divided into two subsections. One is devoted to the case of the fixed point of a primitive substitution, and the other to the general case.

5.1 The case of the fixed point of a primitive substitution

Proposition 19 *Let $\zeta = (\zeta, \mathcal{A}, \alpha)$ be a primitive substitution and u be a non-empty prefix of X_ζ . The derived sequence $\mathcal{D}_u(X_\zeta)$ is the fixed point of a primitive substitution $\tau_{X_\zeta, u} = (\tau_{X_\zeta, u}, \mathcal{N}_{X_\zeta, u}, 1)$.*

Proof. Let i be an element of $\mathcal{N}_{X_\zeta, u}$ and $v = \Theta_{X_\zeta, u}(i)$. The word v belongs to $\mathcal{H}_{X_\zeta, u}$, hence vu is a factor of X_ζ and $\zeta(v)\zeta(u)$ too. The word u is a prefix of X_ζ and u is a prefix of vu . Therefore u is a prefix of $\zeta(u)$ and $\zeta(v)\zeta(u)$. And finally $\zeta(v)$ belongs to $\mathcal{H}_{X_\zeta, u}^+$. So, we can define the map $\tau_{X_\zeta, u}$ from $\mathcal{N}_{X_\zeta, u}$ to $\mathcal{N}_{X_\zeta, u}^+$ by:

$$\tau_{X_\zeta, u}(i) = \Lambda_{X_\zeta, u}(\zeta(\Theta_{X_\zeta, u}(i))), \quad i \in \mathcal{N}_{X_\zeta, u}.$$

In this way we have

$$\begin{aligned}\tau_{X_\zeta, u}(\mathcal{D}_u(X_\zeta)) &= \Lambda_{X_\zeta, u}(\zeta(\Theta_{X_\zeta, u}(\mathcal{D}_u(X_\zeta)))) = \Lambda_{X_\zeta, u}(\zeta(X_\zeta)) = \Lambda_{X_\zeta, u}(X_\zeta) \\ &= \mathcal{D}_u(X_\zeta).\end{aligned}$$

The map $\tau_{X_\zeta, u} = (\tau_{X_\zeta, u}, \mathcal{N}_{X_\zeta, u}, 1)$ is a substitution because 1 is a prefix of $\tau_{X_\zeta, u}(1)$ and

$$\lim_{n \rightarrow +\infty} |\tau_{X_\zeta, u}^n(i)| = \lim_{n \rightarrow +\infty} |\Lambda_{X_\zeta, u}(\zeta^n(\Theta_{X_\zeta, u}(i)))| = +\infty, \forall i \in \mathcal{N}_{X_\zeta, u}.$$

And $X_{\tau_{X_\zeta, u}} = \mathcal{D}_u(X_\zeta)$ is its fixed point. Finally, it is primitive because ζ is primitive. \square

Theorem 20 *Let $\zeta = (\zeta, \mathcal{A}, \alpha)$ be a primitive substitution. The number of different derivated sequences of X_ζ is finite.*

Proof. Let X_ζ be non-periodic. Let i be an element of $\mathcal{N}_{X_\zeta, u}$ and $v = \Theta_{X_\zeta, u}(i)$ an element of $\mathcal{H}_{X_\zeta, u}$. We take the notations of Theorem 18. We have $|v| \leq M|u|$ and $|\zeta(v)| \leq S(\zeta)M|u|$. The length of each element of $\mathcal{H}_{X_\zeta, u}$ is larger than $L|u|$. Then we can decompose $\zeta(v)$ in at the most $S(\zeta)ML^{-1}$ elements of $\mathcal{H}_{X_\zeta, u}$, so $|\tau_{X_\zeta, u}(i)| \leq S(\zeta)ML^{-1}$.

There exists K such that $\text{Card}(\mathcal{H}_{X_\zeta, u}) \leq K$ for all non empty prefixes of X_ζ . Therefore we can deduce there is a finite number of alphabets $\mathcal{N}_{X_\zeta, u}$, and a finite number of substitutions $\tau_{X_\zeta, u} = (\tau_{X_\zeta, u}, \mathcal{N}_{X_\zeta, u}, 1)$: the set $\{\mathcal{D}_u(X_\zeta); u \prec X_\zeta\}$ is finite.

Let X_ζ be periodic. There exists a prefix u of X_ζ satisfying $\text{Card}(\mathcal{H}_{X_\zeta, v}) = 1$ for all words v such that u is a prefix of v and v is a prefix of X_ζ . Then $\mathcal{D}_v(X_\zeta) = 1^\omega$ for all words v such that u is a prefix of v and v is a prefix of X_ζ . The proof is completed. \square

5.2 The case of a substitutive primitive sequence

Here we end the proof of the main theorem.

Theorem 21 *Let Y be a substitutive primitive sequence, the number of its different derivated sequences is finite.*

Proof. If Y is periodic, we treated this case in the proof of Theorem 20.

Let Y be non-periodic. We have $Y = \phi(X_\zeta)$ where ϕ is a projection and ζ a primitive substitution. We proved that there exist three positive constants K ,

L and M such that for all non-empty prefixes u of Y , and for all words v of $\mathcal{H}_{Y,u}$ we have $L|u| \leq |v| \leq M|u|$ and $\text{Card}(\mathcal{H}_{Y,u}) \leq K$.

Let v be a non-empty prefix of Y , and u the prefix of X_ζ of length $|v|$. We have $v = \phi(u)$. Let i be an element of $\mathcal{N}_{X_\zeta,u}$ and $w = \Theta_{X_\zeta,u}(i)$ an element of $\mathcal{H}_{X_\zeta,u}$. As w belongs to $L(X_\zeta)$ the word $\phi(w)v$ belongs to $L(Y)$. Moreover u is a prefix of wu , then v is a prefix of $\phi(w)v$. Therefore $\phi(w)$ belongs to $\mathcal{H}_{Y,v}^+$ and this word has an unique concatenation decomposition in elements of $\mathcal{H}_{Y,v}$. We can define the map $\lambda_u : \mathcal{N}_{X_\zeta,u} \rightarrow \mathcal{N}_{Y,v}^*$ by $\lambda_u = \Lambda_{Y,v} \phi \Theta_{X_\zeta,u}$. So, we have $\lambda_u(\mathcal{D}_u(X_\zeta)) = \mathcal{D}_v(Y)$, $\Theta_{Y,v} \lambda_u = \phi \Theta_{X_\zeta,u}$ and

$$\begin{aligned} |\lambda_u(i)|L|v| &\leq |\lambda_u(i)| \inf\{|w|; w \in \mathcal{H}_{Y,v}\} \leq |\Theta_{Y,v}(\lambda_u(i))| \\ &= |\Theta_{X_\zeta,u}(i)| \leq \sup\{|w|; w \in \mathcal{H}_{X_\zeta,u}\} \leq M|u| \leq M|v|. \end{aligned}$$

Hence $|\lambda_u(i)| \leq M/L$.

The sets $\{\lambda_u; u \prec X_\zeta\}$ and $\{\mathcal{D}_u(X_\zeta); u \prec X_\zeta\}$ are finites. Therefore the set $\{\mathcal{D}_v(Y); v \prec Y\} = \{\lambda_u(\mathcal{D}_u(X_\zeta)); u \prec X_\zeta, |u| = |v|\}$ is finite. This completes the proof. \square

Theorem 5 is proved.

6 Open problems and perspectives

6.1 Topological dynamical system

Let X be a minimal sequence and $\Omega = (\overline{\{T^n X; n \in \mathbb{N}\}}, T)$, where T is the shift, be the dynamical system generated by X (see [7,11] or [12]). In dynamical topological terms, we can formulate Theorem 5 as follows: X is primitive substitutive if and only if the number of induced systems on cylinder generated by a prefix is finite. We don't know whether it is true if we induce on any cylinders, or on any clopen sets.

Our method characterizes only one point of Ω , is it possible to characterize all the points belonging to Ω by analysis methods?

6.2 Minimality and primitivity

Our method allows us to characterize the primitive substitutive sequences, but we do not know if every substitutive minimal sequence is primitive, i.e. generated by a primitive substitution. We can only prove this for the minimal fixed points arising from non-primitive substitutions. The following example is significant and give a sketch of the proof.

Let $\zeta = (\zeta, \{1, 2\}, 1)$ be the non-primitive substitution defined by $\zeta(1) = 1211$ et $\zeta(2) = 2$. The sequence X_ζ is minimal. There exists a prefix u of X_ζ such that $\mathcal{D}_u(X_\zeta)$ is the fixed point of the primitive substitution $\tau_{X_\zeta, u}$. Then X_ζ is primitive substitutive because $\Theta_{X_\zeta, u}(\mathcal{D}_u(X_\zeta)) = X_\zeta$. The sequence X_ζ is the image under $\varphi : \{1, 2, 3\} \rightarrow \{1, 2\}$, defined by $\varphi(1) = \varphi(3) = 1$ and $\varphi(2) = 2$, of the fixed point of the primitive substitution $\tau = (\tau, \{1, 2, 3\}, 1)$, defined by $\tau(1) = 12$, $\tau(2) = 312$ and $\tau(3) = 1233$. The proof of Theorem 9 gives us the morphisms φ and τ .

It remains to treat the case of the substitutive minimal sequences arising from sequences which are not minimals.

6.3 Complexity, return words and S -adic sequences

B. Mossé recently proved that if X is the fixed point of a primitive substitution then the sequence $(p_X(n+1) - p_X(n))_{n \in \mathbb{N}}$ is bounded, where $p_X(n)$ is the number of the words of length n belonging to $L(X)$. With our results we are able to give a short proof of this property, and to extend it to primitive substitutive sequences. Indeed we can prove, with the help of the graph of words of Rauzy [1], the following property:

if Y is a minimal sequence such that $(\text{Card}(\mathcal{H}_{Y, Y_{[0, n]}}))_{n \in \mathbb{N}}$ is bounded then $(p_Y(n+1) - p_Y(n))_{n \in \mathbb{N}}$ is bounded.

A S -adic sequence Y on the alphabet \mathcal{A} is given by an infinite product of substitutions belonging to a finite set \mathcal{S} . B. Host formulated the following conjecture: a minimal sequence Y is S -adic if and only if $(p_Y(n+1) - p_Y(n))_{n \in \mathbb{N}}$ is bounded.

If the matrices of the substitutions of \mathcal{S} (see [12]) are strictly positives then we can prove that $(\text{Card}(\mathcal{H}_{Y, Y_{[0, n]}}))_{n \in \mathbb{N}}$ is bounded, by extending the proof of the primitive substitutive case. It follows that $(p_Y(n+1) - p_Y(n))_{n \in \mathbb{N}}$ is bounded.

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