A NONLINEAR ELLIPTIC PDE WITH THE INVERSE SQUARE POTENTIAL

Louis Dupaigne

0. Introduction

Statement of the problem.

This paper is concerned with the following equation :

$$(P_{t,p}) \qquad \begin{cases} -\Delta u - \dfrac{c}{|x|^2} u = u^p + tf & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Here, Ω is a smooth bounded open set of $\mathbb{R}^n$ ($n \geq 3$) containing the origin, $c > 0$, $p > 1$, $t > 0$ are constants and $f \not\equiv 0$ is a smooth, bounded, nonnegative function.

We assume from now on that

$$(0.1) \qquad 0 < c \leq c_0 := \frac{(n-2)^2}{4}$$

The relevance of the constant c_0 will appear after we clarify the notion of a solution of $(P_{t,p})$.

Three types of solution are defined thereafter : weak solutions, which provide a good setting for non-existence proofs (see Theorem 1 and Proposition 2.1), $H^1_0(\Omega)$ solutions, for which uniqueness results can be established (see Theorem 2) and strong solutions, which set the optimal regularity one can hope for (see Theorem 1 and Lemma 1.5.)

We shall say that $u \in L^1(\Omega)$ is a **weak solution** of $(P_{t,p})$ if $u \geq 0$ a.e. and if it satisfies the two following conditions :

$$\begin{cases} \int_\Omega \left(\dfrac{u}{|x|^2} + u^p \right) dist(x, \partial\Omega)\, dx < \infty \\ \int_\Omega u \left(-\Delta\phi - \frac{c}{|x|^2}\phi \right) = \int_\Omega (u^p + tf)\phi \qquad \text{for } \phi \in C^2(\bar{\Omega}) \ , \ \phi|_{\partial\Omega} = 0 \end{cases}$$

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Observe that the first condition merely ensures that the integrals in the second equation make sense.

An $\mathbf{H_0^1}(\Omega)$ **solution** is a function $u \in H_0^1(\Omega)$ such that $u \geq 0$ a.e., $u^p \in L^{\frac{2n}{n+2}}(\Omega)$ and

$$\int_\Omega \nabla u \nabla \phi - \int_\Omega \frac{c}{|x|^2} u\phi = \int_\Omega (u^p + tf)\,\phi \qquad \text{for all } \phi \in H_0^1(\Omega)$$

All integrals are well defined because of Sobolev's and Hardy's inequalities (see (0.3) for the latter.)

Finally, a **strong solution** u is a $C^2(\bar{\Omega} \setminus \{0\})$ function satisfying the system of equations $(P_{t,p})$ everywhere except possibly at the origin, such that for some $C > 0$,

$$0 \leq u \leq C\,|x|^{-a}$$

where

$$a := \frac{n - 2 - \sqrt{(n-2)^2 - 4c}}{2} > 0 \tag{0.2}$$

Observe that $-a$ is the larger root of $P(X) = X(X-1) + (n-1)X + c = 0$. Also define a' by

(0.2') $\qquad -a'$ is the smaller root of P(X)

• **Why are definitions (0.1), (0.2) important ?**

The constant c_0 defined in (0.1) is the best constant in Hardy's inequality :

$$\int_\Omega |\nabla u|^2 \geq c_0 \int_\Omega \frac{u^2}{|x|^2} \qquad \text{for all } u \in H_0^1(\Omega) \tag{0.3}$$

Consequently, when $c < c_0$, the operator $-\Delta - \frac{c}{|x|^2}$ is coercive in $H_0^1(\Omega)$. This turns out to be crucial since Theorem 2.2 in [BG] implies that if $c > c_0$, there is no nonnegative u, $u \not\equiv 0$ such that $-\Delta u - \frac{c}{|x|^2}u \geq 0$ and hence no solution of $(P_{t,p})$, even in the weak sense. We arrive at the same conclusion if $c > 0$ is arbitrary and the space dimension n is 1 or 2, as can be deduced from the first lines of the proof of Theorem 1.2 in [BC]. We therefore restrict to $n \geq 3$.

The constant a defined in (0.2) plays a central role, even in the linear theory. Indeed, if $f \not\equiv 0$ is say, a smooth nonnegative bounded function on Ω and $u \in H_0^1(\Omega)$ is the unique solution of

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2}u = f \text{ in } \Omega \\ \qquad\qquad u = 0 \text{ on } \partial\Omega \end{cases} \tag{0.4}$$

then $u(x) \geq C|x|^{-a}$ near the origin, for some $C > 0$ (see Lemma 1.5 .) In particular, strong solutions are the nicest one can hope for. In addition, $\psi := |x|^{-a}$ solves $-\Delta\psi - \frac{c}{|x|^2}\psi = 0$ in $\mathbb{R}^n$.

We introduce a third constant, the exponent

$$p_0 := 1 + \frac{n-2+\sqrt{(n-2)^2-4c}}{c} \tag{0.5}$$

which satisfies

$$a + 2 = p_0\, a$$

Roughly speaking, if u behaves like $|x|^{-a}$, then $-\Delta u - \frac{c}{|x|^2}u \sim |x|^{-(a+2)}$ and $u^p \sim |x|^{-ap}$. Hence, p_0 sets the threshold beyond which the nonlinear term produces a stronger singularity at the origin than the differential operator. In fact, we will show that for $p \geq p_0$, $(P_{t,p})$ has no solution, no matter how small $t > 0$ is. See Theorem 1 for details.

This fact is somewhat surprising : one would expect that working with the map $F(u) := -\Delta u - \frac{c}{|x|^2}u - u^p$, which is such that $F'(0) = -\Delta - \frac{c}{|x|^2}$ is formally bijective and $F(0) = 0$, the inverse function theorem would yield solutions for $t > 0$ sufficiently small. Such an argument fails because there is no functional setting in which it may be applied. See section 7 of [BV] or the introduction of [BC] for a similar situation.

Another interesting property of p_0 is its variation as c decreases from $c = c_0$ to $c = 0$: when $c = c_0$, $p_0 = \dfrac{n+2}{n-2}$ is the Sobolev exponent whereas when $c \to 0$, $p_0 \to \infty$. This is natural in view of the case $c = 0$, for which $p > 1$ can be chosen arbitrarily (see e.g. [D],[BCMR],[CR].)

- **How do strong, $\mathbf{H^1_0}(\Omega)$ and weak solutions relate ?**

Proposition 0.1.

Suppose (0.1) holds and recall (0.2), (0.5). Suppose also that $1 < p < p_0$.

- *If u is a strong solution of $(P_{t,p})$, then u is an $H^1_0(\Omega)$ solution of $(P_{t,p})$.*
- *If u is an $H^1_0(\Omega)$ solution of $(P_{t,p})$, then u is a weak solution of $(P_{t,p})$.*
- *If u is a weak solution of $(P_{t,p})$ and $0 \leq u \leq C\,|x|^{-a}$ then u is an $H^1_0(\Omega)$ solution of $(P_{t,p})$.*
- *If u is an $H^1_0(\Omega)$ solution of $(P_{t,p})$ and $0 \leq u \leq C\,|x|^{-a}$ then u is a strong solution of $(P_{t,p})$.*

This will be proved in Section 1.

Remark 0.1. In section 5, we provide examples of both strong and $H^1_0(\Omega)$ solutions. We do not know however if there exist weak solutions that are not $H^1_0(\Omega)$.

With these definitions in mind, we investigate the existence, uniqueness and regularity of solutions of $(P_{t,p})$:

Main results.

Theorem 1.

Suppose (0.1) holds and recall (0.5).

- *If* $1 < p < p_0$, *there exists* $t_0 > 0$ *depending on* n, c, p, f *such that*

  if $t < t_0$ *then* $(P_{t,p})$ *has a minimal strong solution,*
  if $t = t_0$ *then* $(P_{t,p})$ *has a minimal weak solution,*
  if $t > t_0$ *then* $(P_{t,p})$ *has no solution, even in the weak sense and there is complete blow-up.*

- *If* $p \geq p_0$ *then, for any* $t > 0$,

  $(P_{t,p})$ *has no solution, even in the weak sense, and there is complete blow-up.*

This result requires the following definition :

Definition 0.1. Let $\{a_n(x)\}$ and $\{g_n(u)\}$ be increasing sequences of bounded smooth functions converging pointwise respectively to $\dfrac{c}{|x|^2}$ and $u \to u^p$ and let $\underline{u_n}$ be the minimal nonnegative solution of

$$(P_n) \qquad \begin{cases} -\Delta\underline{u_n} - a_n\underline{u_n} = g_n(\underline{u_n}) + tf & \text{in } \Omega \\ \underline{u_n} = 0 & \text{on } \partial\Omega \end{cases}$$

We say that there is **complete blow-up** in $(P_{t,p})$ if, given any such $\{a_n(x)\}$, $\{g_n(u)\}$ and $\{\underline{u_n}\}$,

$$\frac{\underline{u_n}(x)}{\delta(x)} \to +\infty \text{ uniformly on } \Omega,$$

where $\delta(x) := dist(x, \partial\Omega)$.

Theorem 2.

Suppose (0.1) holds and $1 < p < p_0$, $0 < t < t_0$. *Then if* u_t *denotes the minimal strong solution of* $(P_{t,p})$,

- u_t *is stable*
- u_t *is the only stable* $H^1_0(\Omega)$ *solution of* $(P_{t,p})$

If u_{t_0} *denotes the minimal weak solution of* $(P_{t_0,p})$ *and* $0 < c < c_0$ *and*

- *if* u_{t_0} *solves the problem in the strong sense then* $\lambda_1(u_{t_0}) = 0$

Stability is defined as follows :

Definition 0.2. We say that u is **stable** if the generalized first eigenvalue $\lambda_1(u)$ of the linearized operator of equation $(P_{t,p})$ is positive, i.e., if

$$\lambda_1(u) := \inf\{J(\phi) : \phi \in C_c^\infty(\Omega) \setminus \{0\}\} > 0$$

where

$$J(\phi) = \frac{\int_\Omega |\nabla\phi|^2 - \int_\Omega \frac{c}{|x|^2}\phi^2 - \int_\Omega p u^{p-1}\phi^2}{\int_\Omega \phi^2}$$

The proof of Theorem 1 is presented in sections 2 and 3, whereas Theorem 2 is proved in section 4.

In section 5, we study the extremal case $t = t_0$ and provide examples of two distinct behaviors of the extremal solution of $(P_{t_0,p})$.

Finally, in section 6, proofs of all previously announced results pertaining to the case $c = c_0$ are given.

Notation and further definitions.

Dealing with linear equations of the form (0.4) with $f \in L^1(\Omega, dist(x, \partial\Omega)\,dx)$, a weak solution u is one that satisfies the equation $\int_\Omega u\left(-\Delta\phi - \frac{c}{|x|^2}\phi\right) = \int_\Omega f\phi$ with the integrability condition $\int_\Omega \frac{|u|}{|x|^2} < \infty$. Strong solutions are defined as in the nonlinear case.

Of course, Proposition 0.1 need not be true in this setting.

Sometimes we shall refer to inequalities holding in the weak sense or talk about (weak) supersolutions. This means that we integrate the equation with nonnegative test functions.

For example, $-\Delta u - \frac{c}{|x|^2}u \geq f$ holds **in the weak sense**,
given $f \in L^1(\Omega, dist(x, \partial\Omega)\,dx)$, if $\dfrac{u}{|x|^2} \in L^1(\Omega)$ and if

$$\int_\Omega u\left(-\Delta\phi - \frac{c}{|x|^2}\phi\right) \geq \int_\Omega f\phi \qquad \text{for all } \phi \in C^2(\bar\Omega) \text{ with } \phi \geq 0 \text{ and } \phi|_{\partial\Omega} = 0$$

The following L^q weighted spaces will be used in the sequel :

$L^q_\delta = L^q(\Omega, \delta(x)\,dx)$,
$L^q_m = L^q(\Omega, |x|^m\,dx)$,
$L^q_{m,\delta} = L^q(\Omega, |x|^m\,\delta(x)\,dx)$ and
$L^\infty_m = \{u : u \cdot |x|^{-m} \in L^\infty(\Omega)\}$

where $1 \leq q < \infty$, $\delta(x) = dist(x, \partial\Omega)$ and $m \in \mathbb{R}$.

Also, for $\rho > 0$, B_ρ denotes the open ball of radius ρ centered at the origin. The letter C denotes a generic positive constant.

1. Preliminary : linear theory

We construct here a few basic tools to be used later on and start out with the L^2 theory.

Lemma 1.1.
Suppose $0 < c < c_0$ *and let* $f \in H^{-1}(\Omega)$*. There exists a unique* $u \in H_0^1(\Omega)$*, weak solution of*

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2} u = f \ \text{ in } \Omega \\ \qquad\qquad u = 0 \ \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

Furthermore,

$$\|u\|_{H_0^1(\Omega)} \leq C\|f\|_{H^{-1}} \tag{1.2}$$

$$f \geq 0 \ \text{a.e.} \ \Rightarrow u \geq 0 \ \text{a.e.} \tag{1.3}$$

Proof. Hardy's inequality (0.3) implies that $-\Delta - \frac{c}{|x|^2}$ is coercive in $H_0^1(\Omega)$. (1.2) follows from Lax-Milgram's lemma. Observe that, using approximation in $H_0^1(\Omega)$ by smooth functions and integration by parts in $\bar{\Omega} \setminus B_\epsilon$ with $\epsilon \to 0$, our definition of a weak solution and that of Lax-Milgram's lemma coincide in this setting.

For $u \in H_0^1(\Omega)$, it is well known that $u^- \in H_0^1(\Omega)$. Testing the variational formulation of (1.1) against u^- yields (1.3).
□

Next, we consider the L^q theory and restrict ourselves to the radial case.

Lemma 1.2.
Suppose $0 < c < c_0$ *(with* c_0 *defined in (0.1)) and recall (0.2).*
Let $q \in \left(\dfrac{n}{n-a}, \dfrac{n}{2+a}\right)$, $E = W^{2,q}(B_1) \cap W_0^{1,q}(B_1) \cap \{u : \frac{u}{|x|^2} \in L^q(B_1)\}$.

For any radial $f \in L^q(B_1)$*, there exists a unique radial weak solution* $u \in E$ *of*

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2} u = f \ \text{ in } B_1 \\ \qquad\qquad u = 0 \ \text{ on } \partial B_1 \end{cases} \tag{1.4}$$

Furthermore,

$$\|u\|_E \leq C\|f\|_{L^q} \tag{1.5}$$

$$f \geq 0 \; a.e. \quad \Rightarrow u \geq 0 \; a.e. \tag{1.5'}$$

Remark 1.2.

- It can be shown that $u \in W^{2,q} \cap W_0^{1,q} \Rightarrow \frac{u}{|x|^2} \in L^q$ for $1 < q < n/2$, so that the definition of E can be slightly simplified.
- Observe that the interval $\left(\dfrac{n}{n-a}, \dfrac{n}{2+a} \right)$ is nonempty if and only if $c < c_0$.
- The restrictions on the range of q are optimal. If $q \leq \frac{n}{n-a}$, uniqueness is lost (see Remark 1.4), whereas if the lemma were to hold for some $q \geq \frac{n}{2+a}$, one could construct solutions of $(P_{t,p})$ for some p, $p \geq p_0$ by means of the inverse function theorem, contradicting Theorem 1 (see the methods of Proposition 4.1 .)
- It would be natural to extend Lemma 1.2 to the nonradial case. The problem remains open.

Proof.
Uniqueness will follow from the maximum principle (Lemma 1.4) proved in this section, provided we can show that $E \subset L^1_{-a-2}$.

If $u \in E$, $\frac{u}{|x|^2} \in L^q$ and using Hölder's inequality, $u \in L^1_{-a-2}$ if $|x|^{-a} \in L^{\frac{q}{q-1}}$, which is equivalent to asking $q > \dfrac{n}{n-a}$.

For existence, we suppose (without loss of generality in view of estimate 1.5) that $f \in C_c^\infty(0,1)$, $f \geq 0$ and define

$$u(r) := \Phi(f)(r) = \frac{r^{-a}}{\alpha} \int_0^1 f(s) \cdot s^{\frac{n+\alpha}{2}} \left[\max(s,r)^{-\alpha} - 1 \right] ds$$

where $\alpha = \sqrt{(n-2)^2 - 4c}$, $r \in (0,1)$.

(1.5') follows from the definition of u.

Since f is supported away from the origin, it is quite clear that $\dfrac{u}{r^{-a}}$ is smooth everywhere on $[0,1]$ so that $|u| \leq Cr^{-a}$ and $|u'| \leq Cr^{-a-1}$. Also, $u(1) = 0$. Differentiating u, we get

$$-u'' - \frac{n-1}{r} u' - \frac{c}{r^2} u = f \tag{1.6}$$

This equality holds for every $r \neq 0$ and also in the weak sense, using integration by parts in $B_1 \setminus B_\epsilon$ with $\epsilon \to 0$ and the above estimate on u and u'.

So, we just have to prove (1.5), which we shall do using Hardy-inequality-type arguments. Using the definition of u, we see that

$$\begin{aligned} 0 \leq C\frac{u}{r^2} \leq\ & r^{-(1+\frac{n+\alpha}{2})} \int_0^r f(s) \cdot s^{\frac{n+\alpha}{2}} \, ds + r^{-(1+n/2)+\alpha/2} \int_r^1 f(s) s^{\frac{n-\alpha}{2}} \, ds \\ \equiv\ & A \qquad\qquad\qquad\qquad\qquad\qquad + B \end{aligned}$$

Letting $g(s) = f(s)s^{\frac{n+\alpha}{2}}$ for $0 \le s \le 1$ and $G(r) = \int_0^r g(s)\,ds$ for $0 \le r \le 1$, integration by parts yields

$$I := \int_0^1 r^{-(1+\frac{n+\alpha}{2})q} G^q(r) r^{n-1}\,dr =$$

$$\frac{1}{n-(1+\frac{n+\alpha}{2})q} G^q(1) - \frac{q}{n-(1+\frac{n+\alpha}{2})q} \int_0^1 r^{n-(1+\frac{n+\alpha}{2})q} G^{q-1}(r) g(r)\,dr$$

$$\le C \int_0^1 r^{n-(1+\frac{n+\alpha}{2})q} G^{q-1}(r) g(r)\,dr$$

The last inequality results from the fact that when $q > \frac{n}{n-a}$, $\frac{1}{n-(1+\frac{n+\alpha}{2})q} < 0$.

Applying Hölder,

$$I \le I^{\frac{q-1}{q}} \left(\int_0^1 r^\gamma g^q(r)\,dr \right)^{1/q}$$

where $\gamma = q(n-(1+\frac{n+\alpha}{2}))$. But $r^\gamma g^q(r) = r^{q(n-1)} f^q(r) \le r^{(n-1)} f^q(r)$ so

$$\left(\int_{B_1} A^q \right)^{1/q} = C \cdot I^{1/q} \le C \|f\|_{L^q} \tag{1.7}$$

To bound B, we introduce similarly $h(s) = s^{\frac{n-\alpha}{2}} f(s)$ and $H(r) = \int_r^1 h(s)\,ds$. Then, since $H(1) = 0$ and $(-a-2)q + n > 0$, integration by parts yields

$$\int_0^1 r^{-(a+2)q} H^q(r) r^{n-1}\,dr \le C \int_0^1 r^{-(a+2)q+n} H^{q-1}(r) h(r)\,dr$$

$$\le C \left(\int_0^1 r^{-(a+2)q+n-1} H^q(r)\,dr \right)^{\frac{q-1}{q}} \left(\int_0^1 r^\gamma h^q(r)\,dr \right)^{\frac{1}{q}}$$

where $\gamma = -(a+2)q + n + q - 1$. Now, $r^\gamma h^q(r) = r^{n-1} f^q(r)$ and it follows that

$$\left(\int_{B_1} B^q \right)^{1/q} \le C \|f\|_{L^q} \tag{1.8}$$

Combining (1.7) and (1.8) gives $\|u/r^2\|_{L^q} \le C\|f\|_{L^q}$.

To get (1.5), using equation (1.6), it suffices to show that $u'/r \in L^q$. From the definition of $u = \Phi(f)$, we see that

$$u'/r = -a \cdot u/r^2 - \alpha A$$

and the estimate follows from our previous analysis.

□

Existence or uniqueness hold in other functional spaces, as the two following lemmas show :

Lemma 1.3.

Recall (0.1), (0.2), (0.2').
Let f *be such that* $\int_\Omega |f| \cdot |x|^{-a} dist(x, \partial\Omega)\, dx < \infty$.
There exists at least one weak solution u *with* $u \cdot |x|^{-2} \in L^1(\Omega)$, *of*

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2} u = f \text{ in } \Omega \\ \qquad u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.9}$$

Furthermore,

$$\|u\|_{L^1_{-2}} \le C\|f\|_{L^1_{-a,\delta}} \tag{1.10}$$

$$\|u\|_{L^\infty_{-a}} \le C\|f\|_{L^\infty} \tag{1.11}$$

$$\|u\|_{L^\infty_{-b}} \le C\|f\|_{L^\infty_{-b-2}} \qquad \text{for } a < b < a' \tag{1.11'}$$

Proof. (Case $0 < c < c_0$)

We assume, without loss of generality, that $f \ge 0$ (for the general case, apply the result to the positive and negative parts of f).
Let $f_k = \min(f, k)$ for $k \in \mathbb{N}$. Then, $f_k \nearrow f$ in $L^1_{-a,\delta}$.
By Lemma 1.1, there exists u_k, unique solution in $H^1_0(\Omega)$ of (1.9) with f_k in place of f. Clearly, $\{u_k\}$ is monotone increasing.

Let ζ_0 be the $H^1_0(\Omega)$ solution of

$$\begin{cases} -\Delta \zeta_0 - \dfrac{c}{|x|^2} \zeta_0 = 1 \text{ in } \Omega \\ \qquad \zeta_0 = 0 \text{ on } \partial\Omega \end{cases} \tag{1.12}$$

When $\Omega = B_1$, $\zeta_0 = \zeta_0^1 := C(|x|^{-a} - |x|^2)$, for some $C > 0$. Otherwise, $\Omega \subset B_R$ for some $R > 0$ and $C \cdot \zeta_0^1(x/R)$ is a supersolution of problem (1.12), for some $C > 0$. So,

$$0 \le \zeta_0 \le C|x|^{-a}\, \delta(x) \qquad \text{in } \Omega \tag{1.13}$$

Since u_k and $\zeta_0 \in H^1_0(\Omega)$, they are valid test functions in their respective equations and

$$\int_\Omega \nabla u_k \nabla \zeta_0 - \int_\Omega \frac{c}{|x|^2} u_k \zeta_0 = \int_\Omega u_k = \int_\Omega f_k \zeta_0$$

Since $f \ge 0$, so are f_k and u_k and

$$\|u_k\|_{L^1} = \int_\Omega f_k \zeta_0 \le C\|f_k\|_{L^1_{-a,\delta}} \tag{1.14}$$

Let ζ_1 be the smooth solution of

$$\begin{cases} -\Delta\zeta_1 = 1 \text{ in } \Omega \\ \zeta_1 = 0 \text{ on } \partial\Omega \end{cases} \tag{1.15}$$

and integrate in the equation satisfied by u_k :

$$\int_\Omega u_k - \int_\Omega \frac{c}{|x|^2} u_k \zeta_1 = \int_\Omega f_k \zeta_1 \tag{1.16}$$

Using (1.14) and (1.16) and the inequality $m\delta(x) \leq \zeta_1 \leq M\delta(x)$, where m, M are some positive constants, we get

$$\|u_k\|_{L^1_{-2}} \leq C\|f_k\|_{L^1_{-a,\delta}}$$

It is then easy to construct by monotonicity a solution of (1.9) satisfying (1.10).

For estimate (1.11), one should just check that if $f \in L^\infty$, $\|f\|_{L^\infty}\zeta_0$ is a supersolution of (1.9) and apply the maximum principle (see e.g. Lemma 1.4). Hence,

$$u \leq \|f\|_{L^\infty}\zeta_0$$

Applying this estimate to $-u$ yields (1.11).

For estimate (1.11'), $\|f\|_{L^\infty_{-b-2}}\zeta_2$ provides a supersolution of (1.9) where

$$\begin{cases} -\Delta\zeta_2 - \dfrac{c}{|x|^2}\zeta_2 = |x|^{-b-2} \text{ in } \Omega \\ \zeta_2 = 0 \text{ on } \partial\Omega \end{cases} \tag{1.15}$$

Observe that in the radial case $\zeta_2 = C(|x|^{-b} - |x|^{-a})$ so that in general $0 \leq \zeta_2 \leq C|x|^{-b}$ and that Lemma 1.4 may be applied because $a < b < a'$.

□

Remark 1.3.

- In view of Lemma 1.5, for equation (1.9) to have a solution with $f \in L^1_\delta$, it may be necessary that $f \in L^1_{-a,\delta}$.
- In the case $0 < c < c_0$, if $\int_\Omega |f| \cdot |x|^{-a} \cdot |\ln(x)| \cdot \delta(x)\, dx < \infty$, that is, if we ask a little more regularity on f, then $u \in L^1_{-a-2}$ and is therefore unique (using Lemma 1.4 .) For a proof, use the methods of the lemma with ζ_2 solving

$$\begin{cases} -\Delta\zeta_3 - \dfrac{c}{|x|^2}\zeta_3 = |x|^{-a-2} & \text{in } \Omega \\ \zeta_3 = 0 & \text{on } \partial\Omega \end{cases}$$

When $\Omega = B_1$, $\zeta_3 = C|x|^{-a}\ln(1/|x|)$.

Proof of Proposition 0.1 (case $0 < c < c_0$).

Suppose first that u is a strong solution of $(P_{t,p})$.
Let $\zeta_n \in C_c^\infty(\Omega \setminus \{0\})$ be such that $0 \le \zeta_n \le 1$, $|\nabla \zeta_n| \le Cn$, $|\Delta \zeta_n| \le Cn^2$ and

$$\zeta_n = \begin{cases} 0 & \text{if} \quad |x| \le 1/n \\ 1 & \text{if} \quad |x| \ge 2/n \end{cases}$$

Multiplying $(P_{t,p})$ by $u\zeta_n$ and integrating by parts, it follows that

$$\int_\Omega \left(\frac{c}{|x|^2} u + u^p + tf \right) u\zeta_n = -\int_\Omega \Delta u \, u\zeta_n = \int_\Omega |\nabla u|^2 \zeta_n + \int_\Omega u \nabla u \nabla \zeta_n$$

Since $u \le C|x|^{-a}$ and $p < p_0$, $u^p \le C|x|^{-a-2}$. Hence, on the one hand,$u^p \in L^{\frac{2n}{n+2}}(\Omega)$ and on the other hand, the left-hand-side integral in the above equation is bounded by $C \int_\Omega |x|^{-2a-2} \le C$, whereas $\left|\int_\Omega u \nabla u \nabla \zeta_n\right| = \left| \frac{1}{2} \int_\Omega u^2 \Delta \zeta_n \right| \le Cn^2 \int_{1/n<|x|<2/n} |x|^{-2a} \to 0$ as $n \to \infty$. Hence $\int_\Omega |\nabla u|^2 \zeta_n \le C$ and $u \in H_0^1(\Omega)$.

Multipying $(P_{t,p})$ by $\phi\zeta_n$ for $\phi \in C_c^\infty(\Omega)$ yields

$$\int_\Omega \left(\frac{c}{|x|^2} u + u^p + tf \right) \phi\zeta_n = -\int_\Omega \Delta u \, \phi\zeta_n = \int_\Omega \zeta_n \nabla u \nabla \phi + \int_\Omega \phi \nabla u \nabla \zeta_n$$

The last term in the right-hand-side can be rewritten as

$$\int_\Omega \phi \nabla u \nabla \zeta_n = \int_\Omega \nabla(u\phi) \nabla \zeta_n - \int_\Omega u \nabla \phi \nabla \zeta_n = -\int_\Omega u\phi \Delta \zeta_n - \int_\Omega u \nabla \phi \nabla \zeta_n$$

and converges to zero as in the previous case when $n \to \infty$.
It follows that u is an $H_0^1(\Omega)$ solution of $(P_{t,p})$.

Approximating $u \in H_0^1(\Omega)$ by smooth functions and integrating by parts implies that $H_0^1(\Omega)$ solutions are weak solutions.

Suppose now that u is a weak solution satisfying the estimate $u \le C|x|^{-a}$.
Then as before, $u^p \le C|x|^{-a-2} \in L^{\frac{2n}{n+2}}(\Omega) \subset H^{-1}(\Omega)$.
Letting $g = u^p + tf$, it follows from Lemma 1.1 that there exists a weak solution $v \in H_0^1(\Omega)$ of (1.9) with g in place of f. u is also a weak solution of (1.9) and by Remark 1.3, we must have $u = v \in H_0^1(\Omega)$. Hence, u is an $H_0^1(\Omega)$ solution.

Finally if u is an $H_0^1(\Omega)$ solution satisfying the estimate $u \le C|x|^{-a}$, using local elliptic regularity theorems in $\Omega \setminus B_\epsilon$ for an arbitrary $\epsilon > 0$, we may conclude that $u \in C^\infty(\bar{\Omega} \setminus \{0\})$ and satisfies $(P_{t,p})$ in the strong sense.

Lemma 1.4 (Maximum Principle).

If $\int_\Omega |u| \cdot |x|^{-a-2} < \infty$ *and if*

$$-\Delta u - \frac{c}{|x|^2} u \geq 0 \qquad \textit{in the weak sense.} \tag{1.16}$$

then

$$u \geq 0 \ a.e.$$

Proof (case $0 < c < c_0$*).*

It is enough to show that $\int_\Omega u\phi \geq 0$ for $\phi \in C_c^\infty(\Omega \setminus \{0\})$, $\phi \geq 0$.

For such a ϕ and $\epsilon > 0$, construct $v_\epsilon \in C^2(\bar{\Omega})$, $v_\epsilon \geq 0$, solving

$$\begin{cases} -\Delta v_\epsilon - \dfrac{c}{|x|^2 + \epsilon} v_\epsilon = \phi \text{ in } \Omega \\ \qquad\qquad v_\epsilon = 0 \text{ on } \partial\Omega \end{cases}$$

Also let $v \in H_0^1(\Omega)$ be the solution of

$$\begin{cases} -\Delta v - \dfrac{c}{|x|^2} v = \phi \text{ in } \Omega \\ \qquad\qquad v = 0 \text{ on } \partial\Omega \end{cases}$$

Using Lemma 1.1, since $-\Delta(v_\epsilon - v) - \frac{c}{|x|^2}(v_\epsilon - v) \leq 0$,

$$0 \leq v_\epsilon \leq v \qquad \text{a.e. in } \Omega \tag{1.17}$$

Applying (1.11) in Lemma 1.3 to v,

$$0 \leq v \leq C|x|^{-a} \qquad \text{a.e. in } \Omega \tag{1.18}$$

Combining (1.17) and (1.18),

$$0 \leq v_\epsilon \leq C|x|^{-a} \qquad \text{a.e. in } \Omega \tag{1.19}$$

Applying (1.16) with $\phi = v_\epsilon$,

$$\int_\Omega u \left(-\Delta v_\epsilon - \frac{c}{|x|^2} v_\epsilon \right) \geq 0$$

Since $-\Delta v_\epsilon - \frac{c}{|x|^2} v_\epsilon = \phi - c \left\{ \dfrac{1}{|x|^2} - \dfrac{1}{|x|^2 + \epsilon} \right\}$,

$$\int_\Omega u\phi \geq \int_\Omega c \left\{ \frac{1}{|x|^2} - \frac{1}{|x|^2 + \epsilon} \right\} u\, v_\epsilon.$$

Clearly, $\{v_\epsilon\}$ is monotone increasing and converges pointwise to a finite value a.e. in Ω by (1.19).

So the integrand in the right hand side of the previous equation converges a.e. to 0. Using (1.19) and $u \in L^1_{-a-2}$, this integrand is dominated by an L^1 function.

By Lebesgue's theorem, we conclude that

$$\int_\Omega u\phi \geq 0.$$

□

Remark 1.4.

- This maximum principle is sharp in the following sense :
  if $q > -a$ then there exists $u \in L^1_{q-2}$ such that $-\Delta u - \frac{c}{|x|^2} u = 0$ yet $u \not\equiv 0$.
  Just take $\Omega = B_1$ and $u := |x|^{-a'} - |x|^{-a}$, with $-a'$ and $-a$ defined in (0.2), (0.2').

We conclude this section with a lemma giving necessary conditions for the existence of a solution to the linear problem.

Lemma 1.5.
Suppose $f \geq 0$ a.e. , $f \not\equiv 0$, $\int_\Omega f(x)\, dist(x, \partial\Omega)\, dx < \infty$.
If u is a nonnegative weak solution of

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2} u = f \ \text{ in } \Omega \\ \qquad\qquad u = 0 \ \text{ on } \partial\Omega \end{cases} \tag{1.20}$$

Then there exists a constant $C > 0$ depending only on Ω such that

$$u \geq C \left(\int_\Omega f\zeta_0 \right) \zeta_0 \qquad a.e. \ in \ \Omega$$

with ζ_0 defined by (1.12). In particular, for some $m > 0$

$$u \geq m|x|^{-a} \ \text{ a.e. near the origin}$$

Furthermore, for any $\epsilon > 0$, if u denotes the minimal solution of (1.20) then

$$\int_\Omega u \cdot |x|^{-a-2+\epsilon} dx \leq C_\epsilon \int_\Omega f \cdot |x|^{-a} dist(x, \partial\Omega)\, dx < \infty$$

Most of the results of this lemma are a direct consequence of a more general theorem on the associated evolution equation, established by Baras and J. Goldstein (see [BG] Th 2.2 page 124.) We give here a simple proof for convenience of the reader.

Proof (case $0 < c < c_0$).

Step 1. $\mathbf{u \geq m|x|^{-a}}$ near the origin.

Let $f_1 = \min(f, k)$ with $k > 0$ such that $f_1 \not\equiv 0$ and $u_1 \geq 0$ be the minimal solution of

$$\begin{cases} -\Delta u_1 - \dfrac{c}{|x|^2} u_1 = f_1 \text{ in } \Omega \\ u_1 = 0 \text{ on } \partial\Omega \end{cases}$$

Since u is a supersolution of the above problem, u_1 is well defined and $0 \leq u_1 \leq u$ so it suffices to prove the result for u_1.

Since $f_1 \in L^\infty(\Omega)$, on the one hand $0 \leq u_1 \leq C|x|^{-a}$ by (1.11) and on the other hand the equation has a solution $v \in H_0^1(\Omega)$. By Lemma 1.4, we must have $u_1 = v$.

Now, since $u_1 \not\equiv 0$, $u_1 \geq 0$ and $-\Delta u_1 \geq 0$ in the connected set Ω, we have for some $\epsilon > 0$ and $\eta > 0$,

$$u_1 \geq \epsilon \qquad \text{a.e. in } B_{2\eta}$$

Choose $C > 0$ so that $\epsilon \geq Cr^{-a}$ for $r \geq \eta$ and let $z = (u_1 - C|x|^{-a})^-$. We claim that $z \in H_0^1(B_\eta)$. Indeed, if $u_n = u_1 * \rho_n$, with ρ_n a mollifier, then $u_n \geq \epsilon$ in B_η for n sufficiently large and $(u_n - C|x|^{-a})^- \in H_0^1(B_\eta)$. Passing to the limit with $n \to \infty$ proves the claim.

Next, we multiply $u_1 - C|x|^{-a}$ by z and integrate by parts :

$$\begin{aligned} 0 \geq -\int_\Omega |\nabla z|^2 + \int_\Omega \frac{c}{|x|^2} z^2 &= \int_\Omega \nabla(u_1 - C|x|^{-a})\nabla z - \int_\Omega \frac{c}{|x|^2}(u_1 - C|x|^{-a})z \\ &= \int_\Omega fz - C\left(\int_{B_\eta} \nabla|x|^{-a}\nabla z - \int_{B_\eta} \frac{c}{|x|^2}|x|^{-a} z\right) \\ &\geq -C\int_{\partial B_\eta} z\partial_\nu |x|^{-a} \geq 0 \end{aligned}$$

And hence $z \equiv 0$ in B_η.

Step 2. $\mathbf{u \geq C(K,\Omega)\int_\Omega f\zeta_0}$ in $\mathbf{K \subset\subset \Omega}$ when $\mathbf{f \in L^\infty(\Omega)}$

The proof is an adaptation of Lemma 3.2 in [BC]. Observe that up to replacing u by the minimal nonnegative solution of the problem, we may assume u to be an $H_0^1(\Omega)$ solution satisfting $0 \leq u \leq C|x|^{-a}$.

Let $\rho = dist(K, \partial\Omega)/2$ and take m balls of radius ρ such that

$$K \subset B_\rho(x_1) \cup \cdots \cup B_\rho(x_m) \subset \Omega$$

Let $\zeta_1, \ldots, \zeta_m$ be the solutions (given, say, by Lemma 1.1) of

$$\begin{cases} -\Delta \zeta_i - \dfrac{c}{|x|^2}\zeta_i = \chi_{B_\rho(x_i)} \text{ in } \Omega \\ \zeta_i = 0 \text{ on } \partial\Omega \end{cases}$$

where χ_A denotes the characteristic function of A. There is a constant $C > 0$ such that

$$\zeta_i(x) \geq C\zeta_0(x) \qquad \text{in } \Omega \qquad \text{for } 1 \leq i \leq m$$

Indeed, by Step 1, this inequality must hold near the origin and by Hopf's boundary lemma, we also have $\zeta_i \geq c\delta \geq C\zeta_0$ away from the origin.

Let now $x \in K$, and take a ball $B_\rho(x_i)$ containing x. Then $B_\rho(x_i) \subset B_{2\rho}(x) \subset \Omega$ and, since $-\Delta u \geq 0$ in Ω, we conclude

$$\begin{aligned} u(x) &\geq \fint_{B_{2\rho}(x)} u = C \int_{B_{2\rho}(x)} u \geq C \int_{B_\rho(x_i)} u \\ &= C \int_\Omega u \left(-\Delta\zeta_i - \frac{c}{|x|^2}\zeta_i \right) = C \int_\Omega f\zeta_i \\ &\geq C \int_\Omega f\zeta_0 \end{aligned}$$

Step 3. $\mathbf{u \geq C(\Omega) \left(\int_\Omega f\zeta_0\right) \zeta_0}$ **in** Ω **when** $\mathbf{f \in L^\infty(\Omega)}$

Suppose without loss of generality that $B_1 \subset \Omega$ and let $K = \bar{B_1} \setminus B_{1/2}$. By Step 2, it suffices to prove the inequality in $\Omega \setminus K$. Let w be the solution of

$$\begin{cases} -\Delta w - \dfrac{c}{|x|^2} w = 0 \text{ in } \Omega \setminus B_1 \\ \qquad\qquad w = 0 \text{ on } \partial\Omega \\ \qquad\qquad w = 1 \text{ on } \partial B_1 \end{cases}$$

and extend w by $w := (2|x|)^{-a}$ in $B_{1/2}$, so that the above equation still holds in $\Omega \setminus K$ with $w|_{\partial K} \equiv 1$. By Hopf's boundary Lemma applied in $\Omega \setminus B_1$, we conclude that

$$w \geq C\zeta_0 \qquad \text{in } \Omega \setminus K$$

u is assumed to be a strong solution so we can apply the maximum principle (Lemma 1.4) in $\Omega \setminus K$ to conclude that

$$u \geq C \left(\int_\Omega f\zeta_0 \right) w \geq C \left(\int_\Omega f\zeta_0 \right) \zeta_0 \qquad \text{in } \Omega \setminus K$$

Step 4. $\mathbf{\int_\Omega |x|^{-a} f \delta(x) < \infty}$**.**

We assume for now that $f \in L^\infty(\Omega)$ and that $u \geq 0$ is the minimal solution of (1.20).

We let $\{\phi_n\}$ be a sequence of smooth, nonnegative and bounded functions converging pointwise and monotonically to $c|x|^{-a-2}$ and construct v_n as the (smooth) solution of

$$\begin{cases} -\Delta v_n = \phi_n \text{ in } \Omega \\ \quad v_n = 0 \text{ on } \partial\Omega \end{cases}$$

Testing v_n in (1.20) yields

$$\int_\Omega f v_n = \int_\Omega u \left(-\Delta v_n - \frac{c}{|x|^2} v_n \right) = \int_\Omega u \left(\phi_n - \frac{c}{|x|^2} v_n \right) \tag{1.21}$$

Now $\phi_n \nearrow c|x|^{-a-2}$ pointwise and in L^1, so, by Lemma 2.1, $v_n \nearrow |x|^{-a} - w$ pointwise and in L^1, where w solves

$$\begin{cases} -\Delta w = 0 \text{ in } \Omega \\ \quad w = |x|^{-a} \text{ on } \partial\Omega \end{cases}$$

Since u is minimal, $0 \le u \le C|x|^{-a}$ by (1.11) and we can safely pass to the limit in (1.21) to obtain

$$\int_\Omega (|x|^{-a} - w) f = \int_\Omega u \left(c|x|^{-a-2} - \frac{c}{|x|^2}(|x|^{-a} - w) \right) = c \int_\Omega \frac{u}{|x|^2} w$$

Observe that w is bounded and that $|x|^{-a} - w \ge C|x|^{-a}\delta(x)$, hence

$$\int_\Omega |x|^{-a} f \delta(x) \le C \int_\Omega |x|^{-2} u$$

This estimate holds when $f \in L^\infty$ and u is minimal but also in the general case, as approximation of f by $f_n = \min(n, f)$ shows.

Step 5. $\mathbf{u \ge C(\Omega)\left(\int_\Omega f\zeta_0\right)\zeta_0}$ **in** Ω **when** $\mathbf{f \in L^1_{-a,\delta}}$

Let $k > 0$ be so large that $f_k = \min(f, k) \not\equiv 0$. Then u is a supersolution of (1.20) with f_k in place of f and by Step 3, we have

$$u \ge C(\Omega) \left(\int_\Omega f_k \zeta_0 \right) \zeta_0$$

Letting $k \to \infty$, Lebesgue's theorem yields the desired result.

Step 6. $\int_\Omega \mathbf{|x|^{-a-2+\epsilon} u < \infty}$.

We proceed as in Step 4, only this time we let $\phi_n \nearrow -P(-a+\epsilon)|x|^{-a-2+\epsilon}$, where $P(X) = X(X-1) + (n-1)X + c$ and construct v_n solving

$$\begin{cases} -\Delta v_n - \dfrac{c}{|x|^2 + 1/n} v_n = \phi_n \text{ in } \Omega \\ \qquad\qquad v_n = 0 \text{ on } \partial\Omega \end{cases}$$

Hence,

$$\int_\Omega f v_n = \int_\Omega u \phi_n + \int_\Omega \left(\frac{c}{|x|^2 + 1/n} - \frac{c}{|x|^2} \right) v_n u \tag{1.22}$$

If ζ solves

$$\begin{cases} -\Delta \zeta - \dfrac{c}{|x|^2} \zeta = -P(-a+\epsilon)|x|^{-a-2+\epsilon} \text{ in } \Omega \\ \qquad \zeta = 0 \text{ on } \partial\Omega \end{cases}$$

then we have $0 \le v_n \le \zeta \le C|x|^{-a}$. Indeed, if $\Omega = B_1$, then $\zeta = \zeta^1 := C(|x|^{-a} - |x|^{-a+\epsilon})$. Otherwise, $\Omega \subset B_R$ for some $R > 0$ and $C\zeta^1(x/R)$ is a supersolution of the problem, for some $C > 0$.

By Step 4, $\int_\Omega f v_n \le \int_\Omega f\zeta < \infty$. Assuming first that f is bounded (whence $u \le C|x|^{-a}$) and then working by approximation, it follows from Lebesgue's theorem and from (1.22) that

$$\int_\Omega |x|^{-a-2+\epsilon} u \leq C_\epsilon \int_\Omega f\zeta < \infty$$

Remark. More results about the linear theory of our operator, with $c \in \mathbb{R}$ arbitrary have been detailed by F. Pacard in unpublished work (see [P].)

2. Existence vs. complete blow-up

In this section, we will prove existence or nonexistence of weak solutions of $(P_{t,p})$, using the tools we have just constructed and monotonicity arguments.

2.1. Case $\mathbf{p < p_0}$, $\mathbf{c < c_0}$: existence for small $\mathbf{t > 0}$.

p_0 has been defined so that $p_0 a = a + 2$. So, for $p < p_0$, $ap < a + 2$ and for some $b \in (a, a')$, the inequality $bp < b + 2$ still holds. We fix such a b and prove that for an appropriate choice of $A > 0$ and for $t > 0$ small,

$$w := A|x|^{-b} \in H^1(\Omega) \text{ is a supersolution of } (P_{t,p}).$$

Observe that $w \in H^1(\Omega)$ as long as b is close enough to a, which may be assumed. We have

$$-\Delta w - \frac{c}{|x|^2} w = -AP(-b)|x|^{-b-2} \qquad \text{where } P(X) = X(X-1) + (n-1)X + c$$

Observe that $P(-b) < 0$ since $b \in (a', a)$ and a' and a are the roots of $P(X)$.
We would like to have $-AP(-b)|x|^{-b-2} \geq A^p|x|^{-pb} + tf$ in Ω. This will be true as soon as

$$\begin{cases} -\dfrac{1}{2}AP(-b)|x|^{-b-2} \geq A^p|x|^{-pb} & \text{and} \\ -\dfrac{1}{2}AP(-b)|x|^{-b-2} \geq tf \end{cases}$$

The first inequality amounts to

$$A \leq \left[-\frac{1}{2}P(-b)|x|^{pb-b-2}\right]^{\frac{1}{p-1}}$$

which will be satisfied, taking $R > 0$ such that $\Omega \subset B_R$, if

$$A \leq \left[-\frac{1}{2}P(-b)R^{pb-b-2}\right]^{\frac{1}{p-1}}$$

since $pb - b - 2 < 0$.
With such a choice of A, pick any $t > 0$ such that

$$-\frac{1}{2}AP(-b)R^{-b-2} \geq t\|f\|_{L^\infty}$$

We have just constructed $w \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta w - \dfrac{c}{|x|^2} w \geq w^p + tf & \text{in } \Omega \\ \qquad\qquad w \geq 0 & \text{on } \partial\Omega \end{cases}$$

Finally we construct an $H_0^1(\Omega)$ supersolution of $(P_{t,p})$. We let w_1 be a smooth extension inside Ω of $w|_{\partial\Omega}$ which is also supported away from the origin. Then $g = \Delta w_1 + \dfrac{c}{|x|^2} w_1$ is smooth and bounded and using Lemma 1.1, there is a unique strong solution z of

$$\begin{cases} -\Delta z - \dfrac{c}{|x|^2} z = g \text{ in } \Omega \\ \qquad\qquad z = 0 \text{ on } \partial\Omega \end{cases} \tag{2.1}$$

Letting $w_2 = z + w_1$, it follows that

$$\begin{cases} -\Delta w_2 - \dfrac{c}{|x|^2} w_2 = 0 \text{ in } \Omega \\ \qquad\qquad w_2 = w \text{ on } \partial\Omega \end{cases} \tag{2.2}$$

Multiplying by w_2^-, it follows that $w_2 \geq 0$ a.e. in Ω.

It is now clear that $\tilde{w} = w - w_2$ is an $H_0^1(\Omega)$ supersolution of $(P_{t,p})$. For convenience, we drop the superscript $\tilde{}$ thereafter.

Construction of a minimal solution u of $(P_{t,p})$ is now just a matter of monotone iteration. For this purpose we recall the following lemma, proved in [BCMR] :

Lemma 2.1. *Suppose $\int_\Omega |f(x)|\, dist(x, \partial\Omega) < \infty$. Then there exists a unique $v \in L^1(\Omega)$ which is a weak solution of*

$$\begin{cases} -\Delta v = f \;\; in \; \Omega \\ \quad v = 0 \;\; on \; \partial\Omega \end{cases}$$

Moreover,

$$\|v\|_{L^1} \leq C\|f\|_{L^1_\delta}$$

Moreover if $v \in L^1(\Omega)$ and $-\Delta v \geq 0$ weakly, i.e. if

$$\int_\Omega (-\Delta\phi)\, v \geq 0 \qquad for\ all\ \phi \in C^2(\bar{\Omega}),\ \phi|_{\partial\Omega} \equiv 0,\ \phi \geq 0\ in\ \Omega$$

then

$$v \geq 0 \qquad a.e.\ in\ \Omega$$

Define $\{u_k\}$ by induction to be the L^1 weak solutions of

$$\begin{cases} -\Delta u_0 = tf & \text{in } \Omega \\ \quad u_0 = 0 & \text{on } \partial\Omega \end{cases} \qquad \text{for } k = 0$$

$$\begin{cases} -\Delta u_k = \dfrac{c}{|x|^2} u_{k-1} + u_{k-1}^p + tf & \text{in } \Omega \\ \quad u_k = 0 & \text{on } \partial\Omega \end{cases} \quad \text{for } k \geq 1$$

We now check that this definition makes sense and that (u_k) is monotone and satisfies $0 \leq u_k \leq w$ a.e. in Ω.

For u_0 there is nothing to prove. Suppose the result true up to order $k-1$. Then

$$0 \leq \frac{c}{|x|^2} u_{k-1} + u_{k-1}^p + tf \leq \frac{c}{|x|^2} w + w^p + tf$$
$$\leq C|x|^{-a-2} \in L^1(\Omega)$$

So u_k is well defined using the previous lemma, $u_k \geq 0$ a.e. and since

$$-\Delta(u_k - u_{k-1}) = \frac{c}{|x|^2}(u_{k-1} - u_{k-2}) + u_{k-1}^p - u_{k-2}^p \geq 0 \text{ by induction hypothesis}$$

and similarly $-\Delta(w - u_k) \geq 0$, we conclude using Lemma 2.1 that

$$0 \leq u_{k-1} \leq u_k \leq w \qquad \text{a.e. in } \Omega$$

By a standard monotone convergence argument, $\{u_k\}$ converges to a weak solution of $(P_{t,p})$.

2.2. Pushing t to $\mathbf{t_0}$.

We let $t_0 = \sup\{t : (P_{t,p}) \text{ has a weak solution.}\}$ and adapt the methods of [BCMR].

If ϕ_1 is a positive eigenvector of $-\Delta$ (with zero Dirichlet condition) associated to its first eigenvalue λ_1, in other words if $\phi_1 > 0$ in Ω and, for some $\lambda_1 > 0$,

$$\begin{cases} -\Delta\phi_1 = \lambda_1 \, \phi_1 \text{ in } \Omega \\ \quad \phi_1 = 0 \text{ on } \partial\Omega \end{cases}$$

and if u is a weak solution of $(P_{t,p})$, testing against ϕ_1 yields

$$\int_\Omega \frac{c}{|x|^2} u \, \phi_1 + \int_\Omega u^p \phi_1 + t \int_\Omega f \, \phi_1 = \lambda_1 \int_\Omega u \, \phi_1$$

and, by Young's inequality,

$$\int_\Omega u \, \phi_1 \leq \frac{1}{2} \int_\Omega u^p \phi_1 + C \int_\Omega \phi_1.$$

Thus,

$$t \int_\Omega f \, \phi_1 + \int_\Omega \frac{c}{|x|^2} u \, \phi_1 + \int_\Omega u^p \phi_1 \leq C \tag{2.3}$$

which implies $t_0 < \infty$. In particular, there are no weak solutions of $(P_{t,p})$ for $t > t_0$. This implies complete blow-up (see Definition 0.1), as the following proposition shows.

Proposition 2.1.
Suppose (0.1) holds, $p > 1$ and $t > 0$.
If $(P_{t,p})$ has no weak solution then there is complete blow-up.

Proof.
The proof is an easy adaptation of Theorem 3.1 in [BC].
Suppose indeed that $(P_{t,p})$ has no weak solution and by contradiction that $\int_\Omega g_n(\underline{u_n})\,\delta + \int_\Omega a_n\underline{u_n}\,\delta \leq C$, where $\{a_n\}, \{g_n\}, \{u_n\}$, are given in Definition 0.1 .
Then, multiplying (P_n) by ζ_1, solution of (1.15) we get
$\int_\Omega \underline{u_n}(-\Delta z) - \int_\Omega a_n\underline{u_n}z = \int_\Omega g_n(\underline{u_n})z + \int_\Omega tf\,z.$
Hence, $\int_\Omega \underline{u_n} \leq C$ and there exists a u such that $\underline{u_n} \nearrow u$ in $L^1(\Omega)$, by monotone convergence.
Since $\{a_n\}$ and $\{g_n\}$ converge monotonically, we can pass to the limit in (P_n), using monotone convergence again and obtain a solution u of $(P_{t,p})$, which is a contradiction.
We have just proved that $\int_\Omega g_n(\underline{u_n})\,\delta + \int_\Omega a_n\underline{u_n}\,\delta \to \infty$. Now, using (P_n) and Lemma 3.2 in [BC], it follows that

$$\frac{\underline{u_n}(x)}{\delta(x)} \geq C(\Omega)\left(\int_\Omega g_n(\underline{u_n})\,\delta + \int_\Omega a_n\underline{u_n}\,\delta\right) \to \infty \quad \square$$

Next, we want to prove that if $(P_{\tau,p})$ has a solution then so does $(P_{t,p})$ for $0 < t \leq \tau$. This is true because u_τ is a supersolution of $(P_{t,p})$ in the sense that, weakly,

$$-\Delta u_\tau \geq \frac{c}{|x|^2}u_\tau + u_\tau^p + tf$$

and with the help of Lemma 2.1, we may construct a solution of $(P_{t,p})$ by monotone iteration.

Finally, we prove that $(P_{t_0,p})$ has a weak solution. Choose a nondecreasing sequence $\{t_n\}$ converging to t_0 and for each $n \in \mathbb{N}$, let u_n be a (weak) solution of $(P_{t_n,p})$. Since $\phi_1 \geq m\delta(x)$ for some $m > 0$, equation (2.3) implies that

$$\int_\Omega \frac{c}{|x|^2}u_n\,\delta(x) + \int_\Omega u_n^p\delta(x) \leq C$$

Multiplying by ζ_1, solution of (1.15) then implies boundedness of $\{u_n\}$ in L^1 and hence monotone convergence to a solution of $(P_{t_0,p})$ as $t_n \to t_0$.

2.3. Case $0 < c < c_0$, $p \geq p_0$: blow-up for all $t > 0$.

By Proposition 2.1, we just need to prove that there are no weak solutions of $(P_{t,p})$ for $p \geq p_0$. Assume by contradiction there exists one and call it u. If we apply Lemma 1.5 with $u^p + tf$ in place of f, it follows that

$$\int_\Omega u^p|x|^{-a}\,\delta(x) < \infty \qquad \text{and} \qquad u \geq m|x|^{-a} \qquad \text{a.e. near the origin.}$$

Using Hölder's inequality,

$\int_\Omega u|x|^{-a-2}\,\delta(x) \le \left(\int_\Omega u^p|x|^{-a}\,\delta(x)\right)^{1/p} \cdot \left(\int_\Omega |x|^{-a-2\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$.

If $p \ge p_0$ and $c < c_0$ then $-a - 2\dfrac{p}{p-1} > -n$, hence, since $u \in L^1(\Omega)$,

$$\int_\Omega u\,|x|^{-a-2} < \infty \tag{2.4}$$

Suppose without loss of generality, that $\Omega \subset B_1$ and define $w = A|x|^{-a}\ln(\frac{1}{|x|})$ for some $A > 0$.

Then $-\Delta w - \frac{c}{|x|^2}w = A\sqrt{(n-2)^2-4c}\,|x|^{-a-2}$. Also,

$$-\Delta u - \frac{c}{|x|^2}u \ge u^p \ge m|x|^{-ap} \ge m|x|^{-a-2} \qquad \text{in } B_\eta\text{, for a fixed small } \eta > 0.$$

Let $A = m(\sqrt{(n-2)^2-4c} + c\ln\frac{1}{\eta})^{-1}$ and $C = A\eta^{-a}\ln\frac{1}{\eta}$.

Finally define $z = u + C - w$. Using (2.4), $z \in L^1(B_\eta, |x|^{-a-2}dx)$. Furthermore,

$$\begin{aligned}
-\Delta z - \frac{c}{|x|^2}z &\ge u^p - \frac{cC}{|x|^2} - A\sqrt{(n-2)^2-4c}\,|x|^{-a-2}\\
&\ge m|x|^{-ap} - cC|x|^{-2} - A\sqrt{(n-2)^2-4c}\,|x|^{-a-2}\\
&\ge |x|^{-2}\left[m|x|^{-a} - cC - A\sqrt{(n-2)^2-4c}\,|x|^{-a}\right]\\
&\ge |x|^{-2}\left[(m - A\sqrt{(n-2)^2-4cy})\,\eta^{-a} - cC\right]\\
&\ge 0
\end{aligned}$$

All these inequalities hold in the weak sense in B_η (since our choice of constants implies $z|_{\partial B_\eta} \ge C - w|_{\partial B_\eta} \ge 0$.)

Applying Lemma 1.4, we conclude

$$u \ge A|x|^{-a}\ln\frac{1}{|x|} - C \qquad \text{a.e. in } B_\eta$$

Choosing A and η smaller, we may assume that

$$u \ge A|x|^{-a}\ln\frac{1}{|x|} \ge 1 \qquad \text{a.e. in } B_\eta$$

The next step is to consider the function $\Phi \in C^1(\mathbb{R})$ defined by

$$\Phi(x) = \begin{cases} \ln x & \text{if } x \ge 1\\ x - 1 & \text{otherwise.}\end{cases}$$

and apply Lemma 1.7 in [BC] to conclude that in B_η

$$\begin{aligned}
-\Delta(\ln u) &\ge \frac{-\Delta u}{u} \ge u^{p-1} \ge A^{p-1}|x|^{-a(p-1)}\left(\ln\frac{1}{|x|}\right)^{p-1}\\
&\ge A^{p-1}|x|^{-2}\left(\ln\frac{1}{|x|}\right)^{p-1}
\end{aligned}$$

Now if $v = \left(\ln \frac{1}{|x|}\right)^p$, a computation yields

$$-\Delta v \leq C|x|^{-2}\left(\ln \frac{1}{|x|}\right)^{p-1}$$

And by the L^1 maximum principle (Lemma 2.1),

$$\ln u \geq d\left(\ln \frac{1}{|x|}\right)^p - C \qquad \text{for some } d>0 \text{ and } C>0$$

This clearly violates $u \in L^1_{loc}(\Omega)$.

3. Regularity

We start out with a result in the spirit of Lemma 5.3 in [BC] :

Lemma 3.1. *Let $f \in L^1_{-a,\delta}$ and $v = |x|^{-a}$. Then if $u \in L^1_{-2}$ is the solution given by Lemma 1.3 of*

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2}u = f & \text{in } \Omega \\ \qquad\qquad u = 0 & \text{on } \partial\Omega \end{cases}$$

and if $\Phi \in C^1(\mathbb{R})$ is concave, $\Phi' \in L^\infty$ and $\Phi(1) = 0$, then $v\Phi\left(\frac{u}{v}\right) \in L^1_{-2}$ and

$$-\Delta\left(v\Phi\left(\frac{u}{v}\right)\right) - \frac{c}{|x|^2}\left(v\Phi\left(\frac{u}{v}\right)\right) \geq \Phi'\left(\frac{u}{v}\right)f \qquad \textit{in the weak sense.}$$

Proof (case $0 < c < c_0$). Suppose first $u, v \in C^2(\bar\Omega)$, $v > 0$ in Ω and $\Phi \in C^2(\mathbb{R})$. Applying Lemma 5.3 in [BC], it follows that a.e. in Ω,

$$\begin{aligned} -\Delta w - \frac{c}{|x|^2}w \geq& \Phi'(u/v)(-\Delta u) + [\Phi(u/v) - \Phi'(u/v)u/v](-\Delta v) - \frac{c}{|x|^2}\Phi(u/v)v \\ \geq& \Phi'(u/v)f + [\Phi(u/v) - \Phi'(u/v)u/v]\left(-\Delta v - \frac{c}{|x|^2}v\right) \\ \geq& \Phi'(u/v)\left(f - -\Delta v - \frac{c}{|x|^2}v\right)[\Phi(u/v) - \Phi'(u/v)u/v + \Phi'(u/v)]\left(-\Delta v - \frac{c}{|x|^2}v\right) \end{aligned}$$

Since Φ is concave,

$$\Phi(s) + (1-s)\Phi'(s) \geq \Phi(1) \qquad \text{for all } s \in \mathbb{R}$$

Hence, if $w = v\Phi(u/v)$,

$$-\Delta w - \frac{c}{|x|^2}w \geq \Phi'(u/v)\left(f - -\Delta v - \frac{c}{|x|^2}v\right) \qquad \text{a.e. in } \Omega \tag{3.1}$$

Since Φ' is bounded, we see, as in [BC], that

$$|v\Phi(u/v)| = |v\,(\Phi(u/v) - \Phi(0)) + \Phi(0)v| \le C(u+v) \tag{3.2}$$

Hence, w vanishes on $\partial\Omega$ and integrating by parts, (3.1) holds in the weak sense. By approximation of Φ, we can also say that (3.1) holds even when Φ is only C^1.

In the general case, let $a_n = c/(|x|+1/n)^2$ and f_n be a smooth bounded function increasing pointwise and respectively to $c/|x|^2$,f and u_n be the solution of the associated equation. Also write $w_n = v_n\Phi(u_n/v_n)$ where $v_n = (|x|+1/n)^{-a}$.

We can then apply (3.1) to obtain

$$-\Delta w_n - a_n(x)w_n \ge \Phi'(u_n/v_n)f_n \qquad \text{weakly}$$

Clearly, $v\Phi(u/v)$ is well defined a.e. Moreover, it is clear that $u_n \nearrow u$ in L^1 and that $a_n(x)u_n(x) \nearrow \dfrac{c}{|x|^2}u(x)$ in L^1_δ and similarly for v. So that, using the above equation and Lebesgue's theorem

$$w_n \to w \qquad \text{in } L^1 \text{ and} \qquad a_n(x)w_n \to \frac{c}{|x|^2}w \qquad \text{in } L^1_\delta$$

Since Φ' is bounded, we can also easily pass to the limit in the right-hand side and obtain the desired result. □

Lemma 3.2.
Let u be the minimal weak solution of $(P_{t,p})$ for $t < t_0$ (and $p < p_0$).

Then u is a strong solution of $(P_{t,p})$

Remark 3.2.

- By Proposition 0.1, we only need to show that $0 < u \le C|x|^{-a}$
- By Lemma 1.5, we also have the lower bound $u \ge m|x|^{-a}\, dist(x, \partial\Omega)$.

Proof.
Recall that ζ_0 solving, for f as in the definition of $(P_{t,p})$,

$$\begin{cases} -\Delta\zeta_0 - \dfrac{c}{|x|^2}\zeta_0 = f & \text{in } \Omega \\ \zeta_0 = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies $0 < \zeta_0 \le C|x|^{-a}$. For $u \in \mathbb{R}^+$, let

$$g(u) = (u + t_0\|\zeta_0/v\|_{L^\infty})^p \qquad \text{and} \qquad \tilde{g}(u) = (u + t\|\zeta_0/v\|_{L^\infty})^p$$

and construct $\Phi \in C^1(\mathbb{R})$ with $\Phi(0) = 0$ and

$$\Phi'(u) = \frac{\tilde{g}(\Phi(u))}{g(u)} \tag{3.3}$$

as in Lemma 4 of [BCMR].

Next, if u_0 is the minimal solution of $(P_{t_0,p})$ then $z := u_0 - t_0\zeta_0$ is the minimal solution of

$$\begin{cases} -\Delta z - \dfrac{c}{|x|^2} z = (z + t_0\zeta_0)^p & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega \end{cases}$$

Applying Lemma 3.1 to z with the above function Φ and $v = |x|^{-a}$,

$$-\Delta\left(v\Phi\left(\frac{z}{v}\right)\right) - \frac{c}{|x|^2}\left(v\Phi\left(\frac{z}{v}\right)\right) \geq \Phi'\left(\frac{z}{v}\right)(z + t_0\zeta_0)^p \geq$$
$$\left(\frac{\Phi\left(\frac{z}{v}\right) + t\|\zeta_0/v\|_{L^\infty}}{\frac{z}{v} + t_0\|\zeta_0/v\|_{L^\infty}}\right)^p (z + t_0\zeta_0)^p$$

We need the following easy lemma :

Lemma 3.3. *Let $A, B > 0$ such that $A \leq \dfrac{t}{t_0}B$.*

Then $F(C) := \dfrac{A + tC}{B + t_0 C}$ is increasing with C.

Observe that, since Φ is concave and Φ' is defined by (3.3), $\Phi'(u) \leq \Phi'(0) = \dfrac{t}{t_0}$ for $u \in \mathbb{R}^+$. Hence, since $\Phi(0) = 0$, $\Phi(u) \leq \dfrac{t}{t_0}u$ for $u \in \mathbb{R}^+$. Applying Lemma 3.3 with $A = \Phi(\frac{z}{v})$ and $B = \frac{z}{v}$, we get

$$\frac{\Phi\left(\frac{z}{v}\right) + t\frac{\zeta_0}{v}}{\frac{z}{v} + t_0\frac{\zeta_0}{v}} \leq \frac{\Phi\left(\frac{z}{v}\right) + t\|\frac{\zeta_0}{v}\|_{L^\infty}}{\frac{z}{v} + t_0\|\frac{\zeta_0}{v}\|_{L^\infty}}$$

and

$$-\Delta\left(v\Phi\left(\frac{z}{v}\right)\right) - \frac{c}{|x|^2}\left(v\Phi\left(\frac{z}{v}\right)\right) \geq \left(\frac{\Phi\left(\frac{z}{v}\right) + t\zeta_0/v}{\frac{z}{v} + t_0\zeta_0/v}\right)^p (z + t_0\zeta_0)^p$$
$$\geq \left(v\Phi\left(\frac{z}{v}\right) + t\zeta_0\right)^p$$

We finally define $w = v\Phi\left(\frac{z}{v}\right) + t\zeta_0$, which satisfies

$$\begin{cases} -\Delta w - \dfrac{c}{|x|^2} w \geq w^p + tf & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

We have just constructed a supersolution of problem $(P_{t,p})$ satisfying $0 < w \leq C|x|^{-a}$ (since $\Phi(\infty) < \infty$ by Lemma 4 in [BCMR]) and, of course, the same estimate holds for u, the minimal solution of $(P_{t,p})$.

□

This completes the proof of Theorem 1 (in the case $0 < c < c_0$.)

4. Stability

We show first that $\lambda_1(u_t) > -\infty$ (recall Definition 0.2) and study the corresponding eigenfunction ϕ_1.

Indeed, if u_t is the minimal solution of $(P_{t,p})$ with $t < t_0$, then $0 \leq u_t \leq C|x|^{-a}$ and

$$\int_\Omega u_t^{p-1}\phi^2 \leq C\int_\Omega |x|^{-a(p-1)}\phi^2 \leq C\left(\int_\Omega |x|^{-2}\phi^2\right)^{\frac{a(p-1)}{2}} \cdot \left(\int_\Omega \phi^2\right)^{1-\frac{a(p-1)}{2}}$$
$$\leq C\|\phi\|_{H_0^1(\Omega)}^{a(p-1)}\,\|\phi\|_{L^2}^{2-a(p-1)}$$

So $\lambda_1 > -\infty$ and if $\{\phi_n\}$ is a minimizing sequence of J (see Definition 0.2), $\{\phi_n\}$ is bounded in $H_0^1(\Omega)$ and converges (weakly and up to a subsequence) to $\phi_1 \in H_0^1(\Omega)$ solving

$$\begin{cases} -\Delta\phi_1 - \dfrac{c}{|x|^2}\phi_1 = pu_t^{p-1}\,\phi_1 + \lambda_1\phi_1 & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

Claim. $0 \leq \phi_1 \leq C|x|^{-a}$

Testing equation (4.1) against ϕ_1^+, it follows that

$$\int_\Omega |\nabla\phi_1^+|^2 - \int_\Omega \frac{c}{|x|^2}{\phi_1^+}^2 - \int_\Omega pu^{p-1}{\phi_1^+}^2 = \lambda_1\int_\Omega {\phi_1^+}^2$$

Hence ϕ_1^+ is also a minimizer of J and up to replacing ϕ_1 by ϕ_1^+, we may assume that $\phi_1 \geq 0$.

Next, using local elliptic regularity, $\phi_1 \in C^\infty(\bar\Omega \setminus \{0\})$. Also, pick $\tilde c \in (c, c_0)$ and $\eta > 0$ so small that

$$\frac{\tilde c - c}{|x|^2} \geq pu_t^{p-1} + \lambda_1 \qquad \text{a.e. in } B_\eta.$$

Let $z = \phi_1 - M|x|^{-\tilde a}$ and $M = \|\phi_1\|_{L^\infty(\partial B_\eta)}\,\eta^{\tilde a}$ ($-\tilde a$ being the greater root of $P(X) = X(X-1) + (n-1)X + \tilde c = 0$). Then,

$$\begin{cases} -\Delta z - \dfrac{\tilde c}{|x|^2}z \leq 0 & \text{in } B_\eta \\ z \leq 0 & \text{on } \partial B_\eta \end{cases} \tag{4.3}$$

Testing (4.3) against z^+ (which is permitted since $z^+ \in H_0^1(B_\eta)$),

$$\phi_1 \leq M \cdot |x|^{-\tilde a} \qquad \text{a.e. in } B_\eta.$$

With $\tilde c$ close enough to c, it follows that $pu_t^{p-1}\phi_1 + \lambda_1\phi_1 \leq C|x|^{-a-2+\epsilon}$, for some $\epsilon > 0$. Let $\zeta \in H_0^1(\Omega)$ be the solution of

$$\begin{cases} -\Delta\zeta - \dfrac{c}{|x|^2}\zeta = |x|^{-a-2+\epsilon} & \text{in } \Omega \\ \zeta = 0 & \text{on } \partial\Omega \end{cases} \tag{4.4}$$

As in the proof of Lemma 1.5,

$$0 \le \phi_1 \le C\zeta \le C|x|^{-a} \qquad \text{a.e. in } \Omega \tag{4.5}$$

Next, we prove that there exists $0 < t_1 \le t_0$ such that u_t is stable for $t < t_1$.

Fix $b \in (a, a')$ such that $pb < b+2$ and $b + a(p-1) < a+2$, and define $F : X \times \mathbb{R} \to Y$, by

- X is the space of functions $v \in C(\bar\Omega \setminus \{0\})$ such that there exist a constant $C > 0$ and a function $g \in C(\bar\Omega \setminus \{0\})$ satisfying $|v| \le C|x|^{-b}$, $|g| \le C|x|^{-b-2}$ and

$$\begin{cases} -\Delta v - \dfrac{c}{|x|^2}v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

  in the weak sense.
- $Y = \{f \in C(\bar\Omega \setminus \{0\}) \,:\, |x|^{b+2}f \in L^\infty(\Omega)\}$
- $F(v,t) = -\Delta v - \frac{c}{|x|^2}v - |v|^p - tf$

Observe that F is well defined with our choice of b, that $F \in C^1$ and that $F(u_t,t) = 0$. Also $L := F_u(0,0)$ is an isomorphism between X and Y. Indeed L is injective by Lemma 1.4 and surjective with continuous inverse by Lemma 1.3. These facts and a global form of the implicit function theorem (see e.g. Cor. 3 in [BN]) imply the existence of a maximal $t_1 > 0$ such that $t \to u_t$ is a C^1 map from $(0,t_1)$ to X and $F_u(u_t,t) \in Iso(X,Y)$.

In particular, since $\phi_1 \in X$, $\lambda_1(u_t) \ne 0$ for $t < t_1$. It can also be shown that $t \to \lambda_1(u_t)$ is continuous : if $\tau_n \to \tau < t_1$ and λ_1^n and ϕ_1^n are the corresponding eigenvalues and eigenfunctions with $\|\phi_1^n\|_{L^2} = 1$, looking carefully at the previous claim, we obtain that ϕ_1^n is bounded in $H_0^1(\Omega)$ and that

$$0 \le \phi_1^n \le C|x|^{-a}$$

Passing to a subsequence, it is then easy to show that $\lambda_1^n \to \lambda_1(u_\tau)$ and therefore that λ_1 is continuous.

Hence, since $\lambda_1(0) > 0$ and λ_1 cannot vanish, we have $\lambda_1 > 0$ for $t < t_1$.

We now prove that $t_1 = t_0$. If not, we would have for $t_1 < t < t_0$,

$$-\Delta(u_t - u_{t_1}) - \frac{c}{|x|^2}(u_t - u_{t_1}) - pu_{t_1}^{p-1}(u_t - u_{t_1}) =$$
$$u_t^p - u_{t_1}^p - pu_{t_1}^{p-1}(u_t - u_{t_1}) + (t - t_1)f \ge (t-t_1)f$$

And testing against ϕ_1, solution of (4.1) with t_1 in place of t, we would obtain

$$0 \geq (t - t_1) \int_\Omega f\phi_1$$

which is impossible. Hence, $t_1 = t_0$.

Next, we prove that if v is another stable $H_0^1(\Omega)$ then it must coincide with u_t. Suppose indeed v is another $H_0^1(\Omega)$ solution such that $\lambda_1(v) \geq 0$.Then $v \geq u_t$ and

$$\begin{aligned}\int_\Omega pv^{p-1}(v - u_t)^2 &\leq \int_\Omega |\nabla(v - u_t)|^2 - \int_\Omega \frac{c}{|x|^2}(v - u_t)^2 \\ &\leq \int_\Omega (v^p + tf - u_t^p - tf)(v - u_t)\end{aligned}$$

So that,

$$\int_\Omega (v - u_t)(v^p - u_t^p - pv^{p-1}(v - u_t)) \geq 0$$

Since $u \to u^p$ is strictly convex and $v \geq u_t$, we must have $v = u_t$.

Finally, stability of strong extremal solutions is determined through the following proposition :

Proposition 4.1.

Suppose that $0 < c < c_0$ and $1 < p < p_0$.

If u, the minimal solution of $(P_{t_0,p})$, solves the problem in the strong sense then

$$\lambda_1(u) = 0$$

Proof.

Arguing by contradiction, our general strategy is to use the implicit function theorem to extend the curve $t \to u_t$ of minimal solutions of $(P_{t,p})$ beyond t_0 if $\lambda_1(u) > 0$.

More precisely assume that $\lambda_1(u) > 0$ and define $F : X \times \mathbb{R} \to Y$ as before. If we can prove that $F_u(u, t_0) \in Iso(X, Y)$, the implicit function theorem will yield the desired contradiction.

We first claim that $F_u(u, t_0)$ is injective. If not, there would be a weak solution $\phi_1 \in X$ of

$$\begin{cases} -\Delta\phi_1 - \dfrac{c}{|x|^2}\phi_1 = pu^{p-1}\,\phi_1 & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega \end{cases}$$

Since $b + a(p - 1) < a + 2$, $u^{p-1}\phi_1 \in L^{\frac{2n}{n+2}}(\Omega)$ and, using the methods of Proposition 0.1, ϕ_1 is an $H_0^1(\Omega)$ solution. Testing the above equation against ϕ_1

would then imply $J(\phi_1) = 0$, which contradicts $\lambda_1(u) > 0$. Thus $F_u(u, t_0)$ is injective.

Next we prove that $F_u(u, t_0)$ is surjective.

First observe that $L := F_u(0, 0)$ is an isomorphism between X and Y. Indeed L is injective by Lemma 1.4 and surjective with continuous inverse by Lemma 1.3.

Let $Z := \{f : |x|^{-a(p-1)} f \in Y\}$ and define $K \in \mathcal{L}(Z)$ by

$$K : \begin{cases} Z \to & Y \to Z \\ \phi \mapsto pu^{p-1}\phi \mapsto & L^{-1}(pu^{p-1}\phi) \end{cases}$$

K is compact in Z. Indeed if $\{\phi_n\}$ is a bounded sequence in Z then $u_n := K\phi_n$ is bounded in X, by continuity of L^{-1}. It follows from standard elliptic theory that up to a subsequence, $u_n \to u$ uniformly on compacts of $\bar{\Omega} \setminus \{0\}$ for some $u \in X$. Also, letting $\gamma = 2 - a(p-1) > 0$, we have for $\epsilon > 0$ small

$$\|u_n - u\|_Z \leq C\|u_n - u\|_{L^\infty(\Omega \setminus B_\epsilon)} + \epsilon^\gamma \|u_n - u\|_{L^\infty_{-b}} \leq C(\|u_n - u\|_{L^\infty(\Omega \setminus B_\epsilon)} + \epsilon^\gamma)$$

so that

$$\limsup_{n\to\infty} \|u_n - u\|_Z \leq C\epsilon^\gamma$$

Letting $\epsilon \to 0$, we obtain that K is compact in Z.

With these notations, our problem reduces to showing that $Id - K$ is surjective. By Fredholm's alternative, we just need to prove that $Id - K$ is injective. Now if for some $\phi \in Z, \phi = K\phi$ then $\phi \in X$ by definition of K, and $F_u(u, t_0)\phi = 0$. But we just showed that $F_u(u, t_0)$ is injective so $\phi \equiv 0$.

□

5. What happens in the extremal case $t = t_0$?

In this section, we look at two specific sets of conditions on c, p, f and Ω.

In one case, the minimal solution u of $(P_{t_0,p})$ solves the problem in the strong sense. It then follows from Proposition 5.1 that $\lambda_1(u) = 0$.

In the other case, the minimal solution u is not a strong one and its singularity at the origin is worse than $|x|^{-a}$. Moreover, u is stable, i.e., $\lambda_1(u) > 0$.

Situation 1.

Suppose $\Omega = B_1$, $c < c_0$ close to c_0, f radial and $p > 1$ close to 1.

Then u, the minimal solution of $(P_{t_0,p})$, solves the problem in the strong sense and $\lambda_1(u) = 0$. Furthemore, $u = u(r)$ is radial and

$$\begin{aligned} &u = r^{-a}\, w && \textit{where } w \in C[0,1] \cap C^\infty(0,1] \\ &w' \sim m\, r^{-a(p-1)+1} && \textit{for some } m < 0 \textit{ as } r \to 0 \\ &w' < 0 && \textit{in } (0,1) \end{aligned}$$

Proof.
We suppose for simplicity that $f \equiv 1$.

First, we note that for any rotation of the space $A \in SO(n,\mathbb{R})$, $u \circ A$ is a solution of $(P_{t_0,p})$ and since u is minimal, we must have $u \le u \circ A$. This inequality holds almost everywhere in B_1, hence for almost all $y = A^{-1}x$ with $x \in \mathbb{R}^n$ so that u must be radial.

Next, define $\alpha := \sqrt{(n-2)^2 - 4c}$ and for $r \in (0,1)$,

$$\Phi(u)(r) := \frac{r^{-a}}{\alpha} \int_0^1 s^{1+\alpha+a} u^p(s) [\max(s,r)^{-\alpha} - 1]\, ds + \frac{t_0}{2n+c} [r^{-a} - r^2] \tag{5.1}$$

In view of Lemma 1.5, $\Phi(u)(r)$ is well defined for $r \neq 0$ and it follows from Lebesgue's theorem that $w := r^a\, \Phi(u) \in C(0,1]$.

Using Lebesgue's theorem again, it is also true that $w \in C^1(0,1]$ and that for $r \in (0,1]$,

$$w'(r) = -r^{-1-\alpha} \left(\int_0^r s^{1+\alpha+a} u^p(s)\, ds \right) - (2+a) \frac{t_0}{2n+c} r^{1+a}$$

Using the fundamental theorem of calculus, w is twice differentiable a.e. in $(0,1)$ and

$$w''(r) = -r^a u^p(r) - (1+\alpha)\frac{1}{r}\left[w'(r) + (2+a)\frac{t_0}{2n+c} r^{1+a} \right] - (2+a)(1+a)\frac{t_0}{2n+c} r^a$$

So that

$$-(w'' + (1+\alpha)\frac{1}{r} w') = r^a u^p(r) + t_0\, r^a \qquad \text{a.e. in } (0,1) \tag{5.2}$$

Using the fundamental theorem of calculus again, this equation also holds in the sense of distributions in $(0,1)$. Furthermore, since u is a weak solution of $(P_{t_0,p})$, it is not hard to see that $\tilde{w} := r^a\, u$ solves (5.2) in $\mathcal{D}'(0,1)$.

So if $z = \tilde{w}' - w'$, it follows from (5.2) and this last remark that

$$z' + (1+\alpha)\frac{1}{r} z = 0 \qquad \text{in } \mathcal{D}'(0,1).$$

And by a straightforward computation, we see that

$$\left[r^{1+\alpha}\, z \right]' = 0 \qquad \text{in } \mathcal{D}'(0,1).$$

Hence $z = A r^{-(1+\alpha)}$ for some $A \in \mathbb{R}$ and, for some $B \in \mathbb{R}$,

$$\tilde{w} = w + \frac{A}{\alpha} r^{-\alpha} + B \tag{5.3}$$

Since w is C^1 away from $r = 0$ (and hence, so must be $\tilde{w}$), we must have, on the one hand, using the boundary condition of $(P_{t_0,p})$ and equation (5.1), that

$w(1) = \tilde{w}(1) = 0$ and $B = 0$ and on the other hand that u is C^1 away from the origin. Bootstrapping this result with the help of (5.1) and (5.3), it follows that $w, \tilde{w} \in C^\infty(0,1]$.

Let us now prove that $A = 0$. Suppose by contradiction that $A > 0$ and let $u_1(x) := |x|^{-a} w(|x|)$ for $x \in B_1$. Then,

$$\begin{aligned} -\Delta u_1 - \frac{c}{|x|^2} u_1 = u^p + t_0 &= \left[u_1 + \frac{A}{\alpha}(|x|^{-a'} - |x|^{-a}) \right]^p + t_0 \\ &\geq u_1^p + t_0 \end{aligned}$$

This equation holds at every $x \neq 0$ and also in the weak sense, as integration by parts on $B_1 \setminus B_\epsilon$ with $\epsilon \to 0$ shows. But then u_1 would be a nonnegative supersolution of problem $(P_{t_0,p})$, contradicting minimality of u.

We have just shown that $A \leq 0$. We now prove that $A = 0$. Recall that

$$\tilde{w}'(r) = -r^{-1-\alpha} \int_0^r s^{1+\alpha-a} u^p(s)\, ds - (2-a) \frac{t_0}{2n+c} r^{1+a} - A r^{-1-\alpha} \tag{5.4}$$

By Hopf's boundary lemma, $u'(1) = \tilde{w}'(1) < 0$. We claim that

$$\tilde{w}'(r) < 0 \qquad \text{for all } r \in (0,1]$$

Suppose not and let $r_0 = \sup\{r \in (0,1) : \tilde{w}'(r) = 0\}$. Then $\tilde{w}' < 0$ on $(r_0, 1]$ and (5.2) implies that

$$\tilde{w}''(r_0) = -(1+\alpha) \frac{1}{r_0} \tilde{w}'(r_0) - (r_0^a u^p(r_0) + t_0\, r_0^a) < 0$$

So $\tilde{w}$ has a local maximum at r_0. Suppose by contradiction that $\tilde{w}$ has another critical point and let $r_1 < r_0$ so that

$$\tilde{w}'(r_1) = 0 \qquad \text{and} \qquad \tilde{w}'(r) > 0 \qquad \text{for } r \in (r_1, r_0)$$

From (5.2), it follows as before that $\tilde{w}''(r_1) < 0$ and r_1 would be a local maximum of $\tilde{w}$, contradicting $\tilde{w}' > 0$ on (r_1, r_0).

Hence $\tilde{w}$ has an absolute maximum at r_0 and must therefore be bounded, which forces $A = 0$.

But then, using (5.4), $\tilde{w}'(r) < 0$ in $(0,1]$, contradicting $\tilde{w}'(r_0) = 0$.

So, we have proved that $\tilde{w}' < 0$ in $(0,1]$.

From (5.4), it follows that if $A < 0$, $\tilde{w}'(r) = -Ar^{-1-\alpha}(1 + o(1))$ as $r \to 0$ and we cannot have at the same time $\tilde{w}' < 0$ and $A < 0$. Hence $A = 0$.

So far we know that :

$$\begin{aligned} \tilde{w} &= w \\ w' < 0 &\qquad \text{in } (0,1] \end{aligned}$$

We now prove that $u \leq Cr^{-a}$. From equation (5.1), we already know that $u = \Phi(u) \leq Cr^{-a-\alpha}$. Plugging this result into (5.1) again, we only need to show that the right-hand-side integral is bounded as $r \to 0$, which holds as soon as

$$1 - a(p-1) - \alpha p > -1$$

This last condition is satisfied for αp small and in particular when c is close to c_0 and p close to 1. This result, combined with (5.4) yields the asymptotic behaviour of w' at the origin.

Finally, by Proposition 4.1, we have that $\lambda_1(u) = 0$.

When p is chosen close to the critical exponent p_0, the minimal solution u of $(P_{t_0,p})$ may become more singular than when $t < t_0$, in such a way that u^{p-1} has a singularity at the origin of same order as $\frac{1}{|x|^2}$:

Situation 2.

Suppose $0 < c < c_0$ *and* p *close to* p_0*. Then there exists a smooth nonnegative nonzero data* f *such that* u*, the minimal solution of* $(P_{t_0,p})$*, is stable and such that, near the origin,*

$$u = m\,|x|^{-\gamma} \qquad \text{,where } m > 0 \qquad \text{and} \qquad \gamma = \frac{2}{p-1} > a > 0$$

Proof.
We adapt a proof given in [D].
Let $v = |P(-\gamma)|^{\frac{1}{p-1}}\,|x|^{-\gamma}$, where $P(X) = X(X-1) + (n-1)X + c$.
Then, $-\Delta v - \frac{c}{|x|^2}v = v^p$ in $\mathbb{R}^n$ and, since when $p \to p_0$, $\gamma \to a$, we may assume that $v \in H^1$.
Lemma 5 in [D] constructs a function $\psi \in C^\infty(\bar{\Omega})$ with the following properties:

(1) $\psi \geq 0$ in $\bar{\Omega}$
(2) $\Delta\psi + \dfrac{c}{|x|^2}\psi \geq 0$ in Ω
(3) $\psi \equiv 0$ in a neighbourhood of 0, and
(4) $\psi = v$ on $\partial\Omega$

We then let $u = v - \psi$ and see that

$$\begin{aligned} -\Delta u - \frac{c}{|x|^2}u &= -\Delta v - \frac{c}{|x|^2}v + \Delta\psi + \frac{c}{|x|^2}\psi \\ &= v^p + \Delta\psi + \frac{c}{|x|^2}\psi \\ &\geq 0 \end{aligned}$$

and $u = 0$ on $\partial\Omega$, so, by Lemma 1.1 say, $u \geq 0$.
Taking $f = \Delta\psi + \dfrac{c}{|x|^2}\psi + v^p - u^p$, we then have

$$-\Delta u - \frac{c}{|x|^2}u = u^p + f.$$

Observe that $f \geq 0$ and is smooth since $u \leq v$ and $u \equiv v$ near the origin.

Next, we prove that $\lambda_1(u) > 0$. Given $\phi \in H_0^1(\Omega)$,

$$\begin{aligned}\int_\Omega pu^{p-1}\phi^2 &\leq \int_\Omega pv^{p-1}\phi^2 \\ &= p|P(-\gamma)|\int_\Omega \frac{\phi^2}{|x|^2} \\ &\leq \int_\Omega |\nabla\phi|^2 - \int_\Omega \frac{c}{|x|^2}\phi^2 - \epsilon\int_\Omega \phi^2\end{aligned}$$

The last inequality holds, using Hardy's inequality (0.3), provided $c+p|P(-\gamma)| < c_0$ and $\epsilon > 0$ small. This condition is readily satisfied since as $p \to p_0$, $\gamma \to a$ and $P(-\gamma) \to P(-a) = 0$. Hence, we get that $\lambda_1(u) \geq \epsilon > 0$.

We still need to prove that, for our choice of f, $t_0 = 1$ and u is the minimal solution of $(P_{t_0,p})$.

If u_1 denotes the minimal solution of $(P_{1,p})$, it is clear that $0 \leq u_1 \leq u$, hence $u_1^p \leq \dfrac{C}{|x|^2}$ and using this inequality and $(P_{1,p})$, $u_1 \in H_0^1(\Omega)$.

Since $\lambda_1(u) \geq 0$, it follows that

$$\begin{aligned}\int_\Omega pu^{p-1}(u-u_1)^2 &\leq \int_\Omega |\nabla(u-u_1)|^2 - \int_\Omega \frac{c}{|x|^2}(u-u_1)^2 \\ &\leq \int_\Omega (u^p + f - u_1^p - f)(u-u_1)\end{aligned}$$

So that,

$$\int_\Omega (u-u_1)(u^p - u_1^p - pu^{p-1}(u-u_1)) \geq 0$$

Since $u \to u^p$ is strictly convex and $u \geq u_1$, we must have $u = u_1$. And since u is not a strong solution of $(P_{1,p})$, we must have $1 = t_0$.

□

6. The case $c = c_0$

When $c = c_0$, the operator $-\Delta - \frac{c}{|x|^2}$ is no longer coercive in $H_0^1(\Omega)$. However, one can still make use of the improved Hardy inequality (see [BV] or [VZ])

$$\int_\Omega |\nabla u|^2 - c_0\int_\Omega \frac{u^2}{|x|^2} \geq C(\Omega)\int_\Omega u^2 \qquad \text{for all } u \in C_c^\infty(\Omega) \tag{6.1}$$

to define a new Hilbert space H in which the operator is coercive, even when $c = c_0$.

Definition.
H is the space obtained by completing $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_H^2 := \int_\Omega |\nabla u|^2 - c_0 \int_\Omega \frac{u^2}{|x|^2}$$

By analogy with the case $c < c_0$, an H solution u will be one such that $u^p \in H^*$ and such that the equation holds in the sense of Lax-Milgrams lemma in H.

We now list the modifications needed to prove Theorem 1 when $c = c_0$. When no proof is given, just replace $H_0^1(\Omega)$ by H in the original demonstration.

Lemma 1.1'. *Lemma 1.1 still holds if $c = c_0$, $H_0^1(\Omega)$ is replaced by H and H^{-1} by H^*, the dual of H.*

Proof. Only the proof of (1.3) needs to be clarified in this setting.

Let $f \in H^*$, $f \geq 0$ and $u \in H$ be the corresponding solution of (1.1).

By definition of H, there exists a sequence $\{u_n\}$ in $C_c^\infty(\Omega)$ converging to u in H. Letting $f_n = -\Delta u_n - \frac{c}{|x|^2} u_n$, it follows that $f_n \in H^{-1}(\Omega)$ and $f_n \to f$ in H^*.

Now, $u_n \in H_0^1(\Omega) \Rightarrow u_n^- \in H_0^1(\Omega)$ and integrating the equation satisfied by u_n against u_n^- yields

$$-\|u_n^-\|_H^2 = \langle f_n, u_n^- \rangle_{H*,H}$$

To pass to the limit in this last equation, we just need to prove that $\{u_n^-\}$ remains bounded in H. But

$$\begin{aligned}
\|u_n^-\|_H^2 &= \int_\Omega |\nabla u_n^-|^2 - c_0 \int_\Omega \frac{(u_n^-)^2}{|x|^2} \\
&= \int_\Omega |\nabla u_n^-|^2 - c_0 \int_\Omega \frac{u_n^2}{|x|^2} + \int_\Omega \frac{c_0}{|x|^2}(u_n^+)^2 \\
&\leq \int_\Omega |\nabla u_n^-|^2 - c_0 \int_\Omega \frac{u_n^2}{|x|^2} + \int_\Omega |\nabla u_n^+|^2 = \int_\Omega |\nabla u_n|^2 - c_0 \int_\Omega \frac{u_n^2}{|x|^2} \\
&= \|u_n\|_H^2
\end{aligned} \tag{6.2}$$

where we've used (0.3) in the inequality.

Proposition 0.1'. *Proposition 0.1 still holds when $c = c_0$ and $H_0^1(\Omega)$ solutions are replaced by H solutions.*

Proof. Suppose first that u is a strong solution of $(P_{t,p})$.

Let $\zeta_n \in C_c^\infty(\Omega \setminus \{0\})$ be such that $0 \leq \zeta_n \leq 1$, $|\Delta \zeta_n| \leq Cn^2$ and

$$\zeta_n = \begin{cases} 0 & \text{if} \quad |x| \leq 1/n \\ 1 & \text{if} \quad |x| \geq 2/n \end{cases}$$

Multiplying $(P_{t,p})$ by $u\zeta_n$ and integrating by parts, it follows that

$$\int_\Omega (u^p + tf)\, u\zeta_n = -\int_\Omega \Delta u\, u\zeta_n - \int_\Omega \frac{c}{|x|^2} u^2 \zeta_n$$
$$= \int_\Omega |\nabla u|^2 \zeta_n - \int_\Omega \frac{c}{|x|^2} u^2 \zeta_n + \int_\Omega u \nabla u \nabla \zeta_n$$

Since $u \le C|x|^{-a}$ and $p < p_0$, $u^p \le C|x|^{-a-2+\epsilon}$, for some $\epsilon > 0$, so that the first integral in the above equation is bounded by $C\int_\Omega |x|^{-2a-2+\epsilon} \le C$ whereas $\left|\int_\Omega u\nabla u \nabla\zeta_n\right| = \left|\frac{1}{2}\int_\Omega u^2 \Delta\zeta_n\right| \le Cn^2 \int_{1/n<|x|<2/n} |x|^{-2a} \le C$ as $n \to \infty$.

Hence $\int_\Omega |\nabla u|^2 \zeta_n - \int_\Omega \frac{c}{|x|^2} u^2 \zeta_n \le C$ and $u \in H$. Approximating $u \in H$ by smooth functions and integrating by parts in $\Omega \setminus B_\epsilon$ with $\epsilon \to 0$, it follows that u is a weak solution of $(P_{t,p})$. For u to be an H solution, we only need to prove the following :

Claim. Suppose u is a weak solution satisfying the estimate $u \le C|x|^{-a}$. Then

$$u^p \in H^*$$

For $\phi \in C_c^\infty(\Omega)$, $1 < q < 2$, it follows from Hölder's inequality that

$$\left|\int_\Omega |x|^{-ap}\phi\right| \le \left(\int_\Omega |x|^{-(ap+1)\frac{q}{q-1}}\right)^{\frac{q-1}{q}} \left\|\frac{\phi}{|x|}\right\|_{L^q}$$

On the one hand, since $p < p_0$, the integral in the right hand side will be finite if q is chosen close enough to 2.

On the other hand, using Hardy's inequality in L^q and the inclusion $H \hookrightarrow W_0^{1,q}$ (see section 4 of [VZ]),

$$\left\|\frac{\phi}{|x|}\right\|_{L^q} \le C\|\phi\|_{W_0^{1,q}} \le C\|\phi\|_H$$

and $u^p \in H^*$.

Hence, strong solutions are also H solutions.

Showing that H solutions are weak solutions is similar to the case $c < c_0$, whereas, starting from a weak solution u, we observe as above that $u^p \in H^*$ and define $u_n \ge 0$ to be the minimal weak solution of

$$\begin{cases} -\Delta u_n - \dfrac{c-1/n}{|x|^2} u_n = u_n^p + tf & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

u is a supersolution of this equation so u_n is well defined and $0 \le u_n \le u \le C|x|^{-a}$. By Proposition 0.1 (case $c < c_0$), it follows that $u_n \in H_0^1(\Omega)$ and testing in the above equation against u_n,

$$\|u_n\|_H^2 \le (\|u_n^p\|_{H^*} + C)\|u_n\|_H$$

Letting $n \to \infty$, we get $u \in H$. Since $H \hookrightarrow W_0^{1,q}$ for $1 \le q < 2$, elliptic regularity can be applied to complete the proof.

Lemma 1.3'. *Lemma 1.3 still holds when* $c = c_0$

Lemma 1.4'. *Lemma 1.4 still holds when* $c = c_0$

Lemma 1.5'. *Lemma 1.5 still holds when* $c = c_0$

Proof. We assume first that u is the minimal (weak) solution of (1.20) and can therefore be written as the pointwise limit of an increasing sequence $\{u_\epsilon\}$, where u_ϵ solves (1.20) with $c - \epsilon$ in place of c. Since Lemma 1.5 can be applied to u_ϵ, an argument of monotone convergence yields the result.

If u isn't minimal, the above discussion yields all the results up to the conclusion of Step 6 in the original proof. That step can be applied -as is- in our context, which finishes the proof.

2.1' : Existence of a solution of $(\mathbf{P_{t,p}})$ for $\mathbf{1 < p < p_0}$, small $\mathbf{t > 0}$.

For simplicity, we assume without loss of generality that $\Omega \subset B_{\frac{1}{2}}$. Consider $w = A|x|^{-a}\left(\ln \frac{1}{|x|}\right)^{\frac{1}{4p}}$, with $A > 0$ to be fixed later. Then $-\Delta w - \frac{c}{|x|^2} w = \frac{4p-1}{16p^2} A\,|x|^{-a-2}\left(\ln \frac{1}{|x|}\right)^{\frac{1}{4p}-2}$ and w will be a supersolution of $(P_{t,p})$ as soon as

$$\begin{cases} \dfrac{4p-1}{32p^2} A|x|^{-a-2}\left(\ln \dfrac{1}{|x|}\right)^{\frac{1}{4p}-2} \geq A^p|x|^{-ap}\left(\ln \dfrac{1}{|x|}\right)^{p/4} \\ \dfrac{4p-1}{32p^2} A|x|^{-a-2}\left(\ln \dfrac{1}{|x|}\right)^{\frac{1}{4p}-2} \geq tf \end{cases}$$

The first inequality amounts to

$$A \leq C \min_{r\in(0,1/2]} \left\{ r^{-a-2+pa}\left(\ln \frac{1}{r}\right)^{\frac{1}{4p}-2-p/4} \right\}^{\frac{1}{p-1}}$$

and the second to

$$t \leq C \cdot A \min_{r\in(0,1/2]} \left\{ r^{-a-2}\left(\ln \frac{1}{r}\right)^{\frac{1}{4p}-2} \right\}$$

Under these conditions, w is a supersolution of $(P_{t,p})$.

We now just have to construct a supersolution in H.

Let w_1 be a smooth extension inside Ω of $w|_{\partial\Omega}$ such that $w_1 = w$ in $\Omega \setminus B_{1/4}$ and $w_1 = 0$ in $B_{1/8}$. Next, we let $g = \Delta w_1 + \dfrac{c}{|x|^2} w_1$ and construct $z \in H$ solving (2.1) and $w_2 = z + w_1$ solving (2.2).

We would like to show that $w_2 \geq 0$ and remark that $w_2^- \in H$. Indeed, let $\phi_k \in C_c^\infty(\Omega) \to z$ in H. Then $(\phi_k + w_1)^- \in H_0^1(\Omega) \subset H$ and

$$
\begin{aligned}
\|(\phi_k+w_1)^-\|_H^2 &= \int_{\{\phi_k+w_1<0\}} \left(|\nabla(\phi_k+w_1)|^2 - \frac{c}{|x|^2}(\phi_k+w_1)^2 \right) \\
&\leq \|\phi_k\|_H^2 + C + 2\int_{\{\phi_k+w_1<0\}} \left(\nabla\phi_k\cdot\nabla w_1 - \frac{c}{|x|^2}\phi_k w_1 \right) \\
&\leq \|\phi_k\|_H^2 + C + \frac{1}{2}\|(\phi_k+w_1)^-\|_H^2
\end{aligned}
$$

Hence $\|(\phi_k+w_1)^-\|_H \leq C$ and passing to the limit (in the weak topology and for a subsequence), it follows that $w_2^- \in H$.

Letting $\psi_k \in C_c^\infty(\Omega) \to w_2^-$ in H, integration by parts then yields

$$
(w_2|\psi_k)_H = \int_\Omega \left(\nabla z\nabla\psi_k - \frac{c}{|x|^2}z\psi_k \right) + \int_\Omega \left(\nabla w_1\nabla\psi_k - \frac{c}{|x|^2}w_1\psi_k \right) = \int_{\partial\Omega} \psi_k\partial_\nu w
$$

and letting $k \to \infty$ in H,

$$
\|w_2^-\|_H^2 = (w_2|w_2^-)_H = \int_{\partial\Omega} \partial_\nu w\, w^- = 0
$$

Hence $w_2 \geq 0$.

Finally, letting $\tilde{w} = w - w_2 = w - z - w_1$, we only need to prove that $\tilde{w} \in H$ and the rest of the proof will remain unchanged. Since $z \in H$, it is enough to show that $w - w_1 \in H$.

If $H(\omega)$ denotes the space H relative to the open set ω of $\mathbb{R}^n$, it has been shown in [VZ] (see 5.2) that f defined for $0 < r < r_0 < 1$ by

$$
f(r) = r^{-a}(\ln(1/r))^\alpha
$$

and continued smoothly up to the boundary of the ball B_1, where $f = 0$, belongs to $H(B_1)$ as long as $\alpha < 1/2$.

$(w-w_1)|_{B_{1/4}}$ precisely satisfies these conditions, hence belongs to $H(B_{1/4})$. Since $w - w_1 \equiv 0$ in $\Omega \setminus B_{1/4}$, it follows that $w - w_1 \in H(\Omega)$.

2.3' : Case $\mathbf{p \geq p_0}$: blow-up.

By Proposition 2.1, we just need to prove that $(P_{t,p})$ has no weak solution if $p \geq p_0$. Assume by contradiction there exists one and call it u. If we apply Lemma 1.5 with $u^p + tf$ in place of f, it follows that

$$
\int_\Omega u^p|x|^{-a}\,\delta(x) < \infty \qquad \text{and} \qquad u \geq m|x|^{-a} \qquad \text{a.e. near the origin.}
$$

This is impossible since near the origin,

$$
|x|^{-a}u^p \geq m|x|^{-a(p+1)} \geq m|x|^{-n}
$$

Lemma 3.1'. *Lemma 3.1 still holds when* $c = c_0$

Lemma 3.2'. *Lemma 3.2 still holds when* $c = c_0$ *and* $t < t_0$

Theorem 2'. *Theorem 2 still holds when* $c = c_0$ *with* $H_0^1(\Omega)$ *solutions replaced by* H *solutions.*

Proof. For $t < t_0$, u_t the strong minimal solution of $(P_{t,p})$ can be written as the monotone limit of u_n, where u_n is the strong minimal solution of the same problem with c replaced by $c_n = c - 1/n$.By our analysis in the case $c < c_0$, we know that $\lambda_1(u_n) > 0$. Passing to the limit, we easily get that $\lambda_1(u_t) \geq 0$.

To obtain a strict inequality, it is enough to show that $t \to \lambda_1(u_t)$ is a strictly decreasing function. It should be clear from its definition that $t \to \lambda_1(u_t)$ is nonincreasing.

Suppose that $\lambda_1(u_t) = \lambda_1(u_s)$ for some $s \leq t$. Call ϕ_1^t and ϕ_1^s the corresponding eigenfunctions, which can be constructed as in the case $c < c_0$. Then,

$$\begin{aligned}
\lambda_1(u_t) &= \int_\Omega |\nabla \phi_1^t|^2 - \int_\Omega \frac{c}{|x|^2} {\phi_1^t}^2 - \int_\Omega p u_t^{p-1} {\phi_1^t}^2 \\
&\leq \int_\Omega |\nabla \phi_1^s|^2 - \int_\Omega \frac{c}{|x|^2} {\phi_1^s}^2 - \int_\Omega p u_t^{p-1} {\phi_1^s}^2 \\
&\leq \int_\Omega |\nabla \phi_1^s|^2 - \int_\Omega \frac{c}{|x|^2} {\phi_1^s}^2 - \int_\Omega p u_s^{p-1} {\phi_1^s}^2 \\
&= \lambda_1(u_s)
\end{aligned}$$

Hence,

$$\int_\Omega |\nabla \phi_1^s|^2 - \int_\Omega \frac{c}{|x|^2} {\phi_1^s}^2 - \int_\Omega p u_t^{p-1} {\phi_1^s}^2 = \int_\Omega |\nabla \phi_1^s|^2 - \int_\Omega \frac{c}{|x|^2} {\phi_1^s}^2 - \int_\Omega p u_s^{p-1} {\phi_1^s}^2$$

and

$$u_t = u_s \qquad \text{,which implies} \qquad t = s\ .$$

Hence u_t is a stable solution of $(P_{t,p})$.

To prove that u_t is the only stable H solution, we can argue exactly as in the case $c < c_0$.

The results of section 5 extend in the following way (we skip the proof) :

Situation 1'. *Suppose* $c = c_0$, $\Omega = B_1$, f *radial and* $1 < p < p_0$.

Then u*, the minimal solution of* $(P_{t_0,p})$*, solves the problem in the strong sense.*

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References

[B] H. Brezis, *Analyse fonctionnelle, théorie et applications*, Masson, 1983.

[BC] H. Brezis and X. Cabré, *Some simple nonlinear PDE's without solutions*, Boll. Unione Mat. Ital. Sez. **1** (1998), 223–262.

[BCMR] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa, *Blow-up for $u_t - \Delta u = g(u)$ revisited*, Adv. Diff. Eq. **1** (1996), 73–90.

[BE] J. Bebernes and D. Eberly, *Mathematical problems from combustion theory*, Math. Sci. Vol. 83, Springer Verlag, New York, 1989.

[BG] P. Baras and J.A. Goldstein, *The heat equation with a singular potential*, Trans. Amer. Math. Soc. **284** (1984), 121–139.

[BN] H. Brezis and L. Nirenberg, *Nonlinear elliptic equations. Chapter 1*, in preparation.

[BV] H. Brezis and J.L. Vázquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10** (1997), no. 2, 443-469.

[CR] M.G. Crandall and P. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Rational Mech. Anal. **58** (1975), no. 3, 207–218.

[D] J. Davila, *Some extremal singular solutions of a nonlinear elliptic equation*, to appear in Diff. Int. Eq..

[P] F. Pacard, *Mapping properties of $r^2\Delta + \kappa$ in weighted spaces*, unpublished notes (2001).

[RS] M. Reed and B. Simon, *Methods of modern mathematical physics,* Vol. II, Academic Press, New York.

[VZ] J.L. Vazquez and E. Zuazua, *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential*, Jour. Func. Anal. **173** (2000), 103–153.

Department of Mathematics, Hill Center, 110 Frelinghuysen Rd Rutgers, The State University of New Jersey, Piscataway, NJ 08854-8019, USA

E-mail address: `dupaigne@ math.rutgers.edu`