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Hardy-type inequalities

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1. Introduction

The well-known Hardy–Sobolev inequality states that for any given domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$ and any $u \in C_c^\infty(\Omega)$,

$$K^2 \int_\Omega \frac{u^2}{|x|^2} \leq \int_\Omega |\nabla u|^2, \tag{1}$$

where $K = (n-2)/2$. Though the constant K^2 is optimal, in the sense that

$$K^2 = \inf_{u \in C_c^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2/|x|^2},$$

equality in (1) is never achieved (by any $u \in H_0^1(\Omega)$). This fact has led to the improvement of the inequality in various ways: Brezis and Vázquez [BV] first showed that if Ω is bounded then for some $\gamma > 0$,

$$\gamma \left(\int_\Omega |u|^p \right)^{2/p} + K^2 \int_\Omega \frac{u^2}{|x|^2} \leq \int_\Omega |\nabla u|^2, \tag{2}$$

with $1 \leq p < 2n/(n-2)$. Vázquez and Zuazua [VZ] were then able to replace the L^p norm on the left hand side of (2) by a $W^{1,q}$ norm for $q < 2$. Various improvements (involving e.g. weighted L^p or $W^{1,p}$ norms) were also obtained and we refer the interested reader to [Da], [ACR], [FT], [BFT] and the references therein.

One of the consequences of inequality (2) is that the operator $L_0 := -\Delta - \mu/|x|^2$ has a positive first eigenvalue, in the sense that

$$\inf_{\|u\|_{L^2(\Omega)=1}} \int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) > 0,$$

whenever $\mu \leq K^2$.

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In the first part of this work, given a compact smooth boundaryless manifold $\Sigma \subset \Omega$ of codimension $k \neq 2$, we look at operators of the form

$$L = -\Delta - \frac{\mu}{d(x)^2},$$

where $d(x) = \text{dist}(x, \Sigma)$ and $\mu \in \mathbb{R}$, and wonder whether an inequality similar to (2) holds.

The first results in this direction are due to Marcus, Mizel and Pinchover [MMP] and Matskewich and Sobolevskii [MSo]. They showed that if Ω is a convex domain and $\Sigma = \partial\Omega$ then

$$\frac{1}{4}\int_\Omega \frac{u^2}{d(x)^2} \leq \int_\Omega |\nabla u|^2. \tag{3}$$

The same authors showed that (3) did not hold in a general domain Ω and provided examples of smooth domains Ω such that

$$\inf_{u \not\equiv 0} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2/d^2} < \frac{1}{4}.$$

Alternatively, Brezis and Marcus showed in [BM] that the following inequality remains true on a general (smooth bounded) domain Ω:

$$\frac{1}{4}\int_\Omega \frac{u^2}{d^2} \leq \int_\Omega |\nabla u|^2 + C\int_\Omega u^2,$$

where C is some positive constant.

Finally, among many other results, Barbatis, Filippas and Tertikas [BFT, FT] extended (3) to the case where $\Sigma \subset \Omega$ is a smooth compact manifold of codimension k, satisfying some geometric condition: they showed that if $\Delta d^{2-k} \leq 0$ in $\mathcal{D}'(\Omega \setminus \Sigma)$ then

$$\gamma\left(\int_\Omega |u|^p\right)^{2/p} + H^2\int_\Omega \frac{u^2}{d^2} \leq \int_\Omega |\nabla u|^2,$$

where $H = (k-2)/2$ and $1 \leq p < 2n/(n-2)$.

Our goal here is to drop the assumption $\Delta d^{2-k} \leq 0$. Our results are summarized in the following two theorems:

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $\Sigma \subset \Omega$ be a compact smooth manifold without boundary of codimension $k \neq 2$. Let $H = (k-2)/2$. Then there exist ~~$p > 2$ and~~ $C > 0$, $\gamma > 0$ independent of u such that for any $u \in C_c^\infty(\Omega \setminus \Sigma)$,*

$$\gamma\left(\int_\Omega |u|^p\right)^{2/p} + H^2\int_\Omega \frac{u^2}{d^2} \leq \int_\Omega |\nabla u|^2 + C\int_\Omega u^2, \tag{4}$$

where $d(x) = \text{dist}(x, \Sigma)$, $1 \leq p < p_k$ and p_k is given by

$$\frac{1}{p_k} = \frac{1}{2} - \frac{2}{k(n-k+2)} \quad \text{for } k > 2, \qquad \frac{1}{p_1} = \frac{1}{2} - \frac{1}{n+1} \quad \text{if } k = 1.$$

Theorem 2. *Under the assumptions of Theorem 1, there exist $\beta > 0$ and a neighborhood $\Omega_\beta := \{x \in \Omega : d(x, \Sigma) < \beta\}$ of Σ in Ω such that for any $u \in C_c^\infty(\Omega_\beta \setminus \Sigma)$,*

$$H^2 \int_\Omega \frac{u^2}{d^2} \leq \int_\Omega |\nabla u|^2. \tag{5}$$

Remark 1. • If $k \geq 3$ it follows by density that (4) and (5) hold for all $u \in C_c^\infty(\Omega)$.

- The exponent p_k appearing in Theorem 1 is probably not optimal and we expect that (4) holds for all $1 \leq p < 2n/(n-2)$. In fact Maz'ja [Ma, Corollary 3, Section 2.1.6] proved this result when $\Sigma = \{x \in \mathbb{R}^n : x_1 = x_2 = \cdots = x_k = 0\}$.

As a direct consequence of Theorem 1, we see that the first eigenvalue of the operator $L = -\Delta - \mu\, d^{-2}$ is finite, i.e.

$$\lambda_1 := \inf_{\|u\|_{L^2(\Omega)=1}} \int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{d^2} \right) > -\infty,$$

whenever $\mu \leq H^2$. We proved in [DD] that in such circumstances there exists an eigenfunction φ_1 associated to λ_1, i.e. a solution (in a sense which we shall make precise soon) of

$$\begin{cases} -\Delta\varphi_1 - \dfrac{\mu}{d^2}\varphi_1 = \lambda_1\varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Normalizing φ_1 by $\|\varphi_1\|_{L^2(\Omega)} = 1$ and $\varphi_1 > 0$, we then investigate the behavior of φ_1 near Σ and show that in a neighborhood of Σ, there exist constants $C_1, C_2 > 0$ such that

$$C_1 d(x)^{-\alpha(\mu)} \leq \varphi_1 \leq C_2 d(x)^{-\alpha(\mu)}, \tag{6}$$

where $\alpha(\mu) = H - \sqrt{H^2 - \mu}$.

This result enables us to treat two model applications. First we consider the quantity

$$J_\lambda := \inf_{u \not\equiv 0} \frac{\int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2}{\int_\Omega u^2/d^2}$$

and extend a result of Brezis and Marcus [BM] stating that J_λ is achieved if and only if $J_\lambda < H^2$.

Our second application is a nonexistence result for positive solutions of the equation

$$-\Delta u - \frac{\mu}{d^2} u = u^p + \lambda,$$

completing a study started in [DN]. See Section 4.2 for details.

The last purpose of this article is to extend some results in [DD]. This generalization is necessary to include the case of potentials $a(x) = \mu\, \mathrm{dist}(x, \Sigma)^{-2}$. More precisely, we shall derive estimates for solutions of the linear equation

$$\begin{cases} -\Delta u - a(x)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

under the assumptions that $a \in L^1_{\mathrm{loc}}(\Omega)$, a is bounded below, i.e.

$$\operatorname*{ess\,inf}_{\Omega}\ a > -\infty,$$

and

$$\gamma\left(\int_\Omega |u|^r\right)^{2/r} + \int_\Omega a(x)u^2 \le \int_\Omega |\nabla u|^2 + M\int_\Omega u^2, \tag{8}$$

for some $r > 2$, $\gamma > 0$, $M > 0$.

Let us now clarify what we mean by a solution of (7).

We first define the Hilbert space $\mathcal{H}$ as the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{\mathcal{H}} = (u|u)_{\mathcal{H}} := \int_\Omega (|\nabla u|^2 - a(x)u^2 + Mu^2),$$

where M is the same constant that appears in (8). Observe that the definition of $\mathcal{H}$ does not change if we replace M by any larger constant.

Given $f \in \mathcal{H}^*$, we then say that $u \in \mathcal{H}$ is a solution of (7) if

$$(u|v)_{\mathcal{H}} = \langle f, v\rangle_{\mathcal{H}^*,\mathcal{H}} + M(u|v)_{L^2(\Omega)} \quad \forall v \in \mathcal{H}.$$

It is convenient at this point to recall some facts that were proved in [DD]. We start by mentioning that $\mathcal{H}$ embeds compactly in $L^2(\Omega)$. In particular

$$L = -\Delta - a(x)$$

has a first eigenvalue λ_1, which is simple. λ_1 is not necessarily positive (Theorem 1 provides examples of potentials $a(x) = H^2/d(x)^2$ for which in general λ_1 can be nonpositive), but when it is, then for $f \in \mathcal{H}^*$ problem (7) has a unique solution $u \in \mathcal{H}$.

We note here that uniqueness fails if one considers other classes of solutions (see an example in [DD]).

The first eigenvalue λ_1 has an associated positive eigenfunction φ_1 (it is not only positive a.e. but it also satisfies $\varphi_1 \ge c\,\mathrm{dist}(x, \partial\Omega)$ for some $c > 0$).

Solutions in $\mathcal{H}$ of an equation like (7) are typically unbounded (see examples in [D, DD, DN]). In [DD] we showed that if $\lambda_1 > 0$ and $f \ge 0$, $f \not\equiv 0$ then the solution $u \in \mathcal{H}$ of (7) is bounded below by a positive constant times φ_1. We also proved that if $\lambda_1 > 0$ and $f = 1$, then the solution u of (7) satisfies $u \le C\varphi_1$ for some $C > 0$.

Our main result is the following:

Theorem 3. *Let $0 < m < r$ and suppose that*

$$p > \frac{2r}{m(r-2)} \quad \text{and} \quad p \ge \frac{r}{r-m}.$$

Assume that $f \in \mathcal{H}^*$ satisfies $\|\varphi_1^{1-m} f\|_{L^p(\Omega)} < \infty$ and that $u \in \mathcal{H}$ is a solution of (7). Then

$$|u(x)| \le C(\|\varphi_1^{1-m} f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)})\varphi_1(x), \quad \text{a.e. } x \in \Omega.$$

A simple corollary of this result, obtained by choosing $m = 2$, is the following:

Corollary 1. *Assume $f \in \mathcal{H}^*$ satisfies $f/\varphi_1 \in L^p(\Omega)$ for some $p > r/(r-2)$ and that $u \in \mathcal{H}$ is a solution of (7). Then*

$$|u(x)| \leq C(\|f/\varphi_1\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)})\varphi_1(x), \quad a.e.\ x \in \Omega.$$

The paper is organized as follows: Section 2 is devoted to the proof of Theorems 1 and 2. In Section 3 we derive (6) whereas Section 4 is dedicated to the aforementioned applications. Finally, we prove Theorem 3 in Section 5.

2. Hardy inequalities

2.1. Proof of Theorem 1

The object of this subsection is to prove Theorem 1. Our arguments are based on improvements of the one-dimensional Hardy inequality inspired by [BM] and a decomposition of L^2 functions in spherical harmonics taken from [VZ].

We start with a series of three lemmas, which yield a refined version of the classical Hardy inequality in $\mathbb{R}^k$ (see (3)). The first lemma deals with radial functions:

Lemma 1. *Let $k \neq 2$ and $H = (k-2)/2$. There exists a constant $C > 0$ depending only on k such that*

$$\int_0^{1/2} \left[\left(\frac{du}{dr} \right)^2 - H^2 \frac{u^2}{r^2} \right] r^{k-1}\, dr + C \int_0^{1/2} u^2 r^{k-1}\, dr \geq \int_0^{1/2} \left[\left(\frac{du}{dr} \right)^2 + H^2 \frac{u^2}{r^2} \right] r^k\, dr + \frac{1}{4} \int_0^{1/2} r \left(\frac{d}{dr}(r^H u(r)) \right)^2 dr \tag{9}$$

for all $u \in C_c^\infty(0, 1/2)$.

Proof. Let $u \in C_c^\infty(0, 1/2)$ and $v(r) = r^H u(r)$. A standard computation yields

$$\left[\left(\frac{du}{dr} \right)^2 - H^2 \frac{u^2}{r^2} \right] r^{k-1} = r \left(\frac{dv}{dr} \right)^2 - H \frac{d(v^2)}{dr}. \tag{10}$$

Integrating, it follows that

$$A := \int_0^{1/2} \left[\left(\frac{du}{dr} \right)^2 - H^2 \frac{u^2}{r^2} \right] r^{k-1}\, dr = \int_0^{1/2} r \left(\frac{dv}{dr} \right)^2 dr. \tag{11}$$

Similarly, by (10) and an integration by parts,

$$\begin{aligned} B &:= \int_0^{1/2} \left[\left(\frac{du}{dr} \right)^2 + H^2 \frac{u^2}{r^2} \right] r^k\, dr \\ &= \int_0^{1/2} r^2 \left(\frac{dv}{dr} \right)^2 dr + 2H^2 \int_0^{1/2} \frac{u^2}{r^2} r^k\, dr - H \int_0^{1/2} r \left[\frac{d(v^2)}{dr} \right] dr \\ &= \int_0^{1/2} r^2 \left(\frac{dv}{dr} \right)^2 dr + (2H^2 + H) \int_0^{1/2} v^2\, dr. \end{aligned} \tag{12}$$

Using integration by parts again, it follows that for given $\varepsilon > 0$, there exists $C > 0$ such that

$$\int_0^{1/2} v^2\,dr = -2\int_0^{1/2} rv\frac{dv}{dr}\,dr \le C\int_0^{1/2} v^2 r\,dr + \varepsilon\int_0^{1/2}\left(\frac{dv}{dr}\right)^2 r\,dr$$
$$= C\int_0^{1/2} u^2 r^{k-1}\,dr + \varepsilon\int_0^{1/2}\left(\frac{dv}{dr}\right)^2 r\,dr. \qquad (13)$$

Collecting (11), (12) and (13), we obtain for ε small enough

$$A - B \ge \int_0^{1/2} r(1 - r - C\varepsilon)\left(\frac{dv}{dr}\right)^2 dr - C\int_0^{1/2} u^2 r^{k-1}\,dr$$
$$\ge \frac{1}{4}\int_0^{1/2} r\left(\frac{dv}{dr}\right)^2 dr - C\int_0^{1/2} u^2 r^{k-1}\,dr. \qquad \square$$

The next lemma will help us deal with the nonradial part of a given function $u : \mathbb{R}^k \to \mathbb{R}$.

Lemma 2. *Let $k \ne 2$, $H = (k-2)/2$ and $c > \bar{c} > 0$. There exist constants $C, \tau > 0$ depending only on k and $\bar{c}$ such that*

$$\int_0^{1/2}\left[\left(\frac{du}{dr}\right)^2 - (H^2 - c)\frac{u^2}{r^2}\right] r^{k-1}\,dr + C\int_0^{1/2} u^2 r^{k-1}\,dr$$
$$\ge \int_0^{1/2}\left[\left(\frac{du}{dr}\right)^2 + (H^2 + c)\frac{u^2}{r^2}\right] r^k\,dr + \tau\int_0^{1/2}\left[\left(\frac{du}{dr}\right)^2 + c\frac{u^2}{r^2}\right] r^{k-1}\,dr \qquad (14)$$

for all $u \in C_c^\infty(0, 1/2)$.

Proof. It follows from (9) that if

$$D := \int_0^{1/2}\left[\left(\frac{du}{dr}\right)^2 - (H^2 - c)\frac{u^2}{r^2}\right] r^{k-1}\,dr + C\int_0^{1/2} u^2 r^{k-1}\,dr \qquad (15)$$

and

$$E := \int_0^{1/2}\left[\left(\frac{du}{dr}\right)^2 + (H^2 + c)\frac{u^2}{r^2}\right] r^k\,dr \qquad (16)$$

then

$$D - E \ge c\int_0^{1/2}\frac{u^2}{r^2}(1 - r)r^{k-1}\,dr + \frac{1}{4}\int_0^{1/2} r\left(\frac{dv}{dr}\right)^2 dr$$
$$\ge \frac{c}{2}\int_0^{1/2}\frac{u^2}{r^2}r^{k-1}\,dr + \frac{1}{4}\int_0^{1/2} r\left(\frac{dv}{dr}\right)^2 dr. \qquad (17)$$

We can also rewrite (10) as

$$r^{k-1}\left(\frac{du}{dr}\right)^2 = H^2\frac{u^2}{r^2}r^{k-1} + r\left(\frac{dv}{dr}\right)^2 - H\frac{d(v^2)}{dr}$$

so that if $\tau = \min\left(\frac{\bar{c}}{4H^2}, \frac{1}{4}\right)$, then

$$\tau\int_0^{1/2} r^{k-1}\left(\frac{du}{dr}\right)^2 dr \leq \frac{\bar{c}}{4}\int_0^{1/2}\frac{u^2}{r^2}r^{k-1}\,dr + \frac{1}{4}\int_0^{1/2} r\left(\frac{dv}{dr}\right)^2 dr. \tag{18}$$

It then follows from (17) and (18) that

$$D - E \geq \tau\int_0^{1/2} r^{k-1}\left(\frac{du}{dr}\right)^2 dr + \frac{c}{4}\int_0^{1/2}\frac{u^2}{r^2}r^{k-1}\,dr. \tag{19}$$

Hence (2) holds. □

Finally, the following lemma yields the improved Hardy inequality (in $\mathbb{R}^k$) that we will be using in what follows.

Lemma 3. *Let $k \neq 2$, $H = (k-2)/2$ and $\beta > 0$. Let B^k_β denote the ball of $\mathbb{R}^k$ centered at the origin and of radius β. There exist positive constants $C = C(\beta, k)$, $\tau = \tau(k)$ and $\alpha = \alpha(\beta, k)$ such that*

$$\int_{B^k_\beta}\left(|\nabla u|^2 - H^2\frac{u^2}{|y|^2}\right)dy + C\int_{B^k_\beta} u^2\,dy$$

$$\geq \frac{1}{2\beta}\int_{B^k_\beta}|y|\left(|\nabla u|^2 + H^2\frac{u^2}{|y|^2}\right)dy + \tau\int_{B^k_\beta}|\nabla(u-u_0)|^2\,dy + \alpha\int_0^\beta r\left(\frac{dv_0}{dr}\right)^2 dr \tag{20}$$

for all $u \in C^\infty_c(B^k_\beta \setminus \{0\})$, where $u_0(r) = u_0(|y|) = ⨍_{\partial B^k_r} u\,d\sigma$ and $v_0(r) = r^H u_0(r)$.

Proof. Let $\{f_i\}_{i=0}^\infty$ be an orthonormal basis of $L^2(S^{k-1})$, composed of eigenvectors of the Laplace–Beltrami operator $\Delta|_{S^{k-1}}$. The corresponding eigenvalues are given by $c_i = i(k+i-2)$, $i = 0, 1, 2, \ldots$ (see e.g. [St]). Any $u \in C^\infty_c(B^k_{1/2} \setminus \{0\})$ can then be written as

$$u(x) = \sum_{i=0}^\infty u_i(r) f_i(\theta)$$

where $1/2 > r > 0$, $\theta \in S^{k-1}$ and $x = r\theta$.

Furthermore, for $g \in C(\mathbb{R}^+, \mathbb{R})$,

$$\int_{B^k_{1/2}}|\nabla u|^2 g(|y|)\,dy = \int_0^{1/2} r^{k-1}g(r)\,dr\int_{S^{k-1}}\left[\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}|\nabla_\theta u|^2\right]d\theta$$

$$= \sum_{i=0}^\infty\int_0^{1/2} r^{k-1}g(r)\left[\left(\frac{du_i}{dr}\right)^2 + \frac{c_i}{r^2}u_i^2\right]dr. \tag{21}$$

For $i = 0$, it follows from (9) that if $v_0(r) = r^H u_0(r)$, then

$$\int_0^{1/2} \left[\left(\frac{du_0}{dr} \right)^2 - H^2 \frac{u_0^2}{r^2} \right] r^{k-1}\, dr + C \int_0^{1/2} u_0^2 r^{k-1}\, dr \geq \int_0^{1/2} \left[\left(\frac{du_0}{dr} \right)^2 + H^2 \frac{u_0^2}{r^2} \right] r^k\, dr + \frac{1}{4} \int_0^{1/2} r \left(\frac{dv_0}{dr} \right)^2 dr, \tag{22}$$

while (2) implies that for $i \geq 1$,

$$\int_0^{1/2} \left[\left(\frac{du_i}{dr} \right)^2 - (H^2 - c_i) \frac{u_i^2}{r^2} \right] r^{k-1}\, dr + C \int_0^{1/2} u_i^2 r^{k-1}\, dr \geq \int_0^{1/2} \left[\left(\frac{du_i}{dr} \right)^2 + (H^2 + c_i) \frac{u_i^2}{r^2} \right] r^k\, dr + \tau \int_0^{1/2} \left[\left(\frac{du_i}{dr} \right)^2 + c_i \frac{u_i^2}{r^2} \right] r^{k-1}\, dr. \tag{23}$$

Using (22), (23) and (21) with $g(r) \equiv 1$ for terms involving r^{k-1} and $g(r) = r$ for terms with r^k, we deduce (3) for $\beta = 1/2$. The general case is obtained by scaling. □

Next, we introduce some geometric notation that will be needed in the proof of Theorem 1. Define

$$\Omega_\beta = \{ x \in \Omega \mid \operatorname{dist}(x, \Sigma) < \beta \}.$$

We will work only with β small enough so that the projection $\pi : \Omega_\beta \to \Sigma$ given by $|\pi(x) - x| = \operatorname{dist}(x, \Sigma)$ is well defined and smooth.

Let $\{V_i\}_{i=1,\ldots,m}$ be a family of open disjoint subsets of Σ such that

$$\Sigma = \bigcup_{i=1}^m \overline{V}_i, \qquad |\overline{V}_i \cap \overline{V}_j| = 0 \quad \forall i \neq j.$$

We can also assume that:

(a) $\forall i = 1, \ldots, m$ there exists a smooth diffeomorphism

$$p_i : B_1^{n-k} \to U_i,$$

where $U_i \subset \Sigma$ is open and $\overline{V}_i \subset U_i$;

(b) $p_i^{-1}(V_i)$, which is an open set in $\mathbb{R}^{n-k}$, has a Lipschitz boundary; and

(c) there is a smooth choice of unit vectors $N_1^i(\sigma), \ldots, N_k^i(\sigma)$ for $\sigma \in U_i$ which form an orthonormal frame for Σ on $U_i \subset \mathbb{R}^n$, i.e. for all $\sigma \in U_i$,

$$N_j^i(\sigma) \in \mathbb{R}^n, \qquad N_j^i(\sigma) \cdot N_k^i(\sigma) = \delta_{jk}, \qquad N_j^i(\sigma) \cdot v = 0 \quad \forall v \in T_\sigma \Sigma.$$

Let $W_i = p_i^{-1}(V_i)$. For $z \in W_i$ we will also write (abusing the notation) $N_j^i(z) = N_j^i(p_i(z))$. Let

$$F_i(y, z) = p_i(z) + \sum_{j=1}^k y_j N_j^i(z),$$

where $y = (y_1, \ldots, y_k) \in B_\beta^k$ and $z \in W_i$, so that F_i is a smooth diffeomorphism between $B_\beta^k \times W_i$ and T_β^i, where

$$T_\beta^i = \pi^{-1}(V_i) \cap \Omega_\beta. \tag{24}$$

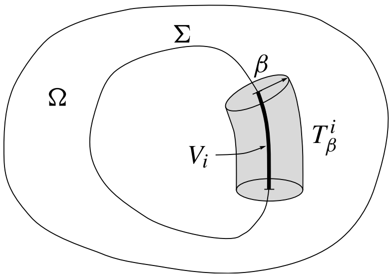

Fig. 1.

It follows from the condition $|\overline{V}_i \cap \overline{V}_j| = 0 \ \forall i \neq j$ that $|\overline{T}^i_\beta \cap \overline{T}^j_\beta| = 0 \ \forall i \neq j$, and hence, for any $f \in L^1(\Omega_\beta)$ we have

$$\int_{\Omega_\beta} f = \sum_{i=1}^{m} \int_{T^i_\beta} f = \sum_{i=1}^{m} \int_{W_i \times B^k_\beta} f \circ F_i(y,z)\, JF_i(y,z)\, dy\, dz, \tag{25}$$

where $JF_i(y,z)$ stands for the Jacobian of F_i at (y,z). We claim that

$$JF_i(y,z) = G_i(z)(1 + O(|y|)), \tag{26}$$

where $O(|y|)$ denotes a quantity bounded by $|y|$ (uniformly for $z \in W_i$) and $G_i(z)$ is a smooth function which is bounded away from zero. More precisely

$$G_i(z) = Jp_i(z) = \sqrt{(Dp_i(z))^* Dp_i(z)}.$$

To prove (26) it suffices to observe that $JF_i(y,z)$ is smooth and to compute it at $y = 0$:

$$\begin{aligned} JF_i(0,z)^2 &= \det(DF_i(0,z)^* DF_i(0,z)) \\ &= \det([D_z p_i | N^i_1, \ldots, N^i_k]^* [D_z p_i | N^i_1, \ldots, N^i_k]) \\ &= \det\begin{bmatrix} (D_z p_i)^* D_z p_i & 0 \\ 0 & I \end{bmatrix}. \end{aligned}$$

Proof of Theorem 1. First, observe that it is sufficient to prove the theorem for u with support near Σ. Indeed, following an idea of Vázquez and Zuazua [VZ], let $\eta \in C^\infty_c(\mathbb{R}^n)$ be such that $\eta \equiv 1$ in $\Omega_{\beta/2}$ and $\operatorname{supp}(\eta) \subset \Omega_\beta$. Let $u \in C^\infty_c(\Omega \setminus \Sigma)$ and write $u = u_1 + u_2$ where $u_1 = \eta u$, $u_2 = (1-\eta)u$. Suppose that the conclusion of the theorem holds for u_1. Then

$$\begin{aligned} \int_\Omega \left(|\nabla u|^2 - H^2 \frac{u^2}{d^2} \right) &= \int_\Omega \left(|\nabla u_1|^2 - H^2 \frac{u_1^2}{d^2} \right) + \int_\Omega \left(|\nabla u_2|^2 - H^2 \frac{u_2^2}{d^2} \right) \\ &\quad + 2 \int_\Omega \left(\nabla u_1 \cdot \nabla u_2 - H^2 \frac{u_1 u_2}{d^2} \right). \end{aligned} \tag{27}$$

Since $1/d$ is bounded away from Σ we have

$$\int_\Omega \left(\frac{u_2^2}{d^2} + \frac{u_1 u_2}{d^2}\right) \le C \int_\Omega u^2.$$

Also note that

$$\begin{aligned}
\int_\Omega \nabla u_1 \cdot \nabla u_2 &= \int_\Omega [\eta(1-\eta)|\nabla u|^2 - |\nabla \eta|^2 u^2 + u\nabla u \cdot \nabla \eta(1-2\eta)] \\
&= \int_\Omega [\eta(1-\eta)|\nabla u|^2 - |\nabla \eta|^2 u^2] - \frac{1}{2}\int_{\Omega_\beta \setminus \Omega_{\beta/2}} u^2 \nabla \cdot (\nabla \eta(1-2\eta)) \\
&\ge -C \int_\Omega u^2. \qquad (28)
\end{aligned}$$

It follows from (27), (28) that

$$\int_\Omega \left[|\nabla u|^2 - H^2 \frac{u^2}{d^2}\right] \ge \int_\Omega \left[|\nabla u_1|^2 - H^2 \frac{u_1^2}{d^2}\right] + \int_\Omega |\nabla u_2|^2 - C \int_\Omega u^2.$$

Using (4) with u_1 we conclude that

$$\int_\Omega \left[|\nabla u|^2 - H^2 \frac{u^2}{d^2}\right] + C \int_\Omega u^2 \ge \gamma \left(\int_\Omega |u_1|^p\right)^{2/p} + \int_\Omega |\nabla u_2|^2,$$

for some $\gamma > 0$ independent of u. Hence the conclusion of the theorem for u follows easily.

Let

$$I_i = \int_{T_\beta^i} \left[|\nabla u|^2 - H^2 \frac{u^2}{d^2} + u^2\right], \qquad (29)$$

where T_β^i was defined in (24). In what follows we will fix i and show that there are $p > 2$ and $C > 0$ independent of u such that

$$\left(\int_{T_\beta^i} |u|^p\right)^{2/p} \le C I_i.$$

For simplicity, and since i is fixed, we will drop the index i from all the notation that follows.

Let us introduce some additional notation:

$$\tilde{u}(y,z) = u(F(y,z)), \qquad (30)$$

$$\tilde{u}_0(r,z) = \fint_{\partial B_r} \tilde{u}(y,z)\, ds(y), \qquad (31)$$

$$v_0(r,z) = r^H \tilde{u}_0(r,z). \qquad (32)$$

Let us write

$$\nabla u = \nabla_N u + \nabla_T u$$

where $\nabla_N u$ is the gradient of u in the normal direction and $\nabla_T u$ is orthogonal to $\nabla_N u$. More precisely, for a point $x = F(y, z)$,

$$\nabla_N u(x) = \sum_{j=1}^{k} \nabla u(x) \cdot N_j(z)\, N_j(z).$$

Step 1. There exists $C > 0$ independent of u such that

$$\begin{aligned} CI \geq & \int_{W \times B_\beta^k} |\nabla_y \tilde{u}|^2 |y|\, dy\, dz + \int_{W \times B_\beta^k} |\nabla_y (\tilde{u}(y,z) - \tilde{u}_0(y,z))|^2\, dy\, dz \\ & + \int_W \int_0^\beta \left(\frac{\partial v_0}{\partial r}\right)^2 r\, dr\, dz + \int_{W \times B_\beta^k} |(\nabla_T u) \circ F|^2\, dy\, dz. \end{aligned} \tag{33}$$

First note that by (25), there is a constant $C > 0$ such that

$$\begin{aligned} I \geq & \int_{W \times B_\beta^k} \left(|\nabla_N u(F(y,z))|^2 - H^2 \frac{\tilde{u}^2}{|y|^2} \right) G(z)\, dy\, dz \\ & - C \int_{W \times B_\beta^k} \left(|\nabla_N u(F(y,z))|^2 + H^2 \frac{\tilde{u}^2}{|y|^2} \right) G(z) |y|\, dy\, dz \\ & + \int_{W \times B_\beta^k} (|\nabla_T u(F(y,z))|^2 + \tilde{u}^2)(1 - C|y|) G(z)\, dy\, dz. \end{aligned} \tag{34}$$

For fixed z we can apply Lemma 3 to the function $\tilde{u}(\cdot, z)$. Observe that

$$\frac{\partial \tilde{u}(y,z)}{\partial y_j} = \nabla u(F(y,z)) \cdot N_j(z)$$

and thus

$$|\nabla_y \tilde{u}(y,z)|^2 = |\nabla_N u(F(y,z))|^2.$$

Lemma 3 then yields

$$\begin{aligned} & \int_{B_\beta^k} \left(|\nabla_N u(F(y,z))|^2 - H^2 \frac{u^2}{|y|^2} \right) dy + C \int_{B_\beta^k} \tilde{u}^2\, dy \\ & \qquad \geq \frac{1}{2\beta} \int_{B_\beta^k} |y| \left(|\nabla_N u(F(y,z))|^2 + H^2 \frac{\tilde{u}^2}{|y|^2} \right) dy \\ & \qquad\quad + \tau \int_{B_\beta^k} \left| \nabla_y (\tilde{u} - \tilde{u}_0) \right|^2 dy + \alpha \int_0^\beta r \left(\frac{dv_0}{dr} \right)^2 dr. \end{aligned} \tag{35}$$

We choose (and fix once for all) $\beta > 0$ small enough so that $1/(2\beta) \geq C + 1$. Then multiplying (35) by $G(z)$, integrating over W and combining the result with (34) we conclude that (33) holds.

Step 2.

$$\|\nabla v_0\|^2_{L^2(W\times B^2_\beta)} \le CI. \tag{36}$$

By (33) the partial derivative $\partial v_0/\partial r$ is bounded in $L^2(W \times B^2_\beta)$ by CI. We just have to control the derivatives $\partial v_0/\partial z_i$, $i = 1, \dots, n-k$. But

$$\frac{\partial v_0}{\partial z_i}(r,z) = r^H \fint_{\partial B_r} \frac{\partial \tilde u}{\partial z_i}(y,z)\, ds(y)$$

and

$$\frac{\partial \tilde u}{\partial z_i}(y,z) = \nabla u(F(y,z)) \cdot \left[\frac{\partial p}{\partial z_i} + \sum_{j=1}^k y_j \frac{\partial N_j}{\partial z_i} \right].$$

But note that $\partial p/\partial z_i$ is a tangent vector, hence

$$|\nabla_z \tilde u(y,z)| \le C|\nabla_T u(F(y,z))| + C|y|\,|\nabla_N u(F(y,z))|.$$

Integrating over $W \times B^k_\beta$ we have

$$\int_{W\times B^k_\beta} |\nabla_z \tilde u(y,z)|^2\, dy\, dz \le CI, \tag{37}$$

for some C independent of u by (33). It follows that

$$\begin{aligned}
\int_{W\times B^2_\beta} |\nabla_z v_0|^2\, dy\, dz &= \int_W \int_0^\beta r^{2H+1} \left| \fint_{\partial B_r} \nabla_z \tilde u(y,z)\, ds(y) \right|^2 dr\, dz \\
&\le \int_W \int_0^\beta r^{k-1} \fint_{\partial B_r} |\nabla_z \tilde u(y,z)|^2\, ds(y)\, dr\, dz \\
&\le C \int_{W\times B^k_\beta} |\nabla_z \tilde u(y,z)|^2\, dy\, dz \le CI \qquad (38)
\end{aligned}$$

by (37).

Step 3. There is $p > 2$ such that

$$\|\tilde u_0\|^2_{L^p(W\times B^k_\beta)} \le CI. \tag{39}$$

More precisely, for $k \ge 3$ one can take any $2 < p < p_k$ where p_k is given by

$$\frac{1}{p_k} = \frac{1}{2} - \frac{2}{k(n-k+2)},$$

and for $k = 1$ one can take $2 < p \le p_1$ where p_1 is given by

$$\frac{1}{p_1} = \frac{1}{2} - \frac{1}{n+1}.$$

Using Sobolev's inequality (on $W \times B^2_\beta$) combined with (36) we obtain

$$\int_W \int_0^\beta |v_0|^q r \, dr \, dz \leq C I^{q/2},$$

with q given by $1/q = 1/2 - 1/(n-k+2)$. That is, in terms of $\tilde{u}_0$ we have

$$\int_W \int_0^\beta |\tilde{u}_0|^q r^{qH+1} \, dr \, dz \leq C I^{q/2}. \tag{40}$$

We want an estimate for $\int |\tilde{u}_0|^p r^{k-1} \, dr \, dz$ for some suitable $2 < p < q$ and for this we use Hölder's inequality, distinguishing two cases:

Case $k \geq 3$. We have

$$\begin{aligned}\int_W \int_0^\beta |\tilde{u}_0|^p r^{k-1} \, dr \, dz &= \int_W \int_0^\beta |\tilde{u}_0|^p r^\alpha r^{k-2-\alpha} r \, dr \, dz \\ &\leq C \left(\int_W \int_0^\beta |\tilde{u}_0|^q \, r^{\alpha q/p+1} \, dr, dz \right)^{p/q} \left(\int_0^\beta r^{\frac{k-2-\alpha}{1-p/q}+1} \, dr \right)^{1-p/q}. \end{aligned} \tag{41}$$

We then choose α so that

$$\frac{\alpha}{p} = H = \frac{k-2}{2}.$$

In order to have the second factor on the right hand side of (41) finite we need to impose

$$\frac{k-2-\alpha}{1-p/q} > -2,$$

which is equivalent to the condition

$$\alpha < \frac{k}{1 + \frac{4}{q(k-2)}}.$$

Thus we need $p = \alpha/H < p_k$, where p_k is given by

$$p_k = \frac{2k}{(k-2)\left(1 + \frac{4}{q(k-2)}\right)},$$

i.e.

$$\frac{1}{p_k} = \frac{1}{2} - \frac{2}{k(n-k+2)}.$$

Observe that $p_k > 2$. Combining then (40) and (41) finishes this case.

Case $k = 1$. In this case q is given by $1/q = 1/2 - 1/n + 1$, and we can choose $p = q$:

$$\begin{aligned}\int_W \int_0^\beta |\tilde{u}_0|^q r^{k-1}\, dr\, dz &= \int_W \int_0^\beta |\tilde{u}_0|^q\, dr\, dz \\ &\le \int_W \int_0^\beta |\tilde{u}_0|^q r^{-q/2+1}\, dr\, dz \\ &= \int_W \int_0^\beta |\tilde{u}_0|^q r^{Hq+1}\, dr\, dz\end{aligned}$$

because $-q/2 + 1 < 0$.

Step 4.

$$\|\tilde{u} - \tilde{u}_0\|^2_{L^{2^*}(W\times B^k_\beta)} \le CI. \tag{42}$$

This is a consequence of Sobolev's inequality applied to the function $\tilde{u} - \tilde{u}_0$ on the domain $W \times B^k_\beta$. (33) already provides a bound in $L^2(W \times B^k_\beta)$ for $\nabla_y(\tilde{u} - \tilde{u}_0)$. Hence we only need to obtain a bound for the derivative of $\tilde{u} - \tilde{u}_0$ with respect to z. In the case of the function $\tilde{u}$ we have it already in (37). For $\tilde{u}_0$ it is derived by a computation very similar to that at the end of Step 2. Indeed,

$$\int_{W\times B^k_\beta} |\nabla_z \tilde{u}_0|^2\, dy\, dz = \int_W \int_0^\beta r^{k-1} \left| ⨍_{\partial B_r} \nabla_z \tilde{u}(y,z)\, ds(y)\right|^2 dr\, dz \le CI,$$

which we obtain as in (38).

Conclusion. By (39) and (42) we see that

$$\|\tilde{u}\|^2_{L^p(W\times B^k_\beta)} \le CI$$

for some C independent of u. Changing variables and reintroducing the index i we have

$$\|u\|^2_{L^p(T^i_\beta)} \le C \int_{T^i_\beta} \left(|\nabla u|^2 - H^2 \frac{u^2}{d^2} + u^2 \right).$$

Adding these inequalities over i proves the statement of the theorem. □

2.2. *A local version of the Hardy inequality*

In this section, we show how to adapt the proof of Theorem 1 to obtain Theorem 2. We first derive variants of Lemmas 1, 2, 3.

Lemma 4. *Let* $k \ne 2$ *and* $H = (k-2)/2$. *There exist constants* $C, \beta_0 > 0$ *such that for* $0 < \beta \le \beta_0$,

$$\int_0^\beta \left[\left(\frac{du}{dr}\right)^2 - H^2 \frac{u^2}{r^2} \right] r^{k-1}\, dr \ge \int_0^\beta \left[\left(\frac{du}{dr}\right)^2 + H^2 \frac{u^2}{r^2} \right] r^k\, dr \tag{43}$$

for all $u \in C^\infty_c(0, \beta)$.

Proof. Given $v \in C_c^\infty(0, 1/2)$, we have

$$\int_0^{1/2} v^2\, dr = -2\int_0^{1/2} rv\frac{dv}{dr}\, dr \le C\int_0^{1/2} r^2\Big(\frac{dv}{dr}\Big)^2\, dr + \frac{1}{2}\int_0^{1/2} v^2\, dr.$$

Using this and (12) we obtain

$$\int_0^{1/2}\Big[\Big(\frac{du}{dr}\Big)^2 + H^2\frac{u^2}{r^2}\Big]r^k\, dr \le C\int_0^{1/2} r^2\Big(\frac{dv}{dr}\Big)^2\, dr.$$

Changing variables, it then follows that for $u \in C_c^\infty(0, \beta)$,

$$\int_0^{\beta}\Big[\Big(\frac{du}{dr}\Big)^2 + H^2\frac{u^2}{r^2}\Big]r^k\, dr \le C\beta^{k-2}\int_0^{\beta} r^2\Big(\frac{dv}{dr}\Big)^2\, dr, \tag{44}$$

while (11) becomes

$$\int_0^{\beta}\Big[\Big(\frac{du}{dr}\Big)^2 - H^2\frac{u^2}{r^2}\Big]r^{k-1}\, dr = \beta^{k-2}\int_0^{\beta} r\Big(\frac{dv}{dr}\Big)^2\, dr. \tag{45}$$

If we pick β small, (43) follows from (44) and (45). □

A straightforward corollary of the above lemma is:

Lemma 5. *Let $k \neq 2$, $H = (k-2)/2$ and $c > 0$. There exist constants $C, \beta_0 > 0$ such that for $0 < \beta \le \beta_0$,*

$$\int_0^{\beta}\Big[\Big(\frac{du}{dr}\Big)^2 - (H^2 - c)\frac{u^2}{r^2}\Big]r^{k-1}\, dr \ge \int_0^{\beta}\Big[\Big(\frac{du}{dr}\Big)^2 + (H^2 + c)\frac{u^2}{r^2}\Big]r^k\, dr \tag{46}$$

for all $u \in C_c^\infty(0, \beta)$.

Combining these two lemmas, we then obtain:

Lemma 6. *Let $k \neq 2$, $H = (k-2)/2$ and $\beta > 0$. Let B_β^k denote the ball of $\mathbb{R}^k$ centered at the origin and of radius β. There exist positive constants C, β_0 such that for $\beta \le \beta_0$,*

$$\int_{B_\beta^k}\Big(|\nabla u|^2 - H^2\frac{u^2}{|y|^2}\Big)dy \ge \frac{C}{\beta}\int_{B_\beta^k}|y|\Big(|\nabla u|^2 + H^2\frac{u^2}{|y|^2}\Big)dy \tag{47}$$

for all $u \in C_c^\infty(B_\beta^k \setminus \{0\})$, where $u_0(r) = u_0(|y|) = \fint_{\partial B_r^k} u\, d\sigma$ and $v_0(r) = r^H u_0(r)$.

As in Lemma 3, for a fixed value $\beta = \beta_0 > 0$ the proof is an application of the decomposition of a function in spherical harmonics. A simple scaling then yields the β-dependence of the constant appearing in (47).

Proof of Theorem 2. Instead of (29), we now consider

$$J_i := \int_{T_\beta^i}\Big[|\nabla u|^2 - H^2\frac{u^2}{d^2}\Big]. \tag{48}$$

Using the notation of (30) we then have, by (25) and (26),

$$J_i \geq \int_{W\times B_\beta^k} \left(|\nabla_N u(F_i(y,z))|^2 - H^2 \frac{\tilde{u}^2}{|y|^2} \right) G(z)\, dy\, dz$$
$$- C \int_{W\times B_\beta^k} |y| \left(|\nabla_N u(F_i(y,z))|^2 + H^2 \frac{\tilde{u}^2}{|y|^2} \right) G(z)\, dy\, dz \geq 0,$$

where we used Lemma 6 with $\beta > 0$ small in the last inequality. Adding the above estimates over i yields the desired result. □

3. Remarks on the potential $a(x) = \mu\, \mathrm{dist}(x, \Sigma)^{-2}$

For $0 < \mu \leq H^2$ we consider the potential

$$a(x) = \mu / d(x)^2$$

and let L denote the operator

$$Lu = -\Delta u - a(x) u.$$

Note that $a(x)$ and L depend on μ but we will omit this dependence from the notation.

Recall that we defined the Hilbert space $\mathcal{H}$ as the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{H}}^2 = \int_\Omega (|\nabla u|^2 - a(x)u^2 + Mu^2). \tag{49}$$

If $\mu < H^2$ then by Theorem 1, $\mathcal{H}$ coincides with $H_0^1(\Omega)$.

The main concern in this section is to obtain a precise description of the behavior near Σ of the first eigenfunction φ_1 of the operator L. Indeed, we shall prove:

Lemma 7. *There are positive constants C_1, C_2 such that*

$$C_1 d(x)^{-\alpha(\mu)} \leq \varphi_1(x) \leq C_2 d(x)^{-\alpha(\mu)} \tag{50}$$

for x in a neighborhood of Σ, where $\alpha(\mu)$ is given by

$$\alpha(\mu) = H - \sqrt{H^2 - \mu}. \tag{51}$$

Note that when $\mu = H^2$ we have $-\alpha(\mu) = 1 - k/2$. Thus $\varphi_1 \notin H_0^1(\Omega)$ in this case. Before proving the above lemma it will be necessary to show that if $\mu = H^2$ then $d^{1-k/2}$ (appropriately modified so that it is zero on $\partial\Omega$) belongs to $\mathcal{H}$. We prove this and a little more next.

Lemma 8. *Let $\mu = H^2$ and define*

$$v_s(x) = \eta(x) d(x)^{1-k/2} (-\log d(x))^{-s},$$

where $\eta \in C_c^\infty(\Omega)$ is a cut-off function such that $\eta \equiv 1$ in a neighborhood of Σ and $\eta(x) = 0$ for $d(x) \geq \mathrm{dist}(\Sigma, \partial\Omega)/2$. Then $v_s \in \mathcal{H}$ if and only if $s > -1/2$.

Remark 2. This lemma was stated in [VZ] in the case where Σ is a point.

Proof. Let us recall and also introduce some notation:

$$\Omega_r = \{x \in \mathbb{R}^N \mid d(x) < r\}, \qquad \Sigma_r = \partial\Omega_r = \{x \in \mathbb{R}^N \mid d(x) = r\}.$$

By the Pappus theorems, the $(N-1)$-dimensional area of Σ_r is given by

$$|\Sigma_r|_{n-1} = \omega_{k-1} r^{k-1} |\Sigma|_{n-k},$$

where ω_{k-1} is the area of the unit sphere in $\mathbb{R}^k$ and $|\cdot|_j$ denotes the j-dimensional Lebesgue measure.

First we prove that $v_s \in \mathcal{H}$ for $s > -1/2$. For this purpose it is enough to exhibit a sequence $f_\varepsilon \in \mathcal{H}$ such that $\|f_\varepsilon\|_{\mathcal{H}} \le C$ with C independent of ε and such that $f_\varepsilon \to v_s$ a.e. as $\varepsilon \to 0$; we take

$$f_\varepsilon = \eta d^{1-k/2+\varepsilon}(-\log d)^{-s}, \qquad \varepsilon > 0.$$

Clearly $f_\varepsilon \in H_0^1(\Omega) \subset \mathcal{H}$, $\int_\Omega f_\varepsilon^2 \le C$ and f_ε is smooth away from Σ. Thus to estimate $\|f_i\|_{\mathcal{H}}$ it is sufficient to verify that for a fixed $R > 0$ small

$$\int_{\Omega_R} |\nabla f_\varepsilon|^2 - a(x) f_\varepsilon^2 \le C \tag{52}$$

with C independent of ε.

Near Σ, $\eta \equiv 1$ and

$$\begin{aligned}|\nabla f_\varepsilon|^2 = d^{-k+2\varepsilon}\big((1-k/2+\varepsilon)^2(-\log d)^{-2s} \\ + s(2-k+2\varepsilon)(-\log d)^{-2s-1} + s^2(-\log d)^{-2s-2}\big).\end{aligned}$$

so that

$$\begin{aligned}\frac{1}{\omega_{k-1}|\Sigma|_{n-k}} \int_{\Omega_R} |\nabla f_\varepsilon|^2 &= (1-k/2+\varepsilon)^2 \int_0^R r^{2\varepsilon-1}(-\log r)^{-2s}\,dr \\ &\quad + s(2-k+2\varepsilon)\int_0^R r^{2\varepsilon-1}(-\log r)^{-2s-1}\,dr \\ &\quad + s^2 \int_0^R r^{2\varepsilon-1}(-\log r)^{-2s-2}\,dr.\end{aligned}$$

Note that the last integral on the right hand side above is bounded independently of ε for $s > -1/2$, that is,

$$\int_0^R r^{2\varepsilon-1}(-\log r)^{-2s-2}\,dr = O(1).$$

Therefore

$$\begin{aligned}\frac{1}{\omega_{k-1}|\Sigma|_{n-k}}\int_{\Omega_R}\left(|\nabla f_\varepsilon|^2 - H^2\frac{f_\varepsilon^2}{d^2}\right)\\ &= \varepsilon(2-k+\varepsilon)\int_0^R r^{2\varepsilon-1}(-\log r)^{-2s}\,dr\\ &\quad + s(2-k+2\varepsilon)\int_0^R r^{2\varepsilon-1}(-\log r)^{-2s-1}\,dr + O(1). \qquad (53)\end{aligned}$$

Integrating by parts gives

$$\int_0^R r^{2\varepsilon-1}(-\log r)^{-2s}\,dr = \frac{1}{2\varepsilon}R^{2\varepsilon}(-\log R)^{-2s} - \frac{s}{\varepsilon}\int_0^R r^{2\varepsilon-1}(-\log r)^{-2s-1}\,dr$$

and substituting in (53) yields

$$\begin{aligned}\frac{1}{\omega_{k-1}|\Sigma|_{n-k}}\int_{\Omega_R}\left(|\nabla f_\varepsilon|^2 - H^2\frac{f_\varepsilon^2}{d^2}\right) &= \frac{2-k+\varepsilon}{2}R^{2\varepsilon}(-\log R)^{-2s}\\ &\quad + \varepsilon s\int_0^R r^{2\varepsilon-1}(-\log r)^{-2s-1}\,dr + O(1)\\ &= \varepsilon s\int_0^R r^{2\varepsilon-1}(-\log r)^{-2s-1}\,dr + O(1). \qquad (54)\end{aligned}$$

Integrating by parts again shows that

$$\begin{aligned}&\int_0^R r^{2\varepsilon-1}(-\log r)^{-2s-1}\,dr\\ &\quad = \frac{1}{2\varepsilon}R^{2\varepsilon}(-\log R)^{-2s-1} - \frac{2s+1}{2\varepsilon}\int_0^R r^{2\varepsilon-1}(-\log r)^{-2s-2}\,dr = O\left(\frac{1}{\varepsilon}\right).\end{aligned}$$

After substitution in (54) we finally obtain the estimate (52). Hence $v_s \in \mathcal{H}$ for $s > -1/2$.

Our argument to show that $v_s \notin \mathcal{H}$ for $s \le -1/2$ relies on the intuitive idea that $\int(-\Delta v - a(x)v + Mv)v = \|v\|_{\mathcal{H}}^2$. To exploit this idea, let us first compute Δv_s near Σ, where $\eta \equiv 1$. Write

$$y(t) = t^{1-k/2}(-\log t)^{-s}.$$

Then near Σ, since $|\nabla d|^2 = 1$,

$$\Delta v_s = y''(d) + y'(d)\Delta d.$$

We recall here the fact (see [DN]) that

$$\Delta d = \frac{k-1}{d} + g,$$

where $g \in L^\infty$. Hence,

$$\begin{aligned}\Delta v_s = & -H^2 d^{-k/2-1}(-\log d)^{-s} + s(s+1)d^{-k/2-1}(-\log d)^{-s-2} \\ & + (1-k/2)gd^{-k/2}(-\log d)^{-s} + sgd^{-k/2}(-\log d)^{-s-1}\end{aligned}$$

so that

$$\begin{aligned}\Delta v_s + H^2\frac{v_s}{d^2} = & \, s(s+1)d^{-k/2-1}(-\log d)^{-s-2} \\ & + (1-k/2)gd^{-k/2}(-\log d)^{-s} + sgd^{-k/2}(-\log d)^{-s-1}. \qquad (55)\end{aligned}$$

Observe that $\Delta v_s, v_s/d^2 \in L^1(\Omega)$ and that equation (55) holds in the sense of distributions. Since we also have $\nabla v_s \in L^1(\Omega)$, it follows that for any $\varphi \in C_c^\infty(\Omega)$,

$$\begin{aligned}(v_s|\varphi)_{\mathcal{H}} = & \int_\Omega \left(-\Delta v_s - H^2\frac{v_s^2}{d^2}\right)\varphi + M\int_\Omega v_s\varphi \\ & - \int_{\Omega_R} \big(s(s+1)d^{-k/2-1}(-\log d)^{-s-2} \\ & + (1-k/2)gd^{-k/2}(-\log d)^{-s} + sgd^{-k/2}(-\log d)^{-s-1}\big)\varphi \\ & + \int_{\Omega\setminus\Omega_R}\left(-\Delta v_s - H^2\frac{v_s}{d^2}\right)\varphi + M\int_\Omega v_s\varphi. \qquad (56)\end{aligned}$$

By density (56) also holds if φ is Lipschitz and $\varphi = 0$ on $\partial\Omega$.

Let us consider first the case $s \neq -1$, so that $s(s+1) \neq 0$, and suppose that $s < -1/2$ and $v_s \in \mathcal{H}$. Then there exist $v_n \in C_c^\infty(\Omega)$ such that $v_n \to v_s$ in $\mathcal{H}$. Note that since the injection $\mathcal{H} \subset L^2(\Omega)$ is continuous, by passing to a subsequence we also have $v_n \to v_s$ a.e. Recall from [DN, inequality (1.4) of Lemma 1.1] that for $u \in \mathcal{H}$, we have $u^+ \in \mathcal{H}$ and $\|u^+\|_{\mathcal{H}} \leq \|u\|_{\mathcal{H}}$. As a consequence $v_n^+ \to v_s$ in $\mathcal{H}$ and a.e. Using v_n^+ in (56) we conclude that

$$\int_{\Omega_R} d^{-k/2-1}(-\log d)^{-s-2}v_n^+ \leq C$$

with C independent of n. But then Fatou's lemma implies that

$$\int_{\Omega_R} d^{-k}(-\log d)^{-2s-2} < \infty,$$

which is impossible for $s < -1/2$.

For the case $s = -1$ the argument above does not work. We see that in this case, if Σ is flat and $\eta \equiv 1$ in an open set then actually

$$\Delta w + H^2\frac{w}{d^2} = 0$$

in that open set, where

$$w := v_{(-1)} = \eta d^{1-k/2}(-\log d).$$

So we argue as follows: let $-1/2 < s < 0$. We are going to show that $(L + M)v_s \geq (L + M)w$ near Σ. If we assume that $w \in \mathcal{H}$, then we can apply the maximum principle and deduce that $v_s \geq \varepsilon w$ near Σ, which is impossible. Indeed, by formula (55),

$$(L + M)w = -(1 - k/2)gd^{-k/2}(-\log d) + gd^{-k/2} + Md^{1-k/2}(-\log d),$$

and

$$\begin{aligned}(L + M)v_s = & -s(s + 1)d^{-k/2-1}(-\log d)^{-s-2} - (1 - k/2)gd^{-k/2}(-\log d)^{-s} \\ & - sgd^{-k/2}(-\log d)^{-s-1} + Md^{1-k/2}(-\log d)^{-s}.\end{aligned}$$

Thus, ~~for any $\varepsilon > 0$~~ there is a neighborhood Ω_R of Σ such that

$$(L + M)(\varepsilon w - v_s) \leq 0 \quad \text{in } \Omega_R. \tag{57}$$

Pick $\varepsilon > 0$ such that $\varepsilon w - v_s \leq 0$ in $\partial\Omega_R$. Under the hypothesis $w \in \mathcal{H}$ we can use a version of the maximum principle to deduce that

$$\varepsilon w - v_s \leq 0 \quad \text{in } \Omega_R.$$

Indeed, assuming $w \in \mathcal{H}$, we have $(\varepsilon w - v_s)^+ \in \mathcal{H}$. Hence the function

$$z = \begin{cases} (\varepsilon w - v_s)^+ & \text{in } \Omega_R, \\ 0 & \text{in } \Omega \setminus \Omega_R, \end{cases}$$

also belongs to $\mathcal{H}$. Let $z_n \in C_c^\infty(\Omega)$ be such that $z_n \to z$ in $\mathcal{H}$. Note that (57) holds in the sense of distributions and hence testing (57) with z_n^+ we see that

$$(\varepsilon w - v_s | z_n^+)_{\mathcal{H}} \leq 0.$$

Letting $n \to \infty$ we get

$$\|z\|_{\mathcal{H}} = (\varepsilon w - v_s | z)_{\mathcal{H}} \leq 0.$$

Thus $z \equiv 0$, which implies that $\varepsilon w \leq v_s$ in Ω_R, concluding the proof of Lemma 8. $\square$

Remark 3. To show that $v_s \in \mathcal{H}$ for $s > -1/2$ one may be tempted to use other approximating sequences, and a very natural one is

$$f_i = \min(v_s, i), \quad i = 1, 2, \ldots.$$

Again it would be sufficient to establish that for a fixed $R > 0$ small

$$\int_{\Omega_R} (|\nabla f_i|^2 - a(x) f_i^2) \leq C$$

with C independent of i. For i large let $r_i > 0$ be such that

$$r_i^{1-k/2}(-\log r_i)^{-s} = i$$

so that $r_i \to 0$ as $i \to \infty$. A computation (that we omit) shows that

$$\frac{1}{w_{k-1}} \int_{\Omega_R} (|\nabla f_i|^2 - a(x) f_i^2) = \frac{k-2}{4}(-\log r_i)^{-2s} - \frac{k-2}{2}(-\log R)^{-2s} + \frac{s^2}{2s+1}((-\log R)^{-2s-1} - (-\log r_i)^{-2s-1}).$$

We see that the above quantity remains bounded as $i \to \infty$ only for $s \geq 0$!

Remark 4. The above example shows that for $m \in \mathbb{N}$ there exists $v_m \in \mathcal{H}$ with $\|v_m\|_{\mathcal{H}} = 1$ and

$$\|\min(v_m, m)\|_{\mathcal{H}} \to \infty \qquad \text{as } m \to \infty.$$

Since $v = \min(v, m) + (v - m)^+$ we also have

$$\|(v_m - m)^+\|_{\mathcal{H}} \to \infty \qquad \text{as } m \to \infty.$$

$\mathcal{H}$ is thus quite different from $H_0^1(\Omega)$, in the sense that truncation operators like the one above are not uniformly bounded in $\mathcal{H}$ whereas it is always true that for any $v \in H_0^1(\Omega)$, $\min(v, m) \to v$ in the H^1 topology.

Proof of Lemma 7. We will give a proof using a comparison argument with a suitable function. First let us recall that in a neighborhood of Σ,

$$\Delta d = \frac{k-1}{d} + g,$$

where g is a bounded function. Hence

$$L d^{-\alpha} = -\Delta d^{-\alpha} - \mu \frac{d^{-\alpha}}{d^2} = -d^{-\alpha-2}(\alpha^2 - \alpha(k-2) + \mu - \alpha g d). \tag{58}$$

Let $\alpha = \alpha(\mu)$ as given by (51). This implies that $\alpha^2 - \alpha(k-2) + \mu = 0$. Then

$$L(d^{-\alpha} + C_1 d^{-\alpha+1}) = -d^{-\alpha-1}[-\alpha g + C_1((\alpha-1)^2 - (\alpha-1)(k-2) + \mu - (\alpha-1) g d)].$$

Instead of [illegible]ing with the operator $L = -\Delta - a(x)$ consider $L + M$, where M is so large that (4) holds (with C replaced by M, this is the same M that we use in the definition of the space $\mathcal{H}$). Then, since $(\alpha-1)^2 - (\alpha-1)(k-2) + \mu > 0$ we conclude that for $C_1 > 0$ large enough

$$\begin{aligned}(L+M)(d^{-\alpha} + C_1 d^{-\alpha+1}) &= -d^{-\alpha-1}[-\alpha g + C_1((\alpha-1)^2 - (\alpha-1)(k-2) + \mu - (\alpha-1) g d)] \\ &\quad + M(d^{-\alpha} + C_1 d^{-\alpha+1}) \\ &\leq 0 \end{aligned} \tag{59}$$

in some fixed neighborhood Ω_R, $R > 0$, of Σ. On the other hand, the first eigenfunction φ_1 of L satisfies

$$(L + M)\varphi_1 = (\lambda_1 + M)\varphi_1 \geq 0. \tag{60}$$

Now, both functions φ_1 and $d^{-\alpha} + C_1 d^{-\alpha+1}$ are smooth away from Σ and near Σ so that one can find $\varepsilon > 0$ such that $\varepsilon(d^{-\alpha} + C_1 d^{-\alpha+1}) \le \varphi_1$ in $\partial\Omega_R$. We can use now the same version of the maximum principle as in the previous lemma to deduce that

$$\varepsilon(d^{-\alpha} + C_1 d^{-\alpha+1}) \le \varphi_1 \quad \text{in } \Omega_R.$$

For the estimate $\varphi_1 \le C_2 d^{-\alpha(\mu)}$ we need a result from [DD].

Theorem 4. *Let Ω be a bounded smooth domain. Assume that $\tilde a \in L^1_{\mathrm{loc}}(\Omega)$, $\tilde a$ is bounded below (i.e. $\inf_\Omega \tilde a > -\infty$) and that it satisfies*

$$\gamma\left(\int_\Omega |u|^r\right)^{2/r} + \int_\Omega \tilde a(x) u^2 \le \int_\Omega |\nabla u|^2$$

for some $\gamma > 0$ and $r > 2$. Let $\varphi_1 > 0$ denote the first eigenfunction for the operator $L = -\Delta - \tilde a(x)$ with zero Dirichlet boundary condition, normalized by $\|\varphi_1\|_{L^2(\Omega)} = 1$, and let ζ_0 denote the solution of

$$\begin{cases} -\Delta\zeta_0 - \tilde a(x)\zeta_0 = 1 & \text{in } \Omega, \\ \zeta_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Then there exists $C = C(\Omega, \gamma(a), r) > 0$ such that

$$C^{-1}\zeta_0 \le \varphi_1 \le C\zeta_0.$$

Proof of Lemma 7 continued. We use the above theorem with $\tilde a = a - M$. In view of this result it suffices to show that

$$\zeta_0 \le C d^{-\alpha(\mu)}.$$

Using (58) and taking $\alpha = \alpha(\mu)$ we have

$$\begin{aligned} &(L + M)(d^{-\alpha} - C d^{-\alpha+1}) \\ &\qquad = -d^{-\alpha-1}[-\alpha g - C((\alpha-1)^2 - (\alpha-1)(k-2) + \mu - (\alpha-1)gd)] \\ &\qquad\quad + M(d^{-\alpha} - C d^{-\alpha+1}) \\ &\qquad \ge 1 \end{aligned}$$

in Ω_R if we choose $R > 0$ small and $C > 0$ large enough. Now take C_1 large so that $\zeta_0 \le C_1(d^{-\alpha} - C d^{-\alpha+1})$ in $\partial\Omega_R$. Using the maximum principle as before we deduce that $\zeta_0 \le C_1(d^{-\alpha} - C d^{-\alpha+1})$, which finishes the proof. □

Remark 5. The fact that $d^{1-k/2} \in \mathcal{H}$ for $\mu = H^2$ was used in the proof above at the point where the maximum principle was applied. That argument requires that both functions that one would like to compare are in $\mathcal{H}$. In general, if one of these functions does not belong to $\mathcal{H}$ then the maximum principle cannot be applied; see [DD] for an example.

4. Some applications

4.1. Minimizers for the Hardy inequality

We start this section by extending a result of Brezis and Marcus [BM] regarding the quantity

$$J_\lambda = \inf_{u \in C_c^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 - \lambda u^2}{\int_\Omega u^2/d(x)^2}, \tag{61}$$

where as usual $d(x) = \operatorname{dist}(x, \Sigma)$.

The case studied in [BM] corresponds to $\Sigma = \partial\Omega$, and an interesting feature that the authors found in that work is the following, which we state in our situation:

Theorem 5. *Fix $\lambda \in \mathbb{R}$. Then the infimum in* (61) *is achieved (in $H_0^1(\Omega)$) if and only if*

$$J_\lambda < H^2.$$

Proof. To prove that the condition $J_\lambda < H^2$ is sufficient for the infimum in (61) to be achieved, one just needs to mimic the arguments in [BM] so we skip this step.

We prove the converse, that is, the claim that if $J_\lambda = H^2$ then the infimum is not achieved, with an argument similar in spirit to that of [BM]. Suppose that the infimum is achieved by a function $u \in H_0^1(\Omega)$, which we can assume to be nonnegative and not identically zero. Assume also that $J_\lambda = H^2$. Then u satisfies

$$-\Delta u - H^2 \frac{u}{d(x)^2} = \lambda u.$$

It follows that λ is the first eigenvalue for the operator $-\Delta - H^2/d^2$ and that $u > 0$. Moreover u has to be a multiple of φ_1 (for this result see e.g. [DD, Lemma 2.3]). But by (50) we know that $\varphi_1 \sim d^{1-k/2}$. This shows on the one hand that $\int_\Omega u^2/d^2 = \infty$. But Hardy's inequality (4) implies on the other hand that $\int_\Omega u^2/d^2 < \infty$. □

4.2. Study of a semilinear problem

In this section, we return to the study of a semilinear problem studied in [DN]. For $p > 1$, $0 < \mu \le H^2$ and $\lambda > 0$ consider the equation

$$\begin{cases} -\Delta u - \dfrac{\mu}{d(x)^2} u = u^p + \lambda & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{62}$$

where as usual $d(x) = \operatorname{dist}(x, \Sigma)$. We showed in [DN] that (at least for small values of $\mu > 0$) there exists a critical exponent

$$p_0 = 1 + \frac{2}{\alpha(\mu)} \quad \text{with} \quad \alpha(\mu) = H - \sqrt{H^2 - \mu}$$

such that (62) admits no solution (in any reasonable sense) for $p > p_0$ and $\lambda > 0$, whereas for some $\lambda^* = \lambda^*(p)$ solutions exist when $p < p_0$ and $0 < \lambda \leq \lambda^*$ (and again no solution exists when $\lambda > \lambda^*$). However, the critical case $p = p_0$ remained open. Using Lemma 7 in combination with Theorem 4, and following the proof of Proposition 6.1 of [DN], one can prove the following:

Theorem 6. *Given any $\lambda > 0$, problem (62) with $p = p_0$ admits no solution.*

5. Estimate for solutions of some singular equations

In what follows we will use the method developed in [DD] to prove Theorem 3. The idea is to work with $w = u/\varphi_1$, which satisfies an elliptic equation to which Moser's iteration technique can be applied. In the argument it is desirable to approximate the potential $a(x)$ by bounded ones. In order to get the convergence of the corresponding solutions, it is convenient to rewrite the equation (7) as

$$-\Delta u - \tilde{a}(x)u = C_0 u + f,$$

where

$$\tilde{a} = a - C_0$$

and C_0 is chosen large enough, larger than C in (7) (although it will be taken even larger at one point below). We observe that now for any $h \in \mathcal{H}^*$ the equation

$$\begin{cases} -\Delta v - \tilde{a} v = h & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{63}$$

has a unique solution $v \in \mathcal{H}$. Let us also note that the first eigenfunction for the operator $-\Delta - \tilde{a}$ is still φ_1.

Let us state a result which is a kind of Sobolev inequality with weight (see a proof in [DD]).

Lemma 9. *Assume that a satisfies (8). Then for any $2 \leq q \leq r$ there is a constant C depending only Ω, r and $\gamma(a)$ such that*

$$\left(\int_\Omega \varphi_1^s |w|^q \right)^{2/q} \leq C \int_\Omega \varphi_1^2 (|\nabla w|^2 + w^2) \tag{64}$$

for all $w \in C^1(\overline{\Omega})$, where s is given by the relation

$$\frac{s}{r} = \frac{q-2}{r-2}.$$

Lemma 10. *Let* $0 < m < r$ *and suppose that*

$$p > \frac{2r}{m(r-2)} \tag{65}$$

and

$$p \geq \frac{r}{r-m}. \tag{66}$$

Then for $f \in \mathcal{H}^*$*, the unique solution* v *to* (63) *satisfies*

$$|v(x)| \leq C\|\varphi_1^{1-m}h\|_p\,\varphi_1(x), \quad \textit{a.e. } x \in \Omega.$$

Remark 6. If $m \geq 1$, the assumption $h \in \mathcal{H}^*$ can be dropped since one can prove that $\|h\|_{\mathcal{H}^*} \leq C\|\varphi_1^{1-m}h\|_p$.

Proof of Remark 6. If $\|\varphi_1^{1-m}h\|_p = +\infty$, there is nothing to prove. Otherwise h is locally integrable and for $\varphi \in C_c^\infty(\Omega)$,

$$\left|\int_\Omega h\varphi\right| \leq \|h\varphi_1^{1-m}\|_p\|\varphi\varphi_1^{m-1}\|_{p'} \leq \|h\varphi_1^{1-m}\|_p\|\varphi_1\|_{mp'}^{m/m'}\|\varphi\|_{mp'},$$

where we used Hölder's inequality twice. Now (66) implies that $mp' \leq r$, so we end up with

$$\left|\int_\Omega h\varphi\right| \leq C\|h\varphi_1^{1-m}\|_p\|\varphi\|_{\mathcal{H}} \quad \forall\varphi \in C_c^\infty(\Omega),$$

which is the desired result. □

Proof of Lemma 10. First we note that it is sufficient to prove this result for a bounded potential a, as long as the constants that appear in the estimates only depend on the constants r, γ, C appearing in (7) and Ω. This is the same argument employed in [DD] and we will just sketch it here. Consider $\tilde{a}_k = \min(\tilde{a}, k)$, and the first eigenfunction φ_1^k and solution v_k of (63) with the potential a replaced by a_k. Then $\varphi_1^k \to \varphi_1$ in $\mathcal{H}$ and $v_k \to v$. Furthermore, $\tilde{a}_k$ satisfies

$$\gamma\left(\int_\Omega |u|^r\right)^{2/r} + \int_\Omega \tilde{a}(x)u^2 \leq \int_\Omega |\nabla u|^2, \quad \forall u \in C_c^\infty(\Omega).$$

So it is enough to establish the results for $\tilde{a}_k$. We will assume then that $\tilde{a}$ is bounded. Then all functions involved belong to $C^{1,\alpha}(\overline{\Omega})$.

By working with h^+ and h^- we can assume that $h \geq 0$ and hence $v \geq 0$. Set

$$w = v/\varphi_1.$$

Then w satisfies the equation

$$-\nabla\cdot(\varphi_1^2\nabla w) = \varphi_1 h - (C_0 + \lambda_1)\varphi_1 v.$$

Multiplying the equation by w^{2j-1} and integrating in Ω we find

$$\begin{aligned}\frac{2j-1}{j^2}\int_\Omega \varphi_1^2|\nabla w^j|^2 &= \int_\Omega \varphi_1 w^{2j-1}h - (C_0+\lambda_1)\int_\Omega \varphi_1 v w^{2j-1}\\ &= \int_\Omega \varphi_1 w^{2j-1}h - (C_0+\lambda_1)\int_\Omega \varphi_1^2 w^{2j}. \end{aligned} \tag{67}$$

Using the variant of Sobolev's inequality (64) applied to w^j with $s = mp'$ we obtain

$$\left(\int_\Omega \varphi_1^{mp'} w^{qj}\right)^{2/q} \le C\int_\Omega \varphi_1^2(|\nabla w^j|^2 + w^{2j}), \tag{68}$$

where q is given by

$$q = 2 + mp'\frac{r-2}{r}.$$

We note that by (66) we have $mp' \le r$ and therefore we can indeed apply Lemma 9. Combining (67) with (68) we get

$$\left(\int_\Omega \varphi_1^{mp'} w^{qj}\right)^{2/q} \le \frac{Cj^2}{2j-1}\int_\Omega \varphi_1 w^{2j-1}h + C\left(1 - \frac{j^2}{2j-1}(C_0+\lambda_1)\right)\int_\Omega \varphi_1^2 w^{2j}.$$

We make C_0 larger if necessary, so that for $j \ge 1$ the second term on the right hand side is negative. Therefore

$$\left(\int_\Omega \varphi_1^{mp'} w^{qj}\right)^{2/q} \le Cj\int_\Omega \varphi_1 w^{2j-1}h.$$

By Hölder's inequality

$$\int_\Omega \varphi_1 w^{2j-1}h \le \left(\int_\Omega \varphi_1^{mp'} w^{(2j-1)p'}\right)^{1/p'} \|\varphi_1^{1-m}h\|_p$$

and therefore

$$\left(\int_\Omega \varphi_1^{mp'} w^{qj}\right)^{1/(qj)} \le (Cj)^{1/(2j)}\left(\int_\Omega \varphi_1^{mp'} w^{(2j-1)p'}\right)^{1/(2jp')} \|\varphi_1^{1-m}h\|_p^{1/(2j)}. \tag{69}$$

Observe now that condition (65) is equivalent to $q > 2p'$ and therefore a standard iteration argument yields the result. Let indeed $j_0 = 1/2 + m/2$ and for $k \ge 1$, define j_k inductively by

$$(2j_k - 1)p' = qj_{k-1}. \tag{70}$$

One can easily show that $\{j_k\}$ is increasing and converges to $+\infty$ as $k \to +\infty$, so that if

$$\theta_k = \left(\int_\Omega \varphi_1^{mp'} w^{qj_k}\right)^{1/(qj_k)}\left(\int_\Omega \varphi_1^{mp'}\right)^{(-1/qj_k)},$$

then $\{\theta_k\}_k$ is increasing and converges to $\|w\|_\infty$ as $k \to \infty$. Observe in passing that since $\varphi_1 \in \mathcal{H}$ and $mp' \leq r$, we have

$$\left(\int_\Omega \varphi_1^{mp'}\right) < \infty.$$

Equation (69) then yields

$$\theta_k \leq (Cj_k)^{1/(2j_k)} \left(\int_\Omega \varphi_1^{mp'}\right)^{-1/(qj_k)+1/(2j_k p')} \theta_{k-1}^{qj_{k-1}/(2j_k p')} \left(\int_\Omega \varphi_1^{(1-m)p} h^p\right)^{1/(2j_k p)}. \tag{71}$$

Now, either $\{\theta_k\}_k$ remains bounded by $(\int_\Omega \varphi_1^{(1-m)p} h^p)^{1/p}$ for all k, in which case passing to the limit provides the desired inequality, or there exists a smallest integer k_0 such that for $k \geq k_0 - 1$,

$$\theta_k \geq \left(\int_\Omega \varphi_1^{(1-m)p} h^p\right)^{1/p}.$$

Using this inequality in (71), we obtain for $k \geq k_0$,

$$\theta_k \leq (Cj_k)^{1/(2j_k)} \left(\int_\Omega \varphi_1^{mp'}\right)^{\frac{1}{j_k}(-\frac{1}{q}+\frac{1}{2p'})} \theta_{k-1}. \tag{72}$$

Applying (72) inductively, it follows that

$$\|w\|_\infty \leq \prod_{k=k_0}^{\infty} \left[(Cj_k)^{1/(2j_k)} \left(\int_\Omega \varphi_1^{mp'}\right)^{\frac{1}{j_k}(-\frac{1}{q}+\frac{1}{2p'})}\right] \theta_{k_0-1}. \tag{73}$$

Starting from (70), a straightforward computation shows that for some $c > 0$,

$$j_k = \left(\frac{q}{2p'}\right)^k j_0 + \frac{\left(\frac{q}{2p'}\right)^k - 1}{\frac{q}{2p'} - 1} \sim c \left(\frac{q}{2p'}\right)^k \quad \text{as } k \to \infty.$$

Since $q/(2p') > 1$, we then conclude that the infinite product on the right hand side of (73) converges to some finite constant.

If $k_0 \geq 2$, applying again (71) for $k = k_0 - 1$, we also have

$$\theta_{k_0-1} \leq C\theta_{k_0-2}^{qj_{k_0-2}/(2j_{k_0-1}p')} \left(\int_\Omega \varphi_1^{(1-m)p} h^p\right)^{1/(2j_{k_0-1}p)} \leq C\left(\int_\Omega \varphi_1^{(1-m)p} h^p\right)^{1/p}, \tag{74}$$

where we used the minimality of k_0 in the last inequality. Combining (74) and (73) yields the desired result.

If $k_0 = 1$ then by (69),

$$\left(\int_\Omega \varphi_1^{mp'} w^{qj_0}\right)^{1/(qj_0)} \leq C\left(\int_\Omega v^r\right)^{m/(2j_0 r)} \left(\int_\Omega \varphi_1^{(1-m)p} h^p\right)^{1/(2j_0 p)}, \tag{75}$$

where we used Hölder's inequality and the fact that $mp' \le r$, which follows from (66). Now,

$$\gamma \|v\|_r^2 \le \|v\|_{\mathcal{H}}^2 = \int_\Omega hv \le \|h\varphi_1^{1-m}\|_p \|\varphi_1^{m-1} v\|_{p'}$$
$$\le \|h\varphi_1^{1-m}\|_p \|w\|_\infty \|\varphi_1^m\|_{p'} \le C\|h\varphi_1^{1-m}\|_p \|w\|_\infty. \tag{76}$$

Using (73), (75) and (76), we obtain

$$\|w\|_\infty \le C\|h\varphi_1^{1-m}\|_p^{\frac{m}{2(m+1)}+\frac{1}{m+1}} \|w\|_\infty^{\frac{m}{2(m+1)}},$$

which after simplification yields the desired result. □

Lemma 11. *Let $0 < m < r$ and suppose that*

$$p < \frac{2r}{m(r-2)} \quad \textit{and} \quad p \ge \frac{r}{r-m}.$$

Then given $h \in \mathcal{H}^*$, the unique solution v to

$$\begin{cases} -\Delta v - \tilde{a} v = h & \textit{in } \Omega, \\ v = 0 & \textit{on } \partial\Omega, \end{cases} \tag{77}$$

satisfies

$$\left(\int_\Omega \varphi_1^{mp'} \left| \frac{v}{\varphi_1} \right|^\alpha \right)^{1/\alpha} \le C\|\varphi_1^{1-m} h\|_p. \tag{78}$$

for any $\alpha \ge 1$ such that

$$\frac{1}{\alpha} \ge \frac{1}{p} - \left(1 - \frac{2}{q}\right), \tag{79}$$

where

$$q = 2 + mp' \frac{r-2}{r}.$$

Proof. The computations of the previous lemma are valid up to (69). Now observe that (69) yields an estimate for

$$\left(\int_\Omega \varphi_1^{mp'} w^{qj} \right)^{1/(qj)}$$

if $qj \ge (2j-1)p'$, which is equivalent to

$$j \le \frac{p'}{2p' - q}.$$

Take α satisfying (79) and $j = \alpha/q$. By Hölder's inequality

$$\int_\Omega \varphi_1^{mp'} w^{(2j-1)p'} \le \left(\int_\Omega \varphi_1^{mp'} w^\alpha \right)^{(2j-1)p'/\alpha} \left(\int_\Omega \varphi_1^{mp'} \right)^{1-(2j-1)p'/\alpha},$$

but observe that $\int_\Omega \varphi_1^{mp'} < \infty$ because $mp' \le r$ and $\varphi_1 \in L^r$. The previous inequality together with (69) yields the result. □

Remark 7. A direct consequence of the above lemma is that if instead of assuming

$$p < \frac{2r}{m(r-2)}$$

we assume that

$$p = \frac{2r}{m(r-2)},$$

then the conclusion is that for all $1 \leq \alpha < \infty$,

$$\left(\int_\Omega \varphi_1^{mp'} \left| \frac{v}{\varphi_1} \right|^\alpha \right)^{1/\alpha} \leq C \|\varphi_1^{1-m} h\|_p,$$

where the constant C may depend on α.

Remark 8. In contrast with what we observed in Remark 6, h need not be in $\mathcal{H}^*$ for $\|h\varphi_1^{1-m}\|_p$ to be finite. Hence, in light of inequality (78), one can define by density an operator

$$T = (-\Delta - \tilde{a}(x))^{-1} : \; L^p(\Omega, \varphi_1^{1-m}\, dx) \to L^\alpha(\Omega, \varphi_1^{mp'/\alpha - 1}\, dx),$$

which restricted to $h \in \mathcal{H}^*$ assigns the corresponding solution $v =: T(h) \in \mathcal{H}$ of (77).

On the other hand, given $h \in L^1(\Omega)$, one can consider a weak solution $u \in L^1(\Omega)$ of equation (77) in the sense that $\int_\Omega a(x)|u| \mathrm{dist}(x, \partial\Omega) < \infty$ and

$$\int_\Omega u(-\Delta\varphi - \tilde{a}(x)\varphi) = \int_\Omega f\varphi$$

for all $\varphi \in C^2(\bar{\Omega})$ with $\varphi|_{\partial\Omega} \equiv 0$. If $h \in L^p(\Omega, \varphi_1^{1-m}\, dx)$ and $u \in L^\alpha(\Omega, \varphi_1^{mp'/\alpha - 1}\, dx)$, is it true that $u = T(h)$?

Proof of Theorem 3. Consider now $u \in H$ satisfying (7) and let u_1 be the solution of

$$\begin{cases} -\Delta u_1 - \tilde{a} u_1 = f & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

We remark that Lemma 10 implies that $\|u_1/\varphi_1\|_\infty \leq C\|\varphi_1^{1-m} f\|_p$. Thus

$$\left(\int_\Omega \varphi_1^2 \left| \frac{u_1}{\varphi_1} \right|^l \right)^{1/l} \leq C \|\varphi_1^{1-m} f\|_p \tag{80}$$

for any $l \geq 1$. Define $u_2 = u - u_1$ so that $u = u_1 + u_2$ and $u_2 \in H$ is the unique solution of

$$\begin{cases} -\Delta u_2 - \tilde{a} u_2 = C_0 u & \text{in } \Omega, \\ u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Starting with $p_1 = 2$ we shall construct a finite increasing sequence p_k which will stop at some $\bar{k}$ such that

$$p_{\bar{k}} \geq \frac{r}{r-2}$$

and such that for each $k = 1, \dots, \bar{k}$ the following inequality holds:

$$\left(\int_\Omega \varphi_1^2 \left| \frac{u}{\varphi_1} \right|^{p_k} \right)^{1/p_k} \leq C(\|\varphi_1^{1-m} f\|_p + \|u\|_2). \tag{81}$$

Indeed, Lemma 11 applied to u_2 with $p = p_1 = 2$ and $m_1 = 1$ implies

$$\left(\int_\Omega \varphi_1^2 \left| \frac{u_2}{\varphi_1} \right|^{p_2} \right)^{1/p_2} \leq C\|u\|_{L^2}, \tag{82}$$

where p_2 is given by

$$\frac{1}{p_2} = \frac{1}{p_1} - \left(1 - \frac{2}{q}\right)$$

and $q = 2+2(r-2)/r$. Inequality (80) combined with (82) shows that (81) holds for p_2.

We continue this process using Lemma 11 repeatedly with p and m in that lemma given by

$$\frac{1}{p_{k+1}} = \frac{1}{p_k} - \left(1 - \frac{2}{q}\right), \qquad m_k = \frac{2(p_k - 1)}{p_k}$$

and $q = 2 + 2(r-2)/r$. At each step we obtain (inductively)

$$\left(\int_\Omega \varphi_1^2 \left| \frac{u_2}{\varphi_1} \right|^{p_{k+1}} \right)^{1/p_{k+1}} \leq C\left(\int_\Omega \varphi_1^2 \left| \frac{u}{\varphi_1} \right|^{p_k} \right)^{1/p_k} \leq C(\|\varphi_1^{1-m} f\|_p + \|u\|_2).$$

This together with (80) proves that (81) holds for p_{k+1}. We can continue in this way provided

$$\frac{1}{p_k} - \left(1 - \frac{2}{q}\right) > 0,$$

or equivalently

$$p_k < \frac{r}{r-2}.$$

Let $\bar{k}$ be the first time that we find

$$p_{\bar{k}} \geq \frac{r}{r-2}$$

so that (81) still holds for $\bar{k}$. If $p_{\bar{k}} > r/(r-2)$ then we can apply Lemma 10 directly and conclude that

$$\|u_2/\varphi_1\|_\infty \leq C(\|\varphi_1^{1-m} f\|_p + \|u\|_2),$$

which would finish the proof of the theorem.

In the case $p_{\bar{k}} = r/(r-2)$ we first use Remark 7 and then Lemma 10. □

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