Numerical solution of KdV and KP equations by using finite differences compact schemes

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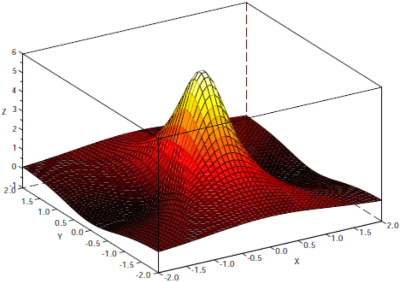
Outline



Motivation

Describe, model and understand some aspects of *water* drops and sea waves using numerical models.

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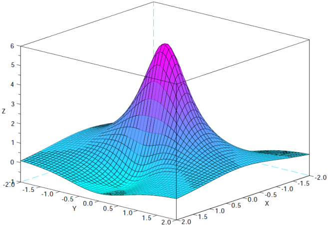


Ingredients:

- The KdV and KP equations: Models
- Numerical discretization: Finite differences (centered and compact themes), spatial discretization
- Fixed-point Methods: Time discretization

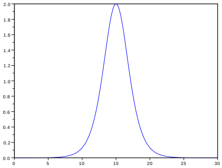
Korteweg-de Vries equation

The Korteweg-de Vries equation (KdV)

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} = 0$$

$$u : \Omega \subset \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}, \qquad u \in C_1^3\left([0, L] \times [0, \infty)\right)$$

describes the theory of water waves in shallow channels, such as a canal. It is a non-linear equation which exhibits special solutions, known as *Solitons*, which are stable and do not disperse with time. Furthermore there as solutions with more than one soliton which can move towards each other, interact and then emerge at the same speed with no change in shape.



1D Model: Korteweg-de Vries

Steps to solve numerically the KdV equation:

1. Discretize Ω
2. Construct the space differentiation operators
3. Construct the time discretization form
4. Solve

Assuming $u(x, t > 0)$ is periodic in $\Omega = [0, L]$

$$u(L + x, t) = u(x, t)$$

Discretizing Ω

$$\Omega_d = \text{mesh}(\Omega, n) \qquad\qquad h = h_x = h_y = \frac{L}{n}$$

1D Model: Korteweg-de Vries

The discretization of space requieres two difference operators

$$\frac{\partial u}{\partial x} \approx D_x u(x,t) \quad \text{and} \quad \frac{\partial^3 u}{\partial x^3} \approx D_{xxx} u(x,t)$$

These are not unique, we propose to construct them using:

- Centered finite differences schemes
- Compact finite differences schemes
- ...

Centered finite differences schemes

$$\frac{\partial u}{\partial x}(x_i, t) \approx \frac{u(x_{i+1}, t) - u(x_{i-1}, t)}{2h}$$
$$\frac{\partial^3 u}{\partial x^3}(x_i, t) \approx \frac{u(x_{i+2}, t) - u(x_{i-2}, t) - 2(u(x_{i+1}, t) - u(x_{i-1}, t))}{2h^3}$$

$$D_x = \frac{1}{2h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ -1 & & & -1 & 0 \end{bmatrix}, \quad D_{xxx} = \frac{1}{2h^3}\begin{bmatrix} 0 & -2 & -1 & & & -1 & 2 \\ 2 & \ddots & \ddots & \ddots & & & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & -1 \\ 1 & & & \ddots & \ddots & \ddots & -2 \\ -2 & -1 & & & 1 & 2 & 0 \end{bmatrix}$$

Compact Finite Difference Schemes

Compact Finite Difference Schemes are discretization techniques with appealing properties to our problem, they usually provide a higher order aproximation and are of the form

- $\partial_x : \frac{\partial u}{\partial x} = P^{-1}Qu = D_x u$
- $\partial_x^3 : \frac{\partial^3 u}{\partial x^3} = P^{-1}\widetilde{Q}u = D_{xxx} u$

Where

$$P : \frac{\alpha(u'(x_{i+1}, t) + u'(x_{i-1}, t) + \beta(u'(x_{i+2}, t) + u'(x_{i-2}, t))}{2h}$$

$$Q : \frac{a(u(x_{i+1}, t) - u(x_{i-1}, t))}{2h} + \frac{b(u(x_{i+2}, t) - u(x_{i-2}, t))}{4h} + \frac{c(u(x_{i+3}, t) - u(x_{i-3}, t))}{6h}$$

See details in Lele 1991.

Space/Time Discretization of KdV

The space discretized KdV equation is

$$\frac{\partial u}{\partial t} + D_{xxx}u + D_x\left(\frac{1}{2}u^2\right) = 0$$

Now, the time discretization is done with a Cranck-Nicolson theme:

$$\frac{u^{k+1} - u^k}{\Delta t} + D_{xxx}\left(\frac{u^{k+1} + u^k}{2}\right) + \frac{1}{2}\left(\frac{1}{2}D_x(u^{k+1})^2 + \frac{1}{2}D_x(u^k)^2\right) = 0$$

Advantages:

1. Invariant
2. Numerically stable
3. Easy to implement

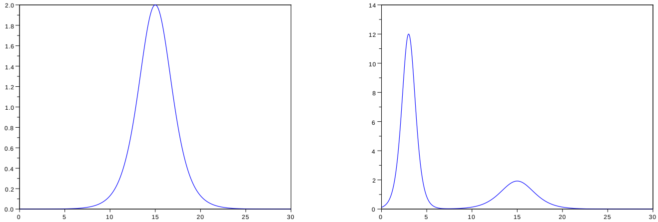
Space/Time Discretization of KdV

The CN equation can be solve for u^{k+1} using a fixed-point method

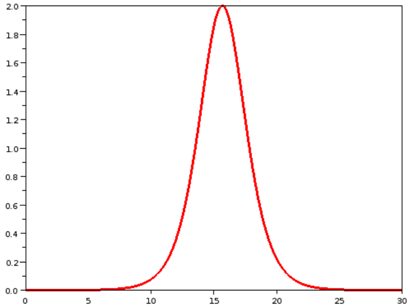
$$\left(I + \frac{\Delta t}{2} D_{xxx}\right) u^{*} = \left(I - \frac{\Delta t}{2} D_{xxx}\right) u^{k} - \frac{\Delta t}{2}\left(\frac{1}{2} D_x (u^{*})^2 + \frac{1}{2} D_x (u^{k})^2\right)$$

and the following initial solution for the soliton

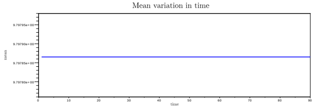
$$u(x, t = t_0) = \sum_{i=1}^{d} A_i \text{sech}\left(K_i \left(x - x_0\right)^{\frac{2}{p_i}}\right)$$



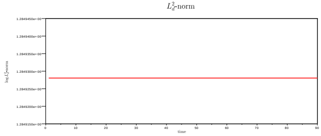
Numerical Results



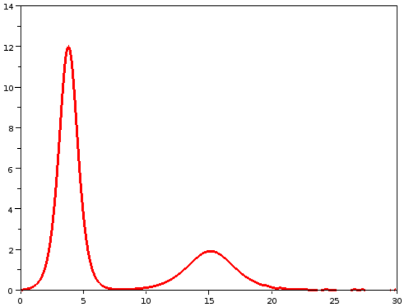
Invariance in Time



Conservation of E



Numerical Results



Invariance in Time

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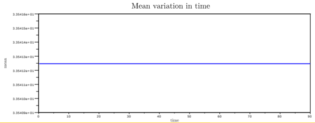
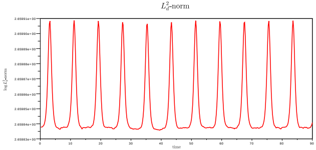
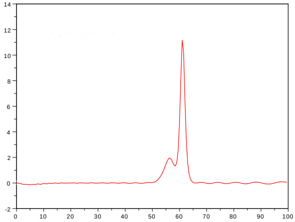




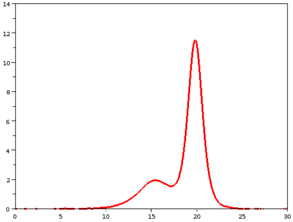
Numerical Results

Advantages in using compact themes

Using Centered (2nd order) FD



Using Compact (6th order) FD



2D Model: Kadomtsev Petviashvili

The Kadomtsev Petviashvili equation (KP) is a partial differential equation to describe nonlinear wave motion that is considered as generalization of KdV equation to two dimensions. The KP equation is:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+u\frac{\partial u}{\partial x}\right)+\lambda\frac{\partial^2 u}{\partial y^2}=0$$

where

$$\lambda = \pm 1, u = u(x, y, t) : \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}, t > 0, u \in \mathcal{C}_1^3$$

Like the KdV equation, the KP equation is completely integrable.

2D Model: Kadomtsev Petviashvili

If $\Omega \subset \mathbb{R}^2$ is the region of interest, then

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} + \lambda \int_{\Omega_x} \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + \frac{\partial}{\partial x}(\frac{1}{2}u^2) + \lambda \int_{\Omega_x} \frac{\partial^2 u}{\partial y^2} = 0$$

can be solve in a similar way that the 1D case:

1. Discretize Ω
2. Construct the space differentiation operators
3. Construct the time discretization form
4. Solve

2D Model: Kadomtsev Petviashvili

Assuming $u(x, y, t > 0)$ is periodic in $\Omega = [0, L] \times [0, L]$

$$u(L + x, L + y, t) = u(x, y, t)$$

Discretizing Ω

$$\Omega_d = \text{mesh}(\Omega, n) \qquad h = h_x = h_y = \frac{L}{n}$$

we can generalize the differentiation operators D of the 1D case using the Kronecker operator ($\otimes$)

$$H_x = I \otimes D_x, H_{xxx} = I \otimes D_x xx, H_{yy} = D_{yy} \otimes I$$

Note: $H \in \mathbb{R}^{n^2 \times n^2}$ $\rightarrow$ time to move to sparse-mode...

2D Model: Kadomtsev Petviashvili

The discretization of Ω, Ω_d, has an assosiated map between (x_i, y_j, t) and (p_k, t).

Notation: $u_k \equiv u(p_k, t) \equiv u(x_i, y_i, t), k = \mathrm{map}(i, j)$

Space discretized form of KP

$$\frac{\partial}{\partial t} + H_{xxx} u + H_x(\frac{1}{2} u^2) + \lambda H_x^{-1} H_{yy} u = 0$$

2D Model: Kadomtsev Petviashvili

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Space discretized form of KP

$$\frac{\partial}{\partial t} + H_{xxx} u + H_x(\frac{1}{2}u^2) + \lambda H_x^{-1} H_{yy} u = 0$$

Notes:

1. Operator H_x is singular (!)
2. Replace $H^{-}1_x$ by an anti-derivative (restrictions!)
3. Proposed anti-derivative, "M" $D_x : [-\mathbf{1}, \mathbf{0}, +\mathbf{1}]$

$$M = I \otimes \text{tril}(\mathbf{1}), \quad \mathbf{1} \in \mathbb{R}^{n \times n}$$

Space discretized form of KP

$$\frac{\partial}{\partial t} + H_{xxx}u + H_x(\frac{1}{2}u^2) + \lambda M H_{yy}u = 0$$

Time discretization is done with the Crank-Nicolson theme for 2D:

$$\frac{u^{k+1} - u^k}{\Delta t} + \frac{1}{2}H_{xxx}\left(u^{k+1} + u^k\right) + \frac{1}{2}\left[H_x\left(\frac{1}{2}u^k\right)^2 + H_x\left(\frac{1}{2}u^{k+1}\right)^2\right]$$
$$+\lambda h M D_{yy}\left(\frac{u^{k+1} + u^k}{2}\right) = 0$$

Space discretized form of KP

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$$+\lambda h M D_{yy}\left(\frac{u^{k+1} + u^k}{2}\right) = 0$$

Solving it by using a fixed-point method $(\lambda = -1, \Delta t = t_0)$

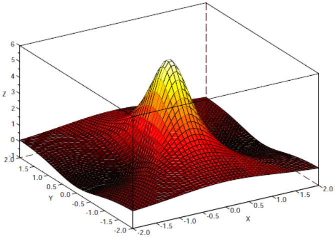
$$I + \frac{\Delta t}{2} H_{xxx} u^\star = (I - \frac{\Delta t}{2} H_{xxx}) u^k - \frac{1}{2}\left[H_x\left(\frac{1}{2}u^k\right)^2 + H_x\left(\frac{1}{2}u^\star\right)^2\right]$$
$$-h\Delta t M H_{yy}\left(\frac{u^2 + u^\star}{2}\right)$$

2D Model: Kadomtsev Petviashvili

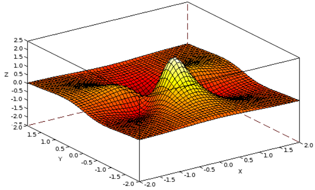
The solution to $u(x, y, t > 0$ is known as the *Soliton 2d*

$$u(x,y,t) = \frac{4\left[-(x-Vt)^2 + \omega y^2 + \frac{3}{A^2}\right]}{\left[(x-Vt)^2 + \omega y^2 + \frac{3}{A^2}\right]^2}$$

$u(x, y, t)$



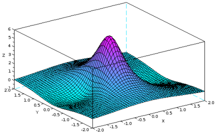
$\frac{\partial^2 u}{\partial x \partial y}(x, y, t)$



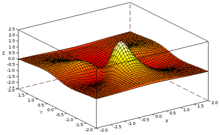
2D Model: Preliminary Results

- M anti-derivative operator is *stable*: cond(M), etc. . .

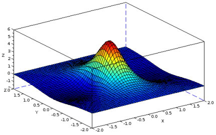
$u(x, y, t)$



$H_x u(x, y, t)$



$MH_x u(x, y, t)$

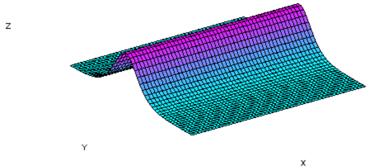


- Numerical algorithms are based in sparse techniques
- Work in progress (!)

2D Model: Preliminary Results

Another solution of the KP equation is the extension of the 1D soliton to 2D. The following animation is the evolution in time of such solution.

$$u(x, y, t)$$



Final Remarks

1. KdV 1D modeled successfully, showing its evolution in time
2. Numerical condition can deteriorate under simple conditions.
3. Noise propagates fast due to imposed periodicity.
4. High resolution required in space
5. KP equation discretized in space and time, no experimentation yet
6. For space discretization: substitute low order finite diferentiation for higher order compact themes. Experiment with lower mesh densities.