Progressive waves of the spatial spread of a pathogen

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Abstract. The aim of the project is to obtain numerically the progressive waves of the systems. We use Freefem++ and Scilab to perform the numerical simulations.

Keywords. reaction-diffusion system, epidemics, finite elements, finite differences.

Introduction

The disease cycle is described as follows: susceptible leaves (denoted by S) inoculated with spores first become latent (L), then turn infectious (I) and produce spores (U_s short-range dispersed spores and U_l long-range) during some infectious period after which they are removed (R) as they cannot be infected again. The total density of leaves is (N). Each segment is solution of the following differential system:

$$\begin{aligned}
\frac{dS}{dt} &= -e\delta(U_s+U_l)\frac{S}{N}+\alpha N\left(1-\frac{N}{k}\right)\\
\frac{dL}{dt} &= e\delta(U_s+U_l)\frac{S}{N}-\frac{1}{j}L\\
\frac{dI}{dt} &= \frac{1}{j}L-\frac{1}{i}I\\
\frac{dR}{dt} &= \frac{1}{i}I\\
\frac{\partial U_s}{\partial t} &= D_s\Delta U_s-\delta U_s+\gamma p I\\
\frac{\partial U_l}{\partial t} &= D_l\Delta U_l-\delta U_l+\gamma(1-p)I.
\end{aligned}$$

Here $i, j, \delta, e, D_s, D_l, \gamma, p, \alpha, k$ are positive constants.

This system of equations was studied in [1]. In particular, we know that this system is well-posed.

1 Progressive waves

We consider progressive waves: for $Y = (S, L, I, R, U_s, U_l)$

$$Y(x, y, t) = \widetilde{Y}(w, y), \text{ with } w = x - ct.$$

The system becomes time-independant and is written as follows:

$$\begin{aligned}
-c\frac{d\tilde{S}}{dw} &= -e\delta(\tilde{U}_s + \tilde{U}_l)\frac{\tilde{S}}{\tilde{N}} + \alpha\tilde{N}\left(1 - \frac{\tilde{N}}{k}\right) \\
-c\frac{d\tilde{L}}{dw} &= e\delta(\tilde{U}_s + \tilde{U}_l)\frac{\tilde{S}}{\tilde{N}} - \frac{1}{j}\tilde{L} \\
-c\frac{d\tilde{I}}{dw} &= \frac{1}{j}\tilde{L} - \frac{1}{i}\tilde{I} \\
-c\frac{d\tilde{R}}{dw} &= \frac{1}{i}\tilde{I} \\
-c\frac{\partial\tilde{U}_s}{\partial w} &= D_s\Delta\tilde{U}_s - \delta\tilde{U}_s + \gamma p\tilde{I} \\
-c\frac{\partial\tilde{U}_l}{\partial w} &= D_l\Delta\tilde{U}_l - \delta\tilde{U}_l + \gamma(1-p)\tilde{I}
\end{aligned}$$

Let $\phi = (\phi_1, ..., \phi_6) \in \left(\mathcal{C}^\infty(\mathbb{R}^2)\right)^6$. To multiply equations by ϕ and integrate over space provide the variational formulation of the problem

$$\begin{aligned}
\int_\Omega (cS_X - e\delta(U_s + U_l)\frac{S}{N})\Phi_1 + \int_\Omega \alpha N\left(1 - \frac{N}{k}\right)\phi_1 &= 0 \\
\int_\Omega (cL_X + e\delta(U_s + U_l)\frac{S}{N} - \frac{1}{j}L)\phi_2 &= 0 \\
\int_\Omega (cI_X + \frac{1}{j}L - \frac{1}{i}I)\phi_3 &= 0 \\
\int_\Omega (cR_X + \frac{1}{i}I)\phi_4 &= 0 \\
\int_\Omega (cU_{s,x} - \delta U_s + \gamma pI)\phi_5 - \int_\Omega D_s\nabla U_s.\nabla\phi_5 + \int_{\partial\Omega} D_s\partial_n U_s\,\phi_5 &= 0 \\
\int_\Omega (cU_{l,x} - \delta U_l + \gamma(1-p)I)\phi_6 - \int_\Omega D_l\nabla U_l.\nabla\phi_6 + \int_{\partial\Omega} D_l\partial_n U_l\,\phi_6 &= 0.
\end{aligned}$$

To simplify the computations, we choose boundary conditions such that the above integrals on the boundary vanish.

Theorem 1.1 *Let $H(\Omega)$ be the space of functions with the first space derivative in $L^2(\Omega)$ and null Dirichlet boundary conditions. There exists a unique weak solution $(S, L, I, R) \in (L^2(\Omega))^4$ and $(U_s, U_l) \in (H(\Omega))^2$.*

Proof. We can prove that the operator is continuous and coercive in $(H(\Omega))^3$, the Lax-Milgram lemma provides the result. Let $\phi_5 \in H(\Omega)$, the Cauchy-Schwarz inequality provides

$$\left|\int_\Omega (cU_{s,x} - \delta U_s)\phi_5 - \int_\Omega D_s\nabla U_s.\nabla\phi_5\right| \leq C(||U_{s,X}|| + ||U_s||)||\phi_5|| + C(||\nabla U_s||\,||\nabla\phi_5||.$$

Lemma 1.2 *(Exercise) We have*

$$||\phi||_\infty \leq 2(||\phi||\ ||\phi_X||\ ||\phi_y||\ ||\phi_{Xy}||)^{1/4}.$$

In the same way, we obtain such inequalities for the other integrals, and the continuity is proved.

We choose $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) = (S, L, I, R, U_s, U_l) \in (L^2(\Omega))^4 \times (H(\Omega))^2$, and sum the integrals defining the bilinear form give the coercivity. $\square$

2 Space discretization

2.1 Finite elements method

To solve the variationnal method, we choose the test functions ϕ in finite dimension subspace V_h of $(L^2(\Omega))^4 \times (H(\Omega))^2$ and solve the problem

$$a(Y_h, \phi_h)) = 0,\ \forall \phi_h \in V_h.$$

Such methods are called Galerkin method. Here, we choose V_h as finite elements. We prove that the solution exists and is unique for sufficiently smooth data. We need P1 Lagrange finite elements (dense in the space $L^2(\Omega)$ and $H(\Omega)$):

$$V_h := \left\{\phi \in L^2(\Omega); \forall K \in \mathcal{T}_h, v_{|K} \in \mathbb{P}_1\right\}.$$

To solve the non-linear problem, a fixed point method is chosen. We have

$$a(Y_h, \phi_h)) = 0 \iff Y_h = a(Y_h, \phi_h)) - Y_h.$$

The algorithm reads as follows: let $\varepsilon > 0$,
while $||Y_h^{n+1} - Y_h^n)|| > \varepsilon$
do $Y_h^{n+1} = a(Y_h^n, \phi_h)) - Y_h^n$.

2.2 Finite differences method

Let us denotes $M_x > 0$, $M_y > 0$ the number of discretizations, and $\Delta w = \Delta y > 0$ the space steps. We choose explicit Euler method to approximate the first derivative, and the scheme is written as follows: for $m \in \{1, ..., M_x\}$ and $n \in \{1, ..., M_y\}$,

$$\begin{aligned}
\tilde{S}^{m+1,n} &= \tilde{S}^{m,n} + \frac{e\delta}{c}\Delta w(\tilde{U}_s^{m,n} + \tilde{U}_l^{m,n})\frac{\tilde{S}^{m,n}}{\tilde{N}^{m,n}} - \frac{\alpha}{c}\Delta w \tilde{N}^{m,n}\left(1 - \frac{\tilde{N}^{m,n}}{k}\right) \\
\tilde{L}^{m+1,n} &= \tilde{L}^{m,n} - \frac{e\delta}{c}\Delta w(\tilde{U}_s^{m,n} + \tilde{U}_l^{m,n})\frac{\tilde{S}^{m,n}}{\tilde{N}^{m,n}} + \frac{1}{jc}\Delta w \tilde{L}^{m,n} \\
\tilde{I}^{m+1,n} &= \tilde{I}^{m,n} - \frac{1}{jc}\Delta w \tilde{L}^{m,n} + \frac{1}{ic}\Delta w \tilde{I}^{m,n} \\
\tilde{R}^{m+1,n} &= \tilde{R}^{m,n} - \frac{1}{ic}\Delta w \tilde{I}^{m,n} \\
\tilde{U}_s^{m+1,n} &= \tilde{U}_s^{m,n} - \frac{D_s}{c}\left(\frac{\tilde{U}_s^{m+1,n} + \tilde{U}_s^{m,n+1} - 4\tilde{U}_s^{m,n} + \tilde{U}_s^{m-1,n} + \tilde{U}_s^{m,n-1}}{\Delta w}\right) - \frac{\delta \Delta w}{c}\tilde{U}_s^{m,n} + \frac{\gamma \Delta w}{c} p \tilde{I}^{m,n} \\
\tilde{U}_l^{m+1,n} &= \tilde{U}_l^{m,n} - \frac{D_l}{c}\left(\frac{\tilde{U}_l^{m+1,n} + \tilde{U}_l^{m,n+1} - 4\tilde{U}_l^{m,n} + \tilde{U}^{m-1,n} + \tilde{U}_l^{m,n-1}}{\Delta w}\right) - \frac{\delta \Delta w}{c}\tilde{U}_l^{m,n} + \frac{\gamma \Delta w}{c}(1-p)\tilde{I}^{m,n}
\end{aligned}$$

We can write this scheme as system of linear equations as follows:

$$\begin{aligned}
A_l \tilde{U}_l &= \frac{\gamma \Delta w}{c}(1-p)\tilde{I} \\
A_s \tilde{U}_s &= \frac{\gamma \Delta w}{c} p \tilde{I} \\
B_S \tilde{S} &= \frac{\alpha \Delta w}{c} \tilde{N} \left(1 - \frac{\tilde{N}}{k}\right) \\
B_L \tilde{L} &= \frac{e\delta \Delta w}{c}(\tilde{U}_s + \tilde{U}_l)\frac{\tilde{S}}{\tilde{N}} \\
B_I \tilde{I} &= \frac{\Delta w}{jc} \tilde{L} \\
B_R \tilde{R} &= \frac{\Delta w}{ic} \tilde{I},
\end{aligned}$$

with

$$\begin{aligned}
A_s &= \frac{D_s}{c\Delta w} \begin{pmatrix} -4 & 1 & \dots & 1 & \dots & & \\ 1 & -4 & 1 & \dots & 1 & & \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \\ 1 & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ 0 & & 1 & \dots & \dots & 1 & -4 \end{pmatrix} + \left(\frac{\gamma \Delta w}{c} - 1\right) I_d \\
A_l &= \frac{D_l}{c\Delta w} \begin{pmatrix} -4 & 1 & \dots & 1 & \dots & & \\ 1 & -4 & 1 & \dots & 1 & & \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \\ 1 & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ 0 & & 1 & \dots & \dots & 1 & -4 \end{pmatrix} + \left(\frac{\gamma \Delta w}{c} - 1\right) I_d \\
B_S &= \begin{pmatrix} 1 + \frac{e\delta\Delta w}{c}(\tilde{U}_s^{1,1} + \tilde{U}_l^{1,1})\frac{1}{\tilde{N}^{1,1}} & -1 & \dots & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & 1 + \frac{e\delta\Delta w}{c}(\tilde{U}_s^{1,1} + \tilde{U}_l^{n,m})\frac{1}{\tilde{N}^{n,m}} \\ & & & \end{pmatrix} \\
B_L &= \begin{pmatrix} 1 + \frac{\Delta w}{jc} & -1 & \dots & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & 1 + \frac{\Delta w}{jc} \\ & & & \end{pmatrix}
\end{aligned}$$

$$B_I = \begin{pmatrix} 1+\frac{\Delta w}{ic} & -1 & \dots & & \\ & \ddots & \ddots & & \\ & & \ddots & -1 & \\ & & & 1+\frac{\Delta w}{ic} & \\ & & & & \end{pmatrix}$$

$$B_R = \begin{pmatrix} 1 & -1 & \dots & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & 1 \\ & & & \end{pmatrix}$$

3 Numerical simulations

We choose $e = 0.001, \delta = 50, \alpha = 0.2, k = 10000, j = 10, i = 10, \gamma = 200, p = 0.8; Ds = 2, Dl = 200$;

3.1 Simulations with Freefem++

The computational domain is $\Omega = [0, l] \times [0, l]$, with $l = 10$ $c = 1$. To simplify the computations of the mini-project, we assume that S_h, L_h, I_h, R_h are known as

$$\begin{aligned} Sh &= 1/(1 + exp(-x + 5)) \\ Ih &= 1/(1 + x) \\ Lh &= 0.5 \\ Rh &= 0.2. \end{aligned}$$

We have to determine only the progressive of long-range and short-range spores U_s and U_l, we will solve only the last equations with the boundary conditions:

$$\begin{aligned} U_s = U_l &= 0 \text{ if } x = 0 \text{ or } x = l. \\ U_s = U_l &= 0.5\exp(-(x - l/2)^2) \text{ if } y = 0 \text{ or } y = l. \end{aligned}$$

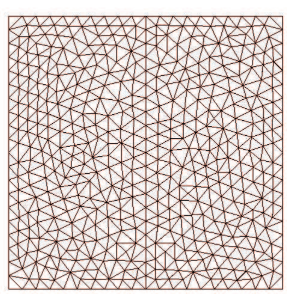

Figure 1: Mesh of the square.

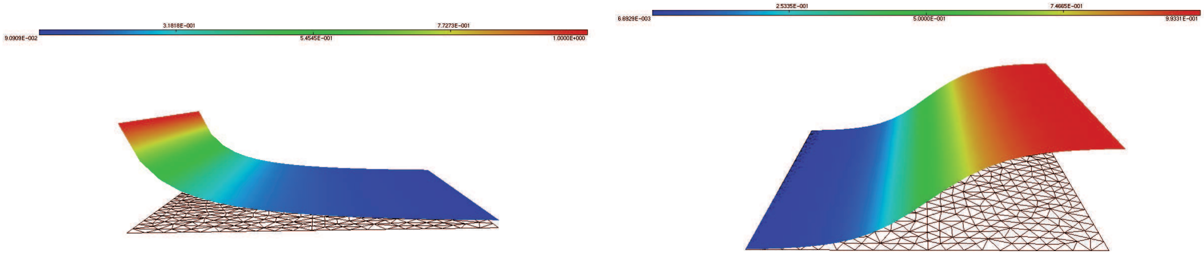

Figure 2: At left, surface of infectious leaves. At right, surface of sensitives.

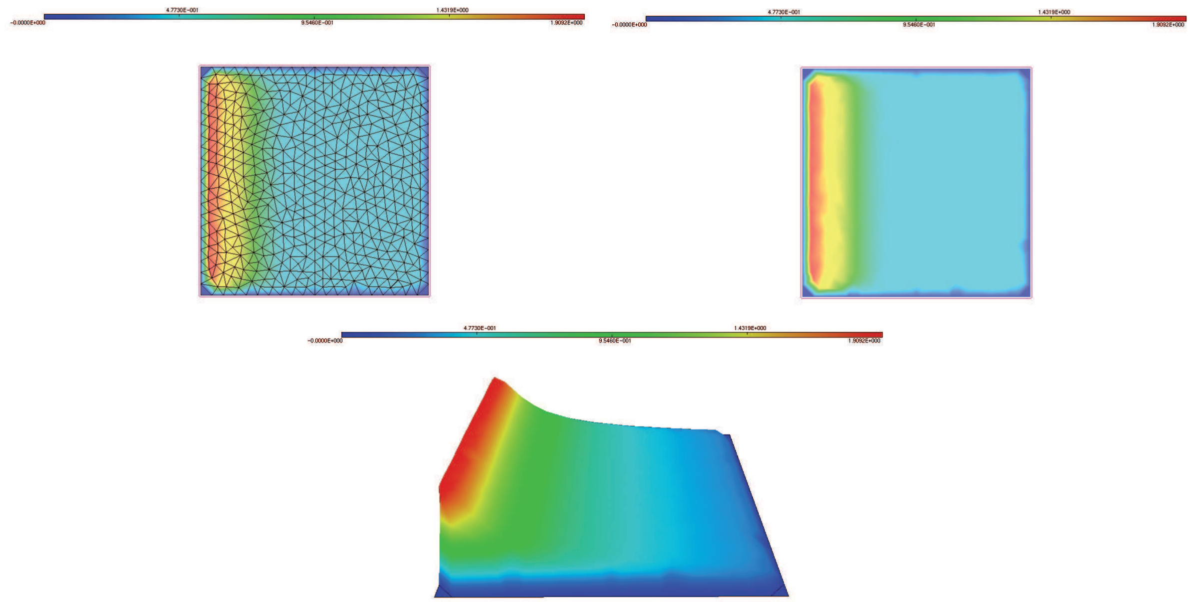

Figure 3: Short-range spores.

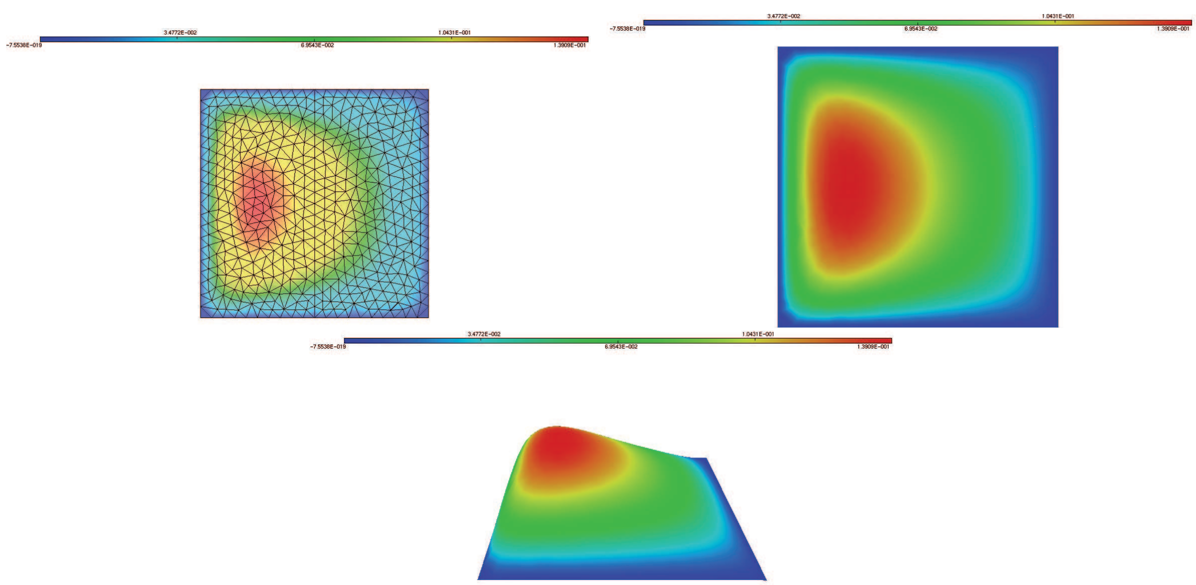

Figure 4: Long-range spores.

3.2 Simulations with Scilab

The computational domain is $\Omega = [0, l] \times [0, l]$, with $l = 10$, with $M_x = M_y = 100$. We start from $I = (1, 1, ..., 1)^T$ and we compute the scheme.

Algorithm 1: Fixed Point

Input M, tol

Output $\tilde{U}_l, \tilde{U}_s, \tilde{S}, \tilde{L}, \tilde{I}, \tilde{R} \in \mathbb{R}^{(M-1)^2}$ $\tilde{I}_0 \leftarrow \mathsf{ones}((M-1)^2, 1)$

compute LU factorization of: A_s, A_l, B_L, B_I, B_R

while $||I^{(n+1)} - I^{(n)}||_2 > tol$ **do**

- solve $A_s \tilde{U}_s^{(n)} = \frac{\gamma \Delta w}{c} p \tilde{I}^{(n)}$
- solve $A_l \tilde{U}_l^{(n)} = \frac{\gamma \Delta w}{c} (1-p) \tilde{I}^{(n)}$
- compute LU factorization of B_S
- solve $B_S \tilde{S}^{(n)} = \frac{\alpha \Delta w}{c} \tilde{N} \left(1 - \frac{\tilde{N}}{k}\right)$
- solve $B_L \tilde{L}^{(n)} = \frac{e \delta \Delta w}{c} \left(\tilde{U}_s^{(n)} + \tilde{U}_l^{(n)}\right) \frac{\tilde{S}^{(n)}}{\tilde{N}}$
- solve $B_L \tilde{I}^{(n+1)} = \frac{\Delta w}{jc} \tilde{L}^{(n)}$
- solve $B_R \tilde{R}^{(n)} = \frac{\Delta w}{ic} \tilde{I}^{(n+1)}$

end

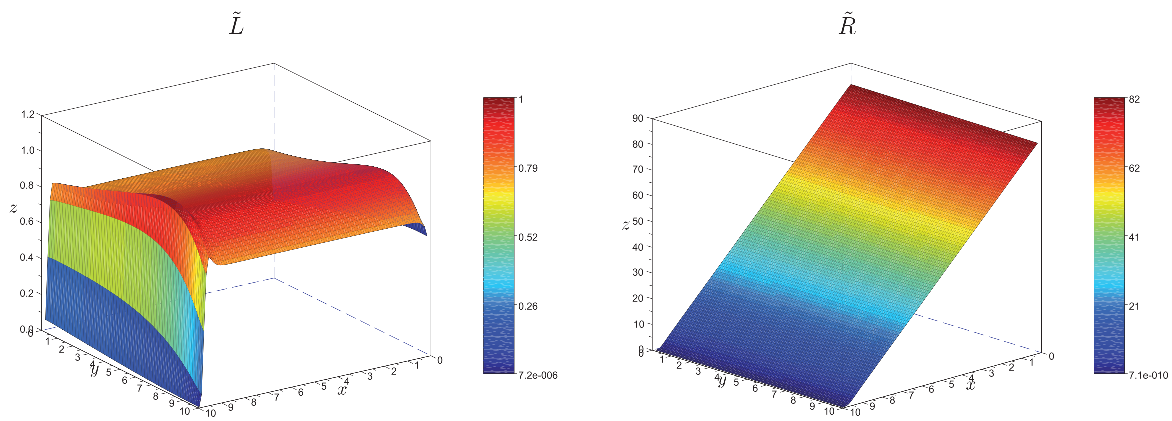

Figure 5: At left, surface of latent leaves and right surface of removed leaves.

In Figure 5, we represent the progressive wave of the spores become latent ($\tilde{L}$) and the spores removed after the infection ($\tilde{R}$).

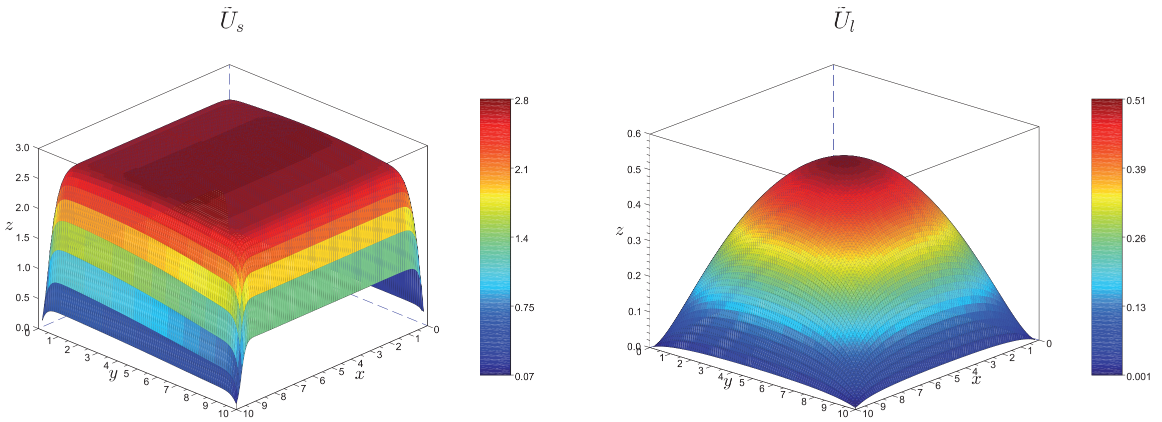

Figure 6: Surface of short and long range spores.

Figure 6 presents results obtained for long-rang ($\tilde{U}_l$) and short-rang ($\tilde{U}_s$) spores, respectively.

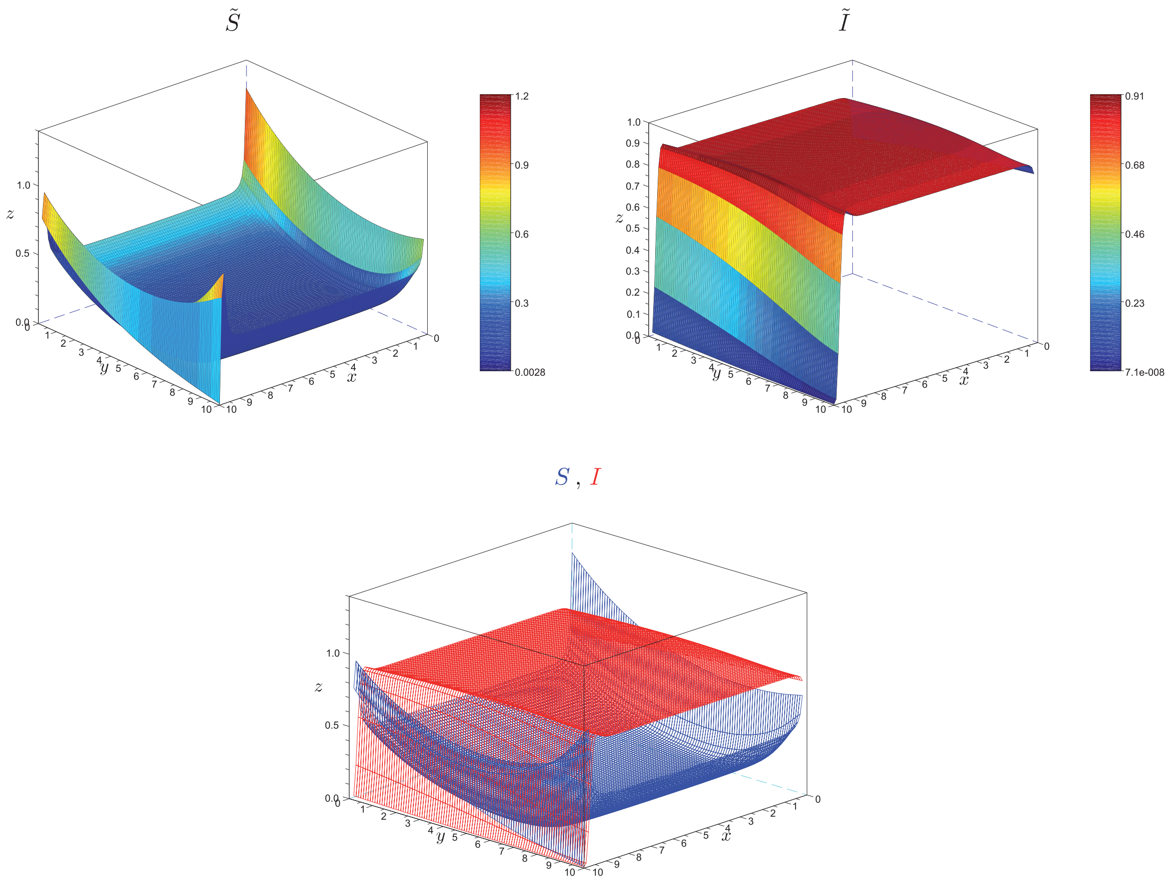

Figure 7: At left, surface of sensitives leaves and right, surface of infectious.

In Figure 7 shows the leaves inoculated ($\tilde{S}$) and the infectious ($\tilde{I}$) produced by the spores.

References

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