Progressive waves of the spatial spread of a pathogen

Group #2

Problem Differential system Progressive waves

Finite Differences Method Space discretization System Iterative Metho Results Simulations

Finite Elements Method Variational problem Well-posedne Posults

Progressive waves of the spatial spread of a pathogen CIMPA - Caracas

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Abril, 2012

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Problem

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Problem



Problem

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Differential system

$$\begin{aligned} \frac{dS}{dt} &= -e\delta(U_s + U_l)\frac{S}{N} + \alpha N\left(1 - \frac{N}{k}\right) \\ \frac{dL}{dt} &= e\delta(U_s + U_l)\frac{S}{N} - \frac{1}{j}L \\ \frac{dI}{dt} &= \frac{1}{j}L - \frac{1}{i}I \\ \frac{dR}{dt} &= \frac{1}{i}I \\ \frac{\partial U_s}{\partial t} &= D_s\Delta U_s - \delta U_s + \gamma pI \\ \frac{\partial U_l}{\partial t} &= D_l\Delta U_l - \delta U_l + \gamma(1 - p)I. \end{aligned}$$

 $i,j,\delta,e,D_s,D_l,\gamma,p,\alpha,k$ are positive constants. $\exists \rightarrow$

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$$\begin{split} Y(x,y,t) &= \widetilde{Y}(w,y), \text{ with } w = x - ct \\ \text{The system becomes time-independant:} \\ &-c\frac{d\widetilde{S}}{dw} = -e\delta(\widetilde{U}_s + \widetilde{U}_l)\frac{\widetilde{S}}{\widetilde{N}} + \alpha\widetilde{N}\left(1 - \frac{\widetilde{N}}{k}\right) \\ &-c\frac{d\widetilde{L}}{dw} = e\delta(\widetilde{U}_s + \widetilde{U}_l)\frac{\widetilde{S}}{\widetilde{N}} - \frac{1}{j}\widetilde{L} \\ &-c\frac{d\widetilde{I}}{dw} = \frac{1}{j}\widetilde{L} - \frac{1}{i}\widetilde{I} \\ &-c\frac{d\widetilde{R}}{dw} = \frac{1}{i}\widetilde{I} \\ &-c\frac{d\widetilde{R}}{dw} = 0_s\Delta\widetilde{U}_s - \delta\widetilde{U}_s + \gamma p\widetilde{I} \end{split}$$

$$-c\frac{\partial U_l}{\partial w} = D_l \Delta \tilde{U}_l - \delta \tilde{U}_l + \gamma (1-p)\tilde{I}$$

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Progressive waves

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$$\tilde{N}' + \frac{\alpha}{c}\tilde{N} = \frac{\alpha}{ck}\tilde{N}^2$$

$$\therefore \tilde{N} = \frac{k}{1 + k e^{\alpha w/e}}$$

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Space discretization

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Well-posedr Results

$$\Delta w = \Delta y$$

$$\left(1 + \frac{e\delta}{c}\Delta w(\tilde{U}_s^{m,n} + \tilde{U}_l^{m,n})\frac{1}{\tilde{N}^{m,n}}\right)\tilde{S}^{m,n} - \tilde{S}^{m+1,n} = -\frac{\alpha}{c}\Delta w\tilde{N}^{m,n}\left(1 - \frac{\tilde{N}^{m,n}}{k}\right) - \frac{1}{\tilde{N}^{m,n}} = -\frac{1}{\tilde{N}^{m,n}}\left(1 - \frac{\tilde{N}^{m,n}}{k}\right) - \frac{1}{\tilde{N}^{m,n}}\left(1 - \frac{\tilde{N}^{m,n}}{k}\right)$$

$$\begin{split} (1+\frac{1}{jc}\Delta w)\tilde{L}^{m,n} &- \tilde{L}^{m+1,n} &= & \frac{e\delta}{c}\Delta w(\tilde{U}^{m,n}_s + \tilde{U}^{m,n}_l)\frac{\tilde{S}^{m,n}}{\tilde{N}^{m,n}} \\ & \left(1+\frac{1}{ic}\Delta w\right)\tilde{I}^{m,n} - \tilde{I}^{m+1,n} &= & \frac{1}{jc}\Delta w\tilde{L}^{m,n} \\ & \tilde{R}^{m,n} - \tilde{R}^{m+1,n} &= & \frac{1}{ic}\Delta w\tilde{I}^{m,n} \end{split}$$

$$\begin{split} &-\frac{D_s}{c\Delta w}\tilde{U}_s^{m-1,n} - \frac{D_s}{c\Delta w}\tilde{U}_s^{m,n-1} + \left(1 + \frac{4D_s}{c\Delta w} + \frac{\delta\Delta w}{c}\right)\tilde{U}_s^{m,n} \\ &\quad -\frac{D_s}{c\Delta w}\tilde{U}_s^{m,n+1} + \left(-1 - \frac{D_s}{c\Delta w}\right)\tilde{U}_s^{m+1,n} &= \frac{\gamma\Delta w}{c}p\tilde{I}^{m,n} \\ &-\frac{D_l}{c\Delta w}\tilde{U}_l^{m-1,n} - \frac{D_l}{c\Delta w}\tilde{U}_l^{m,n-1} + \left(1 + \frac{4D_l}{c\Delta w} + \frac{\delta\Delta w}{c}\right)\tilde{U}_l^{m,n} \\ &\quad -\frac{D_l}{c\Delta w}\tilde{U}_l^{m,n+1} + \left(-1 - \frac{D_l}{c\Delta w}\right)\tilde{U}_l^{m+1,n} &= \frac{\gamma\Delta w}{c}(1-p)\tilde{I}^{m,n} \end{split}$$

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System

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Well-posed Results We can write this scheme as system of linear equations as follows:

$$A_s \tilde{U}_s = \frac{\gamma \Delta w}{c} p \tilde{I} \tag{1}$$

$$A_l \tilde{U}_l = \frac{\gamma \Delta w}{c} (1-p) \tilde{I}$$
⁽²⁾

$$B_S \tilde{S} = \frac{\alpha \Delta w}{c} \tilde{N} \left(1 - \frac{\tilde{N}}{k} \right)$$
(3)

$$B_L \tilde{L} = \frac{e\delta \Delta w}{c} (\tilde{U}_s + \tilde{U}_l) \frac{\tilde{S}}{\tilde{N}}$$
(4)

$$B_L \tilde{I} = \frac{\Delta w}{jc} \tilde{L} \tag{5}$$

$$B_R \tilde{R} = \frac{\Delta w}{ic} \tilde{I}, \tag{6}$$

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Finite Elements Method

Variational problem Well-posedness Results

The matrix B has the form:

1	′ *	**	0		0	0	0	0
	0	*	**	0		0	0	0
	0	0	*	**	0		0	0
	÷		·	·	·			÷
	0	0		0	*	**	0	0
	0	0	0		0	*	**	0
	0	0	0	0		0	*	** /

Iterative Method

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Variational problem Well-posednes Results

Algorithm 1: Fixed Point

1 Input M, tol**2** Output $\tilde{U}_l, \tilde{U}_s, \tilde{S}, \tilde{L}, \tilde{I}, \tilde{R} \in \mathbb{R}^{(M-1)^2}$ 3 $\tilde{I}_0 \leftarrow ones((M-1)^2, 1)$ compute LU factorization of: A_s, A_l, B_L, B_I, B_R 5 while $||I^{(n+1)} - I^{(n)}||_2 > tol$ do solve $A_s \tilde{U}_s^{(n)} = \frac{\gamma \Delta w}{2} p \tilde{I}^{(n)}$ 6 solve $A_l \tilde{U}_l^{(n)} = \frac{\gamma \Delta w}{r} (1-p) \tilde{I}^{(n)}$ 7 compute LU factorization of B_S 8 solve $B_S \tilde{S}^{(n)} = \frac{\alpha \Delta w}{c} \tilde{N} \left(1 - \frac{\tilde{N}}{k} \right)$ 9 solve $B_L \tilde{L}^{(n)} = \frac{e\delta\Delta w}{c} \left(\tilde{U}_s^{(n)} + \tilde{U}_l^{(n)} \right) \frac{\tilde{S}^{(n)}}{\tilde{N}}$ 10solve $B_L \tilde{I}^{(n+1)} = \frac{\Delta w}{ic} \tilde{L}^{(n)}$ 11 solve $B_R \tilde{R}^{(n)} = \frac{\Delta w}{i \pi} \tilde{I}^{(n+1)}$ 12 13 end

Results

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Finite Difference Method

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Simulations

Finite Elements Method Variational problem

Well-posedı Results The computational domain is $\Omega = [0, l] \times [0, l]$, with l = 10. We choose:

- e = 0.001
- $\delta = 50$
- $\alpha = 0.2$
- k = 10000
- *j* = 10
- *i* = 10

- $\gamma = 200$
- *p* = 0.8
- $D_s = 2$
- $D_l = 200$
- $M_x = M_y = 100$
- tol = 0.00001

(Iter: 9)

At left, surface of sensitives leaves and right surface of latent leaves



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Finite Difference Method

discretization System Iterative Metho Results

Finite Elements Method Variational problem Well-posedne



At left, surface of infectious leaves and right surface of removed leaves

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Results



Long and short range spores

 \tilde{U}_l



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0.4

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0.2

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Finite Differences Method Space discretization System Iterative Metl Results

Finite Elements Method Variational problem Well-posednes Results



Sensitives leaves - Infectious



Simulations

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Simulations...

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Variational problem

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Finite Elements Method

Variational problem Well-posednes: Results

$$\int_{\Omega} (cS_X - e\delta(U_s + U_l)\frac{S}{N})\Phi_1 + \int_{\Omega} \alpha N\left(1 - \frac{N}{k}\right)\phi_1 = 0$$

$$\int_{\Omega} (cL_X + e\delta(U_s + U_l)\frac{S}{N} - \frac{1}{j}L)\phi_2 = 0$$

$$\int_{\Omega} (cI_X + \frac{1}{j}L - \frac{1}{i}I)\phi_3 = 0$$

$$\int_{\Omega} (cR_X + \frac{1}{i}I)\phi_4 = 0$$

$$\int_{\Omega} (cU_{s,x} - \delta U_s + \gamma pI)\phi_5 - \int_{\Omega} D_s \nabla U_s \cdot \nabla \phi_5 + \int_{\partial\Omega} D_s \partial_n U_s \phi_5 = 0$$

$$\int_{\Omega} (cU_{l,x} - \delta U_l + \gamma(1 - p)I)\phi_6 - \int_{\Omega} D_l \nabla U_l \cdot \nabla \phi_6 + \int_{\partial\Omega} D_l \partial_n U_l \phi_6 = 0.$$

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Well-posedness

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Theorem

Let $H(\Omega)$ be the space of functions with the first space derivative in $L^2(\Omega)$ and null Dirichlet boundary conditions. There exists a unique weak solution $(S, L, I, R) \in (L^2(\Omega))^4$ and $(U_s, U_l) \in (H(\Omega))^2$.

We can prove that the operator is continuous and coercive in $(H(\Omega))^3$, the Lax-Milgram lemma provides the result.

Results

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Finite Elements Method Variational problem Well-posednes Results • We choose the test functions ϕ in finite dimension subspace V_h of $(L^2(\Omega))^4 \times (H(\Omega))^2$ and solve the problem

$$a(Y_h,\phi_h)) = 0, \ \forall \phi_h \in V_h$$

 P1: Lagrange finite elements (dense in the space L²(Ω) and H(Ω)):

1

$$V_h := \left\{ \phi \in L^2(\Omega); \forall K \in \mathcal{T}_h, v_{|K} \in \mathbb{P}_1 \right\}$$

• To solve the non-linear problem, a fixed point method is chosen. We have

$$a(Y_h, \phi_h)) = 0 \iff Y_h = a(Y_h, \phi_h)) - Y_h$$

Mesh of the square

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Surface of infectious leaves



Surface of sensitives



Short-range spores



Short-range spores



Short-range spores



Long-range spores



Long-range spores



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Long-range spores

