Progressive waves of the spatial spread of a pathogen

CIMPA - Caracas

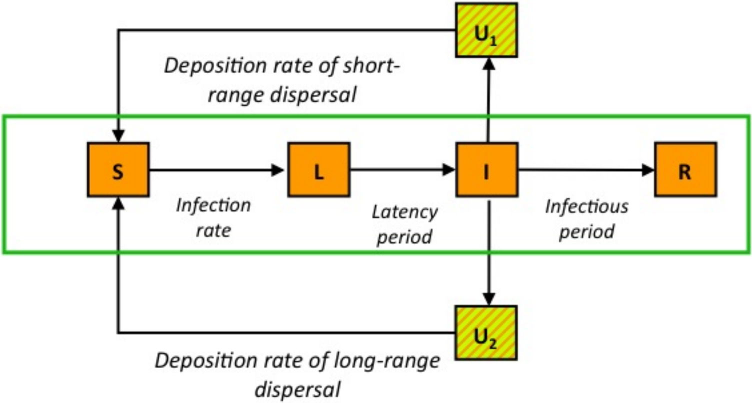
Group #2

Abril, 2012

Problem

The disease cycle is described as follows: susceptible leaves (denoted by S) inoculated with spores first become latent (L), then turn infectious (I) and produce spores (U_s short-range dispersed spores and U_l long-range) during some infectious period after which they are removed (R) as they cannot be infected again. The total density of leaves is (N).

Problem



Each segment is solution of the following differential system:

Problem

$$
\begin{aligned}
\frac{dS}{dt} &= -e\delta(U_s+U_l)\frac{S}{N} + \alpha N\left(1-\frac{N}{k}\right)\\
\frac{dL}{dt} &= e\delta(U_s+U_l)\frac{S}{N} - \frac{1}{j}L\\
\frac{dI}{dt} &= \frac{1}{j}L - \frac{1}{i}I\\
\frac{dR}{dt} &= \frac{1}{i}I\\
\frac{\partial U_s}{\partial t} &= D_s\Delta U_s - \delta U_s + \gamma p I\\
\frac{\partial U_l}{\partial t} &= D_l\Delta U_l - \delta U_l + \gamma(1-p)I.
\end{aligned}
$$

$i, j, \delta, e, D_s, D_l, \gamma, p, \alpha, k$ are positive constants.

Progressive waves

$$Y(x,y,t) = \widetilde{Y}(w,y), \text{ with } w = x - ct$$

The system becomes time-independant:

$$\begin{aligned}
-c\frac{d\tilde{S}}{dw} &= -e\delta(\tilde{U}_s + \tilde{U}_l)\frac{\tilde{S}}{\tilde{N}} + \alpha\tilde{N}\left(1 - \frac{\tilde{N}}{k}\right) \\
-c\frac{d\tilde{L}}{dw} &= e\delta(\tilde{U}_s + \tilde{U}_l)\frac{\tilde{S}}{\tilde{N}} - \frac{1}{j}\tilde{L} \\
-c\frac{d\tilde{I}}{dw} &= \frac{1}{j}\tilde{L} - \frac{1}{i}\tilde{I} \\
-c\frac{d\tilde{R}}{dw} &= \frac{1}{i}\tilde{I} \\
-c\frac{\partial\tilde{U}_s}{\partial w} &= D_s\Delta\tilde{U}_s - \delta\tilde{U}_s + \gamma p\tilde{I} \\
-c\frac{\partial\tilde{U}_l}{\partial w} &= D_l\Delta\tilde{U}_l - \delta\tilde{U}_l + \gamma(1-p)\tilde{I}
\end{aligned}$$

Since $N = S + L + I + R$ and sum the first four equation:

$$\tilde{N}' + \frac{\alpha}{c}\tilde{N} = \frac{\alpha}{ck}\tilde{N}^2$$

$$\therefore \tilde{N} = \frac{k}{1 + ke^{\alpha w/e}}$$

Space discretization

$$\Delta w = \Delta y$$

$$\begin{aligned}
\left(1 + \frac{e\delta}{c}\Delta w(\tilde{U}_s^{m,n} + \tilde{U}_l^{m,n})\frac{1}{\tilde{N}^{m,n}}\right)\tilde{S}^{m,n} - \tilde{S}^{m+1,n} &= \frac{\alpha}{c}\Delta w \tilde{N}^{m,n}\left(1 - \frac{\tilde{N}^{m,n}}{k}\right) \\
(1 + \frac{1}{jc}\Delta w)\tilde{L}^{m,n} - \tilde{L}^{m+1,n} &= \frac{e\delta}{c}\Delta w(\tilde{U}_s^{m,n} + \tilde{U}_l^{m,n})\frac{\tilde{S}^{m,n}}{\tilde{N}^{m,n}} \\
\left(1 + \frac{1}{ic}\Delta w\right)\tilde{I}^{m,n} - \tilde{I}^{m+1,n} &= \frac{1}{jc}\Delta w \tilde{L}^{m,n} \\
\tilde{R}^{m,n} - \tilde{R}^{m+1,n} &= \frac{1}{ic}\Delta w \tilde{I}^{m,n}
\end{aligned}$$

$$\begin{aligned}
-\frac{D_s}{c\Delta w}\tilde{U}_s^{m-1,n} - \frac{D_s}{c\Delta w}\tilde{U}_s^{m,n-1} + \left(1 + \frac{4D_s}{c\Delta w} + \frac{\delta\Delta w}{c}\right)\tilde{U}_s^{m,n} & \\
-\frac{D_s}{c\Delta w}\tilde{U}_s^{m,n+1} + \left(-1 - \frac{D_s}{c\Delta w}\right)\tilde{U}_s^{m+1,n} &= \frac{\gamma\Delta w}{c} p\tilde{I}^{m,n} \\
-\frac{D_l}{c\Delta w}\tilde{U}_l^{m-1,n} - \frac{D_l}{c\Delta w}\tilde{U}_l^{m,n-1} + \left(1 + \frac{4D_l}{c\Delta w} + \frac{\delta\Delta w}{c}\right)\tilde{U}_l^{m,n} & \\
-\frac{D_l}{c\Delta w}\tilde{U}_l^{m,n+1} + \left(-1 - \frac{D_l}{c\Delta w}\right)\tilde{U}_l^{m+1,n} &= \frac{\gamma\Delta w}{c}(1-p)\tilde{I}^{m,n}
\end{aligned}$$

System

We can write this scheme as system of linear equations as follows:

$$A_s\tilde{U}_s = \frac{\gamma\Delta w}{c}p\tilde{I} \tag{1}$$

$$A_l\tilde{U}_l = \frac{\gamma\Delta w}{c}(1-p)\tilde{I} \tag{2}$$

$$B_S\tilde{S} = \frac{\alpha\Delta w}{c}\tilde{N}\left(1-\frac{\tilde{N}}{k}\right) \tag{3}$$

$$B_L\tilde{L} = \frac{e\delta\Delta w}{c}(\tilde{U}_s+\tilde{U}_l)\frac{\tilde{S}}{\tilde{N}} \tag{4}$$

$$B_L\tilde{I} = \frac{\Delta w}{jc}\tilde{L} \tag{5}$$

$$B_R\tilde{R} = \frac{\Delta w}{ic}\tilde{I}, \tag{6}$$

System

The matrix A has the form:

$$\begin{pmatrix}
* & ** & 0 & \dots & 0 & *** & 0 & \dots & 0 & 0 & 0 & 0 \\
** & * & ** & 0 & \dots & 0 & *** & 0 & \dots & 0 & 0 & 0 \\
0 & ** & * & ** & 0 & \dots & 0 & *** & 0 & \dots & 0 & 0 \\
\vdots & & \ddots & \ddots & \ddots & & & & \ddots & & & \vdots \\
0 & & & 0 & * & 0 & & & & *** & 0 & 0 \\
\vdots & & & & \ddots & \ddots & \ddots & & & & \ddots & \vdots \\
** & 0 & \dots & 0 & 0 & ** & * & ** & 0 & \dots & 0 & *** \\
 & \ddots & & \dots & & & \ddots & \ddots & \ddots & & \dots & \\
0 & 0 & ** & 0 & \dots & 0 & 0 & ** & * & ** & 0 & 0 \\
0 & 0 & 0 & ** & 0 & \dots & 0 & 0 & ** & * & ** & 0 \\
0 & 0 & 0 & 0 & ** & 0 & \dots & 0 & 0 & ** & * & **
\end{pmatrix}$$

System

The matrix B has the form:

$$\begin{pmatrix} * & ** & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & * & ** & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & * & ** & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ 0 & 0 & \dots & 0 & * & ** & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & * & ** & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & * & ** \end{pmatrix}$$

Iterative Method

Algorithm 1: Fixed Point

Input M, tol

Output $\tilde{U}_l, \tilde{U}_s, \tilde{S}, \tilde{L}, \tilde{I}, \tilde{R} \in \mathbb{R}^{(M-1)^2}$

$\tilde{I}_0 \leftarrow$ones$((M-1)^2, 1)$

compute LU factorization of: A_s, A_l, B_L, B_I, B_R

while $||I^{(n+1)} - I^{(n)}||_2 > tol$ **do**

  solve $A_s \tilde{U}_s^{(n)} = \frac{\gamma \Delta w}{c} p \tilde{I}^{(n)}$

  solve $A_l \tilde{U}_l^{(n)} = \frac{\gamma \Delta w}{c} (1-p) \tilde{I}^{(n)}$

  compute LU factorization of B_S

  solve $B_S \tilde{S}^{(n)} = \frac{\alpha \Delta w}{c} \tilde{N} \left(1 - \frac{\tilde{N}}{k}\right)$

  solve $B_L \tilde{L}^{(n)} = \frac{e \delta \Delta w}{c} \left(\tilde{U}_s^{(n)} + \tilde{U}_l^{(n)}\right) \frac{\tilde{S}^{(n)}}{\tilde{N}}$

  solve $B_L \tilde{I}^{(n+1)} = \frac{\Delta w}{jc} \tilde{L}^{(n)}$

  solve $B_R \tilde{R}^{(n)} = \frac{\Delta w}{ic} \tilde{I}^{(n+1)}$

end

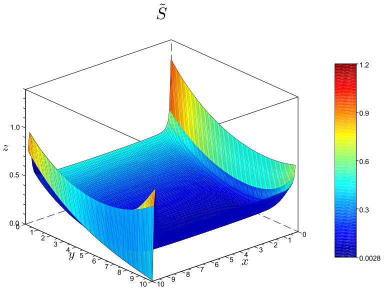
Results

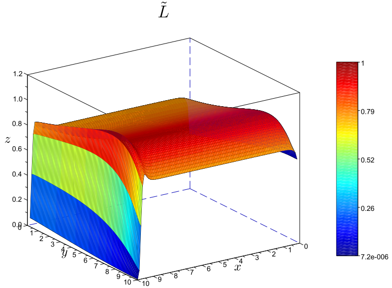
The computational domain is $\Omega = [0, l] \times [0, l]$, with $l = 10$.
We choose:

- $e = 0.001$
- $\delta = 50$
- $\alpha = 0.2$
- $k = 10000$
- $j = 10$
- $i = 10$
- $\gamma = 200$
- $p = 0.8$
- $D_s = 2$
- $D_l = 200$
- $M_x = M_y = 100$
- $tol = 0.00001$

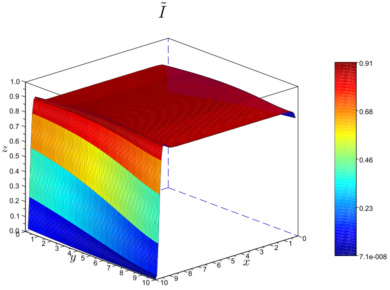
(Iter: 9)

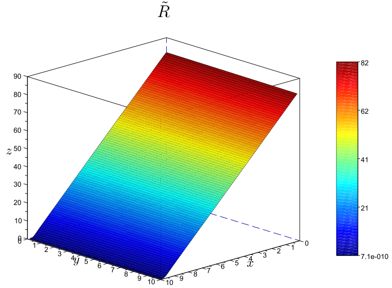
At left, surface of sensitives leaves and right surface of latent leaves





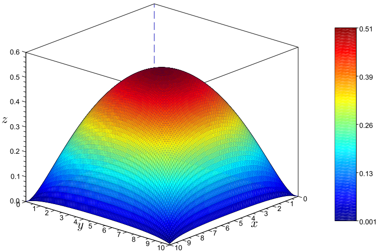
At left, surface of infectious leaves and right surface of removed leaves



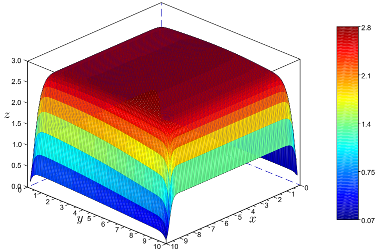


Long and short range spores

$\tilde{U}_l$

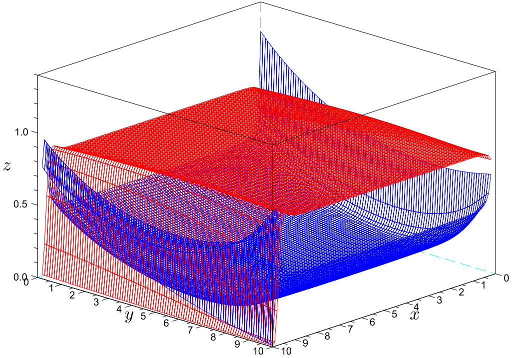


$\tilde{U}_s$



Sensitives leaves - Infectious

S , I



Simulations...

Variational problem

$$
\begin{aligned}
\int_\Omega (cS_X - e\delta(U_s+U_l)\frac{S}{N})\Phi_1 + \int_\Omega \alpha N\left(1-\frac{N}{k}\right)\phi_1 &= 0\\
\int_\Omega (cL_X + e\delta(U_s+U_l)\frac{S}{N} - \frac{1}{j}L)\phi_2 &= 0\\
\int_\Omega (cI_X + \frac{1}{j}L - \frac{1}{i}I)\phi_3 &= 0\\
\int_\Omega (cR_X + \frac{1}{i}I)\phi_4 &= 0\\
\int_\Omega (cU_{s,x} - \delta U_s + \gamma p I)\phi_5 - \int_\Omega D_s\nabla U_s.\nabla\phi_5 + \int_{\partial\Omega} D_s\partial_n U_s\,\phi_5 &= 0\\
\int_\Omega (cU_{l,x} - \delta U_l + \gamma(1-p)I)\phi_6 - \int_\Omega D_l\nabla U_l.\nabla\phi_6 + \int_{\partial\Omega} D_l\partial_n U_l\,\phi_6 &= 0.
\end{aligned}
$$

Well-posedness

Theorem

Let $H(\Omega)$ be the space of functions with the first space derivative in $L^2(\Omega)$ and null Dirichlet boundary conditions. There exists a unique weak solution $(S, L, I, R) \in (L^2(\Omega))^4$ and $(U_s, U_l) \in (H(\Omega))^2$.

We can prove that the operator is continuous and coercive in $(H(\Omega))^3$, the Lax-Milgram lemma provides the result.

Results

- We choose the test functions ϕ in finite dimension subspace V_h of $(L^2(\Omega))^4 \times (H(\Omega))^2$ and solve the problem

$$a(Y_h, \phi_h)) = 0, \ \forall \phi_h \in V_h$$

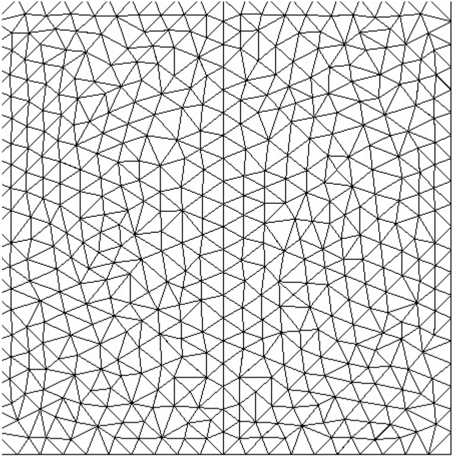
- P1: Lagrange finite elements (dense in the space $L^2(\Omega)$ and $H(\Omega)$):

$$V_h := \left\{ \phi \in L^2(\Omega); \forall K \in \mathcal{T}_h, v_{|K} \in \mathbb{P}_1 \right\}$$

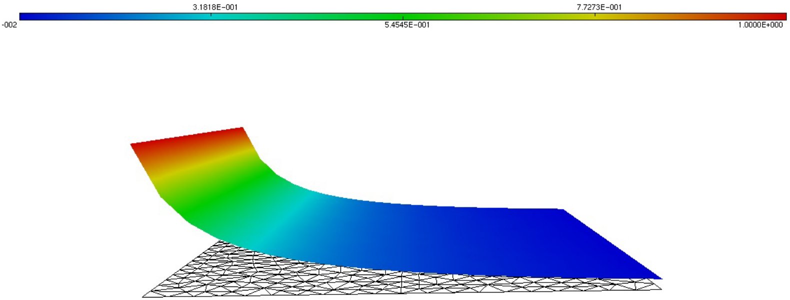
- To solve the non-linear problem, a fixed point method is chosen. We have

$$a(Y_h, \phi_h)) = 0 \iff Y_h = a(Y_h, \phi_h)) - Y_h$$

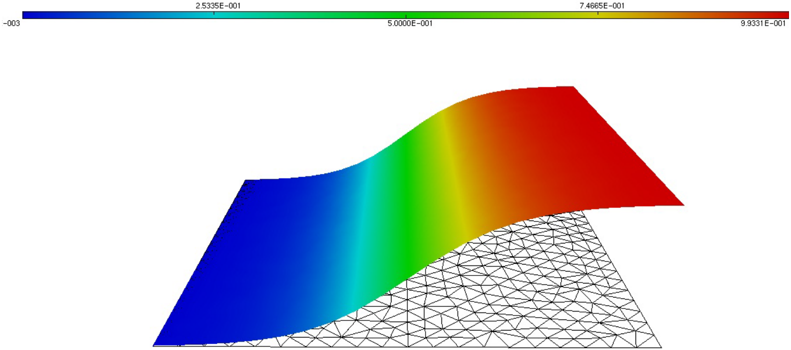
Mesh of the square



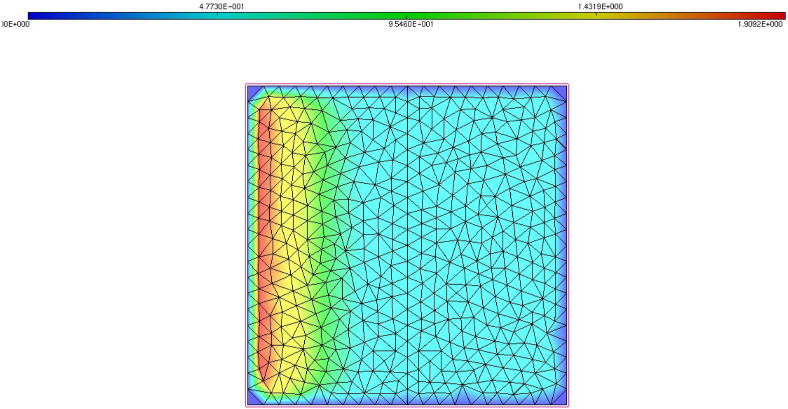
Surface of infectious leaves



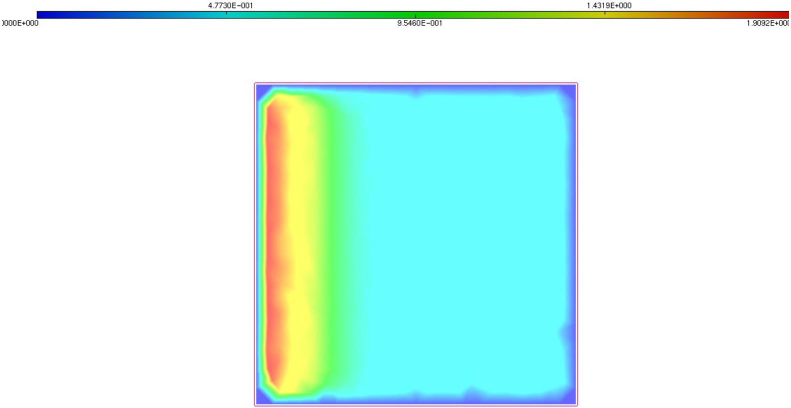
Surface of sensitives



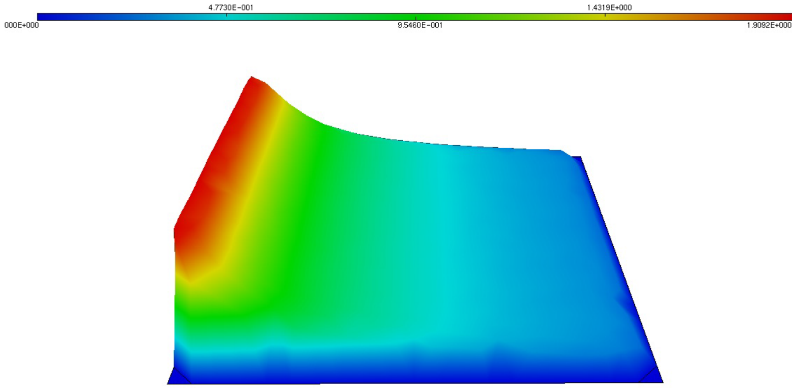
Short-range spores



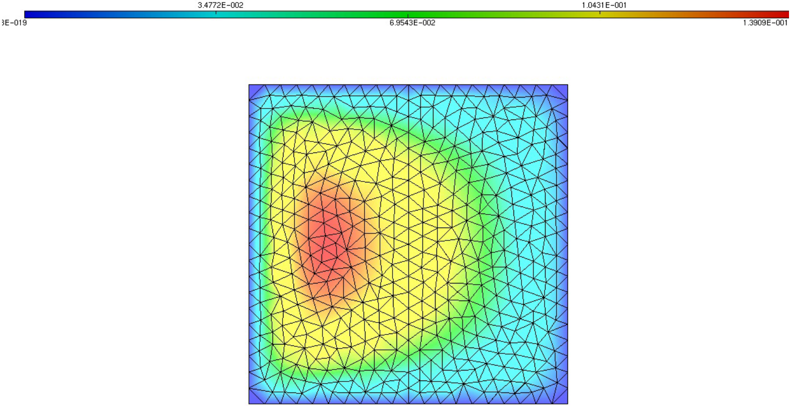
Short-range spores



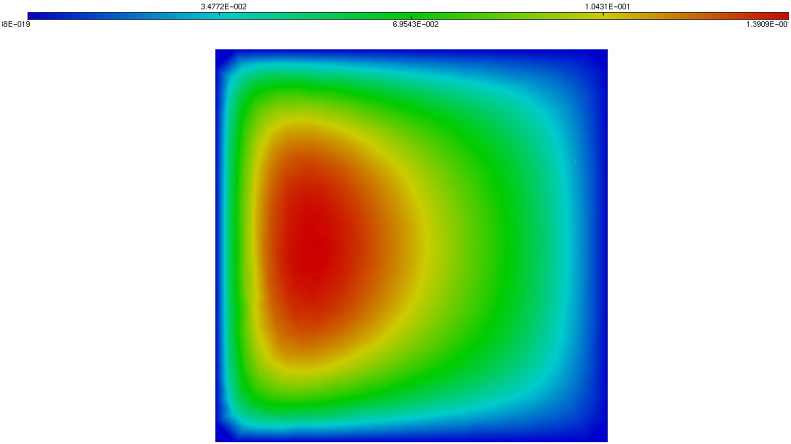
Short-range spores



Long-range spores



Long-range spores



Long-range spores