SOLITARY WAVES OF BOUSSINESQ SYSTEM

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CIMPA 2012

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Outline

1. Acknowledgements
2. Introduction
   Boussinesq System
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4. Finite Element Method
5. Numerical Experiments
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Acknowledgements

We want to thank

- The CIMPA organizer committee.
- Youcef Mammeri and Georges Sadaka for their advice on the mini-project.
- All the professors for their lectures in this CIMPA Research School.

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Introduction

Boussinesq Systems

$$\begin{aligned} \eta_t + \nabla.V + \nabla.(\eta V) + a\Delta\nabla.V - b\Delta\eta_t &= 0 \\ V_t + \nabla\eta + \frac{1}{2}\nabla|V|^2 + c\Delta\nabla\eta - d\Delta V_t &= 0 \end{aligned}$$

These systems models the two-way propagation of small amplitude, long wavelength, gravity waves in shallow water, described by its surface η(x,y,t) and its velocity $V(x,y,t) = (u(x,y,t), v(x,y,t))$.

- if b=d=0:
  KdV-KdV (Korteweg-de Vries)
- if $a = c = 0$:
  BBM-BBM (Benjamin-Bona-Mahony)
- if $a = 0$:
  BS (Bona-Smith)

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Solitary Waves

We consider solitary wave as

$$(\eta, u, v)(x, y, t) = (\widetilde{\eta}, \widetilde{u}, \widetilde{v})(X, y) \text{ where } X = x - st.$$

Thus, Boussinesq system become in the stationary problem

Stationary Problem

$$\begin{aligned} -s\eta_X + \nabla.V + \nabla.(\eta V) + a\Delta\nabla.V + sb\Delta\eta_X &= 0 \\ -sV_X + \nabla\eta + \frac{1}{2}\nabla|V|^2 + c\Delta\nabla\eta + sd\Delta V_X &= 0 \end{aligned}$$

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FINITE ELEMENT METHOD

Variational Formulation

Find $\eta \in H^2(\Omega)$ and $V \in \left(H^2(\Omega)\right)^2$, such that

$$a([\eta, V], [w, \mathbf{q}]) = f([w, \mathbf{q}])$$

for all $w \in H^2(\Omega)$ and $\mathbf{q} \in \left(H^2(\Omega)\right)^2$.

Where

$$a([\eta, V], [w, \mathbf{q}]) = a_1([\eta, V], [w, \mathbf{q}]) + a_2([\eta, V], [w, \mathbf{q}])$$

and

$$f([w, \mathbf{q}]) = f_1(w) + f_2(\mathbf{q})$$

$$
\begin{aligned}
a_1([\eta, V],[w,\mathbf{q}]) &= \int_\Omega (-s\eta_X + \nabla\cdot V)\, w + \int_\Omega \nabla\cdot(\eta V)\, w \\
&- \int_\Omega \nabla(a\nabla\cdot V + sb\eta_X)\cdot\nabla w
\end{aligned}
$$

$$
a_2([\eta, V],[w,\mathbf{q}]) = \int_\Omega (-sV_X + \nabla\eta)\cdot\mathbf{q} \quad + \quad \int_\Omega \left(\frac{1}{2}\nabla|V|^2\right)\cdot\mathbf{q}
$$

$$
+ \int_\Omega \nabla\cdot(c\Delta\nabla\eta + sd\Delta V_X)\,\nabla\cdot\mathbf{q}
$$

$$
f_1(w) = \int_{\partial\Omega} \partial_n(a\nabla\cdot V + sb\eta_X)\cdot w
$$

$$
f_2(\mathbf{q}) = \int_{\partial\Omega} \partial_n\,(c\nabla\eta + sdV_X)\cdot\mathbf{q}
$$

Discrete Non Linear System

The discrete non linear system is

$$a\left([\eta_h, V_h], \Phi \times \Phi^2\right) = F(\Phi \times \Phi^2)$$

where Φ is a basis for the finite dimensional space of approximation.
This system can be solved using a fixed point method as

$$(\eta_h^{n+1}, V_h^{n+1}) = (\eta_h^n, V_h^n) - \mathcal{F}\left(\eta_h^n, V_h^n\right)$$

where

$$\mathcal{F}\left(\eta_h, V_h\right) = a\left([\eta_h, V_h], \Phi \times \Phi^2\right) - F(\Phi \times \Phi^2)$$

Discrete Non Linear System

In order to solve the non linear system in **FreeFem++** we have to use a semi-implicit method:

$$
\begin{aligned}
a_1([\eta^{n+1}, V^{n+1}], [w, \mathbf{q}]) &= \int_\Omega \left(-s\eta_X^{n+1} + \nabla \cdot V^{n+1}\right) w + \int_\Omega \nabla \cdot (\eta^n V^n)\, w \\
&- \int_\Omega \nabla(a\nabla \cdot V^{n+1} + sb\eta_X^{n+1}) \cdot \nabla w
\end{aligned}
$$

$$
\begin{aligned}
a_2([\eta^{n+1}, V^{n+1}], [w, \mathbf{q}]) = \int_\Omega \left(-sV_X^{n+1} + \nabla\eta^{n+1}\right) \cdot \mathbf{q} \quad + \quad \int_\Omega \left(\frac{1}{2}\nabla|V^n|^2\right) \cdot \mathbf{q} \\
+ \int_\Omega \nabla \cdot \left(c\Delta\nabla\eta^{n+1} + sd\Delta V_X^{n+1}\right) \nabla \cdot \mathbf{q}
\end{aligned}
$$

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Numerical Experiments

Simulations are performed with Freefem++. The computational domain is $\Omega = [0, L] \times [0, L]$, with $L = 10$ $s = 1, a = b = 1$. To simplify the computations of the mini-project, we assume that u_h and v_h are known as

$$
\begin{aligned}
uh &= 0.5 \exp(-(x - L/2)^2 - (y - L/2)^2) \\
vh &= 0.
\end{aligned}
$$

We have to determine only the surface η, we will solve only the first equation with the boundary conditions:

$$
\eta_{|\partial\Omega} = 0.
$$

References

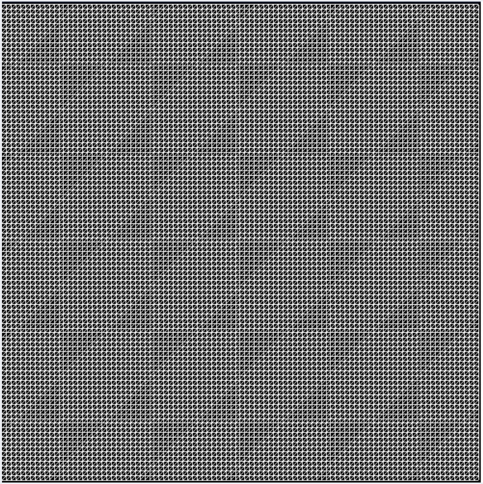

Figure: Mesh

References

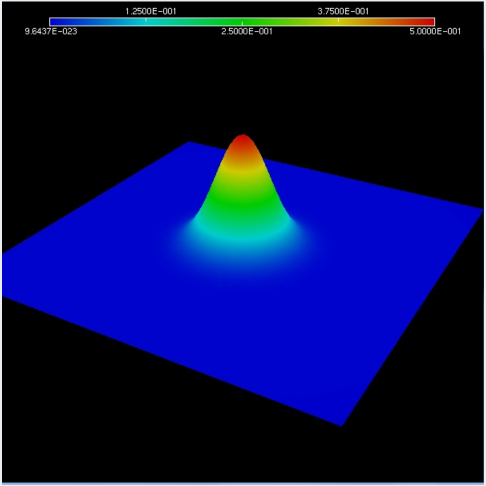

Figure: Velocity uh

References

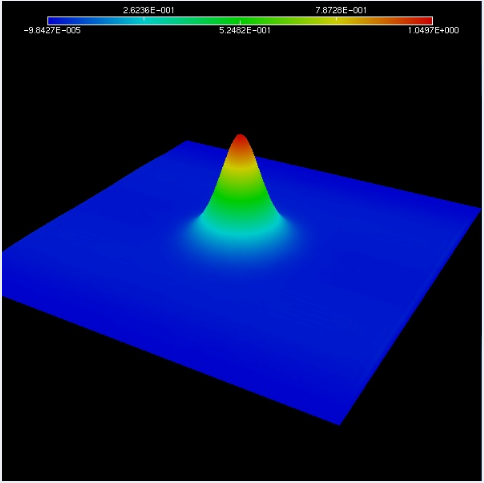

Figure: Surface eh

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References

- J.L. Bona, M. Chen, J.-C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I. Derivation and linear theory. J. Nonlinear Sci. 12(4), (2002), 283-318.
- J. L. Bona, W. G. Pritchard, and L. R. Scott, Numerical schemes for a model for nonlinear dispersive waves, J. Comp. Physics, 60 (1985), 167-186.
- A. Quarteroni, R. Sacco, F. Saleri, Numerical Mathematics, Springer (2000).

THANKS!

ANY QUESTIONS?