Basic Control Theory
Linear Finite Dimensional Systems

Sylvain Ervedoza[1]

[1]Institut de Mathématiques de Toulouse & CNRS

CIMPA Caracas

Outline

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We consider the system

$$x' = Ax + Bu, \quad t \in \mathbb{R}, \qquad x(0) = x_0. \tag{1.1}$$

Proposition

If the finite-dimensional system (1.1) *is approximately controllable a time* $T > 0$ *then it is exactly controllable at time* T.

Proposition

If system (1.1) *is controllable at* 0 *in time* $T > 0$, *it is exactly controllable at time* $T > 0$.

All the notions of control introduced in the previous chapter are equivalent when considering a finite-dimensional system of the form (1.1). Hence, in the following, we will focus on one particular controllability property, the other ones following by the above considerations.
Of course, the weaker the control property is, the easier it is to proved. We will therefore often focus on the approximate controllability or the null-controllability.

Outline

Example 1 The diagonal case

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The corresponding equation is

$$\begin{cases} x_1' = x_1 \\ x_2' = 2x_2 + u(t). \end{cases}$$

The control does not act on the first equation. The system is therefore not controllable. For instance, if $x_1(0) = 0$, whatever the control function is, we will always have $x_1(t) = 0$ for all $t \in \mathbb{R}$ and we cannot achieve the state

$$x_T = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

Example 2 The case of the pendulum $z'' + \alpha^2 z' + \beta^2 z = u(t)$, $\beta > 0$ is the frequency of the oscillator, α is a damping term. After having set $x_1 = z$ et $x_2 = z'$, this equation is equivalent to (1.1) with

$$A = \begin{pmatrix} 0 & 1 \\ -\beta^2 & -\alpha^2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We can therefore restrict ourselves to the study of the null-controllability. For $(z(0), z'(0)) = (z_0, z_1)$ given we set $z(t) = (z_0 + tz_1)\phi(t/T)$ where $\phi : [0,1] \to \mathbb{R}$ is a smooth function satisfying $\phi(0) = 1$, $\phi'(0) = 1$ and $\phi(1) = \phi'(1) = 0$. One easily check that z satisfies $(z(0), z'(0)) = (z_0, z_1)$ and $(z(T), z'(T)) = (0, 0)$. We then set $u(t) = z'' + \alpha^2 z' + \beta^2 z$. The pendulum is therefore exactly controllable at any time $T > 0$.

Outline

We fix $T > 0$ and $x_0 \in \mathbb{R}^N$.
Our goal is to steer the solution x of

$$x' = Ax + Bu, \quad t \in [0, T], \qquad x(0) = x_0.$$

to 0:

$$x[x_0, u](T) = 0$$

The main idea is to introduce the adjoint equation:

$$z' = -A^* z, \quad t \in [0, T], \qquad z(T) = z_T \tag{3.1}$$

and remark the following equivalences:

$$\begin{aligned} & x[x_0, u](T) = 0 \Leftrightarrow \forall z_T \in \mathbb{R}^N, \ \langle x[x_0, u](T), z_T \rangle_{\mathbb{R}^N} = 0 \\ & \Leftrightarrow \forall z_T \in \mathbb{R}^N, \ \int_0^T \langle u(t), B^* z(t) \rangle_{\mathbb{R}^p} \, dt + \langle x_0, z(0) \rangle_{\mathbb{R}^N} = 0, \end{aligned} \tag{3.2}$$

where z denotes the solution of (3.1) with initial data z_T.

A negative result:

Theorem

If there exists $\bar{z}_T \in \mathbb{R}^N \setminus \{0\}$ *such that the solution* $\bar{z}$ *of the adjoint equation* (3.1) *satisfies* $B^* \bar{z}(t) = 0$ *for all* $t \in (0, T)$*, then there exists an initial data* $x_0 \in \mathbb{R}^N$ *that cannot be steered to* 0*. In particular, this is the case for* $x_0 = \bar{z}(0)$*.*

Let us define the set of all null-controls associated to x_0:

$$\mathcal{U}(x_0, T) = \{u \in L^2(0, T; \mathbb{R}^p),$$
$$s.t.\ u \text{ is a null-control associated to } x_0\}.$$

Theorem

Let $x_0 \in \mathbb{R}^N$ and $T > 0$. The set $\mathcal{U}(x_0, T)$ is empty or affine with underlying vector space

$$\mathcal{U}(0, T) = \{B^* z, \text{ s.t. } z \text{ solution of } (3.1)\ \}^{\perp_{L^2(0,T;\mathbb{R}^p)}}$$

In particular, the set $\mathcal{U}(x_0, T)$ satisfies one of the two conditions:

- *$\mathcal{U}(x_0, T)$ is empty.*
- *there exists $Z_T \in \mathbb{R}^p$ such that*

$$\mathcal{U}(x_0, T) = B^* Z + \{B^* z, \text{ s.t. } z \text{ solution of } (3.1)\ \}^{\perp_{L^2(0,T;\mathbb{R}^p)}},$$

where Z is the solution of (3.1) with data Z_T. In particular, in this case, $u(t) = B^* Z(t)$ is the null-control for x_0 of minimal $L^2(0, T; \mathbb{R}^p)$-norm and there is only one control $u \in \mathcal{U}(x_0, T)$ that can be written as $u(t) = B^* z(t)$ for z solution of (3.1).

Proof. Assume $\mathcal{U}(x_0, T) \neq \emptyset$.
First, due to the linearity of system (1.1), it is rather easy to check that, if $\bar{u} \in \mathcal{U}(x_0, T)$, then

$$\mathcal{U}(x_0, T) = \bar{u} + \mathcal{U}(0, T). \tag{3.3}$$

Indeed, if $u \in \mathcal{U}(x_0, T)$, the linearity of the system implies that $u - \bar{u} \in \mathcal{U}(0, T)$. Similarly, if $u \in \mathcal{U}(0, T)$, then $\bar{u} + u \in \mathcal{U}(x_0, T)$. Besides, the set $\mathcal{U}(0, T)$ obviously is a vector space due to the linear structure of system (1.1) which proves the affine structure of $\mathcal{U}(x_0, T)$ with underlying vector space $\mathcal{U}(0, T)$.

We then have to characterize the set $\mathcal{U}(0, T)$. Using the characterization (3.2),

$$u \in \mathcal{U}(0, T) \Leftrightarrow \forall z_T \in \mathbb{R}^p, \int_0^T \langle u(t), B^* z \rangle_{\mathbb{R}^p} = 0.$$

Of course, this is completely equivalent to say that

$$\mathcal{U}(0, T) = \overline{\{B^* z,\ s.t.\ z \text{ solution of } (3.1)\ \}}^{\perp_{L^2(0,T;\mathbb{R}^p)}}$$

But

$$L^2(0, T; \mathbb{R}^p) = \{B^* z,\ s.t.\ z \text{ solution of } (3.1)\ \} \oplus \{B^* z,\ s.t.\ z \text{ solution of } (3.1)\ \}^{\perp_{L^2(0,T;\mathbb{R}^p)}},$$

and thus, $\bar{u}$ in (3.3) can be expanded as $B^* Z + \tilde{u}$, with Z solution of (3.1) with some initial data $Z_T \in \mathbb{R}^p$ and $\tilde{u} \in \mathcal{U}(0, T)$.
Since $\tilde{u} \in \mathcal{U}(0, T)$, we have obtained

$$\mathcal{U}(x_0, T) = B^* Z + \{B^* z,\ s.t.\ z \text{ solution of } (3.1)\ \}^{\perp_{L^2(0,T;\mathbb{R}^p)}},$$

• Minimal $L^2(0, T; \mathbb{R}^p)$-norm: B^*Z is obviously orthogonal to $\{B^*z,\ s.t.\ z \text{ solution of } (3.1)\ \}^{\perp_{L^2(0,T;\mathbb{R}^p)}}$, B^*Z is the control of minimal $L^2(0, T; \mathbb{R}^p)$-norm.

• Uniqueness: Uniqueness of controls u of the form B^*z follows from the structure of $\mathcal{U}(x_0, T)$.

According to the structure of the control set, one can reduce our analysis to the possibility of controlling (1.1) with controls of the form B^*Z for some Z solution of (3.1).

According to (3.2), we are thus looking for a trajectory Z of (3.1) such that

$$\forall z_T \in \mathbb{R}^N, \quad \int_0^T \langle B^*Z, B^*z\rangle_{\mathbb{R}^p}\, dt + \langle x_0, z(0)\rangle_{\mathbb{R}^N} = 0, \tag{3.4}$$

where z is the solution of (3.1) with data z_T.
There are several ways to solve the problem (3.4) that are of course completely equivalent.

Express all the quantities in (3.4) as functions of Z_T, z_T and x_0.

$$\forall z_T,\ \int_0^T \langle B^* e^{-(t-T)A^*} Z_T, B^* e^{-(t-T)A^*} z_T \rangle_{\mathbb{R}^p}\, dt + \langle x_0, e^{TA^*} z_T \rangle_{\mathbb{R}^N} = 0,$$

and thus as

$$\forall z_T,\ \langle \left(\int_0^T e^{-(t-T)A} BB^* e^{-(t-T)A^*}\, dt \right) Z_T, z_T \rangle_{\mathbb{R}^N} + \langle e^{TA} x_0, z_T \rangle_{\mathbb{R}^N} = 0,$$

Therefore $B^* Z$ is a control function for x_0 if and only if

$$\Lambda_T Z_T + e^{TA} x_0 = 0, \text{ where } \Lambda_T = \int_0^T e^{-(t-T)A} BB^* e^{-(t-T)A^*}\, dt. \tag{3.5}$$

The operator Λ_T is the so-called Gramian operator.

Recall

$$\Lambda_T = \int_0^T e^{-(t-T)A} B B^* e^{-(t-T)A^*} \, dt.$$

Theorem

Let $T > 0$ and $x_0 \in \mathbb{R}^N$. If Λ_T is invertible, then system (1.1) is null-controllable.
Besides, if Λ_T is not invertible, there are some initial data that are not null-controllable.

Proof.

• If Λ_T is invertible, one can solve the equation (3.5). By construction, $u(t) = B^* Z$, where Z is the solution of (3.1) with data Z_T, belongs to $\mathcal{U}(x_0, T)$.

• If Λ_T is not invertible, then there exists $\bar{z}_T \in \mathbb{R}^N$ such that $\Lambda_T \bar{z}_T = 0$. Hence:

$$0 = \langle \Lambda_T \bar{z}_T, \bar{z}_T \rangle_{\mathbb{R}^N} = \int_0^T \|B^* \bar{z}(t)\|^2_{\mathbb{R}^p} \, dt. \qquad (3.6)$$

Hence there are some initial data that are not null-controllable.

Remark that (3.4) corresponds to the Euler-Lagrange equation associated to the minimization of the functional

$$J(z_T) = \frac{1}{2}\int_0^T \|B^* z(t)\|_{\mathbb{R}^p}^2 \, dt + \langle z(0), x_0 \rangle_{\mathbb{R}^N}, \tag{3.7}$$

defined for $z_T \in \mathbb{R}^N$, z being the solution of (1.1).

Theorem

Assume that there exists a constant $C_{obs} > 0$ such that the so-called observability property holds: for all data $z_T \in \mathbb{R}^N$ and z the corresponding solution of (3.1),

$$\|z(0)\|_{\mathbb{R}^N}^2 \leq C_{obs}^2 \int_0^T \|B^* z(t)\|_{\mathbb{R}^p}^2 \, dt. \tag{3.8}$$

Then the functional J in (3.7) *admits a unique minimizer $Z_T \in \mathbb{R}^N$ and the function $u(t) = B^* Z(t)$ is the null-control of minimal $L^2(0, T; \mathbb{R}^p)$-norm for x_0.*

Of course, the two approaches are completely equivalent since the goal is to compute the solution Z_T of (3.4), though the assumptions might seem different. Of course, they are not, since easy computations show that

$$\langle \Lambda_T z_T, z_T \rangle_{\mathbb{R}^N} = \int_0^T \|B^* z(t)\|^2_{\mathbb{R}^p} \, dt.$$

Therefore, since Λ_T is self-adjoint and non-negative, the invertibility of Λ_T is completely equivalent to the observability property (3.8).

Beware !! this is true only because we are in a finite dimensional setting and thus the norm of z_T and $z(0) = \exp(TA^*) z_T$ are equivalent norms. Otherwise, one should made precise in which space Λ_T should be invertible.

We have obtained the following results:

1. If there exists $\bar{z}_T \in \mathbb{R}^N \backslash \{0\}$ such that $B^* \bar{z} \equiv 0$ on $[0, T]$, then $x_0 = \bar{z}(0)$ is not null-controllable.
2. If for all $z_T \in \mathbb{R}^N \backslash \{0\}$, $\|B^* z\|_{L^2(0,T;\mathbb{R}^p)} \neq 0$, then system (1.1) is null-controllable at time $T > 0$.

Of course, since we are in finite-dimension, we have the equivalence between the observability property (3.8) and the condition of item 2.

We have thus prove the following:

Theorem

System (1.1) *is null-controllable at time $T > 0$ if and only if the adjoint system* (3.1) *is exactly observable at time $T > 0$, i.e. if there exists C_{obs} such that the observability inequality* (3.8) *hold for all the trajectories of the adjoint system* (3.1).

This is the so-called Hilbert Uniqueness Method - HUM in short - introduced by J.-L. Lions.
In general, easier to study the observability inequality (3.8).

Outline

Assume that the observability property (3.8) holds in time $T > 0$ for the adjoint system (3.1). We can then introduce the control operator:

$$\mathfrak{U}_T : x_0 \in \mathbb{R}^N \mapsto B^* Z \in L^2(0, T; \mathbb{R}^p), \tag{4.1}$$

where $Z = Z[x_0]$ is the solution of (3.1) with data $Z_T[x_0]$ given by HUM.

Then we have the following Theorem:

Theorem

Assume that the adjoint system (3.1) satisfies the exact observability property (3.8) at time $T > 0$ with the constant C_{obs}. Then the map $\mathfrak{U}_T$ is linear and its norm is given by

$$\|\mathfrak{U}_T\|_{\mathfrak{L}(\mathbb{R}^N; L^2(0,T;\mathbb{R}^p))} = C_{obs}, \tag{4.2}$$

where C_{obs} is the best constant in (3.8).

Recall the characterization of the control given by (3.2). If $x_0 \in \mathbb{R}^N$, taking $z = Z[x_0]$, the solution of (3.1) corresponding to x_0 given by HUM, in (3.2), we necessarily have

$$\|\mathfrak{U}_T(x_0)\|^2_{L^2(0,T;\mathbb{R}^p)} = \int_0^T \|B^* Z[x_0](t)\|^2_{\mathbb{R}^p} \, dt = -\langle x_0, Z[x_0](0)\rangle_{\mathbb{R}^N}.$$

Using the observability property, this implies:

$$\|\mathfrak{U}_T(x_0)\|_{L^2(0,T;\mathbb{R}^p)} \leq C_{obs} \, \|x_0\|_{\mathbb{R}^N}.$$

To prove the identity (4.2), let us choose (finite-dimension) $\bar{z}_T \neq 0$ such that

$$\|\bar{z}(0)\|_{\mathbb{R}^N} = C_{obs} \|B^* \bar{z}\|_{L^2(0,T;\mathbb{R}^p)} .$$

Then set $x_0 = -\bar{z}(0)$. Then according to (3.2), the control function $u = \mathfrak{U}_T(x_0)$ should satisfy

$$\int_0^T \langle \mathfrak{U}_T(x_0), B^* \bar{z}(t) \rangle_{\mathbb{R}^p} = \|\bar{z}(0)\|_{\mathbb{R}^N}^2 .$$

Hence,

$$\|\bar{z}(0)\|_{\mathbb{R}^N}^2 \leq \|\mathfrak{U}_T(x_0)\|_{L^2(0,T;\mathbb{R}^p)} \|B^* \bar{z}\|_{L^2(0,T;\mathbb{R}^p)} ,$$

and therefore,

$$\|x_0\|_{\mathbb{R}^N} = \|\bar{z}(0)\|_{\mathbb{R}^N} \leq \frac{1}{C_{obs}} \|\mathfrak{U}_T(x_0)\|_{L^2(0,T;\mathbb{R}^p)}.$$

This of course implies (4.2).

Outline

The previous section do very little use of the finite dimensional setting.
In this section, we shall strongly use it.

Theorem

System (1.1) *is exactly controllable at time* $T > 0$ *if and only if*

$$Rank(B \,|\, AB \,|\, A^2B \cdots |A^{N-1}B) = N, \tag{5.1}$$

where N *is the dimension of the space of the state* x.
In particular, if a finite dimensional system (1.1) *is exactly controllable at some time* $T > 0$, *it is exactly controllable for all time* $T > 0$.

Theorem 9 is known as the Kalman rank condition. It is sometimes written in its dual form, i.e. replacing condition (5.1) by

$$\mathrm{Ker} \begin{pmatrix} B^* \\ B^*A^* \\ \vdots \\ B^*(A^*)^{N-1} \end{pmatrix} = \{0\}. \tag{5.2}$$

System (1.1) is exactly controllable if and only if for all $z_T \in \mathbb{R}^N \backslash \{0\}$, $\|B^* z\|_{L^2(0,T;\mathbb{R}^p)} \neq 0$, z being the solution of (3.1) with initial data z_T.

$$
\begin{aligned}
B^* z(t) \equiv 0 \text{ on } (0,T) &\Leftrightarrow \forall k \in \mathbb{N},\ B^* \left(\left(\frac{d}{dt} \right)^k z \right) (T) == 0 \\
&\Leftrightarrow \forall k \in \mathbb{N},\ B^* (A^*)^k z_T = 0 \\
\text{(Cayley Hamilton)} &\Leftrightarrow \forall k \in \{0, \cdots, N-1\},\ B^* (A^*)^k z_T = 0 \\
&\Leftrightarrow z_T \in \operatorname{Ker} \begin{pmatrix} B^* \\ B^* A^* \\ \vdots \\ B^* (A*)^{N-1} \end{pmatrix}.
\end{aligned}
$$

The Kalman rank condition (5.1) is independent of time, hence the exact controllability property as well.

Outline

We introduce the following linear data assimilation problem (recovering of the state):

Given a measurement B^*Z for Z solution of $Z' = -A^*Z$, find the data $Z(0)$.

• Stability is equivalent to exact observability property at time $T > 0$.

A natural idea is to minimize among $z_0 \in \mathbb{R}^N$

$$J(z_T) = \frac{1}{2} \int_0^T \|B^* z - B^* Z\|^2_{\mathbb{R}^p} \ dt.$$

This is equivalent to minimize

$$J(z_T) = \frac{1}{2} \int_0^T \|B^* z\|^2_{\mathbb{R}^p} \ dt + \langle x_0, z(0) \rangle_X$$

where $x_0 = x(0)$ is given by the solution of

$$x' = Ax + BB^* Z, \quad x(T) = 0.$$

Hence Z is the HUM null-control associated to x_0.
$\rightsquigarrow$ this is the same problem as finding the HUM control.