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Integer-valued polynomial in valued fields with an application to discrete dynamical systems

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Abstract. Integer-valued polynomials on subsets of discrete valuation domains are well studied. We undertake here a systematical study of integer-valued polynomials on subsets S of valued fields and of several connected notions: the polynomial closure of S, the Bhargava's factorial ideals of S and the v-orderings of S. A sequence of numbers is naturally associated to the subset S and a good description can be done in the case where S is regular (a generalization of the regular compact subsets of Y. Amice in local fields). Such a case arises naturally when we consider orbits under the action of an isometry.

Keywords. Integer-valued polynomial, generalized factorials, valued field, valuative capacity, discrete dynamical system.

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1 Introduction

Integer-valued polynomials in a number field K is a natural notion introduced by Pólya [29] and Ostrowski [30]: they are polynomials f with coefficients in K such that $f(\mathcal{O}_K) \subseteq \mathcal{O}_K$ (where $\mathcal{O}_K$ denotes the ring of integers of K). The notion has been generalized, first in [7] by considering any integral domain D and the D-algebra

$$\text{Int}(D) = \{f \in K[X] \mid f(D) \subseteq D\} \tag{1.1}$$

(where K denotes the quotient field of D), and then, in [23] by considering any subset S of the integral domain D and the corresponding D-algebra

$$\text{Int}(S, D) = \{f \in K[X] \mid f(S) \subseteq D\}, \tag{1.2}$$

that is called the ring of *integer-valued polynomials on S with respect to D*.

When D is Noetherian, the notion behaves well by localization: for every maximal ideal $\mathfrak{m}$ of D, one has the equality [8, I.2.7]:

$$\text{Int}(S, D)_{\mathfrak{m}} = \text{Int}(S, D_{\mathfrak{m}}). \tag{1.3}$$

So that, when $D = \mathcal{O}_K$ is the ring of integers of a number field K, the study may be restricted to the case where S is a subset of the ring V of a discrete valuation v (with finite residue field). We then may use the notion of v-ordering introduced by Bhargava ([2] and [3]) that allows an algorithmic construction of bases of the V-module Int(S,V). Finally, the case of subsets of discrete valuation domains is well studied, although it remains some difficult questions. For instance, what happens when we replace the indeterminate X by several indeterminates? (see Mulay [28] and Evrard [18]).

When the field K is no more a number field, but an infinite algebraic extension of $\mathbb{Q}$, the ring of integers $\mathcal{O}_K$ is no more a Dedekind domain, but a Prüfer domain. Then, we are not sure that things still work well under localization. We can find (see [11] or [8, VI.4.13]) characterizations of the equality

$$(\text{Int}(\mathcal{O}_K))_{\mathfrak{m}} = \text{Int}((\mathcal{O}_K)_{\mathfrak{m}}). \tag{1.4}$$

Without any condition on K, we only have the containment:

$$(\text{Int}(\mathcal{O}_K))_{\mathfrak{m}} \subseteq \text{Int}((\mathcal{O}_K)_{\mathfrak{m}}), \tag{1.5}$$

and more generally, for every subset S of K, we have [8, I.2.4]:

$$(\text{Int}(S,\mathcal{O}_K))_{\mathfrak{m}} \subseteq \text{Int}(S,(\mathcal{O}_K)_{\mathfrak{m}}). \tag{1.6}$$

In any case, it may be worth of interest to study the ring Int(S,V) formed by the integer-valued polynomials on a subset S of a (not necessarily discrete) rank-one valuation domain V.

In fact, it is known that many of the results concerning discrete valuation domains may be extended to rank-one valuation domains provided that the completion $\widehat{S}$ of S is assumed to be compact (see [9]) because, in that case we still have a p-adic Stone-Weierstrass theorem [10]. But, here, we wish to remove all restrictions on the subset S (while keeping the assumption that the valuation is rank one).

Classical definitions

Before beginning this study, let us first recall two general notions linked to integer-valued polynomials: the polynomial closure of a subset and the factorial ideals associated to a subset. For every integral domain D with quotient field K and every subset S of D, we may associate to the subset S, and to the corresponding D-algebra

$$\text{Int}(S,D) = \{f \in K[X] \mid f(S) \subseteq D\} \tag{1.7}$$

of integer-valued polynomials on S with respect to D, the following notions:

Definition 1.1. (McQuillan [27])

(i) A subset T of K is said to be *polynomially equivalent* to S if

$$\text{Int}(T,D) = \text{Int}(S,D). \tag{1.8}$$

(ii) The *polynomial closure* $\overline{S}$ of S is the largest subset of K which is polynomially equivalent to S, equivalently,

$$\overline{S} = \{t \in K \mid f(t) \in D \quad \forall f \in \text{Int}(S,D)\}. \tag{1.9}$$

(iii) The subset S is said to be *polynomially closed* if

$$\overline{S} = S \tag{1.10}$$

So that, to study the ring $\mathrm{Int}(S, D)$ we may replace S by its polynomial closure $\overline{S}$ or, when we do not know it, by any subset T which is polynomially equivalent to S. We now generalize to every subset S of an integral domain D the notion of factorial ideal introduced by Bhargava [4] for subsets of Dedekind domains.

Definition 1.2. ([4] and [9]) For every $n \in \mathbb{N}$, the *n-th factorial ideal* of the subset S with respect to the domain D is the inverse $n!_S^D$ (or simply $n!_S$) of the D-module generated by the leading coefficients of the polynomials of $\mathrm{Int}(S, D)$ of degree n, where the inverse of a sub-D-module N of K is $N^{-1} = \{x \in K \mid xN \subseteq D\}$.

Note that, in particular, $K^{-1} = (0)$ and $(0)^{-1} = K$. Bhargava [4] showed that these factorial ideals may have fine properties extending those of the classical factorials. For instance, when D is a Dedekind domain,

$$\forall n, m \in \mathbb{N} \quad n!_S^D \times m!_S^D \ \text{ divides } \ (n+m)!_S^D. \tag{1.11}$$

We fix now the hypotheses and notation for the whole paper.

2 Hypotheses, notation and v-orderings

2.1 Let K be a valued field,

that is, a field endowed with a rank-one valuation v. Then, the values group $\Gamma = v(K^*)$ is a subgroup of the additive group $\mathbb{R}$. We denote by V the corresponding valuation domain, by $\mathfrak{m}$ the maximal ideal and by k the residue field $V/\mathfrak{m}$.

As usual, we define an absolute value on K by letting:

$$\forall\, x \in K^* \quad |x| = e^{-v(x)}. \tag{2.1}$$

For $x \in K$ and $\gamma \in \mathbb{R}$, we denote by $B(x, \gamma)$ the ball of center x and radius $e^{-\gamma}$, that is:

$$B(x, \gamma) = \{y \in K \mid v(x - y) \geq \gamma\}. \tag{2.2}$$

Remark 2.1. With respect to the polynomial closure, we may notice the following:

(i) Since every polynomial $f \in K[X]$ is a continuous function on K, the polynomial closure $\overline{S}$ of any subset S of K obviously contains the topological closure $\widetilde{S}$ of S in K:

$$\widetilde{S} \subseteq \overline{S}. \tag{2.3}$$

(ii) There are subsets S such that $\widetilde{S} \neq \overline{S}$ (Remarks 4.6, 6.4.ii and 11.4.2).

(iii) In general, polynomially closed subsets are stable under intersection [8, IV.1.5], but not under finite union [8, IV.4.Exercise 2]. Nevertheless,we will see that, in a valued field K, a ball is polynomially closed and a finite union of balls is still polynomially closed (Proposition 8.2). (All the balls that we will consider are closed balls of the form $B(x, \gamma)$ unless the contrary is explicitely stated.)

2.2 Now we fix a subset S of K.

Since we are in a valued field K, the factorial ideals $n!_S$ of S are characterized by their valuations. Thus, we introduce the following arithmetical function:

Definition 2.2. [9] The *characteristic function* of S is the function w_S defined by:

$$\forall n \in \mathbb{N} \quad w_S(n) = v(n!_S) \tag{2.4}$$

with the convention that $v((0)) = +\infty$ and $v(K) = -\infty$.

Obviously, $0!_S = V$, and then, $w_S(0) = 0$. In the case of valued fields, factorial ideals have still fine properties. For instance, (1.11) becomes:

$$\forall n, m \in \mathbb{N} \quad w_S(n+m) \geq w_S(n) + w_S(m). \tag{2.5}$$

The following proposition is an obvious consequence of the previous inclusion (2.3):

Proposition 2.3. *Denoting by $\widetilde{S}$ the topological closure of S in K, we have:*

$$\text{Int}(\widetilde{S}, V) = \text{Int}(S, V) \quad \textit{and hence, for every } n \geq 0,\ n!_{\widetilde{S}} = n!_S.$$

Denoting by $\widehat{V}$ and $\widehat{S}$ the completions of V and S with respect to the topology induced by v, we have:

$$\text{Int}(\widehat{S}, \widehat{V}) = \text{Int}(S, \widehat{V}) = \text{Int}(S, V)\widehat{V} \quad \textit{and, for every } n \geq 0,\ n!_{\widehat{S}} = n!_S\widehat{V}.$$

Equivalently,

$$\forall n \in \mathbb{N} \quad w_S(n) = w_{\tilde{S}}(n) = w_{\widehat{S}}(n). \tag{2.6}$$

We will see that most often this characteristic function w_S may be computed by means of the following notion of generalized v-ordering which extends the notion of v-ordering due to Bhargava [2].

2.3 v-orderings

Definition 2.4. [9] Let $N \in \mathbb{N} \cup \{+\infty\}$. A sequence $\{a_n\}_{n=0}^N$ of elements of S is called a *v-ordering* of S if, for $1 \leq n \leq N$, one has:

$$v\left(\prod_{k=0}^{n-1}(a_n - a_k)\right) = \inf_{x \in S} v\left(\prod_{k=0}^{n-1}(x - a_k)\right). \tag{2.7}$$

The proof of [20, Proposition 4] may be extended to these generalized v-orderings, so that we have another characterization of the v-orderings of S:

Proposition 2.5. *The sequence $\{a_n\}_{n=0}^N$ of elements of S is a v-*ordering *of S if and only if:*

$$\forall n \geq 1\ \forall x_0, \ldots, x_n \in S \quad \prod_{0 \leq i < j \leq n}(x_j - x_i) \ \textit{ is divisible by } \prod_{0 \leq i < j \leq n}(a_j - a_i). \tag{2.8}$$

Although these notions of integer-valued polynomial, factorial ideal, polynomial closure and v-ordering have already been partially studied, we wish to undertake here a systematical study of them. The following proposition shows strong links between them.

Proposition 2.6 ([2] and [9]). *Let $N \in \mathbb{N} \cup \{+\infty\}$ and let $\{a_n\}_{n=0}^N$ be a sequence of distinct elements of S. We associate to this sequence of elements a sequence $\{f_n\}_{n=0}^N$ of polynomials:*

$$f_0(X) = 1 \quad \text{and, for } 1 \leq n \leq N,\ f_n(X) = \prod_{k=0}^{n-1} \frac{X - a_k}{a_n - a_k}. \tag{2.9}$$

The following assertions are equivalent:

(i) The sequence $\{a_n\}_{n=0}^N$ is a v-ordering of S.

(ii) The sequence $\{a_n\}_{n=0}^N$ is a v-ordering of $\overline{S}$.

(iii) For each $n \leq N$, f_n belongs to Int(S, V).

(iv) The sequence $\{f_n\}_{n=0}^N$ is a basis of the V-module

$$\text{Int}_N(S, V) = \{f \in \text{Int}(S, V) \mid \deg(f) \leq N\}. \tag{2.10}$$

(v) For $1 \leq n \leq N$, one has:

$$n!_S = \prod_{k=0}^{n-1}(a_n - a_k)V, \text{ that is, } w_S(n) = v\left(\prod_{k=0}^{n-1}(a_n - a_k)\right). \tag{2.11}$$

This proposition shows in particular that, for $0 \leq n \leq N$, the real numbers $v\left(\prod_{k=0}^{n-1}(a_n - a_k)\right)$ do not depend on the v-ordering $\{a_n\}_{n=0}^N$. But, as shown in the following remark, there does not always exist v-orderings for a given subset S although there always exist integer-valued polynomials on S and factorials associated to S.

Remark 2.7. (i) There does not always exist v-orderings when the valuation is not discrete. For instance, assume that the valuation v is not discrete and the subset S is equal to the maximal ideal $\mathfrak{m}$ of V. Then, S does not admit any v-ordering since, for every $s, t \in \mathfrak{m}$, $v(s - t) > 0$ while $\inf_{t \in \mathfrak{m}} v(t - s) = 0$.

(ii) As a v-ordering of S is also a v-ordering of the polynomial closure $\overline{S}$ of S, the relevant question is more likely the existence of a v-ordering in $\overline{S}$. For instance, in the previous example, with v non discrete and $S = \mathfrak{m}$ (the maximal ideal of V), there is no v-ordering in $\mathfrak{m}$. Yet $\overline{S} = V$, since Int$(\mathfrak{m}, V)$ = Int$(V, V) = V[X]$ (see [9, Remark 8] or Theorem 4.3 below). This larger subset $\overline{S}$ may admit v-orderings. Indeed, a sequence $\{a_n\}$ of elements of the non-discrete rank-one valuation domain V is a v-ordering of V if and only if the a_n's are in distinct classes modulo $\mathfrak{m}$, and such a sequence exists if and only if the residue field $k = V/\mathfrak{m}$ is infinite.

Corollary 2.8. *If the sequence $\{a_n\}_{n \in \mathbb{N}}$ is an infinite v-ordering of S, then the subset $T = \{a_n \mid n \in \mathbb{N}\}$ is polynomially equivalent to S.*

2.4 Notation

For $\gamma \in \mathbb{R}$ and $a, b \in K$, we say that:

a and b are *equivalent modulo* γ if $v(a - b) \geq \gamma$.

Then, for our fixed subset S and for every $\gamma \in \mathbb{R}$, we denote by:
– $S(a, \gamma)$ the equivalence class modulo γ of the element $a \in S$, that is,

$$S(a, \gamma) = S \cap B(a, \gamma), \tag{2.12}$$

– $S \ mod \ \gamma$ the set formed by the equivalence classes modulo γ of the elements of S,
– S_γ any set of representatives of $S \ mod \ \gamma$,
– q_γ the cardinality (finite or infinite) of $S \ mod \ \gamma$:

$$q_\gamma = q_\gamma(S) = Card(S \ mod \ \gamma) = Card(S_\gamma). \tag{2.13}$$

Since q_γ may be finite or infinite, it is natural to introduce the following number:

$$\gamma_\infty = \gamma_\infty(S) = \sup\{\gamma \mid q_\gamma \text{ finite}\}. \tag{2.14}$$

Noticing that in the definition of a v-ordering what is important is not the valuation of the elements of S but the valuation of the differences of elements of S, we are led to consider another natural number:

$$\gamma_0 = \gamma_0(S) = \inf\{v(x - y) \mid x, y \in S, x \neq y\}. \tag{2.15}$$

Clearly,

$$-\infty \leq \gamma_0(S) \leq \gamma_\infty(S) \leq +\infty. \tag{2.16}$$

We will see that the fact that one of these three inequalities becomes an equality corresponds to three particular cases:

$\gamma_0 = -\infty$ if and only if S is a non-fractional subset (see Section 3).

$\gamma_0 = \gamma_\infty$ if and only if $\text{Int}(S, V)$ is isomorphic to a generalized polynomial ring (see Section 4).

$\gamma_\infty = +\infty$ if and only if the completion $\widehat{S}$ of S is compact, which corresponds to the well studied case (see Section 6).

2.5 The results

After the study of these three particular cases, it remains to study the case where γ_∞ is finite, and then, we have to distinguish two cases: q_{γ_∞} is finite and q_{γ_∞} is infinite. For instance, in Section 5, when S is a fractional subset, we associate to S a natural sequence of critical valuations $\{\gamma_k\}_{k\in\mathbb{N}}$, this sequence is finite in the first case and infinite in the second case. In Section 7, we prove an inequality concerning the characteristic function w_S:

$$\forall n \in \mathbb{N} \qquad \frac{w_S(n)}{n} \leq \gamma_\infty.$$

In Sections 8 and 9, we establish some containments concerning the polynomial closure: in Section 8, when q_{γ_∞} is finite, we show that $\overline{S} \subseteq S + B(0, \gamma_\infty)$ and, in Section 9, we show that the polynomial closure contains not only the topological closure but also the pseudo-closure, which is a subset that we naturally associate to the pseudo-convergent sequences introduced by Ostrowski [30]. Then, in Section 10, we characterize the case where $S + B(0, \gamma_\infty) \subseteq \overline{S}$: we prove that this containment holds in particular when S is a regular subset, which is a generalization of the notion of regular compact subset introduced in 1964 by Y. Amice [1]. As an application, we show in Section 11 that, when S is any orbit under the action of an isometry, then S is a regular subset and this regular subset is either discrete or precompact. Finally, we end this paper by giving in Section 12 explicit examples. In a forthcoming paper [14], we will study such regular subsets and show that in this case the v-orderings have very strong properties.

3 The non-fractional case ($\gamma_0 = -\infty$)

The following lemma is obvious.

Lemma 3.1. *Let $a \in K^*$ and $b \in K$. Consider $T = aS + b = \{as + b \mid s \in S\}$. Then, the automorphism:*

$$f(X) \in K[X] \mapsto f\left(\frac{X - b}{a}\right) \in K[X]$$

induces an isomorphism between the rings $\mathrm{Int}(S, V)$ *and* $\mathrm{Int}(T, V)$. *Obviously,*

$$n!_T = a^n n!_S, \quad \textit{equivalently}, \quad w_T(n) = nv(a) + w_S(n). \tag{3.1}$$

Moreover, for every $N \in \mathbb{N} \cup \{+\infty\}$, the sequence $\{a_n\}_{n=0}^N$ is a v-ordering of S if and only if the sequence $\{aa_n + b\}_{n=0}^N$ is a v-ordering of T.

Consequently, for our study, we may replace the set S by $aS + b$ for any $a \in K^*$ and $b \in K$. We are then led to consider whether S is a fractional subset of K or not. Recall that:

Definition 3.2. The subset S of K is said to be a *fractional subset* of K if there exists some $d \in K^*$ such that $dS \subseteq V$.

Theorem 3.3. *The following assertions are equivalent:*

(i) *S is not a fractional subset,*

(ii) $\gamma_0 = -\infty$,

(iii) $\gamma_\infty = -\infty$.

(iv) $\mathrm{Int}(S, V) = V$,

(v) $n!_S = K$ *for* $n \geq 1$,

(vi) $w_S(n) = -\infty$ *for* $n \geq 1$.

(vii) The polynomial closure $\overline{S}$ *of* S *is equal to* K.

Proof. Obviously, on the one hand, assertions (i), (ii), and (iii) are equivalent and, on the other hand, assertions (iv), (v) and (vi) are equivalent. The equivalence between (i) and (iv) is known ([27] or [8, Corollary I.1.10]). The equivalence between (iv) and (vii) is obvious. □

Remark 3.4. Clearly, a non-fractional subset does not admit any v-ordering.

From now on, we will assume that S is a fractional subset. Then, Lemma 3.1 allows us to replace S by the subset $dS = \{ds \mid s \in S\}$ where d denotes any element of K such that $v(d) \geq -\gamma_0$, so that, we may assume that $S \subseteq V$. Moreover, replacing S by the subset $S - s_0 = \{s - s_0 \mid s \in S\}$ for some $s_0 \in S$, we may also assume that $0 \in S$.

4 The polynomial ring case ($-\infty < \gamma_0 = \gamma_\infty < +\infty$)

Notation. For $\gamma \in \mathbb{R}$ and $x \in K$, let

$$V\left[(X-x)/\gamma\right] = \left\{ \sum_{k=0}^{n} a_k (X-x)^k \in K[X] \mid v(a_k) \geq -k\gamma \right\}. \tag{4.1}$$

Obviously, if there exists $t \in K$ such that $v(t) = \gamma$, then the ring $V[(X-x)/\gamma]$ is a classical polynomial ring:

$$V[(X-x)/\gamma] = V\left[\frac{X-x}{t}\right]. \tag{4.2}$$

Lemma 4.1. *With the previous notation, for every* $a \in S$ *and* $\gamma \in \mathbb{R}$*, one has:*

$$V[(X-a)/\gamma] \subseteq \mathrm{Int}(B(a,\gamma), V) \subseteq \mathrm{Int}(S(a,\gamma), V). \tag{4.3}$$

Proof. Let $f \in V[(X-a)/\gamma]$ and write

$$f(X) = \sum c_n (X-a)^n \quad \text{where} \quad v(c_n) \geq -n\gamma.$$

For every $b \in B(a,\gamma)$, one has:

$$f(b) = \sum c_n (b-a)^n \quad \text{with} \quad v(c_n(b-a)^n) \geq 0,$$

thus $f(b) \in V$. □

To characterize the ring $\mathrm{Int}(S,V)$ we need a technical lemma that will be useful for other cases. Here, we denote by $\widehat{\Gamma}$ the completion of Γ in $\mathbb{R}$, that is, $\widehat{\Gamma} = \Gamma$ if v is discrete, and $\widehat{\Gamma} = \mathbb{R}$ if v is not discrete.

Lemma 4.2. *Assume that* $\gamma \in \widehat{\Gamma}$, $\delta \in \mathbb{R}$ *and* $a \in S$ *are such that* $\gamma < \delta$ *and* $S(a,\gamma)$ *mod* δ *is infinite. Then, for every* $f \in \mathrm{Int}(S,V)$*, one has:*

$$f \in V[(X-a)/\rho] \quad \textit{where} \quad \rho = \gamma + \frac{n(n+1)}{2}(\delta-\gamma) \textit{ and } n = \deg(f). \tag{4.4}$$

Proof. We do not know whether $\gamma \in \Gamma$ but, for each integer s, there exists $z_s \in K$ with $\gamma - \frac{1}{s} \le v(z_s) \le \gamma$. Fix an integer s and let T be the following subset of V:

$$T = \left\{ \frac{x-a}{z_s} \mid x \in S(a,\gamma) \right\}.$$

Let $g \in \mathrm{Int}(T,V)$ be of degree n. As $S(a,\gamma)$ *mod* δ is infinite, letting $\varepsilon_s = \delta - \gamma + \frac{1}{s}$, one can find $t_0, t_1, \ldots, t_n$ in T such that, for $i \ne j$, $v(t_i - t_j) < \varepsilon_s$. It then follows from Cramer's rule (see for instance [8, Proposition I.3.1]) that the valuation of each coefficient of g is greater or equal to

$$-v\left(\prod_{0\le i<j\le n}(t_j - t_i)\right) > -\frac{n(n+1)}{2}\varepsilon_s.$$

Consequently, if $f(X) = \sum_{m=0}^{n} b_m (X-a)^m$ belongs to $\mathrm{Int}(S,V)$, and hence, to $\mathrm{Int}(S(a,\gamma),V)$, one has:

$$\forall m \ge 1 \quad v(b_m) > -m\gamma - \frac{n(n+1)}{2}\varepsilon_s.$$

Since s may tend to $+\infty$, we obtain the inequality:

$$\forall m \ge 1 \quad v(b_m) \ge -m\gamma - \frac{n(n+1)}{2}(\delta-\gamma) \ge -m\left(\gamma + \frac{n(n+1)}{2}(\delta-\gamma)\right).$$

□

Theorem 4.3. *If* $\gamma_0 = \gamma_\infty > -\infty$*, then*

$$\mathrm{Int}(S,V) = V[X/\gamma_0] \quad \text{and} \quad \overline{S} = B(0,\gamma_0). \tag{4.5}$$

In particular,

$$w_S(n) = n\gamma_0. \tag{4.6}$$

Proof. We may apply the previous lemma. Clearly, $\gamma_0 \in \widehat{\Gamma}$ and $S = S(0,\gamma_0)$. By definition of γ_∞, for every $\delta > \gamma_\infty = \gamma_0$, q_δ is infinite, that is, $S(0,\gamma_0)$ *mod* δ is infinite. So that, if $f \in \mathrm{Int}(S,V)$, then $f \in V[X/\rho]$ where ρ may tend to γ_0 when δ tends to γ_0. Finally, $f \in V[X/\gamma_0]$ and $\mathrm{Int}(S,V) = V[X/\gamma_0]$.

Then, it follows from Lemma 4.1 that $B(0,\gamma_0) \subseteq \overline{S}$. We may conclude because $S = S(0,\gamma_0) \subseteq B(0,\gamma_0)$ and $B(0,\gamma_0)$ is polynomially closed (see the following lemma). □

Lemma 4.4. *For every $x \in K$ and every $\gamma \in \mathbb{R}$, the ball $B(x,\gamma)$ is polynomially closed.*

Proof. If γ belongs to Γ, there exists $t \in K$ such that $v(t) = \gamma$, then the polynomial $f(X) = \frac{1}{t}(X - x)$ belongs to $\text{Int}(B(x,\gamma), V)$ and, for every $y \in K$, $f(y) \in V$ implies $v(y - x) \geq \gamma$, that is, $y \in B(x,\gamma)$.

If the valuation v is discrete, there exists $\delta \in \Gamma$ such that $B(x,\gamma)$ is equal to $B(x,\delta)$, which is polynomially closed. If the valuation v is not discrete, there exists an increasing sequence $\{\delta_n\}_{n\in\mathbb{N}}$ such that $\delta_n \in \Gamma$ for every n and $\lim_n \delta_n = \gamma$. Consequently, $B(x,\gamma) = \cap_{n\in\mathbb{N}} B(x,\delta_n)$. By the first argument, the balls $B(x,\delta_n)$ are polynomially closed and we know that an intersection of polynomially closed subsets is a polynomially closed subset (see [8, IV.1.5]). □

Theorem 4.3 says that, if $\gamma_0 = \gamma_\infty > -\infty$, then $\overline{S}$ is a ball. Conversely:

Proposition 4.5. *Assume that $S = B(x,\gamma)$. Then,*

(i) $\overline{S} = S$.

(ii) $\gamma_0 = \min\{\delta \in \Gamma \mid \delta \geq \gamma\}$ *(in fact, $\gamma_0 = \gamma$ if v is not discrete).*

(iii) If v is discrete and $V/\mathfrak{m}$ is finite with cardinality q, then

$$\gamma_\infty = +\infty \ \text{ and } \ w_S(n) = n\gamma_0 + w_V(n) = n\gamma_0 + \sum_{k\geq 1}\left[\frac{n}{q^k}\right].$$

(iv) If either v is not discrete or $V/\mathfrak{m}$ is infinite, then

$$\gamma_\infty = \gamma_0 \quad , \quad \text{Int}(S,V) = V[(X-x)/\gamma_0] \quad \text{and} \quad w_S(n) = n\gamma_0.$$

(v) S admits infinite v-orderings if and only if:
either v is discrete,
or $\gamma \in \Gamma$ and $V/\mathfrak{m}$ is infinite.

Proof. Using Lemma 3.1, when $\gamma \in \Gamma$, one can replace $S = B(x,\gamma)$ by $V = B(0,0)$.

(i) is Lemma 4.4.

(ii) is obvious.

(iii) follows from [8, Theorem II.2.7].

(iv): the hypothesis implies $\gamma_\infty = \gamma_0$, and then, (iv) follows from Theorem 4.3.

(v) The existence of v-orderings is obvious in both cases. Conversely, assume that v is not discrete. If $\gamma \notin \Gamma$ then, for all $x, y \in S$, one has $v(x - y) > \gamma$ while $\inf_{x,y\in S} = \gamma$ so that, there is no element in S for the second term of a v-ordering. If $\gamma \in \Gamma$, we replace S by V. Then, it follows from (iv) that $w_V(n) = 0$. But, if $Card(V/\mathfrak{m}) = q$ then, for all $x_0, \ldots, x_q \in V$, $v(\prod_{0\leq i<j\leq q}(x_j - x_i)) > 0$, so that, there is no element in V for the $q+1$-th term of a v-ordering. □

Remark 4.6. The polynomial closure of an open ball is the corresponding closed ball. Suppose that $S = \{y \in K \mid v(y-x) > \gamma\}$ where $\gamma \in \Gamma$. Then S is topologically closed but not polynomially closed:

$$\widetilde{S} = S = \{y \in K \mid v(y-x) > \gamma\} \neq \overline{S} = B(x,\gamma) = \{y \in K \mid v(y-x) \geq \gamma\}.$$

5 The critical valuations of a fractional subset ($\gamma_0 < \gamma_\infty$)

Now, we may assume that $\gamma_0 < \gamma_\infty$, which is equivalent to say that S is a fractional subset and $\overline{S}$ is not a ball. Moreover, there exist s_0 and $s_1 \in S$ such that $v(s_0 - s_1) = \gamma_0$ because $S \ mod \ \gamma_0$ is finite. Then, if we replace S by $T = \left\{ \frac{s-s_1}{s_0-s_1} \mid s \in S \right\}$, 0 and 1 belong to T and $\gamma_0(T) = 0$. So that:

From now on, we assume that

$$S \subseteq V \ , \ 0, 1 \in S \ , \ \gamma_0 = 0 \ , \ q_0 = 1 \ \textbf{ and } \ 0 < \gamma_\infty \leq +\infty. \tag{5.1}$$

Moreover, we may choose $S_0 = \{0\}$.

We are interested in the study of the function

$$\gamma \in \mathbb{R} \mapsto q_\gamma \in \mathbb{N} \cup \{+\infty\}. \tag{5.2}$$

This is an increasing function and, by definition of γ_∞, $q_\gamma = +\infty$ for $\gamma > \gamma_\infty$. Moreover, for every $\gamma < \gamma_\infty$, q_γ is finite, thus $\sup\{v(a-b) \mid a, b \in S_\gamma, a \neq b\}$ is a maximum, and hence, is $< \gamma$. Consequently, there exists $\varepsilon > 0$ such that $q_\delta = q_\gamma$ for $\gamma - \varepsilon \leq \delta \leq \gamma$. The function $\gamma \mapsto q_\gamma$ is piecewise constant and left continuous. So that, we have the following proposition:

Proposition 5.1. *For every γ such that q_γ is finite, let*

$$\tilde{\gamma} = \sup \ \{\delta \mid q_\delta = q_\gamma\}. \tag{5.3}$$

These supremum are maximum and the $\tilde{\gamma}$'s may be written as elements of a strictly increasing sequence:

$$\{\gamma_k\}_{0 \leq k \leq l} \ \ \textit{or} \ \ \{\gamma_k\}_{k \geq 0}.$$

The sequence $\{\gamma_k\}$ is finite if and only if q_{γ_∞} is finite, and then $\gamma_l = \gamma_\infty$. The sequence $\{\gamma_k\}$ is infinite if and only if q_{γ_∞} is infinite, and then:

$$\lim_{k \to +\infty} \gamma_k = \gamma_\infty. \tag{5.4}$$

In other words, the γ_k's are characterized by:

$$\gamma_0 = 0 \textit{ and, for } k \geq 1 \ : \gamma_{k-1} < \gamma \leq \gamma_k \ \Leftrightarrow \ q_\gamma = q_{\gamma_k}. \tag{5.5}$$

Note that, when q_{γ_∞} is infinite and γ_∞ is finite, then necessarily the valuation v is not discrete.

Definition 5.2. The sequence $\{\gamma_k\}$, finite or infinite introduced in Proposition 5.1, is called the *sequence of critical valuations* of S.

Remark 5.3. It follows from Proposition 5.1 that it is possible to choose the elements of the S_γ's, the sets of representatives of S modulo γ, for $\gamma < \gamma_\infty$, in such a way that:

$$\gamma < \delta \ \Rightarrow \ S_\gamma \subseteq S_\delta. \tag{5.6}$$

Indeed, for each $k \geq 0$, we just have to choose the elements of $S_{\gamma_{k+1}}$ in such a way that $S_{\gamma_k} \subset S_{\gamma_{k+1}}$. We always assume that this condition is satisfied and also that $S_0 = \{0\}$. With such a choice, when $\gamma_\infty = \lim_{k\to\infty} \gamma_k$, we may also assume:

$$\cup_{\gamma<\gamma_\infty} S_\gamma = \cup_{k\geq 0} S_{\gamma_k} \subseteq S_{\gamma_\infty}. \tag{5.7}$$

This last containment may be strict (see Example 5.4.i below).

Example 5.4.
(i) If $K = \mathbb{Q}_p$ and $S = V = \mathbb{Z}_p$,
then $\gamma_k = k$, $q_{\gamma_k} = p^k$, $S_{\gamma_k} = \{a \in \mathbb{N} \mid 0 \leq a < p^k\}$ and $\cup_{k\geq 0} S_{\gamma_k} = \mathbb{N}$ while $\gamma_\infty = +\infty$ and $S_{\gamma_\infty} = \mathbb{Z}_p = S$.
(ii) If $K = \mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$ and if

$$S = \left\{ \sum_{k=0}^{n} \varepsilon_k p^{1-\frac{1}{k+1}} \mid n \in \mathbb{N}, \varepsilon_k \in \{0,1\} \right\},$$

then $\gamma_k = 1 - \frac{1}{k+1}$, $q_{\gamma_k} = 2^k$, $\gamma_\infty = 1$ and $\cup_{k\geq 0} S_{\gamma_k} = S_{\gamma_\infty} = S$.
(iii) Consider the previous example and let

$$T = S \cup \{S + p^2\}.$$

Then,

$$\gamma_k(T) = \gamma_k(S)\ ,\ \gamma_\infty(T) = 1 \ \text{ and } \ \cup_{k\geq 0} T_{\gamma_k} = T_{\gamma_\infty} \neq T.$$

6 The precompact case ($\gamma_\infty = +\infty$)

The fact that $\gamma_\infty = +\infty$ is clearly equivalent to the fact that S is precompact, that is, the completion $\widehat{S}$ of S is compact (see for instance [10, Lemma 3.1]). We have to distinguish two cases whether S is finite or not.

6.1 S finite

This case is well described by McQuillan [26] for subsets of any integral domain (see also [8, Exercises IV.1, V.2, VI.20, VIII.25 and VIII.28] and [17]). For finite subsets of a valued field K, one can say a bit more. Obviously, there are v-orderings since at each step we just have to choose between a finite number of elements and, clearly, the first assertion of the following proposition is true.

Proposition 6.1. *Assume that S is finite with cardinality N.*

(i) There exist infinite v-orderings $\{a_n\}_{n\in\mathbb{N}}$ of S. Moreover, if $\{a_n\}_{n\in\mathbb{N}}$ is a v-ordering of S, the a_n's for $n = 0$ and $n \geq N$ may be arbitrarily choosen, but necessarily, $\{a_n \mid 0 \leq n < N\} = S$.

(ii) $\overline{S} = S = \widetilde{S}$.

(iii) $w_S(n) = +\infty \quad \Leftrightarrow \quad n \geq N.$

(iv) Let

$$f_n(X) = \prod_{k=0}^{n-1} \frac{X - a_k}{a_n - a_k} \text{ for } 0 \leq n < N \text{ and } \varphi(X) = \prod_{n=0}^{N-1}(X - a_n) = \prod_{s \in S}(X - s).$$

Then,

$$\text{Int}(S, V) = \oplus_{n=0}^{N-1} V f_n(X) \oplus K[X]\varphi(X). \tag{6.1}$$

Proof. Assertion (i) is obvious and assertion (ii) is also well known. Assertion (iii) results from assertion (i). Let us prove assertion (iv). Proposition 2.6 implies:

$$\text{Int}_{N-1}(S, V) = \sum_{n=0}^{N-1} V f_n(X). \tag{6.2}$$

Now, let $f \in \text{Int}(S, V)$ and write $f = \varphi g + h$ where $g, h \in K[X]$ and $\deg(h) < N$. Then, $h = f - \varphi g \in \text{Int}_{N-1}(S, V)$. □

Remark 6.2. The number of sequences such that $\{a_n\}_{n=0}^{N-1}$ is a v-ordering of S is at least N, since a_0 may be arbitrarily choosen in S, and at most $N!$. Note that the upper bound $N!$ is reached for instance when the elements of S are non-congruent modulo $\mathfrak{m}$. On the contrary, the lower bound N is never reached for $N > 2$.

6.2 *S* infinite

This is also a well studied case.

Proposition 6.3 ([9]). *Assume that S is infinite and $\widehat{S}$ is compact. Then,*

(i) There always exist infinite v-orderings in S.

(ii) The polynomial closure $\overline{S}$ of S is equal to its topological closure $\widetilde{S}$ in K.

Proof. The first assertion is [9, Lemma 17], and the second assertion is [9, Theorem 10]. □

Remark 6.4. (i) Note that $\gamma_\infty < +\infty$ implies that either the valuation v is not discrete or the residue field $k = V/\mathfrak{m}$ is infinite (else $\widehat{V}$ would be compact and $\widehat{S}$ as well).

(ii) When $\gamma_\infty < +\infty$, we may have $\widetilde{S} \neq \overline{S}$. For instance, let $t \neq 0$ be such that $v(t) > 0$ and let $S = \{t^{-k} \mid k \in \mathbb{N}\}$. Then, S is not a fractional subset, and hence its polynomial closure $\overline{S}$ is equal to K, while its topological closure $\widetilde{S}$ is equal to S. We will see more interesting cases with Remark 11.4.2.

(iii) In the precompact case ($\gamma_\infty = +\infty$), one has the equality $S = S_{\gamma_\infty}$. This last equality, that may be thought as a generalization of the precompact case, means when $\gamma_\infty < +\infty$ that every class modulo γ_∞ contains only one element: $\forall a \in S,\ S(a, \gamma_\infty) = \{a\}$ (S is uniformly discrete), and hence, $\widetilde{S} = S$. In particular, since

$0 \in S$, for every nonzero element $a \in S$, $v(a) < \gamma_\infty$. Note also that, when $\gamma_\infty < +\infty$, $S = S_{\gamma_\infty}$ is equivalent to $S = \cup_k S_{\gamma_k}$ because q_{γ_∞} is infinite, and hence $\gamma_\infty = \lim_k \gamma_k$.

With respect to Proposition 6.3, note that the first assertion still holds: if $S = S_{\gamma_\infty}$, then S admits infinite v-orderings [12, Proposition 2.3]. On the other hand, the second assertion cannot be extended since we may have $S = S_{\gamma_\infty}$ and $\widetilde{S} \neq \overline{S}$ (see Section 12).

(iv) Another argument that leads to say that $S = S_{\gamma_\infty}$ generalizes the precompact case is the notion of pseudo-convergence introduced by Ostrowki [31, p. 368] and used by Kaplansky [25] in the study of immediate extensions of valued fields. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of K is said to be pseudo-convergent if:

$$\forall\, i,\, j,\, k \quad [\, i < j < k \;\Rightarrow\; v(x_j - x_i) < v(x_k - x_j)\,]. \tag{6.3}$$

One may prove that, if $S_{\gamma_\infty} = S$, then from every infinite sequence of elements of S one can extract a pseudo-convergent subsequence (see [13, § 1.12]).

7 On the characteristic function

In this section we assume that $\gamma_\infty < +\infty$.

Theorem 7.1. *For every subset S, one has:*

$$\forall n \in \mathbb{N} \quad w_S(n) \leq n\gamma_\infty. \tag{7.1}$$

Note that, when $\gamma_\infty = +\infty$, the previous theorem does not give any information on the function w_S, and that, when $\gamma_\infty = -\infty$, that is, when S is not a fractional subset, these inequalities are still true: they mean $w_S(0) = 0$ and, for $n \geq 1$, $w_S(n) = -\infty$, that is, Int$(S, V) = V$.

Proof. Assume that $\gamma \in \Gamma$ and $\delta \in \mathbb{R}$ are such that $\gamma < \delta$, q_γ is finite and q_δ is infinite. Then, there necessarily exits an $a \in S$ such that $S(a, \gamma)$ *mod* δ is infinite. It follows from Lemma 4.2 that, for every n, the valuation of the leading coefficient of a polynomial $f \in$ Int(S, V) of degree n is $\geq -n(\gamma - \frac{n(n+1)}{2}(\delta - \gamma))$. Consequently,

$$w_S(n) \leq n\gamma + \frac{n^2(n+1)}{2}(\delta - \gamma)$$

Assume first that the sequence of critical valuations is finite. Then q_{γ_∞} is finite and, for every $\delta > \gamma_\infty$, q_δ is infinite. Then, the previous inequality, with $\gamma = \gamma_\infty$ and $\delta > \gamma_\infty$, becomes:

$$\forall n \in \mathbb{N} \quad w_S(n) \leq n\gamma_\infty + \frac{n^2(n+1)}{2}(\delta - \gamma_\infty).$$

These inequalities for all $\delta > \gamma_\infty$ imply that $w_S(n) \leq n\gamma_\infty$.

Assume now that the sequence of critical valuations is infinite and that γ_∞ is finite. Then q_{γ_∞} is infinite and, for every $k \in \mathbb{N}$, q_{γ_k} is finite. It follows from the previous inequality, with $\gamma = \gamma_k$ and $\delta = \gamma_\infty$, that

$$\forall n \in \mathbb{N} \quad w_S(n) \leq n\gamma_k + \frac{n^2(n+1)}{2}(\gamma_\infty - \gamma_k).$$

Since $\lim_{k\to\infty}\gamma_k = \gamma_\infty$, we may conclude that $w_S(n) \leq n\gamma_\infty$. □

Recall that, by analogy with the Archimedean case (see for instance [22]), one defines the valuative capacity of S in the following way:

Definition 7.2. [12, §4] The *valuative capacity* of S with respect to v is the limit δ_S (finite or infinite) of the increasing sequence

$$\delta_S(n) = \frac{2}{n(n+1)} \inf_{x_0,\ldots,x_n\in S} v\left(\prod_{0\leq i<j\leq n}(x_j - x_i)\right). \tag{7.2}$$

The link between the sequences $\{\delta_S(n)\}_{n\in\mathbb{N}}$ and $\{w_S(n)\}_{n\in\mathbb{N}}$ is given by the following formulas [12, Theorems 3.13 and 4.2]:

$$\sum_{k=1}^{n} w_S(n) = \frac{1}{2}n(n+1)\delta_S(n) \tag{7.3}$$

and

$$\lim_{n\to\infty}\frac{w_S(n)}{n} = \sup_{n\geq 1}\frac{w_S(n)}{n} = \delta_S. \tag{7.4}$$

Consequently, we always have the inequality:

$$\delta_S \leq \gamma_\infty(S). \tag{7.5}$$

8 On the polynomial closure (when $q_{\gamma_\infty} < +\infty$)

Proposition 8.1. *For every $\gamma < \gamma_\infty$, one has the containment:*

$$\overline{S} \subseteq S + B(0,\gamma). \tag{8.1}$$

Morever, if q_{γ_∞} is finite, one has also:

$$\overline{S} \subseteq S + B(0,\gamma_\infty). \tag{8.2}$$

This is an easy consequence of the fact that a finite union of balls is polynomially closed (Proposition 8.2 below) since

$$S = \cup_{a\in S_\gamma} S(a,\gamma) \subseteq \cup_{a\in S_\gamma} B(a,\gamma) = S + B(0,\gamma). \tag{8.3}$$

Proposition 8.2. *Every finite union of balls is polynomially closed.*

This proposition is itself an easy consequence of the following lemma:

Lemma 8.3. *Let $a, t_1, \ldots, t_r$ be elements of K and let $\gamma, \gamma_1, \ldots, \gamma_r$ be positive real numbers such that the balls $B(a,\gamma), B(t_1,\gamma_1), \ldots, B(t_r,\gamma_r)$ are disjoint. Then, for every $\varepsilon > 0$, there exists $f \in K[X]$ such that:*

$$\forall x \in \cup_{k=1}^{r} B(t_k,\gamma_k) \quad v(f(x)) \geq \varepsilon \quad \textit{and} \quad \forall x \in B(a,\gamma) \quad v(f(x)) = 0. \tag{8.4}$$

Proof. We may assume that a and the t_k's are in V and that

$$\gamma > v(t_1 - a) \geq v(t_2 - a) \geq \cdots \geq v(t_r - a). \tag{8.5}$$

Obviously, we have:

$$\forall k \quad v(a - t_k) < \gamma_k.$$

Now consider

$$f(x) = \prod_{i=1}^{r} \left(\frac{x - t_i}{a - t_i}\right)^{m_i}$$

where the m_i's are integers that we are going to choose.

For $1 \leq k \leq r$ and $x \in B(t_k, \gamma_k)$, one has:

$$v\left(\frac{x - t_k}{a - t_k}\right) \geq \gamma_k - v(a - t_k) = \varepsilon_k > 0$$

and, for $j \in \{k, \ldots, r\}$, one has:

$$v\left(\frac{x - t_j}{a - t_j}\right) = v\left(\frac{t_k - t_j}{a - t_j}\right) \geq 0.$$

Thus, for every $x \in B(t_k, \gamma_k)$,

$$v(f(x)) = \sum_{i=1}^{r} m_i v\left(\frac{x - t_i}{a - t_i}\right) \geq \sum_{i=1}^{k} m_i v\left(\frac{x - t_i}{a - t_i}\right) \geq m_k \varepsilon_k - \sum_{i=1}^{k-1} m_i v(a - t_i).$$

We may choose successively the integers $m_1, \ldots, m_k, \ldots, m_r$ such that

$$\forall k \in \{1, \ldots, r\} \quad m_k \varepsilon_k \geq \varepsilon + \sum_{i=1}^{k-1} m_i v(a - t_i).$$

With such a choice of the m_i's, for every $x \in \cup_{k=1}^{r} B(t_k, \gamma)$, one has $v(f(x)) \geq \varepsilon$. Of course, $f(a) = 1$. Moreover, if $x \in B(a, \gamma)$, then $v(x - t_k) = v(a - t_k)$ for every $k \in \{1, \ldots, r\}$, and hence, $v(f(x)) = 0$. □

Proof. of Proposition 8.2 Assume that $S = \sqcup_{k=1}^{r} B(t_k, \gamma_k)$ where $\sqcup$ denotes a disjoint union. Let $a \in K \setminus S$ and $\delta \in \Gamma_+$. Then, by Lemma 8.3, there exists $f \in K[X]$ such that $v(f(a)) = 0$ and, for every $x \in S$, $v(f(x)) \geq \delta$. Let $d \in V$ be such that $v(d) = \delta$. Then, the polynomial $\frac{1}{d} f(X)$ shows that $a \notin \overline{S}$. □

Theorem 8.4. *For every $\gamma < \gamma_\infty$, one has:*

$$\overline{S} = \cup_{a \in S_\gamma} \left(\overline{S \cap B(a, \gamma)}\right). \tag{8.6}$$

In other words:

$$\overline{\cup_{a \in S_\gamma} S(a, \gamma)} = \cup_{a \in S_\gamma} \overline{S(a, \gamma)}. \tag{8.7}$$

Proof. Obviously, $\cup_{a\in S_\gamma}\overline{S(a,\gamma)} \subseteq \overline{S}$. Let us prove the reverse inclusion. Let b be an element of $\overline{S}$. Then, by Proposition 8.2, b belongs to $\cup_{a\in S_\gamma} B(a,\gamma)$ and there exits a_0 such that $b \in B(a_0,\gamma)$. Assume that $b \notin \overline{S(a_0,\gamma)}$, then there exists $g \in K[X]$ such that $g(S(a_0,\gamma)) \subseteq V$ and $v(g(b)) < 0$. Since the values of a polynomial on a fractional subset is a fractional subset, we may consider $-\varepsilon = \min\{v(g(x)) \mid x \in S\}$. It follows from Lemma 8.3 that there exists $f \in K[X]$ such that

$$\forall x \in \cup_{a\in S_\gamma, a\neq a_0} B(a,\gamma) \;\; v(f(x)) \geq \varepsilon \quad \text{and} \quad \forall x \in B(a_0,\gamma) \;\; v(f(x)) = 0.$$

Then, for $x \in \cup_{a\in S_\gamma, a\neq a_0} B(a,\gamma)$, one has $v(f(x)g(x)) \geq 0$ and, for $x \in S(a_0,\gamma)$, one has $v(f(x)g(x)) = v(g(x)) \geq 0$, while $v(f(b)g(b)) = v(g(b)) < 0$. Consequently, $fg \in \text{Int}(S,V)$ and $f(b)g(b) \notin V$, that is $b \notin \overline{S}$. This is a contradiction. Thus, $\overline{S} \cap B(a_0,\gamma) = \overline{S(a_0,\gamma)}$. □

If $q_{\gamma_\infty} < +\infty$, the previous proof still holds with $\gamma = \gamma_\infty$.

Corollary 8.5. *If q_{γ_∞} is finite, then $\overline{S} = \cup_{a\in S_{\gamma_\infty}} \overline{S(a,\gamma_\infty)}$.*

Now, we consider what happens with respect to v-orderings. We first recall:

Lemma 8.6 ([6], Lemma 3.4). *If $\{a_n\}_{n=0}^N$ is a v-ordering of S then, for every ball B, the (possibly empty) subsequence formed by the a_n's that belong to B is a v-ordering of $S \cap B$.*

Proposition 8.7. *Let γ be such that $\gamma < \gamma_\infty$. Then, S admits an infinite v-ordering if and only if, for every $b \in S_\gamma$, $S(b,\gamma)$ admits an infinite v-ordering.*

Proof. Assume that S admits an infinite v-ordering $\{a_n\}_{n\in\mathbb{N}}$. By Lemma 8.6, for every $b \in S_\gamma$, the subsequence formed by the a_n's that are in $B(b,\gamma)$ is a v-ordering of $S(b,\gamma)$. Let $T = \{a_n \mid n \in \mathbb{N}\}$ and, for every $b \in S_\gamma$, consider $T(b,\gamma) = T \cap B(b,\gamma)$. If $T(b,\gamma)$ is infinite, then $S(b,\gamma)$ admits an infinite v-ordering. Thus, assume that, for some $b \in S_\gamma$, $T(b,\gamma)$ is finite. By Corollary 2.8, $\overline{T} = \overline{S}$ and, by Theorem 8.4, $\overline{T(b,\gamma)} = \overline{S(b,\gamma)}$. Since $T(b,\gamma)$ is assumed to be finite, one has $\overline{T(b,\gamma)} = T(b,\gamma)$ (Proposition 6.1.ii). Consequently, $S(b,\gamma)$ is also finite, and hence, admits an infinite v-ordering (Proposition 6.1.i).

Conversely, assume that, for every $b \in S_\gamma$, $S(b,\gamma)$ admits an infinite v-ordering. We prove the existence of an infinite v-ordering of S by induction on n. Assume that $a_0,\ldots,a_{n-1}$ is a v-ordering of S. The question is: does there exist an element $a_n \in S$ which allows to reach the following infimum

$$\inf_{x\in S} v\left(\prod_{k=0}^{n-1}(x-a_k)\right) = \inf_{b\in S_\gamma} \inf_{x\in S(b,\gamma)} v\left(\prod_{k=0}^{n-1}(x-a_k)\right) ?$$

It is then enough to prove that, for every $b \in S_\gamma$, the following infimum is a minimum:

$$\inf_{x\in S(b,\gamma)} v\left(\prod_{k=0}^{n-1}(x-a_k)\right)$$

$$= v\left(\prod_{a_k\notin S(b,\gamma)}(b-a_k)\right)+\inf_{x\in S(b,\gamma)} v\left(\prod_{a_k\in S(b,\gamma)}(x-a_k)\right).$$

This last infimum is a minimum since the a_k's that belong to $S(b,\gamma)$ form a v-ordering of $S(b,\gamma)$ and, by hypothesis, $S(b,\gamma)$ admits infinite v-orderings. □

9 On the polynomial closure (when $\gamma_\infty < +\infty$)

When $\gamma_\infty = +\infty$, that is, when $\widehat{S}$ is compact, one has $\overline{S} = \widetilde{S}$ (Propositions 6.1 and 6.3). So that, we may assume that $\gamma_\infty < +\infty$, and hence, that either v is not discrete or $k = V/\mathfrak{m}$ is infinite.

We will generalize the fact that the topological closure $\widetilde{S}$ of S in K is contained in the polynomial closure $\overline{S}$ of S by considering the notion of pseudo-convergent sequence previously mentioned (see (6.3)).

Definition 9.1. (i) A sequence $\{x_n\}_{n\geq 0}$ of elements of K is *pseudo-convergent* if

$$\forall\, i,\, j,\, k \quad [\, i<j<k \;\Rightarrow\; v(x_j-x_i) < v(x_k-x_j)\,]. \tag{9.1}$$

(ii) An element x of K is *a pseudo-limit* of a sequence $\{x_n\}_{n\geq 0}$ if

$$\forall\, i,\, j \quad [\, i<j \;\Rightarrow\; v(x-x_i) < v(x-x_j)\,]. \tag{9.2}$$

(iii) The *pseudo-closure* of S in K is the union $\widetilde{\widetilde{S}}$ of S and of the set formed by the pseudo-limits in K of pseudo-convergent sequences of elements of S.

Suppose that x is a pseudo-limit of a sequence $\{x_n\}_{n\geq 0}$ and let

$$\delta = \lim_{n\to+\infty} v(x-x_n).$$

If $\delta = +\infty$, then the sequence $\{x_n\}$ is convergent with x as classical limit-point. In fact, clearly, one has: $\widetilde{S} \subseteq \widetilde{\widetilde{S}}$. If $\delta < +\infty$, then the sequence $\{x_n\}$ is pseudo-convergent since, for $i<j<k$, one has:

$$v(x_j-x_i) = v(x-x_i) < v(x-x_j) = v(x_k-x_j).$$

Note also that, in this case, every y such that $v(x-y)\geq\delta$ is also a pseudo-limit of the sequence $\{x_n\}$.

Theorem 9.2. *The polynomial closure $\overline{S}$ of S satisfies the following containments:*

$$\widetilde{\widetilde{S}} \quad\subseteq\quad \overline{S} \quad\subseteq\quad \cap_{k\geq 0}\,(S+B(0,\gamma_k)) \tag{9.3}$$

where $\widetilde{\widetilde{S}}$ denotes the pseudo-closure of S. Moreover, one has the following equalities:

$$\cap_{k\geq 0}\,(S+B(0,\gamma_k)) \quad=\quad (S+B(0,\gamma_\infty))\cup\widetilde{\widetilde{S}} \quad=\quad \widetilde{\widetilde{S}}+B(0,\gamma_\infty) \tag{9.4}$$

Proof. Let $x \in \widetilde{\widetilde{S}}$. Of course, if $x \in \widetilde{S}$, then $x \in \overline{S}$. Assume that $x \in \widetilde{\widetilde{S}} \setminus \widetilde{S}$, and then, that x is a pseudo-limit of a sequence $\{x_n\}_{n\geq 0}$ of elements of S. For every $n \geq 0$, let $\delta_n = v(x - x_n)$ and let $\delta = \lim_{n\to+\infty} \delta_n < +\infty$. Then, $S(x_n, \delta_n)$ contains all the x_m's for $m \geq n$, so that, $S(x_n, \delta_n)\ mod\ \delta$ is infinite.

Consider now a polynomial $f(X) = \sum_{j=0}^{d} c_j X^j \in \mathrm{Int}(S, V)$ of degree d. It follows from Lemma 4.2 that $f \in V[(X - x_n)/\rho_n]$ where $\rho_n = \delta_n + \frac{d(d+1)}{2}(\delta - \delta_n)$, that is, $v(c_j) \geq -j\rho_n$ for every j. Consequently,

$$v(c_j(x - x_n)^j) \geq j(\delta_n - \rho_n) = -j\frac{d(d+1)}{2}(\delta - \delta_n),$$

and

$$v(f(x)) \geq -\frac{d^2(d+1)}{2}(\delta - \delta_n).$$

Since, $\lim \delta_n = \delta$, one may conclude that $v(f(x)) \geq 0$ and $x \in \overline{S}$. This is the first containment. The second containment is a straightforward consequence of Containment (8.1).

Since $S \subseteq \widetilde{\widetilde{S}}$, it is obvious that $(S + B(0, \gamma_\infty)) \cup \widetilde{\widetilde{S}} \subseteq \widetilde{\widetilde{S}} + B(0, \gamma_\infty)$, and it follows from (9.3) that $\widetilde{\widetilde{S}} + B(0, \gamma_\infty) \subseteq \cap_{k\geq 0} (S + B(0, \gamma_k))$. It remains to prove that $\cap_{k\geq 0} (S + B(0, \gamma_k)) \subseteq (S + B(0, \gamma_\infty)) \cup \widetilde{\widetilde{S}}$.

If q_{γ_∞} is finite, this is obvious. So that, we may assume that $\gamma_\infty = \lim_{k\to+\infty} \gamma_k$. Let x be an element of $\cap_k(S + B(0, \gamma_k))$ which is not in $S + B(0, \gamma_\infty)$. Let $k_1 \geq 0$ and $x_1 \in S_{\gamma_{k_1}}$ be such that $v(x - x_1) \geq \gamma_{k_1}$. Then, $v(x - x_1) < \gamma_\infty$ since $x \notin S + B(0, \gamma_\infty)$. There exists $k_2 > k_1$ such that $v(x - x_1) < \gamma_{k_2}$. Let $x_2 \in S_{\gamma_{k_2}}$ be such that $v(x - x_2) \geq \gamma_{k_2}$. We then have:

$$\gamma_{k_1} \leq v(x - x_1) < \gamma_{k_2} \leq v(x - x_2) < \gamma_\infty.$$

So that, we may contruct two sequences $\{k_n\}_{n\geq 1}$ and $\{x_n\}_{n\geq 1}$ such that

$$\gamma_{k_n} \leq v(x - x_n) < \gamma_{k_{n+1}}.$$

Consequently, $\lim_n \gamma_{k_n} = \gamma_\infty$ and x is a pseudo-limit of the sequence $\{x_n\}$. □

Remark 9.3. (i) When $\gamma_\infty = +\infty$, all the following subsets are equal:

$$\widetilde{S} = \widetilde{\widetilde{S}} = \overline{S} = \cap_{k\geq 0}(S + B(0, \gamma_k)).$$

(ii) When q_{γ_∞} is finite, we have the equality:

$$\cap_{k\geq 0} (S + B(0, \gamma_k)) = S + B(0, \gamma_\infty).$$

When q_{γ_∞} is infinite, we just have a containment:

$$S + B(0, \gamma_\infty) \subseteq \cap_{k\geq 0}(S + B(0, \gamma_k)),$$

and this containment may be strict. If $\gamma_\infty = +\infty$, then $S + B(0, \gamma_\infty) = S$ and the equality means that $S = \widetilde{S}$. If $\gamma_\infty < +\infty$, we have the following counterexample. Assume that $v(K^*) = \mathbb{Q}$ and that the characteristic of K is odd. Consider the subset $S = \{0, 1\} \cup \{2 + u_n\}_{n\geq 2}$ where $v(u_n) = 1 - \frac{1}{n}$. Then, $\gamma_0 = 0, \gamma_n = 1 - \frac{1}{n}, \gamma_\infty = 1$. Let $x = 2 + t$ with $v(t) \geq 1$. Then, x belongs to $\cap_k(S + B(0, \gamma_k))$ but does not belong to $S + B(0, \gamma_\infty)$. This fact still holds even if S is a very regular subset (see Remark 12.3.ii).

(iii) When $\gamma_\infty < +\infty$, the containments in (9.3) may be strict. Let $\gamma, \delta \in \Gamma$ and $b \in V$ be such that $\gamma < \gamma_\infty < \delta$ and $B(b, \gamma) \cap S = \emptyset$. Assume that $k = V/\mathfrak{m}$ is infinite and let $\{t_n\}_{n\geq 0}$ be a sequence of elements of $B(b, \delta)$ such that $v(t_n - t_m) = \delta$ for $n \neq m$. Then, let $T = \{t_n \mid n > 0\}$ and $U = S \cup T$. We have: $\gamma_\infty(U) = \gamma_\infty(S)$, $U \subseteq (S + B(0, \gamma)) \cup B(b, \delta)$ and, by Theorem 8.4, $\overline{U} = \sqcup_{a\in U_\gamma} \overline{U(a, \gamma)} = \sqcup_{a\in S_\gamma} \overline{S(a, \gamma)} \sqcup B(b, \delta)$. So that, on the one hand, $U + B(0, \gamma_\infty) \not\subseteq \overline{U}$, and it follows from Proposition 10.1 below that $\overline{U} \neq \cap_k(U + B(0, \gamma_k))$. On the other hand, by Proposition 4.5, $B(b, \delta) \subseteq \overline{U}$, $t_0 \in \overline{U}$ but $t_0 \notin \widetilde{\widetilde{U}}$, and hence, $\overline{U} \neq \widetilde{\widetilde{U}}$.

Let us now look at containments concerning the polynomial rings. Recall Lemma 4.1 that says that:

$$\forall a \in S\ \forall \gamma \in \mathbb{R}\ \ V[(X - a)/\gamma] \subseteq \text{Int}(S(a, \gamma), V). \tag{9.5}$$

Consequently,

$$\cap_{a\in S_\gamma} V[(X - a)/\gamma] \subseteq \text{Int}(S, V). \tag{9.6}$$

This leads us to recall and slightly generalize a notion introduced in [32]:

Definition 9.4. For every $\gamma \in \mathbb{R}$, the *Bhargava ring* with respect to S and γ is the following domain:

$$\text{Int}_\gamma(S, V) = \cap_{a\in S} V[(X - a)/\gamma] = \cap_{a\in S_\gamma} V[(X - a)/\gamma]. \tag{9.7}$$

In the case where $\gamma \in \Gamma$, then $\gamma = v(t)$ for some $t \in K$ and then:

$$\text{Int}_\gamma(S, V) = \{f \in K[X] \mid \forall s \in S\ \ f(tX + s) \in V[X]\}. \tag{9.8}$$

Yeramian [32] defines only $\text{Int}_\gamma(S, V)$ when $S = V$ and $\gamma = v(t)$ and denotes it by $\mathbf{B}_t(V)$. As previously noticed:

$$\forall \gamma \in \mathbb{R} \quad V[X] \quad \subseteq \quad \text{Int}_\gamma(S, V) \quad \subseteq \quad \text{Int}(S, V). \tag{9.9}$$

Obviously,

$$\gamma < \delta \quad \Rightarrow \quad \text{Int}_\gamma(S, V) \subseteq \text{Int}_\delta(S, V), \tag{9.10}$$

and if $\gamma_\infty = \lim_k \gamma_k < +\infty$, then

$$\cup_k \text{Int}_{\gamma_k}(S, V) \subseteq \text{Int}_{\gamma_\infty}(S, V). \tag{9.11}$$

We may also note that:

Proposition 9.5. *If* $\gamma_\infty < +\infty$*, then:*

$$\forall \gamma \in \mathbb{R} \quad \text{Int}_\gamma(S, V) = \text{Int}(S_\gamma + B(0, \gamma), V). \tag{9.12}$$

Proof. Since $\gamma_\infty < +\infty$, either v is not discrete or $k = V/\mathfrak{m}$ is infinite. Then, Proposition 4.5 says that:

$$\forall a \in K\ \forall \gamma \in \mathbb{R} \quad \text{Int}(B(a, \gamma), V) = V[(X - a)/\gamma]. \tag{9.13}$$

□

Thus, the containment $\overline{S} \subseteq \cap_{k \geq 0}\ (S + B(0, \gamma_k))$ corresponds to the containment

$$\cup_{k \geq 0} \text{Int}_{\gamma_k}(S, V) \subseteq \text{Int}(S, V). \tag{9.14}$$

10 When $\overline{S} = \cap_k\ (S + B(0, \gamma_k))$

In this section we still assume that $\gamma_\infty < +\infty$. We know that

$$\overline{S} \subseteq \cap_{k \geq 0}\ (S + B(0, \gamma_k)). \tag{10.1}$$

When do we have an equality?

Proposition 10.1. *The following assertions are equivalent:*

(i) $\overline{S} = \cap_{k \geq 0}\ (S + B(0, \gamma_k))$.

(ii) $S + B(0, \gamma_\infty) \subseteq \overline{S}$.

(iii) $\text{Int}(S, V) = \text{Int}_{\gamma_\infty}(S, V)$.

Proof. (i) → (ii) since $S + B(0, \gamma_\infty) \subseteq \cap_k (S + B(0, \gamma_k))$.

(ii) → (i): if $S + B(0, \gamma_\infty) \subseteq \overline{S}$, it follows from Theorem 9.2 that $\cap_k (S + B(0, \gamma_k)) \subseteq \overline{S}$.

(ii) ↔ (iii): by Proposition 9.5, we have:

$$\text{Int}_{\gamma_\infty}(S, V) = \text{Int}(S + B(0, \gamma_\infty), V)$$

and, clearly, we have:

$$\text{Int}(S + B(0, \gamma_\infty), V) \subseteq \text{Int}(S, V) = \text{Int}(\overline{S}, V).$$

Thus, the containment $S + B(0, \gamma_\infty) \subseteq \overline{S}$ is equivalent the equality $\text{Int}_{\gamma_\infty}(S, V) = \text{Int}(S, V)$. □

Now, we have to distinguish whether q_{γ_∞} is finite or not.

10.1 When q_{γ_∞} is finite

Recall that, when q_{γ_∞} is finite, one has: $\cap_k\ (S+B(0,\gamma_k)) = S+B(0,\gamma_\infty)$. We begin with a lemma:

Lemma 10.2. *Assume that the sequence $\{\gamma_k\}$ of critical valuations of S is finite. If $a \in S$ is such that, for every $\delta > \gamma_\infty$, $S(a,\gamma_\infty)$ mod δ is infinite, then*

$$\text{Int}(S,V) \subseteq V[(X-a)/\gamma_\infty] \quad \textit{and} \quad B(a,\gamma_\infty) \subseteq \overline{S}. \tag{10.2}$$

Proof. We apply Lemma 4.2 with $\gamma = \gamma_\infty$ and, when δ tends to γ_∞, ρ tends to γ_∞. □

This leads us to the following characterization:

Theorem 10.3. *Assume that q_{γ_∞} is finite. Then, the following three assertions are equivalent:*

$$\forall a \in S \ \ \forall \delta > \gamma_\infty \quad S(a,\gamma_\infty) \ mod\ \delta \ is \ \ infinite. \tag{10.3}$$

$$\overline{S} = S + B(0,\gamma_\infty). \tag{10.4}$$

$$\text{Int}(S,V) = \text{Int}_{\gamma_\infty}(S,V). \tag{10.5}$$

Proof. When assertion (10.3) holds, it follows from Lemma 10.2 that $B(a,\gamma_\infty) \subseteq \overline{S}$ for every $a \in S$, and hence, that $S + B(0,\gamma_\infty) \subseteq \overline{S}$. Since $\overline{S} \subseteq \cap_k(S + B(0,\gamma_k)) = S + B(0,\gamma_\infty)$, we have (10.4).

Now assume that (10.3) does not hold. Then there exist $a \in S$ and $\delta > \gamma_\infty$ such that $S(a,\gamma_\infty)$ *mod* δ is finite. Let $b_1,\ldots,b_s \in S$ be such that $S(a,\gamma_\infty) \subseteq \cup_{i=1}^s B(b_i,\delta)$. Then, $\overline{S(a,\gamma_\infty)} \subseteq \cup_{i=1}^s B(b_i,\delta)$. But $\cup_{i=1}^s B(b_i,\delta)$ cannot be equal to $B(a,\gamma_\infty)$ since either v is not discrete, or $V/\mathfrak{m}$ is infinite (because γ_∞ is finite). Consequently, we have $\overline{S} \cap B(a,\gamma_\infty) = \overline{S(a,\gamma_\infty)} \neq B(a,\gamma_\infty)$. Thus, assertion (10.4) does not hold.

The equivalence between (10.4) and (10.5) follows from Proposition 10.1. □

Remark 10.4. (i) In the case described by Theorem 10.3, $\overline{S}$ is a finite union of balls and, as already said, either the valuation v is not discrete, or the residue field $V/\mathfrak{m}$ is infinite. It follows from Propositions 8.7 and 4.5 that the subset $\overline{S}$ admits an infinite v-ordering if and only if $\gamma_\infty \in \Gamma$ and the residue field $V/\mathfrak{m}$ is infinite.

(ii) Recall that if S is a finite union of balls, that is, if S is of the form:

$$S = T + B(0,\gamma) \quad \text{where} \quad Card(T) = r, \tag{10.6}$$

the study of the characteristic function w_S is done in [6] in the case where the valuation v is discrete. In another paper [15], we will show that we have results that are both similar and different when v is not discrete, in particular the valuative capacity of S is given by the following proposition.

Proposition 10.5. *Let S be a finite union of balls:*

$$S = \cup_{i=1}^r B(t_i,\gamma) \quad \textit{where, for } \ i \neq j,\ v(t_i - t_j) < \gamma. \tag{10.7}$$

Assume that either v is not discrete or the residue field of V is infinite. Consider the following matrix:

$$B = (\beta_{i,j}) \in \mathcal{M}_r(\mathbb{R}) \text{ with } \beta_{i,i} = \gamma \text{ and } \beta_{i,j} = v(t_j - t_i) \text{ for } 1 \leq i \neq j \leq r. \quad (10.8)$$

Denote by B_i the matrix deduced from B by replacing every element of the i-th column by 1. Then,

$$\delta_S = \lim_{n\to+\infty} \frac{w_S(n)}{n} = \frac{\det(B)}{\sum_{1\leq i\leq r} \det(B_i)}. \quad (10.9)$$

Let us now consider the case where q_{γ_∞} is infinite.

10.2 When q_{γ_∞} is infinite

The analog of Lemma 10.2 is:

Lemma 10.6. *Assume that the sequence $\{\gamma_k\}$ of critical valuations of S is infinite. If there exists $a \in S$ such that, for every $\gamma < \gamma_\infty$, $S(a,\gamma)$ mod γ_∞ is infinite, then*

$$\mathrm{Int}(S,V) \subseteq V[(X-a)/\gamma_\infty] \quad \text{and} \quad B(a,\gamma_\infty) \subseteq \overline{S}. \quad (10.10)$$

Proof. We apply Lemma 4.2 with $\gamma = \gamma_k$ and $\delta = \gamma_\infty$ and, when k tends to $+\infty$, ρ tends to γ_∞. □

This lemma leads to the following equivalences:

Theorem 10.7. *Assume that γ_∞ is finite and that q_{γ_∞} is infinite. Then, the following four assertions are equivalent:*

$$\overline{S} = \cap_{k\geq 0}\,(S + B(0,\gamma_k)) \quad (10.11)$$

$$S + B(0,\gamma_\infty) \subseteq \overline{S} \quad (10.12)$$

$$\mathrm{Int}(S,V) = \mathrm{Int}_{\gamma_\infty}(S,V) \quad (10.13)$$

$$\forall a \in S\ \forall\gamma < \gamma_\infty \left\{ [\ S(a,\gamma_\infty) = S(a,\gamma)\ ] \ \Rightarrow\ [\ \overline{S(a,\gamma_\infty)} = B(a,\gamma_\infty)\ ] \right\}. \quad (10.14)$$

These equivalent assertions also hold when the following condition is satisfied:

$$\forall a \in S\ \ \forall \gamma < \gamma_\infty \quad S(a,\gamma) \text{ mod } \gamma_\infty \text{ is infinite}. \quad (10.15)$$

Proof. The equivalences (10.11) ↔ (10.12) ↔ (10.13) are nothing else than Proposition 10.1. Let us prove that (10.12) → (10.14). Assume that $a \in S$ and $\gamma < \gamma_\infty$ are such that $S(a,\gamma_\infty) = S(a,\gamma)$ and that $S + B(0,\gamma_\infty) \subseteq \overline{S}$. Then, by Corollary 8.7:

$$\overline{S(a,\gamma_\infty)} = \overline{S(a,\gamma)} = \overline{S} \cap B(a,\gamma) = B(a,\gamma).$$

Conversely, assume that (10.14) holds and let $a \in S$. Then, by Lemma 10.6, either $B(a,\gamma_\infty) \subseteq \overline{S}$, or there exists $\gamma < \gamma_\infty$ such that $S(a,\gamma)$ *mod* γ_∞ is finite. Assume that $S(a,\gamma)\ mod\ \gamma_\infty$ is finite. Then, $S(a,\gamma)$ is a finite union of balls $\cup_j S(b_j,\gamma_\infty)$. Let $\delta = \min_{j\neq j'} v(b_j - b_{j'})$. Then, $S(a,\delta) = S(a,\gamma_\infty)$. The hypothesis (10.14) implies $\overline{S(a,\gamma_\infty)} = B(a,\gamma_\infty)$, that is assertion (10.12).

Finally, Lemma 10.6 shows that (10.15) → (10.13). Note that assertion (10.15) means that, for every $a \in S$ and every $\gamma < \gamma_\infty$, the sequence formed by the cardinalities of $S(a,\gamma)\ mod\ \gamma_k$ is not a stationary sequence. The following example shows that condition (10.15) is not necessary in order to have (10.12). □

Example 10.8. Let S be a subset satisfying condition (10.15), then it satisfies (10.12). Let $b \in K$ and $\delta < \gamma_\infty$ be such that $B(b,\delta)\cap S = \emptyset$ and let $T = S \cup B(b,\gamma_\infty)$. Then, obviously, $\gamma_\infty(T) = \gamma_\infty(S)$ and $\overline{T} = \overline{S} \cup B(b,\gamma_\infty)$. Consequently,

$$T+B(0,\gamma_\infty) = (S\cup\{b\})+B(0,\gamma_\infty) = (S+B(0,\gamma_\infty))\cup B(b,\gamma_\infty) \subseteq \overline{S}\cup B(b,\gamma_\infty) = \overline{T},$$

while $Card(T(b,\delta)\ mod\ \gamma_\infty) = 1$.

10.3 S regular

The equivalent assertions of Theorems 10.3 and 10.7 are satisfied in particular by the following generalization of the notion of a regular compact subset introduced by Amice [1] in local fields and extended to precompact subsets of discrete valuation domains in [19]:

Definition 10.9. The fractional subset S of K is said to be a *regular subset* if, for every $\gamma < \delta$ such that q_γ is finite, $Card(S(x,\gamma)\ mod\ \delta)$ does not depend on $x \in S$ in the following sense:

(i) if q_δ is finite, then every non-empty ball $S(x,\gamma)$ is the disjoint union of $\frac{q_\delta}{q_\gamma}$ balls $S(y,\delta)$,

(ii) if q_δ is infinite, then every non-empty ball $S(x,\gamma)$ is the disjoint union of infinitely many balls $S(y,\delta)$.

If $\gamma_\infty = \lim_k \gamma_k$, then condition (ii) follows from condition (i). So that, when q_{γ_∞} is infinite, S is a regular subset if and only if one has:

$$\forall k \geq 0,\ q_{\gamma_{k+1}} = \alpha_k q_{\gamma_k}\ \text{(where } \alpha_k \in \mathbb{N}) \tag{10.16}$$

$$\text{and}, \forall a \in S,\ Card\ S(a,\gamma_k)\ mod\ \gamma_{k+1}\ = \alpha_k. \tag{10.17}$$

And, when q_{γ_∞} is finite, one has to add the condition:

$$\forall a \in S,\ \forall \delta > \gamma_\infty \quad S(a,\gamma_\infty)\ mod\ \delta \text{ is infinite.} \tag{10.18}$$

The next section shows that regular subsets appear naturally in discrete dynamical systems.

11 Orbits under the action of an isometry

Let φ be a map from S to S. Then, the pair (S, φ) may be considered as a discrete dynamical system. For every $x \in S$, we may consider the forward orbit $O_+^\varphi(x)$ of x under the action of φ:

$$O_+^\varphi(x) = \{\varphi^n(x) \mid n \in \mathbb{N}\}. \tag{11.1}$$

Proposition 11.1. *Let $\varphi : S \to S$ be an isometry. Fix an $x \in S$ and let*

$$T = O_+^\varphi(x) = \{\varphi^n(x) \mid n \in \mathbb{N}\}.$$

Then, for every $\gamma \in \mathbb{R}$, denoting by $q_\gamma(T)$ the cardinality of T mod γ, we have:

$$\forall n, m \in \mathbb{N} \quad [\, n \equiv m \pmod{q_\gamma(T)} \Leftrightarrow \varphi^n(x) \equiv \varphi^m(x) \pmod{\gamma}\,]. \tag{11.2}$$

In particular, if $q_\gamma(T)$ is finite, $x, \varphi(x), \ldots, \varphi^{q_\gamma(T)-1}$ is a complete system of representatives of $T \bmod \gamma$, and, if $q_\gamma(T)$ is infinite, the $\varphi^k(x)$'s (for $k \in \mathbb{N}$) are non-congruent modulo γ.

Proof. Let $\gamma \in \mathbb{R}$ be such that $q_\gamma(T)$ is finite. Then, there exists $0 \le s < t$ such that $\varphi^s(x) \equiv \varphi^t(x) \pmod{\gamma}$. Consequently, $\varphi^{t-s}(x) \equiv x \pmod{\gamma}$. Let $r > 0$ be the smallest integer such that $\varphi^r(x) \equiv x \pmod{\gamma}$. Then, $x, \varphi(x), \ldots, \varphi^{r-1}(x)$ are non-congruent modulo γ. Moreover,

$$\forall h \in \mathbb{N} \quad \varphi^{(h+1)r} = \varphi^{hr}(\varphi^r(x)) \equiv \varphi^{hr}(x) \pmod{\gamma},$$

and hence,

$$\forall h \in \mathbb{N} \quad \varphi^{hr} \equiv x \pmod{\gamma}.$$

Now, for every $n \in \mathbb{N}$, let n_0 be such that

$$n \equiv n_0 \pmod{r} \quad \text{where} \quad 0 \le n_0 < r,$$

then

$$\varphi^{n-n_0}(x) \equiv x \pmod{\gamma}, \text{ that is, } \varphi^n(x) \equiv \varphi^{n_0}(x) \pmod{\gamma}.$$

Finally, the sequence $x, \varphi(x), \ldots, \varphi^{r-1}(x)$ is a complete system of representatives of $T \, mod \, \gamma$, and $q_\gamma(T) = r$.

Now, let $\delta \in \mathbb{R}$ be such that T mod δ is infinite. Assume that, for some s such that $0 \le s < t$, we had $\varphi^s(x) \equiv \varphi^t(x) \pmod{\delta}$, then the sequence $x, \varphi(x), \ldots, \varphi^{t-s-1}(x)$ would be a complete system of representatives of $T \, mod \, \delta$. This is a contradiction. □

Now we generalize a result obtained in discrete valuation domains with finite residue field (cf. [21, Théorème 7.1] or [16, Theorem 3.3]):

Theorem 11.2. *Let K be a valued field, S be an infinite fractional subset of K, and $\varphi : S \to S$ be an isometry. For every $x \in S$, the forward orbit*

$$O_+^\varphi(x) = \{\varphi^n(x) \mid n \in \mathbb{N}\}$$

is a regular subset.

Proof. Fix an $x \in S$ and let $T = O_+^{\varphi}(x)$. Let $\gamma \in \mathbb{R}$ be such that $Card(T\,mod\,\gamma) = q_\gamma(T) = r$ is finite and consider some $\delta > \gamma$.

Assume first that $Card(T\,mod\,\delta)$ is infinite. Then, the $\varphi^k(x)$, for $k \in \mathbb{N}$, are non-congruent modulo δ. Consequently, in each class $T(\varphi^i(x), \gamma)$ $(0 \leq i < r)$ of $S\,mod\,\gamma$, there are infinitely many elements that are non-congruent modulo δ, namely,

$$T(\varphi^i(x), \gamma) = \{\varphi^{i+kr}(x) \mid k \in \mathbb{N}\}.$$

Assume now that $Card(T\,mod\,\delta) = q_\delta(T) = s$ is finite. It follows from Proposition 11.1 that:

$$T = \bigsqcup_{j=0}^{s-1} T(\varphi^j(x), \delta) = \bigsqcup_{i=0}^{r-1} T(\varphi^i(x), \gamma) \tag{11.3}$$

where $\sqcup$ still denotes a disjoint union. Moreover, for $0 \leq i < r$,

$$T(\varphi^i(x), \gamma) = \{\varphi^{i+rl}(x) \mid l \in \mathbb{N}\} = \varphi^i(\{\varphi^{rl}(x) \mid l \in \mathbb{N}\}) = \varphi^i(T(x, \gamma)).$$

Similarly, for $0 \leq j < s$,

$$T(\varphi^j(x), \delta) = \varphi^j(T(x, \delta)).$$

Clearly,

$$T(x, \gamma) = \bigsqcup_{0 \leq j < s\,;\, r|j} T(\varphi^j(x), \delta).$$

Consequently,

$$T = \bigsqcup_{0 \leq i < r} T(\varphi^i(x), \gamma) = \bigsqcup_{0 \leq i < r} \; \bigsqcup_{0 \leq l < [\frac{s}{r}]} \varphi^{i+rl}(T(x, \delta).$$

On the other hand,

$$T = \bigsqcup_{0 \leq j < s} \varphi^j(T(x, \delta).$$

Since all the unions are disjoint, we necessarily have: $\left[\frac{s}{r}\right] \times r = s$, that is, r divides s and $Card(T(x, \gamma))\ mod\ \delta = \frac{s}{r}$. □

Proposition 11.3. *Let T be the forward orbit of an element x of a valued field K under the action of an isometry φ.*

(i) If $\gamma_\infty(T) = +\infty$, then T is precompact.

(ii) If $\gamma_\infty(T) < +\infty$, then T is discrete.

(iii) If $q_{\gamma_\infty}(T) = +\infty$, then

$$T = T_{\gamma_\infty} = \cup_{\gamma < \gamma_\infty} T_\gamma.$$

(iv) If $q_{\gamma_\infty}(T) < +\infty$, *then*

$$T = T_{\gamma_\infty} + T(x, \gamma_\infty)$$

where

$$T_{\gamma_\infty} = \{\varphi^k(x) \mid 0 \le k < q_{\gamma_\infty}\} \subseteq \{y \in K \mid v(x-y) < \gamma_\infty\}$$

and

$$T(x, \gamma_\infty) = \{\varphi^{rq_{\gamma_\infty}}(x) \mid r \in \mathbb{N}\} \subseteq \{y \in K \mid v(x-y) = \gamma_\infty\}.$$

Proof. It follows from Proposition 11.1 that, for every $\gamma < \gamma_\infty$, $\{\varphi^n(x) \mid 0 \le n < q_\gamma\}$ is a complete set of representatives of T modulo γ, that we may choose as T_γ. Clearly, for $\gamma < \delta < \gamma_\infty$, one has $T_\gamma \subseteq T_\delta$.

Assume that $q_{\gamma_\infty} = +\infty$. Then, $\gamma_k(T) \to \gamma_\infty(T)$, and hence, $T = \cup_{\gamma<\gamma_\infty} T_\gamma$. In particular, $T_{\gamma_\infty} = T$. This is Assertion (iii).

Assume that $q_{\gamma_\infty} < +\infty$ (and hence, $\gamma_\infty < +\infty$) then, for every $\delta > \gamma_\infty$, one has $v(\varphi^n(x) - \varphi^m(x)) < \delta$ whatever $n \neq m$, and hence, $v(\varphi^n(x) - \varphi^m(x)) \le \gamma_\infty$. Consequently, $T(x, \gamma_\infty)$ is not only the intersection of T with the ball $B(x, \gamma_\infty)$, but the intersection with a sphere: $T \cap \{y \in K \mid v(x-y) = \gamma_\infty\}$. Thus, $T \subseteq T_{\gamma_\infty} + \{y \in K \mid v(y) = \gamma_\infty\}$. Clearly, the intersection of T with $B(x, \gamma_\infty)$ is the forward orbit of x under the action of $\varphi^{q_{\gamma_\infty}}$: $T(x, \gamma_\infty) = \{\varphi^{rq_{\gamma_\infty}}(x) \mid r \in \mathbb{N}\}$. This is Assertion (iv).

Assertion (i) follows from Proposition 6.3. Finally, assume that $\gamma_\infty < +\infty$. It follows from Assertions (iii) and (iv) that in both cases ($q_{\gamma_\infty} <$ or $= +\infty$), for all $x \neq y \in T$, $v(x-y) \le \gamma_\infty$. Consequently, T is (uniformly) discrete: for each $t \in T$, $T \cap \{x \in K \mid v(x-t) > \gamma_\infty\} = \{t\}$. This is Assertion (ii). □

Remark 11.4. Let T denote the orbit of an element x of K under the action of an isometry φ.

(1) If $q_{\gamma_\infty}(T)$ is finite, then $V/\mathfrak{m}$ is infinite. Indeed, the previous proof shows that, if q_{γ_∞} is finite, then

$$q_{\gamma_\infty} | n - m \;\Leftrightarrow\; v(\varphi^n(x) - \varphi^m(x)) = \gamma_\infty. \tag{11.4}$$

In particular, $\{y \in K \mid v(y) = \gamma_\infty\}$ is infinite, which is equivalent to the fact that the residue field $V/\mathfrak{m}$ is infinite.

(2) If $\gamma_\infty(T)$ is finite, then T is discrete, and hence, is equal to its topological closure $\widetilde{T}$ in V and also to its completion. If $q_{\gamma_\infty}(T)$ is infinite then, by Theorem 10.7, T is polynomially equivalent to $T + B(0, \gamma_\infty)$. Consequently, $\widetilde{T} \neq \overline{T}$.

Corollary 11.5. *Let S be an infinite fractional subset of K and let $\varphi : S \to S$ be an isometry. If the dynamical system (S, φ) is topologically transitive, that is, if there exists $x \in S$ such that $T = O_+^\varphi(x)$ is dense in S, then S is a regular subset. Moreover, either $\gamma_\infty < +\infty$, S is discrete and $S = S_{\gamma_\infty} = T$, or $\gamma_\infty = +\infty$, S is precompact and $S = S_{\gamma_\infty}$.*

Proof. By hypothesis, for every $\gamma \in \mathbb{R}$, one has T *mod* γ = S *mod* γ, and hence, for every γ such that $q_\gamma(S) < q_{\gamma_\infty}(S)$, $q_\gamma(S) = q_\gamma(T)$ and $(x, \varphi(x), \ldots, \varphi^{q_\gamma - 1}(x))$ is a complete system of representatives of S *mod* γ, that one may choose for S_γ. □

The greatest difference between all the cases is probably due to the fact that q_{γ_∞} is finite or infinite. The case where q_{γ_∞} is finite will be studied in [15]. The regular subsets such that q_{γ_∞} is infinite will be considered in a forthcoming paper [14] where we show that their v-orderings have very strong properties that may be used to describe the dynamics. In particular, extending results of [19], we prove the following proposition.

Proposition 11.6 ([14]). *Let S be an infinite fractional subset of K such that $S = S_{\gamma_\infty}$. The following assertions are equivalent:*

(i) S is a regular subset.

(ii) There exists a sequence $\{b_n\}_{n\in\mathbb{N}}$ of elements of S such that:

$$\forall \gamma \in \Gamma \quad [\, v(b_n - b_m) \geq \gamma \;\Leftrightarrow\; q_\gamma | (n-m)\,]. \tag{11.5}$$

(iii) There exists a sequence $\{c_n\}_{n\in\mathbb{N}}$ of elements of S such that, for every $k \in \mathbb{N}$, $\{c_n\}_{n\geq k}$ is a v-ordering of S.

(iv) The characteristic function w_S of S satisfies the following generalized Legendre formula:

$$w_S(n) = v(n!_S) = n\gamma_0 + \sum_{k=1}^{+\infty} \left[\frac{n}{q_{\gamma_k}(S)}\right](\gamma_k - \gamma_{k-1})\,. \tag{11.6}$$

12 An example

The following example is a generalization of an example given in [12]. Let k be a field and let Γ be a subgroup of $\mathbb{R}$. Let $\Gamma_+ = \{\gamma \in \Gamma \mid \gamma \geq 0\}$ and consider the integral domain

$$A = k[\Gamma_+] = k[\{X^\gamma \mid \gamma \geq 0\}; X^\gamma X^\delta = X^{\gamma+\delta}] \tag{12.1}$$

endowed with the valuation v defined by:

$$\forall k \in \mathbb{N}\ \forall a_k \in k\ \forall \delta_k \in \Gamma_+ \quad v\left(\sum_{k=0}^{n} a_k X^{\delta_k}\right) = \min\{\delta_k \mid a_k \neq 0\}. \tag{12.2}$$

Fix a strictly increasing sequence $\{r_n\}_{n\in\mathbb{N}}$ of elements of Γ_+. For every $n \geq 0$, choose a finite subset C_n of k containing 0 with cardinality $\alpha_n > 1$. Now consider the following subset of A:

$$T = \{c_0 X^{r_0} + c_1 X^{r_1} + \cdots + c_l X^{r_l} \mid l \in \mathbb{N},\, c_h \in C_h, 0 \leq h \leq l\}. \tag{12.3}$$

Then,

$$q_{r_h} = q_{r_h}(T) = Card(T \; mod \; r_h) \quad \text{satisfies} \quad q_0 = 1 \text{ and } q_{r_{h+1}} = \alpha_h q_{r_h}. \tag{12.4}$$

Of course,

$$q_\gamma = q_{r_h} \quad \text{for} \quad r_{h-1} < \gamma \leq r_h. \tag{12.5}$$

In other words, the sequence of critical valuations of T, that is $\{\gamma_k\}_{k\in\mathbb{N}}$, is the sequence $\{r_k\}_{k\in\mathbb{N}}$. Consequently,

$$\gamma_\infty(T) = r_\infty = \sup\{\, r_n \mid n \in \mathbb{N}\,\}. \tag{12.6}$$

Thus, T is a discrete subspace if the sequence $\{r_n\}$ is bounded and T is precompact if the sequence is unbounded. In both cases, the subset T admits infinite v-orderings. We describe now such a v-ordering.

For every $n \geq 0$, the elements of C_n are ordered in a finite sequence:

$$\{c_{n,i}\}_{0\leq i<\alpha_n} \quad \text{with} \quad c_{n,0} = 0. \tag{12.7}$$

For every $n \geq 0$, denoting by $n \mod \alpha$ the unique integer m such that $n \equiv m \pmod{\alpha}$ and $0 \leq m < \alpha$, we consider:

$$n_0 = n \mod \alpha_0 \;,\; n_1 = \frac{n - n_0}{\alpha_0} \mod \alpha_1 \;,\; n_2 = \frac{n - n_0 - n_1\alpha_0}{\alpha_1} \mod \alpha_2 \;,\; \ldots \tag{12.8}$$

So that

$$n = n_0 + n_1\alpha_0 + n_2\alpha_0\alpha_1 + \cdots + n_l\alpha_0\alpha_1\cdots\alpha_{l-1} \tag{12.9}$$

or

$$n = n_0 + n_1 q_{r_1} + n_2 q_{r_2} + \cdots + n_l q_{r_l} \quad \text{with } 0 \leq n_h < \alpha_h \tag{12.10}$$

Then, put

$$a_n = \sum_{h=0}^{l} c_{h,n_h} X^{r_h}. \tag{12.11}$$

Proposition 12.1. *The sequence $\{a_n\}_{n\in\mathbb{N}}$ defined by (12.10) and (12.11) is a v-ordering of the subset T defined by (12.3). Moreover, it satisfies all the properties introduced in Proposition 11.6, in particular:*

$$w_T(n) = v(n!)_T = nr_0 + \sum_{h\geq 1}\left[\frac{n}{q_{r_h}}\right](r_h - r_{h-1}). \tag{12.12}$$

Proof. Denote by $\nu_T(n)$ the greatest integer k such that q_{r_k} divides n. Clearly,

$$\forall n, m \in \mathbb{N} \quad v(a_n - a_m) = r_{\nu_T(n-m)}. \tag{12.13}$$

One easily verifies that for $m \geq n$:

$$v\left(\prod_{k=0}^{n-1}(a_m - a_k)\right) = \sum_{k=0}^{n-1} r_{\nu_T(m-k)} = \sum_{l=1}^{m} r_{\nu_T(l)} - \sum_{l=1}^{m-n} r_{\nu_T(l)}$$

$$= r_0 n + \sum_{k>0}\left(\left[\frac{m}{q_{r_k}}\right] - \left[\frac{m-n}{q_{r_k}}\right]\right)(r_k - r_{k-1}).$$

In particular,

$$v\left(\prod_{k=0}^{n-1}(a_n - a_k)\right) = r_0 n + \sum_{k>0}\left[\frac{n}{q_{r_k}}\right](r_k - r_{k-1}).$$

Thus, the sequence $\{a_n\}$ is a v-ordering of T since

$$\left[\frac{m}{q_{r_k}}\right] - \left[\frac{m-n}{q_{r_k}}\right] \geq \left[\frac{n}{q_{r_k}}\right].$$

□

Corollary 12.2. *The valuative capacity of T is equal to*

$$\delta_T = \sum_{k\geq 0}\frac{1}{\alpha_0\alpha_1\ldots\alpha_{k-1}}\; r_k\;\left(1-\frac{1}{\alpha_k}\right). \tag{12.14}$$

In particular, if $\alpha_k = q$ for every k, then

$$\delta_T = \left(1-\frac{1}{q}\right)\sum_{k\geq 0}\frac{r_k}{q^k}. \tag{12.15}$$

Since the condition on the sequence $\{r_k\}_{k\in\mathbb{N}}$ is just to be a strictly increasing sequence, it is easy to choose the r_k's in order to have δ_T either finite or infinite.

Proof. Recall that the definition of δ_T is given in Section 7. Note first that, if S is a regular subet such that $S = S_{\gamma_\infty}$, then the characteristic function w_S satisfies Formula (11.6) (see Proposition 11.6), and hence, the valuative capacity of S is given by:

$$\delta_S = \lim_{n\to+\infty}\frac{w_S(n)}{n} = \gamma_0 + \sum_{k\geq 1}\frac{1}{q_{\gamma_k}}(\gamma_k - \gamma_{k-1}). \tag{12.16}$$

Replacing γ_k by r_k and q_{γ_k} by $\alpha_0\alpha_1\ldots\alpha_{k-1}$, we easily obtain Formula (12.14). □

Remark 12.3. The map $\varphi : T \to T$ defined by $\varphi(a_n) = a_{n+1}$ for $n \in \mathbb{N}$ is an isometry on T and $O_+^{\varphi}(0) = T \setminus \{0\}$.

(i) If $r_\infty = +\infty$, we consider the subset

$$\widehat{T} = \left\{\sum_{l=0}^{\infty} d_n X^{r_n} \mid d_n \in C_n\right\} \tag{12.17}$$

and any subset S such that

$$T \subseteq S \subseteq \widehat{T}. \tag{12.18}$$

Then, S is precompact, $S_{\gamma_\infty} = T$ and $\overline{S} = \widehat{T}\cap K$. The subsets S and T are polynomially equivalent and the sequence $\{a_n\}$ is a v-ordering of S.

The previous map φ may be extended by continuity to $\widehat{T}$ and the dynamical system $(\widehat{T}, \varphi)$ is transitive since $O_+^{\varphi}(0) = T \setminus \{0\}$ is dense in T, and hence, in $\widehat{T}$.

(ii) If $r_\infty < +\infty$, we consider any subset S such that

$$T \subseteq S \subseteq \cap_k (T + B(0, r_k)). \tag{12.19}$$

Then, $S_{\gamma_\infty} = T$ and $\overline{S} = \cap_k (T + B(0, r_k))$. The subsets S and T are polynomially equivalent and the sequence $\{a_n\}$ is a v-ordering of S.

Note also that in this latter case ($r_\infty < \infty$), $\overline{T}$ may either be equal to $T + B(0, r_\infty)$ or not. Assume that the characteristic of K is $\neq 2$, $r_\infty \in \Gamma$, $C_0 = \{0, 1\}$ and $C_n = \{0, 1, 2\}$ for $n \geq 1$. Then, $2 + X^{r_\infty}$ belongs to $\cap_{k \geq 0}(T + B(0, r_k))$ while $2 + X^{r_\infty}$ does not belong to $T + B(0, r_\infty)$ (in fact, this is quite the example given in Remark 9.3). On the other hand, if all the C_n's are equal, then

$$\overline{T} = T + B(0, r_\infty) = \cap_{k \geq 0}(T + B(0, r_k)).$$

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