The functor of units of Burnside rings

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Overview

1. Burnside rings

2. Units
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3. $p$-groups: combinatorial answer
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The Burnside ring of a finite group

Definition

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In other words $B(G) = \mathbb{Z}^{\text{[finite } G\text{-sets]}} / \langle [X \sqcup Y] - [X] - [Y] \rangle$, where $\mathbb{Z}$ is the set of integers, $\sqcup$ denotes disjoint union, and $[X]$ denotes the equivalence class of $X$ under the relation of isomorphic $G$-sets.
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- As an abelian group, it has a basis $\{[G/H] \mid H \in [s_G]\}$, where $[s_G] = \{H \leq G, \text{ mod. } G\}$.
If $H \leq G$, the map $\phi_H : [X] \mapsto |X^H|$ yields a ring homomorphism $B(G) \to \mathbb{Z}$.
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$$\forall H \in [s_G], \quad e^G_H = \frac{1}{|N_G(H)|} \sum_{K \leq H} |K| \mu(K, H)[G/K].$$
Let $B^\times(G)$ denote the group of multiplicative units (i.e. invertible elements) of $B(G)$. 

Since $B^\times(G) \hookrightarrow \prod_{H \in [sG]} \mathbb{Z} \times = \prod_{H \in [sG]} \{\pm 1\}$, it follows that $B^\times(G)$ is an elementary abelian 2-group. In particular, finding the rank of $B^\times(G)$ should be an easy problem. It is a very hard problem, as shown by the following observation (Tom Dieck (1979)): the statement

If $G$ has odd order, then $B^\times(G) = \{\pm 1\}$

is equivalent to the odd order theorem (Feit-Thompson (1963)). For an arbitrary finite group, not so much is known on this problem. The work of many people (T. Tom Dieck, T. Matsuda, T. Matsuda - T. Miyata, T. Yoshida, E. Yalçın, . . . ) recently led to the solution for finite $p$-groups (more generally for finite nilpotent groups).
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The case of $p$-groups: combinatorial answer

Notation

Let $p$ be a prime number, and $P$ be a finite $p$-group. If $S \leq P$, denote by $Z_P(S)$ the subgroup of $N_P(S)/S = Z(N_P(S)/S)$.

Definition

A finite group $G$ has normal rank 1 if all the abelian normal subgroups of $G$ are cyclic. The finite $p$-groups of normal rank 1 are the cyclic groups $C_p^n (n \geq 0)$, the generalized quaternion $2$-groups $Q_2^n (n \geq 3)$, the dihedral $2$-groups $D_2^n (n \geq 4)$, and the semi-dihedral groups $SD_2^n (n \geq 4)$.
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Genetic subgroups

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**Definition**

A subgroup $S \leq P$ is called **genetic** if the following two conditions are fulfilled:

1. The group $N_P(S)$ has normal rank 1.
2. If $x \in G$ is such that $Sx \cap Z_P(S) \leq S$, then $Sx = S$.

**Example:** If $S \triangleleft P$, then $S$ is a genetic subgroup of $P$ if and only if $P/S$ has normal rank 1.

**Definition**

Define a relation $\hat{\mathcal{P}}$ on the set of subgroups of $P$ by $S \hat{\mathcal{P}} T \iff \exists x \in P, Sx \cap Z_P(T) \leq T$ and $T \cap Z_P(Sx) \leq Sx$. 

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Define a relation $\equiv_P$ on the set of subgroups of $P$ by

$S \equiv_P T \iff \exists x \in P, \ S^x \cap Z_P(T) \leq T \text{ and } T \cap Z_P(S^x) \leq S^x$. 
Theorem (B. (2005))

1. The relation $\equiv_P$ is an equivalence relation on the set of genetic subgroups of $P$. 

Definition

A genetic basis of $P$ is a set of representatives of genetic subgroups of $P$ for the relation $\equiv_P$.

The type of a genetic subgroup $S$ of $P$ is the isomorphism class of $N_P(S)/S$. 

The type of a genetic subgroup of $P$ is one of $C_p^n$ ($n \geq 0$), $Q_2^n$ ($n \geq 3$), $D_2^n$ ($n \geq 4$), or $SD_2^n$ ($n \geq 4$).
Genetic bases

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1. The relation $\mathrel{\preceq}_P$ is an equivalence relation on the set of genetic subgroups of $P$.

2. If $S$ and $T$ are genetic subgroups of $P$ and if $S \preceq_P T$, then $N_P(S)/S \cong N_P(T)/T$. 
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Definition

A genetic basis of \( P \) is a set of representatives of genetic subgroups of \( P \) for the relation \( \preceq_P \).
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An example
An example

![Diagram of a mathematical structure with vertices labeled D8]
An example
An example

\[
\begin{array}{c}
\text{type } C_1 \\ - \rightarrow \\
\bullet \\
\rightarrow \\
\text{type } C_2 \\ - \rightarrow \\
\circ \\
\rightarrow \\
\text{type } C_2 \\ - \rightarrow \\
\circ \\
\rightarrow \\
\end{array}
\]

\[D_8\]
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\begin{array}{c}
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An example
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The functor of units of Burnside rings
An example
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- Type $C_1$ →
- Type $C_2$ → $D_8$
Combinatorial answer

**Theorem**

Let $P$ be a finite $p$-group, and $\mathcal{G}$ be a genetic basis of $P$. Then $B^\times(P) \cong (\mathbb{Z}/2\mathbb{Z})^{u_P}$,
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Let $P$ be a finite $p$-group, and $G$ be a genetic basis of $P$. Then $B^\times(P) \cong (\mathbb{Z}/2\mathbb{Z})^{u_P}$, where $u_P$ is the number of elements of $G$ whose type is trivial, $C_2$, or $D_{2^n}$. 

Examples:

If $P$ is abelian, then $u_P = 1 + |\{S < P | |P:S|=2\}|$ (Matsuda (1982)).
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- If $P$ is abelian, then $u_P = 1 + |\{S < P \mid |P : S| = 2\}|$ (Matsuda (1982)).
Theorem

Let $P$ be a finite $p$-group, and $G$ be a genetic basis of $P$. Then $B^\times(P) \cong (\mathbb{Z}/2\mathbb{Z})^{u_P}$, where $u_P$ is the number of elements of $G$ whose type is trivial, $C_2$, or $D_{2^n}$.

Examples:

- If $P$ is abelian, then $u_P = 1 + |\{S < P \mid |P : S| = 2\}|$ (Matsuda (1982)).
- $B^\times(D_8) \cong (\mathbb{Z}/2\mathbb{Z})^5$. 
Operations on $B^\times(G)$

Let $H \leq G$. Then restriction $B^\times(G) \to B^\times(H)$ induces a group homomorphism $\text{Res}_{G,H} : B^\times(G) \to B^\times(H)$.

Let $H \leq G$. Then tensor induction $B^\times(H) \to B^\times(G)$ induces a group homomorphism $\text{Ten}_{G,H} : B^\times(H) \to B^\times(G)$.

Let $N \trianglelefteq G$. Then inflation from $B^\times(G/N)$ to $B^\times(G)$ induces a group homomorphism $\text{Inf}_{G,G/N} : B^\times(G/N) \to B^\times(G)$.

Let $N \trianglelefteq G$. Then taking fixed points by $N$ induces a group homomorphism called deflation $\text{Def}_{G,G/N} : B^\times(G) \to B^\times(G/N)$.

Let $f : G \to G'$ be a group isomorphism. Then $f$ induces a group isomorphism $\text{Iso}(f) : B^\times(G) \to B^\times(G')$. 

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Operations on $B^\times(G)$

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Operations on $B^\times (G)$

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- Let $H \leq G$. 
- Let $N \unlhd G$. Then inflation from $B(G/N)$ to $B(G)$ induces a group homomorphism $Inf_{G/G/N} : B^\times(G/N) \to B^\times(G)$.
- Let $N \unlhd G$. Then taking fixed points by $N$ induces a group homomorphism called deflation $Def_{G/G/N} : B^\times(G) \to B^\times(G/N)$.
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Serge Bouc (CNRS-Université de Picardie)  The functor of units of Burnside rings  Hokkaido University-24/06/08  11 / 20
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Serge Bouc (CNRS-Université de Picardie)
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Serge Bouc (CNRS-Université de Picardie)
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The functor of units of Burnside rings

Hokkaido University-24/06/08
Faithful elements

Definition

Let $G$ be a finite group.
Faithful elements

Definition

Let $G$ be a finite group. The set of **faithful elements** of $B^\times(G)$ is the subgroup of $B^\times(G)$ defined by

$$
\partial B^\times(G) = \cap \{ N \triangleleft G \mid \ker \text{Def}_G \}
$$

Theorem (Yalçın (2005), B. (2007))

Let $P$ be a $p$-group of normal rank 1. Then $\partial B^\times(P)$ is trivial, except if $P \cong C_1, C_2, D_{2n}$. In these cases $\partial B^\times(P) = \{ 1, \nu_P \}$.

$$
\nu_P = \begin{cases} 
-\frac{P}{P} & \text{if } P \cong C_1 \\
\frac{P}{P} - \frac{P}{I} - \frac{P}{J} & \text{if } P \cong C_2 \\
\frac{P}{P} + \frac{P}{I} - \frac{P}{J} & \text{if } P \cong D_{2n},
\end{cases}
$$

where $|I| = |J| = 2$, $Z(P) \neq I \neq J = Z(P)$. 

Serge Bouc (CNRS-Université de Picardie)
Faithful elements

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Let $G$ be a finite group. The set of faithful elements of $B^\times(G)$ is the subgroup of $B^\times(G)$ defined by

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$\nu_P$ is defined as follows:

- If $P \cong C_1$, then $\nu_P = [P/P]$.
- If $P \cong C_2$, then $\nu_P = [P/P] - [P/I] - [P/J]$.
- If $P \cong D_{2^n}$, then $\nu_P = [P/P] + [P/I] - [P/J]$, where $|I| = |J| = 2$. 

$Z(P) \neq I \neq P \neq J \neq Z(P)$.
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Let $P$ be a finite $p$-group, and $\mathcal{G}$ be a genetic basis of $P$.
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Example: For $P = D_8$, the group $B \times (P \times P)$ has an $F_2$-basis 

$$\begin{cases} \mathbb{F}_2[\mathbb{F}_2/P] \mathbb{F}_2[\mathbb{F}_2/P] \mathbb{F}_2[\mathbb{F}_2/Q] \mathbb{F}_2[\mathbb{F}_2/I] \mathbb{F}_2[\mathbb{F}_2/J] \end{cases}$$


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$$\{ \frac{\partial B \times (P)}{\inf N \times (S) / S} \mid S \in \mathcal{U} \}$$

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\frac{P}{Q} - \frac{P}{Q} - \frac{Q}{P} \\
\forall Q, |P:Q| = 2
\end{cases}$$

(choose $I, J, |I| = |J| = 2$, $I < J$)

Serge Bouc (CNRS-Université de Picardie)

The functor of units of Burnside rings

Hokkaido University-24/06/08
Let $P$ be a finite $p$-group, and $G$ be a genetic basis of $P$. Let $U = \{ S \in G \mid \text{type}(S) \in \{ C_1, C_2, D_{2^n} \}_{n \geq 4} \}$. Then the set

$$\{ \nu_{NP}(S)/S \mid S \in U \}$$

is an $F_2$-basis of $B \times (P)$. In other words the map

$$\oplus S \in G \partial B \times (NP(S)/S) \to B \times (P)$$

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$$\{ \text{Inf}_{N_P(S)/S}^N U_{N_P(S)/S} \mid S \in \mathcal{U} \}$$
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Let $P$ be a finite $p$-group, and $\mathcal{G}$ be a genetic basis of $P$. Let $\mathcal{U} = \{ S \in \mathcal{G} \mid \text{type}(S) \in \{ C_1, C_2, D_{2^n} \}_{n \geq 4} \}$. Then the set

$$\{ \text{Ten}^P_{NP(S)} \text{Inf}^{NP(S)}_{NP(S)/S} \cup_{NP(S)/S} \mid S \in \mathcal{U} \}$$
Let $P$ be a finite $p$-group, and $G$ be a genetic basis of $P$. Let $\mathcal{U} = \{ S \in G \mid \text{type}(S) \in \{ C_1, C_2, D_{2^n} \}_{n \geq 4} \}$. Then the set

$$\{ \text{Ten}_{NP(S)}^P \text{Inf}_{NP(S)}^{NP(S)} \cup_{NP(S)} S \mid S \in \mathcal{U} \}$$

is an $\mathbb{F}_2$-basis of $B^\times(P)$. 
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In other words the map 

$$\bigoplus_{S \in G} \partial B^\times(NP(S)/S) \rightarrow B^\times(P)$$

is an isomorphism.
Theorem (B. (2007))

Let $P$ be a finite $p$-group, and $\mathcal{G}$ be a genetic basis of $P$. Let $\mathcal{U} = \{ S \in \mathcal{G} \mid \text{type}(S) \in \{ C_1, C_2, D_{2^n} \}_{n \geq 4} \}$. Then the set

$$\{ \text{Ten}^P_{N_P(S)} \text{Inf}^{N_P(S)}_{N_P(S)/S} \cup \text{Inf}^{N_P(S)}_{N_P(S)/S} \mid S \in \mathcal{U} \}$$

is an $\mathbb{F}_2$-basis of $B^\times(P)$.

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$$\bigoplus \text{Ten}^P_{N_P(S)/S} \text{Inf}^{N_P(S)}_{N_P(S)/S} : \bigoplus_{S \in \mathcal{G}} \partial B^\times(N_P(S)/S) \to B^\times(P)$$

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**Example:** For $P = D_8$, the group $B^\times(P)$ has an $\mathbb{F}_2$-basis

$$\left\{ -[P/P] \right\}$$
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Let $P$ be a finite $p$-group, and $G$ be a genetic basis of $P$. Let

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$$\{ \text{Ten}^P_{NP(S)}\text{Inf}^N_{NP(S)/S}u_{NP(S)/S} \mid S \in \mathcal{U} \}$$

is an $\mathbb{F}_2$-basis of $B^\times(P)$.

In other words the map

$$\bigoplus_{S \in G} \text{Ten}^P_{NP(S)/S}\text{Inf}^N_{NP(S)/S} : \bigoplus_{S \in G} \partial B^\times(NP(S)/S) \to B^\times(P)$$

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Example: For $P = D_8$, the group $B^\times(P)$ has an $\mathbb{F}_2$-basis

$$\{ [P/P], [P/P] - [P/Q] \}$$
Algebraic answer

**Theorem (B. (2007))**

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$\mathcal{U} = \{ S \in \mathcal{G} \mid \text{type}(S) \in \{ C_1, C_2, D_{2^n} \}_{n \geq 4} \}$. Then the set

$$\{ \text{Ten}^P_{N_P(S)} \text{Inf}^{N_P(S)}_{N_P(S)/S} \cup N_P(S)/s \mid S \in \mathcal{U} \}$$

is an $\mathbb{F}_2$-basis of $B^\times(P)$.

In other words the map

$$\bigoplus \text{Ten}^P_{N_P(S)}/s \text{Inf}^{N_P(S)}_{N_P(S)/s} : \bigoplus_{S \in \mathcal{G}} \partial B^\times(N_P(S)/S) \to B^\times(P)$$

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$$\begin{cases} 
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is an $\mathbb{F}_2$-basis of $B^\times (P)$.

In other words the map

$$\bigoplus \text{Ten}_{NP}(S) \text{Inf}_{NP}(S) / S : \bigoplus_{S \in G} \partial B^\times (NP(S) / S) \to B^\times (P)$$

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Example: For $P = D_8$, the group $B^\times (P)$ has an $\mathbb{F}_2$-basis

$$\begin{cases} 
- [P/P] \\
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[P/P] + [P/1] - [P/I] - [P/J] 
\end{cases} \quad \forall Q, \ |P : Q| = 2$$
Theorem (B. (2007))

Let $P$ be a finite $p$-group, and $G$ be a genetic basis of $P$. Let $U = \{ S \in G \mid \text{type}(S) \in \{ C_1, C_2, D_{2n} \}_{n \geq 4} \}$. Then the set

$$\{ \text{Ten}_{P}^{N_{P}(S)} \text{Inf}_{N_{P}(S)/S}^{N_{P}(S)} \cup N_{P}(S)/S \mid S \in U \}$$

is an $\mathbb{F}_2$-basis of $B^\times(P)$.

In other words the map

$$\bigoplus \text{Ten}_{P}^{N_{P}(S)} \text{Inf}_{N_{P}(S)/S}^{N_{P}(S)} : \bigoplus_{S \in G} \partial B^\times(N_{P}(S)/S) \to B^\times(P)$$

is an isomorphism.

Example: For $P = D_8$, the group $B^\times(P)$ has an $\mathbb{F}_2$-basis

$$\begin{cases} 
- [P/P] \\
[P/P] - [P/Q] \\
[P/P] + [P/1] - [P/I] - [P/J] 
\end{cases} \quad \forall Q, \ |P : Q| = 2$$

(choose $I, J, |I| = |J| = 2, \langle I, J \rangle = P$)
The five types of operations
R: G \rightarrow H,
T: G \rightarrow H,
I: G \rightarrow G / N,
D: G \rightarrow G / N,
I: (f) on B × can be unified using bisets:

Definition
Let G and H be (finite) groups. An (H, G)-biset $U$ is a (finite) set with a left $H$-action and a right $G$-action, which commute, i.e.

$\forall h \in H, \forall u \in U, \forall g \in G, (h \cdot u) \cdot g = h \cdot (u \cdot g)$.

[Equivalently, an (H, G)-biset is an (H × G^{op})-set.]

If G, H, and K are groups, if $U$ is an (H, G)-biset and $V$ a (K, H)-biset, the composition $V \circ U$ is the (K, G)-biset $V \times H U = (V \times U) / \langle (vh, u) \sim (v, hu) \rangle$.

If G is a group, the identity biset $Id_G$ is the (G, G)-biset $G$ with left and right action by multiplication.
The five types of operations $\text{Res}_H^G$, $\text{Ten}_H^G$, $\text{Inf}_{G/N}^G$, $\text{Def}_{G/N}^G$, $\text{Iso}(f)$ on $B^\times$ can be unified using bisets:
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[ Equivalently, an \((H, G)\)-biset is an \((H \times G^{\text{op}})\)-set. ]
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Serge Bouc (CNRS-Université de Picardie) 

The functor of units of Burnside rings

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Bisets

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- If $G$ is a group, the identity biset $Id_G$ is the $(G, G)$-biset $G$ with left and right action by multiplication.
If $U$ is an $\mathcal{(H,G)}$-biset and $X$ is a $G$-set, then $\text{Hom}_G(U^\text{op}, X)$ is an $\mathcal{H}$-set. This extends to a group homomorphism $B \times (U) : B \times (G) \to B \times (H)$. This endows $B$ with a structure of biset functor.

Definition A biset functor $F$ consists of the following data:

1. If $G$ is a group, then $F(G)$ is an abelian group.
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Serge Bouc (CNRS-Université de Picardie)
Biset functors

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6. If $G$ is a group, then $F(\text{Id}_G) = \text{Id}_{F(G)}$. 
Equivalently, a biset functor is an additive functor from the biset category to abelian groups:

**The biset category**

- The objects are finite groups.
- If $G, H$ are finite groups, then $\text{Hom}_C(G, H) = \mathcal{B}(H, G)$, the Burnside group of finite $(H, G)$-bisets.
- The composition of morphisms $G \to H \to K$ is obtained by linearly extending the product $(V, U) \mapsto V \times H U$ of bisets.
- The identity morphism of $G$ is the (class of) the $(G, G)$-biset $\text{Id}_G$.
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The *biset category* $C$ for finite groups is defined as follows:

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A *$p$-biset functor* is a biset functor which is “defined only on $p$-groups”, i.e. an additive functor from the full subcategory $C_p$ of $C$ consisting of $p$-groups, to the category of abelian groups.
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Let $F$ be a biset functor, and $G$ be a finite group.

- If $H \leq G$, let $\text{res}_H^G$ denote the $(H, G)$-biset $G$, 
- If $N \trianglelefteq G$, let $\text{inf}_G G/N$ denote the $(G, G/N)$-biset $G/N$, 
- If $N \trianglelefteq G$, let $\text{def}_G G/N$ denote the $(G/N, G)$-biset $G/N$, 
- If $f : G \to G'$ is a group isomorphism, let $\text{iso}(f)$ denote the $(G', G)$-biset $G'$, 
- If $f : G \to G'$ is a group isomorphism, let $\text{Iso}(f)$ denote the $(G', G)$-biset $G'$. 

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Let $F$ be a biset functor, and $G$ be a finite group.

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Let $F$ be a biset functor, and $G$ be a finite group.

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  $$Ind^G_H = F(ind^G_H) : F(H) \to F(G).$$

If $N \trianglelefteq G$, let $inf^G_G/G$ denote the $(G, G/N)$-biset $G/N$, and set
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- If $N \trianglelefteq G$, let $inf_{G/N}^G$ denote the $(G, G/N)$-biset $G/N$, and set $Inf_{G/N}^G = F(inf_{G/N}^G) : F(G/N) \to F(G)$.
Let $F$ be a biset functor, and $G$ be a finite group.

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- If $H \leq G$, let $res_H^G$ denote the $(H, G)$-biset $G$, and set $Res_H^G = F(res_H^G) : F(G) \to F(H)$.
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- If $N \trianglelefteq G$, let $inf_{G/N}^G$ denote the $(G, G/N)$-biset $G/N$, and set $Inf_{G/N}^G = F(inf_{G/N}^G) : F(G/N) \to F(G)$. 

Serge Bouc (CNRS-Université de Picardie)

The functor of units of Burnside rings

Hokkaido University-24/06/08
Let $F$ be a biset functor, and $G$ be a finite group.

- If $H \leq G$, let $\text{res}_H^G$ denote the $(H, G)$-biset $G$, and set $Res_H^G = F(\text{res}_H^G) : F(G) \rightarrow F(H)$.

- If $H \leq G$, let $\text{ind}_H^G$ denote the $(G, H)$-biset $G$, and set $Ind_H^G = F(\text{ind}_H^G) : F(H) \rightarrow F(G)$.

- If $N \trianglelefteq G$, let $\text{inf}_{G/N}^G$ denote the $(G, G/N)$-biset $G/N$, and set $Inf_{G/N}^G = F(\text{inf}_{G/N}^G) : F(G/N) \rightarrow F(G)$.

- If $N \trianglelefteq G$, let $\text{def}_{G/N}^G$ denote the $(G/N, G)$-biset $G/N$, and set $Def_{G/N}^G = F(\text{def}_{G/N}^G) : F(G) \rightarrow F(G/N)$.
Let $F$ be a biset functor, and $G$ be a finite group.

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- If $N \trianglelefteq G$, let $\text{def}_{G/N}^G$ denote the $(G/N, G)$-biset $G/N$,
Let $F$ be a biset functor, and $G$ be a finite group.

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Let $F$ be a biset functor, and $G$ be a finite group.

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- If $f : G \to G'$ is a group isomorphism, let $\text{iso}(f)$
Let $F$ be a biset functor, and $G$ be a finite group.

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Let $F$ be a biset functor, and $G$ be a finite group.

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- If $N \triangleleft G$, let $\text{def}_{G/N}^G$ denote the $(G/N, G)$-biset $G/N$, and set $\text{Def}_{G/N}^G = F(\text{def}_{G/N}^G) : F(G) \to F(G/N)$.

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Let $F$ be a biset functor, and $G$ be a finite group.
Let $F$ be a biset functor, and $G$ be a finite group. The set of faithful elements of $F(G)$ is defined by $\partial F(G) = \bigcap_{1 \neq N \trianglelefteq G} \text{Def}^G_{G/N}$. 

Example: The functor of rational representations $R^G_Q$ is defined by $G \mapsto R^G_Q(G)$, and the map $R^G_Q(U)$ is induced by $Q \otimes U$. The set $\partial R^G_Q(G)$ consists of linear combinations of faithful irreducible rational representations of $G$. The $p$-biset functor $R^G_Q$ is rational (based on Roquette (1958)).
Faithful elements

Let $F$ be a biset functor, and $G$ be a finite group. The set of **faithful elements** of $F(G)$ is defined by

$$\partial F(G) = \bigcap_{1 \neq N \trianglelefteq G} \text{Def}^G_{G/N}.$$

**Definition**

A $p$-biset functor is called **rational**.
Rational $p$-biset functors

Let $F$ be a biset functor, and $G$ be a finite group. The set of faithful elements of $F(G)$ is defined by $\partial F(G) = \bigcap_{1 \neq N \trianglelefteq G} \text{Def}_{G/N}^G$.

**Definition**

A $p$-biset functor is called rational if for any $p$-group $P$ and any genetic basis $G$ of $P$, the map $\bigoplus_{S \in G} \text{Ind}_{P/N}^P(S) \rightarrow \bigoplus_{S \in G} \partial F(P)_{N/P(S)}$ is an isomorphism.

Example: The functor of rational representations $RQ$ is defined by $G \mapsto RQ(G)$, and the map $RQ(U) \rightarrow RQ(G)$ is induced by $Q \otimes Q - U$. The set $\partial RQ(G)$ consists of linear combinations of faithful irreducible rational representations of $G$.

The $p$-biset functor $RQ$ is rational (based on Roquette (1958)).
Rational $p$-biset functors

Let $F$ be a biset functor, and $G$ be a finite group. The set of **faithful elements** of $F(G)$ is defined by $\partial F(G) = \bigcap_{1 \neq N \trianglelefteq G} \text{Def}^G_G/N$.

**Definition**

A $p$-biset functor is called **rational** if for any $p$-group $P$ and any genetic basis $G$ of $P$, the map

$$\bigoplus_{S \in G} \partial F(N_P(S)/S) \to F(P)$$

is an isomorphism.
Let $F$ be a biset functor, and $G$ be a finite group. The set of \textit{faithful elements} of $F(G)$ is defined by $\partial F(G) = \cap_{1 \neq N \trianglelefteq G} \text{Def}^G_{G/N}$.

\textbf{Definition}

A $p$-biset functor is called \textit{rational} if for any $p$-group $P$ and any genetic basis $\mathcal{G}$ of $P$, the map

$$\bigoplus \text{Ind}_{N_P(S)/S}^P \text{Inf}_{N_P(P)/S}^{N_P(S)} : \bigoplus_{S \in \mathcal{G}} \partial F(N_P(S)/S) \rightarrow F(P)$$

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Rational $p$-biset functors

Let $F$ be a biset functor, and $G$ be a finite group. The set of faithful elements of $F(G)$ is defined by $\partial F(G) = \bigcap_{1 \neq N \leq G} \text{Def}^G_{G/N}$.

**Definition**

A $p$-biset functor is called **rational** if for any $p$-group $P$ and any genetic basis $\mathcal{G}$ of $P$, the map

$$\bigoplus \text{Ind}^P_{NP(S)/S} \text{Inf}^N_{NP(S)/S} : \bigoplus_{S \in \mathcal{G}} \partial F(NP(S)/S) \rightarrow F(P)$$

is an isomorphism.

**Example**: The functor of rational representations $R_Q$
Let $F$ be a biset functor, and $G$ be a finite group. The set of **faithful elements** of $F(G)$ is defined by

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**Definition**

A $p$-biset functor is called **rational** if for any $p$-group $P$ and any genetic basis $\mathcal{G}$ of $P$, the map

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**Example** : The functor of rational representations $R_\mathbb{Q}$ is defined by $G \mapsto R_\mathbb{Q}(G)$,
Rational $p$-biset functors

Let $F$ be a biset functor, and $G$ be a finite group. The set of faithful elements of $F(G)$ is defined by $\partial F(G) = \bigcap_{1 \neq N \trianglelefteq G} \text{Def}^G_{G/N}$.

**Definition**

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Rational $p$-biset functors

Let $F$ be a biset functor, and $G$ be a finite group. The set of faithful elements of $F(G)$ is defined by $\partial F(G) = \bigcap_{1 \neq N \trianglelefteq G} \text{Def}_G^G / N$.

**Definition**

A $p$-biset functor is called **rational** if for any $p$-group $P$ and any genetic basis $\mathcal{G}$ of $P$, the map

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Rational $p$-biset functors

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**Definition**

A $p$-biset functor is called **rational** if for any $p$-group $P$ and any genetic basis $G$ of $P$, the map

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**Example** : The functor of rational representations $R_{\mathbb{Q}}$ is defined by $G \mapsto R_{\mathbb{Q}}(G)$, and the map $R_{\mathbb{Q}}(U)$ is induced by $\mathbb{Q}U \otimes_{\mathbb{Q}G} -$. The set $\partial R_{\mathbb{Q}}(G)$ consists of linear combinations of **faithful irreducible** rational representations of $G$. The $p$-biset functor $R_{\mathbb{Q}}$ is rational (based on Roquette (1958)).
Biset functors form an abelian category $\mathcal{F}$,
Using biset functors

Biset functors form an abelian category $\mathcal{F}$, and $p$-biset functors form an abelian category $\mathcal{F}_p$. 

Theorem (B. (2005))
Rational $p$-biset functors form a Serre subcategory of $\mathcal{F}_p$, stable by duality.

Theorem (B. (2007))
1. The map $\Phi$ induces an injective morphism of biset functors $\epsilon : \mathcal{B} \times \rightarrow \mathcal{F}_{2\mathcal{B}}$.
2. The image by $\epsilon$ of the $p$-biset functor $\mathcal{B} \times$ is contained in the $p$-biset functor $\mathcal{F}_{2\mathcal{B}}^\ast \mathcal{Q}$.
3. Hence $\mathcal{B} \times$ is a rational $p$-biset functor.
Biset functors form an abelian category $\mathcal{F}$, and $p$-biset functors form an abelian category $\mathcal{F}_p$.

**Theorem (B. (2005))**

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Rational $p$-biset functors form a Serre subcategory of $\mathcal{F}_p$, stable by duality.

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1. The map $\Phi$ induces an injective morphism of biset functors $\epsilon : B^\times \to \mathbb{F}_2 B^*$.
Using biset functors

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**Theorem (B. (2005))**

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1. The map $\Phi$ induces an injective morphism of biset functors $\epsilon : B^\times \rightarrow \mathbb{F}_2 B^*$.
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**Theorem (B. (2007))**

1. The map $\Phi$ induces an injective morphism of biset functors $\epsilon : B^\times \rightarrow \mathbb{F}_2 B^*$.  
2. The image by $\epsilon$ of the $p$-biset functor $B^\times$ is contained in the $p$-biset functor $\mathbb{F}_2 R^*_Q$.  
3. Hence $B^\times$ is a rational $p$-biset functor.
Recall that simple $p$-biset functors are indexed by pairs $(H, V)$, where $H$ is a finite $p$-group and $V$ is a simple $\mathbb{Z} \text{Out}(H)$-module (notation $(H, V) \mapsto S^H(V)$).

Theorem (B. (2007))
The functor $B \times$ is a uniserial object of the category $\mathcal{F}_p$ of $p$-biset functors. More precisely:

If $p > 2$, then $B \times$ is a simple object of $\mathcal{F}_p$, isomorphic to $S^1$, $F^2$.

If $p = 2$, then the full lattice of proper subobjects $\{0\} = F_0 \subset F_1 \subset \ldots \subset F_n \subset \ldots$ of $B \times$ is such that $F_1/F_0 \cong S^1$, $F_2/F_1$, and $F_i/F_{i-1} \cong S^{D_2^i + 2}$, $F_2$, for $i \geq 2$. 

Serge Bouc (CNRS-Université de Picardie)
Recall that simple biset functors are indexed by pairs \((H, V)\), where \(H\) is a finite group and \(V\) is a simple \(\mathbb{Z}Out(H)\)-module.
Recall that simple \textit{p}-biset functors are indexed by pairs \((H, V)\), where \(H\) is a finite \textit{p}-group and \(V\) is a simple \(\mathbb{Z}Out(H)\)-module.
Recall that simple $p$-biset functors are indexed by pairs $(H, V)$, where $H$ is a finite $p$-group and $V$ is a simple $\mathbb{Z}Out(H)$-module (notation $(H, V) \mapsto S_{H,V}$).

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Recall that simple $p$-biset functors are indexed by pairs $(H, V)$, where $H$ is a finite $p$-group and $V$ is a simple $\mathbb{Z} \text{Out}(H)$-module (notation $(H, V) \mapsto S_{H,V}$).

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The functor $B^\times$ is a uniserial object of the category $\mathcal{F}_p$ of $p$-biset functors. More precisely:

- If $p > 2$, then $B^\times$ is a simple object of $\mathcal{F}_p$, isomorphic to $S_{1,1}$, $F_2$, and $F_i$ isomorphic to $S_{2i+2}, F_2$, for $i \geq 2$. 

- If $p = 2$, then the full lattice of proper subobjects $\{0\} = F_0 \subset F_1 \subset \ldots \subset F_n \subset \ldots$ of $B^\times$ is such that $F_1/F_0 \cong S_{1,1}$, $F_2/F_0 \cong S_{2,2}$, and $F_i/F_{i-1} \cong S_{2i+2}, F_2$, for $i \geq 2$. 

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Functorial structure

Recall that simple $p$-biset functors are indexed by pairs $(H, V)$, where $H$ is a finite $p$-group and $V$ is a simple $\mathbb{Z}Out(H)$-module (notation $(H, V) \mapsto S_{H,V}$).

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  \[ \{0\} = F_0 \subset F_1 \subset \ldots \subset F_n \subset \ldots \text{ of } B^\times \text{ is such that } F_1/F_0 \cong S_{1,\mathbb{F}_2}, \]
  \[ \text{and } F_i/F_{i-1} \cong S_{D_{2i+2},\mathbb{F}_2}, \text{ for } i \geq 2. \]