# Non-additive exact functors and tensor induction for Mackey functors

Serge Bouc

Abstract. First I will introduce a generalization of the notion of (right)-exact functor between abelian categories to the case of non-additive functors. The main result of this section is an extension theorem: any functor defined on a suitable subcategory can be extended uniquely to a right exact functor defined on the whole category.

Next I use those results to define various functors of generalized tensor induction, associated to finite bisets, between categories attached to finite groups. This includes a definition of tensor induction for Mackey functors, for cohomological Mackey functors, for $p$-permutation modules and algebras. This also gives a single formalism of bisets for restriction, inflation, and ordinary tensor induction for modules.

Contents



## 1. Introduction

[1] The theory of Mackey functors for a finite group $G$ over a ring $R$ provides a single framework for the various representation theories of $G$ and its subgroups. So it looks like an extension of the notion of $RG$-module. The usual notions of

1991 *Mathematics Subject Classification.* Primary 18A22,19A22; Secundary 20J05.

*Key words and phrases.* additive, exact, tensor induction, Burnside, Mackey functor, Green functor.

[1]Received by the editor November 20, 1997, and in revised form November 16, 1998.

induction, restriction, inflation, ... for modules, have their analogues for Mackey functors. I will describe here a missing item in that list: tensor induction.

The first part of this paper is actually more general, and not specific to Mackey functors. It introduces an extension of the notion of right exact functor between abelian categories to non-additive functors: the usual definition of right exactness actually implies additivity, so it has to be modified in order to be extended. The main theorem of this general setting concerns the extension of a (non necessarily additive) functor from a suitable sub-category $\mathcal{P}$ of an abelian category $\mathcal{A}$ to an abelian category $\mathcal{B}$, to a right exact functor from $\mathcal{A}$ to $\mathcal{B}$.

Next I apply those results to various constructions of tensor induction. In all those cases, I will consider two finite groups $G$ and $H$, a finite set $U$ with a $H \times G^{op}$-action (or $H$-set-$G$, or biset), and I will define a (generalized) tensor induction associated to $U$, which will be a functor between categories $\mathcal{C}(G)$ and $\mathcal{C}(H)$ naturally attached to $G$ and $H$.

In the first case, the category $\mathcal{C}(G)$ is the category of Mackey functors for $G$. I will apply the extension theorem to the subcategory of "permutation functors", and this leads to a generalized tensor induction functor $T_U$ from Mackey functors for $G$ to Mackey functors for $H$, associated to a finite biset $U$. This tensor induction behaves well with respect to composition of functors, tensor product of Mackey functors, and disjoint unions of bisets [2]. There is also a kind of binomial formula for the tensor induction of a direct sum.

Next I consider the relations between tensor induction and other functors between categories of Mackey functors, such as induction, restriction, inflation, ... I also define a reasonable notion of direct product of Mackey functors, and study its relations with tensor induction. Finally, I extend those notions to the case of Green functors.

The second case deals with the category $\mathcal{C}(G)$ of cohomological Mackey functors for $G$ over a commutative ring $R$, and uses the subcategory of "permutation cohomological Mackey functors". There is a generalized tensor induction functor associated to finite biset $U$, whenever $U$ is "free enough" with respect to $R$. This cohomological tensor induction is closely related to the tensor induction for Mackey functors.

It leads to the definition of a generalized tensor induction for $p$-permutation modules and $p$-permutation algebras: this was the very starting point of that work, in a conversation with Jacques Thévenaz, who asked me about the possibility of such a generalized construction, giving a suitable functorial structure for the Dade group. In our joint recent preprint [**BT98**], we give an independent exposition of this generalized tensor induction for permutations algebras for $p$-groups, and use it to solve some open questions about the Dade group.

The third case is the case of the category $\mathcal{C}(G)$ of $RG$ modules, using the subcategory of free $RG$-modules. This leads to a generalized tensor induction associated to a finite right-free biset $U$. The case $U = H$, when $G$ is a subgroup of $H$, is the usual tensor induction from $RG$-modules to $RH$-modules. There is no essentially new construction here, since the other cases correspond to restriction and inflation of modules.

---

[2] The construction of a tensor induction for Mackey functors with those properties was a question of T. Yoshida at the Seattle AMS conference (Problem 37 in [**ACPW98**])

It should be noted however that even in those well known cases, the formalism of bisets gives a single natural framework involving restriction, inflation and tensor induction. The classical properties of those construction, such as Mackey formula, transitivity of tensor induction, or composition of inflation and tensor induction, are various aspects of a single simple composition formula. Similarly, the formula giving the tensor induction of a direct sum, which is generally incomplete (and evaluated only "up to terms induced from proper subgroups"), is nothing but a generalization of the binomial formula, and can be written explicitly.

## 2. Non additive exact functors

**2.1. Notations.** If $M$, $N$ and $P$ are objects of an abelian category $\mathcal{A}$, and $f : M \oplus N \to P$ is a morphism, I will denote $f$ by $(f \circ i_M, f \circ i_N)$, where $i_M$ and $i_N$ are the canonical injections from $M$ and $N$ to $M \oplus N$. Similarly, if $g : P \to M \oplus N$ is a morphism, I will denote it by $\binom{s_M \circ g}{s_N \circ g}$, where $s_M$ and $s_N$ are the canonical surjections from $M \oplus N$ to $M$ and $N$. With those notations, the usual rules of matrix multiplication apply.

The identity morphism of $M$ will generally be denoted by 1, and by $Id_M$ if some precision is needed. The zero morphisms will be denoted by 0.

**2.2. Definition.** First I observe that the classical definition of an exact functor actually implies additivity:

LEMMA 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and $F$ be a functor from $\mathcal{A}$ to $\mathcal{B}$, which is not supposed to be additive. Suppose that for any exact sequence in $\mathcal{A}$*

$$M \xrightarrow{\varphi} N \xrightarrow{\psi} L \to 0$$

*the associated sequence*

$$F(M) \xrightarrow{F(\varphi)} F(N) \xrightarrow{F(\psi)} F(L) \to 0$$

*is exact in $\mathcal{B}$. Then $F$ is additive.*

PROOF. Note that I don't suppose that the second exact sequence is the image by $F$ of the first one. In other words, I don't suppose that $F(0) = 0$. But it is a consequence of the exactness of the second sequence: indeed, as $F(\psi) \circ F(\psi) = F(\psi \circ \phi) = F(0)$ has to be zero, and as the identity of the zero object factors through any (zero) morphism, the identity of $F(0)$ has to be zero. Hence $F$ maps the zero object to the zero object.

Let $M$ and $N$ be objects of $\mathcal{A}$. Applying the hypothesis to the sequence

$$M \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} M \oplus N \xrightarrow{(0,1)} N \to 0$$

shows that the sequence

$$F(M) \xrightarrow{F\begin{pmatrix}1\\0\end{pmatrix}} F(M \oplus N) \xrightarrow{F(0,1)} F(N) \to 0$$

is exact. But the morphism $F\binom{1}{0}$ is a split monomorphism, and $F(0,1)$ is a split epimorphism. This proves that there are inverse isomorphisms

$$i_{M,N} = \begin{pmatrix} F(1,0) \\ F(0,1) \end{pmatrix} : F(M \oplus N) \to F(M) \oplus F(N)$$

$$j_{M,N} = \left(F\begin{pmatrix}1\\0\end{pmatrix}, F\begin{pmatrix}0\\1\end{pmatrix}\right) : F(M) \oplus F(N) \to F(M \oplus N)$$

Now if $f, g : M \to N$ are morphisms in $\mathcal{A}$, their sum $f+g$ is obtained by composition

$$M \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} M \oplus M \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}} N \oplus N \xrightarrow{(1,1)} N$$

Taking images by $F$ gives the commutative diagram

$$\begin{array}{ccccccc}
F(M) & \xrightarrow{F\begin{pmatrix}1\\1\end{pmatrix}} & F(M \oplus M) & \xrightarrow{F\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}} & F(N \oplus N) & \xrightarrow{F(1,1)} & F(N) \\
1 \downarrow \uparrow 1 & & i_{M,M} \downarrow \uparrow j_{M,M} & & i_{N,N} \downarrow \uparrow j_{N,N} & & 1 \downarrow \uparrow 1 \\
F(M) & \xrightarrow{\Delta} & F(M) \oplus F(M) & \xrightarrow{\varphi} & F(N) \oplus F(N) & \xrightarrow{\Sigma} & F(N)
\end{array}$$

where the bottom row is obtained through the previous isomorphisms. Thus

$$\Delta = i_{M,M} \circ F\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix} F(1,0) \\ F(0,1) \end{pmatrix} F\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix} F(1) \\ F(1) \end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

and similarly $\Sigma = (1,1)$. Moreover

$$\varphi = \begin{pmatrix} F(1,0) \\ F(0,1) \end{pmatrix} F\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \left(F\begin{pmatrix}1\\0\end{pmatrix}, F\begin{pmatrix}0\\1\end{pmatrix}\right)$$

It follows that

$$\varphi = \begin{pmatrix} F(1,0) \\ F(0,1) \end{pmatrix} \left(F\begin{pmatrix}f\\0\end{pmatrix}, F\begin{pmatrix}0\\g\end{pmatrix}\right) = \begin{pmatrix} F(f) & F(0) \\ F(0) & F(g) \end{pmatrix} = \begin{pmatrix} F(f) & 0 \\ 0 & F(g) \end{pmatrix}$$

Now the composition $\Sigma \circ \varphi \circ \Delta$ is equal to $F(f) + F(g)$. It is also equal to $F(f+g)$, so $F$ is additive. □

I will modify the definition of right exactness to extend it to non-additive functors. First I need the following notation:

NOTATION 2.2. If $\varphi : M \to N$ is a morphism in $\mathcal{A}$, I can build the morphisms $(\varphi, 1)$ and $(0,1)$ from $M \oplus N$ to $N$. So I have morphisms $F(\varphi, 1)$ and $F(0,1)$ in $\mathcal{B}$ from $F(M \oplus N)$ to $F(N)$. I denote by $\Delta F(\varphi)$ their difference

$$\Delta F(\varphi) = F(\varphi, 1) - F(0,1) : F(M \oplus N) \to F(N)$$

If $\psi : N \to L$ is a morphism in $\mathcal{A}$ such that $\psi \circ \varphi = 0$, then of course

$$F(\psi) \circ \Delta F(\varphi) = F\big(\psi \circ (\varphi, 1)\big) - F\big(\psi \circ (0,1)\big) = 0$$

since $\psi \circ (\varphi, 1) = (\psi \circ \varphi, \psi) = (0, \psi) = \psi \circ (0,1)$. This leads to the following definition:

DEFINITION 2.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a (non-necessarily additive) functor between abelian categories. I will say that $F$ is right exact, if for any exact sequence*

$$M \xrightarrow{\varphi} N \xrightarrow{\psi} L \to 0 \tag{2.1}$$

*the associated sequence*

$$F(M \oplus N) \xrightarrow{\Delta F(\varphi)} F(N) \xrightarrow{F(\psi)} F(L) \to 0 \tag{2.2}$$

*is exact.*

In particular, a right exact functor maps epimorphims to epimorphisms.

REMARK 2.4. If $F$ is additive, then $\Delta F(\varphi) = F(\varphi, 0)$, so the previous sequence factors as

$$F(M \oplus N) \xrightarrow{F(1,0)} F(M) \xrightarrow{F(\varphi)} F(N) \xrightarrow{F(\psi)} F(L) \to 0$$

The left morphism is a split epimorphism. So $F$ is right exact for the modified definition if and only if the sequence

$$F(M) \xrightarrow{F(\varphi)} F(N) \xrightarrow{F(\psi)} F(L) \to 0$$

is exact, that is if and only if $F$ is right exact in the usual sense. So the new definition is equivalent to the usual one for additive functors.

REMARK 2.5. Let $P$ be any object of $\mathcal{B}$. Define $F(M) = P$ for any object $M$ of $\mathcal{A}$, and $F(\varphi) = Id_P$ for any map $\varphi$ in $\mathcal{A}$. Then $F$ is a (trivial) example of a right exact functor, which is not additive if $P$ is non-zero.

REMARK 2.6. A functor $F$ is exact if and only if the sequence (2.2) is exact for any short exact sequence

$$0 \to M \xrightarrow{\varphi} N \xrightarrow{\psi} L \to 0 \tag{2.3}$$

Indeed this is obviously a necessary condition. Conversely, if the sequence (2.2) is exact for any short exact sequence (2.3), then in particular, the functor $F$ maps epimorphisms to epimorphisms. Now if

$$M' \xrightarrow{\varphi'} N \xrightarrow{\psi} L \to 0$$

is an exact sequence, denoting by $M$ the cokernel of $\varphi'$, then $\varphi'$ factors through $M$ as $\varphi' = \iota \circ \sigma$, where $\iota$ is a monomorphism and $\sigma$ is an epimorphism. Now the map $\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$ is an epimorphism from $M' \oplus N$ to $M \oplus N$, and so is $F\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$. Moreover

$$\Delta F(\iota) \circ F\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix} = F\Big((\iota, 1)\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}\Big) - F\Big((0,1)\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}\Big) = \dots$$
$$\dots = F(\varphi', 1) - F(0,1) = \Delta F(\varphi')$$

So if $\theta : F(N) \to P$ is any map in $\mathcal{B}$, then $\theta \circ \Delta F(\iota)$ is zero if and only if $\theta \circ \Delta F(\varphi')$ is, since $F\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$ is an epimorphism. Thus $\Delta F(\iota)$ and $\Delta F(\varphi')$ have the same image, and the sequence

$$F(M' \oplus N) \xrightarrow{\Delta F(\varphi')} F(N) \xrightarrow{F(\psi)} F(L) \to 0$$

is also exact.

REMARK 2.7. Suppose that the sequence

$$M \xrightarrow{\varphi} N \xrightarrow{\psi} L \to 0$$

is exact and *split*, in the following sense: there exist a morphism $\alpha : N \to M$ and a morphism $\beta : L \to N$ such that

$$\psi \circ \beta = 1 \qquad\qquad \varphi \circ \alpha + \beta \circ \psi = 1$$

(note that this will be the case in particular if it is exact, and if $M$, $N$ and $L$ are *projective* in $\mathcal{A}$). Then the sequence

$$F(M \oplus N) \xrightarrow{\Delta F(\varphi)} F(N) \xrightarrow{F(\psi)} F(L) \to 0$$

is also exact and split: indeed, set

$$A = -F\begin{pmatrix} -\alpha \\ 1 \end{pmatrix} : F(N) \to F(M \oplus N)$$

$$B = F(\beta) : F(L) \to F(N)$$

Then it is clear that $F(\psi) \circ B = F(\psi \circ \beta) = F(1) = 1$, and that

$$\Delta F(\varphi) \circ A + B \circ F(\psi) = -\Big(F(\varphi, 1) - F(0, 1)\Big) \circ F\begin{pmatrix} -\alpha \\ 1 \end{pmatrix} + F(\beta \circ \psi) = \dots$$

$$\dots = -F(-\varphi \circ \alpha + 1) + F(1) + F(\beta \circ \psi) = F(1) = 1$$

So the condition of the definition of a right exact functor is void on the split exact sequences. In particular, if every exact sequence in $\mathcal{A}$ is split, then every functor from $\mathcal{A}$ to an abelian category is right exact.

REMARK 2.8. Let $p = F(0, 1) : F(M \oplus N) \to F(N)$, and $i = F\binom{0}{1} : F(N) \to F(M \oplus N)$. Then $i \circ p = F\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is an idempotent endomorphism of $F(M \oplus N)$. Moreover

$$F(\varphi, 1)\Big(1 - F\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\Big) = F(\varphi, 1) - F(0, 1) = \Delta F(\varphi)$$

So the functor $F$ is right exact if and only if for any short exact sequence (2.1), the sequence

$$(1 - F\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})F(M \oplus N) \xrightarrow{F(\varphi, 1)} F(N) \xrightarrow{F(\psi)} F(L) \to 0$$

is exact.

### 2.3. Basic properties.

2.3.1. *Composition.* The class of right exact functors is closed by composition:

PROPOSITION 2.9. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be right exact functors between abelian categories. Then $G \circ F$ is right exact.*

PROOF. Let $M \overset{\varphi}{\to} N \overset{\psi}{\to} L \to 0$ be an exact sequence in $\mathcal{A}$. Since $F$ is right exact, the sequence

$$F(M \oplus N) \xrightarrow{\Delta F(\varphi)} F(N) \xrightarrow{F(\psi)} F(L) \to 0$$

is exact. And since $G$ is right exact, the sequence

$$G\Big(F(M \oplus N) \oplus F(N)\Big) \xrightarrow{\Delta G(\Delta F(\varphi))} G \circ F(N) \xrightarrow{G \circ F(\psi)} G \circ F(L) \to 0$$

is exact. Moreover

$$\Delta G(\Delta F(\varphi)) = G\Big(F(\varphi, 1) - F(0, 1), 1\Big) - G(0, 1)$$

On the other hand, the functor $G \circ F$ is right exact if and only if the sequence

$$G \circ F(M \oplus N) \xrightarrow{\Delta(G \circ F)(\varphi)} G \circ F(N) \xrightarrow{G \circ F(\psi)} G \circ F(L) \to 0 \tag{2.4}$$

is exact. Let

$$D = \Delta G(\Delta F(\varphi)) = G\Big(F(\varphi, 1) - F(0, 1), 1\Big) - G(0, 1)$$

$$D' = \Delta(G \circ F)(\varphi) = G \circ F(\varphi, 1) - G \circ F(0, 1)$$

I will show that $\mathrm{Im}\, D = \mathrm{Im}\, D'$.

Let $\alpha$ be the morphism from $F(M \oplus N)$ to $F(M \oplus N) \oplus F(N)$ defined by

$$\alpha = \begin{pmatrix} 1 - F\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ F(0, 1) \end{pmatrix}$$

and let $A = G(\alpha)$. Let $\beta$ be the morphism from $F(M \oplus N) \oplus F(N)$ to $F(M \oplus N)$ defined by

$$\beta = \left(1 - F\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, F\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

and let $B = G(\beta)$. Then $D' \circ B = G(v) - G(v')$, where

$$v = F(\varphi, 1) \circ \beta \qquad\qquad v' = F(0, 1) \circ \beta$$

Note that $v'$ is obtained from $v$ by replacing $\varphi$ by $0$. But

$$v = \left(F(\varphi, 1)(1 - F\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}), F(\varphi, 1)F\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \Big(F(\varphi, 1) - F(0, 1), 1\Big)$$

So $v' = (0, 1)$, and

$$D' \circ B = G\Big(F(\varphi, 1) - F(0, 1), 1\Big) - G(0, 1) = D$$

Moreover $B \circ A = G(s)$, with

$$s = \left(1 - F\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, F\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\begin{pmatrix} 1 - F\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ F(0, 1) \end{pmatrix} = \dots$$

$$\ldots = 1 - F\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + F\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

So $B \circ A = 1$, and in particular $B$ is an epimorphism. So if $\theta : G \circ F(N) \to P$ is any map in $\mathcal{B}$, then

$$\theta \circ D = 0 \Leftrightarrow \theta \circ D' \circ B = 0 \Leftrightarrow \theta \circ D' = 0$$

So $D$ and $D'$ have the same image, and the sequence (2.4) is exact. This completes the proof of the proposition. □

2.3.2. *Products and sums.* If $\mathcal{A}$ and $\mathcal{A}'$ are abelian categories, then their product $\mathcal{A} \times \mathcal{A}'$ is also abelian. If $f : M \to N$ and $f' : M' \to N'$ are maps in $\mathcal{A}$ and $\mathcal{A}'$, I will denote by $[M, M']$ and $[N, N']$ the associated couples in $\mathcal{A} \times \mathcal{A}'$, and $[f, f'] : [M, M'] \to [N, N']$ the associated morphism. The image of $[M, M']$ under a functor $F$ will be denoted by $F[M, M']$ instead of $F([M, M'])$.

PROPOSITION 2.10. *Let $F : \mathcal{A} \to \mathcal{B}$ and $F' : \mathcal{A}' \to \mathcal{B}'$ be right exact functors between abelian categories. Then*

$$F \times F' : \mathcal{A} \times \mathcal{A}' \to \mathcal{B} \times \mathcal{B}'$$

*is right exact.*

PROOF. This is obvious, since a product of exact sequences is exact. □

COROLLARY 2.11. *Let $F, F' : \mathcal{A} \to \mathcal{B}$ be right exact functors between abelian categories. Then $F \oplus F'$ is right exact.*

PROOF. The functor $F \oplus F'$ factors as

$$\mathcal{A} \stackrel{\Delta}{\to} \mathcal{A} \times \mathcal{A} \xrightarrow{F \times F'} \mathcal{B} \times \mathcal{B} \stackrel{\Sigma}{\to} \mathcal{B}$$

where $\Delta$ is the diagonal functor, mapping the object $M$ to $[M, M]$ and the morphism $f$ to $[f, f]$, and $\Sigma$ is the direct sum functor mapping the object $[P, Q]$ to $P \oplus Q$, and the morphism $[f, g]$ to $f \oplus g$. Those two functors are obviously additive and exact, so the corollary follows from proposition 2.9. □

2.3.3. *Pairings.* If $\mathcal{A}$, $\mathcal{A}'$, and $\mathcal{B}$ are abelian categories, a pairing $F : \mathcal{A} \times \mathcal{A}' \to \mathcal{B}$ is just a biadditive functor: for any object $M$ of $\mathcal{A}$, the functor $F[M, -] : \mathcal{A}' \to \mathcal{B}$ is additive, and for any object $M'$ of $\mathcal{A}'$, the functor $F[-, M'] : \mathcal{A} \to \mathcal{B}$ is additive. Note that $F$ itself is not additive in general.

PROPOSITION 2.12. *Let $\mathcal{A}$, $\mathcal{A}'$, and $\mathcal{B}$ be abelian categories, and $F : \mathcal{A} \times \mathcal{A}' \to \mathcal{B}$ be a pairing. The following are equivalent:*

1. *The functor $F$ is right exact.*
2. *For any objects $M$ of $\mathcal{A}$ and $M'$ of $\mathcal{A}'$, the (additive) functors $F[M, -]$ and $F[-, M']$ are right exact.*
3. *For any exact sequences*

$$M \stackrel{\varphi}{\to} N \stackrel{\psi}{\to} L \to 0 \qquad\qquad M' \stackrel{\varphi'}{\to} N' \stackrel{\psi'}{\to} L' \to 0$$

*the sequence*

$$F[M, N'] \oplus F[N, M'] \xrightarrow{\big(F[\varphi, 1], F[1, \varphi']\big)} F[N, N'] \xrightarrow{F[\psi, \psi']} F[L, L'] \to 0$$

*is exact.*

PROOF. Suppose that $F$ is right exact. Fix an object $M'$ of $\mathcal{A}'$. Now the functor $F[-,M'] : M \mapsto F[M,M']$ factors as

$$F[-,M'] = F \circ (Id_{\mathcal{A}} \times c_{M'}) \circ \Delta$$

where $\Delta$ is the diagonal functor $\mathcal{A} \to \mathcal{A} \times \mathcal{A}$ as above, and $c_{M'}$ is the constant functor, equal to $M'$ everywhere. So $F[-,M']$ is composed of three right exact functors, hence it is right exact by proposition 2.9. A similar argument shows that $F[M,-]$ is right exact for any object $M$ of $\mathcal{A}$, so 1) implies 2) (note that this does not depend on the fact that $F$ is a pairing).

Now if 2) holds, and if

$$M \stackrel{\varphi}{\to} N \stackrel{\psi}{\to} L \to 0 \qquad\qquad M' \stackrel{\varphi'}{\to} N' \stackrel{\psi'}{\to} L' \to 0$$

are exact sequences, I have the following commutative diagram

$$\begin{array}{ccccccc}
F[M,M'] & \to & F[M,N'] & \to & F[M,L'] & \to 0 \\
\downarrow & & \downarrow h & & \downarrow & \\
F[N,M'] & \stackrel{g}{\to} & F[N,N'] & \stackrel{b}{\to} & F[N,L'] & \to 0 \\
\downarrow f & & \downarrow a & & \downarrow d & \\
F[L,M'] & \stackrel{e}{\to} & F[L,N'] & \stackrel{c}{\to} & F[L,L'] & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 &
\end{array}$$

where $a = F[\psi,1], \ldots h = F[\varphi,1]$. The rows and columns of this diagram are exact. To prove that 3) holds, I must show that the sequence

$$F[M,N'] \oplus F[N,M'] \xrightarrow{(h,g)} F[N,N'] \xrightarrow{c \circ a} F[L,L'] \to 0$$

is exact. But $c \circ a$ is the product of two epimorphisms, so it is an epimorphism. And if $\theta : F[N,N'] \to P$ is any map in $\mathcal{B}$ such that $\theta \circ (h,g) = 0$, then $\theta \circ h = 0$ and $\theta \circ g = 0$. As the middle column is exact, the map $\theta$ factors as $\theta = \theta' \circ a$. Now

$$0 = \theta \circ g = \theta' \circ a \circ g = \theta' \circ e \circ f$$

As $f$ is an epimorphism, this gives $\theta' \circ e = 0$, and as the bottom row is exact, the map $\theta'$ factors as $\theta' = \theta" \circ c$. So $\theta = \theta" \circ c \circ a$, and the image of $(h,g)$ is the kernel of $c \circ a$. So 2) implies 3).

Now suppose 3) holds. Let

$$M \stackrel{\varphi}{\to} N \stackrel{\psi}{\to} L \to 0 \qquad\qquad M' \stackrel{\varphi'}{\to} N' \stackrel{\psi'}{\to} L' \to 0$$

be exact sequences in $\mathcal{A}$ and $\mathcal{A}'$. The sequence

$$U = F[M,N'] \oplus F[N,M'] \xrightarrow{\left(F[\varphi,1],\, F[1,\varphi']\right)} F[N,N'] \xrightarrow{F[\psi,\psi']} F[L,L'] \to 0$$

is exact, and to prove 1), I must show that the sequence

$$V = F[M \oplus N, M' \oplus N'] \xrightarrow{\Delta F[\varphi,\varphi']} F[N,N'] \xrightarrow{F[\psi,\psi']} F[L,L'] \to 0$$

is exact. I will set $D = \Big(F[\varphi,1], F[1,\varphi']\Big)$.

Let $i : [M, N'] \to [M \oplus N, M' \oplus N']$ be the map in $\mathcal{A} \times \mathcal{A}'$ defined by $i = \Big[\binom{1}{0}, \binom{0}{1}\Big]$. Similarly, let $j : [N, M'] \to [M \oplus N, M' \oplus N']$ be the map defined by $j = \Big[\binom{0}{1}, \binom{1}{0}\Big]$. Let $A : U \to V$ defined by $A = \Big(F(i), F(j)\Big)$.

Define moreover $k : [M \oplus N, M' \oplus N'] \to [M, N']$ by $k = \Big[(1, 0), (\varphi', 1)\Big]$ and $l : [M \oplus N, M' \oplus N'] \to [N, M']$ by $l = \Big[(0, 1), (1, 0)\Big]$. Let $B : V \to U$ defined by $B = \binom{F(k)}{F(l)}$. Now

$$D \circ B = \Big(F[\varphi, 1], F[1, \varphi']\Big) \circ \begin{pmatrix} F(k) \\ F(l) \end{pmatrix} = \dots$$
$$\dots = F\Big([\varphi, 1]\Big[(1, 0), (\varphi', 1)\Big]\Big) + F\Big([1, \varphi']\Big[(0, 1), (1, 0)\Big]\Big) = \dots$$
$$\dots = F\Big([(\varphi, 0), (\varphi', 1)]\Big) + F\Big([(0, 1), (\varphi', 0)]\Big)$$

Since $F$ is biadditive, this is also equal to

$$F\Big([(\varphi, 0), (\varphi', 0)]\Big) + F\Big([(\varphi, 0), (0, 1)]\Big) + F\Big([(0, 1), (\varphi', 0)]\Big)$$

On the other hand, the map $D' = \Delta F[\varphi, \varphi']$ is equal to

$$\Delta F[\varphi, \varphi'] = F\Big[(\varphi, 1), (\varphi', 1)\Big] - F\Big[(0, 1), (0, 1)\Big] = \dots$$
$$\dots = F\Big([(\varphi, 0), (\varphi', 0)]\Big) + F\Big([(\varphi, 0), (0, 1)]\Big) + F\Big([(0, 1), (\varphi', 0)]\Big)$$

Thus $D \circ B = D'$. Moreover

$$B \circ A = \begin{pmatrix} F(k) \\ F(l) \end{pmatrix} \Big(F(i), F(j)\Big) = \begin{pmatrix} F(k \circ i) & F(k \circ j) \\ F(l \circ i) & F(l \circ j) \end{pmatrix}$$

But

$$k \circ i = \Big[(1, 0), (\varphi', 1)\Big]\Big[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big] = [1, 1]$$

So $F(k \circ i) = 1$. Similarly

$$k \circ j = \Big[(1, 0), (\varphi', 1)\Big]\Big[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big] = [0, \varphi']$$

So $F(k \circ j) = 0$, since $F$ is biadditive. Moreover

$$l \circ i = \Big[(0, 1), (1, 0)\Big]\Big[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big] = [0, 0]$$

$$l \circ j = \Big[(0, 1), (1, 0)\Big]\Big[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big] = [1, 1]$$

Finally, I have

$$B \circ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

In particular $B$ is an epimorphism. As $D \circ B = D'$, the images of $D$ and $D'$ are the same. So 3) implies 1), and this completes the proof. □

Corollary 2.13. *Let $[M, N] \mapsto M \otimes N$ be a (biadditive) tensor product from $\mathcal{B} \times \mathcal{B}$ to an abelian category $\mathcal{C}$, which is right exact with respect to $M$ and $N$. Then*

1. *If $F$ and $F'$ are right exact functors from $\mathcal{A}$ to $\mathcal{B}$, so is $F \otimes F' : \mathcal{A} \to \mathcal{C}$.*

2. *In particular if $\mathcal{A} = \mathcal{B} = \mathcal{C}$, then for any positive integer $n$, define inductively the functor $M \mapsto M^{\otimes n}$ by*

$$M^{\otimes 1} = M \qquad\qquad M^{\otimes n} = M^{\otimes (n-1)} \otimes M \quad \text{if} \quad n > 1$$

*Then the functor $M \mapsto M^{\otimes n}$ is right exact.*

Proof. The functor $F \otimes F'$ is the functor from $\mathcal{A}$ to $\mathcal{C}$ defined by the composition

$$\mathcal{A} \overset{\Delta}{\to} \mathcal{A} \times \mathcal{A} \xrightarrow{F \times F'} \mathcal{B} \times \mathcal{B} \overset{\otimes}{\to} \mathcal{B}$$

So it is exact by proposition 2.9.

If moreover $\mathcal{A} = \mathcal{B} = \mathcal{C}$, the functor $M \mapsto M^{\otimes n}$ is right exact by an easy induction argument. □

**2.4. Extension of functors.** The main result concerning right exact functors is the following:

Theorem 2.14. *Let $\mathcal{P}$ be a full subcategory of an abelian category $\mathcal{A}$ with the following properties:*

1. *The objects of $\mathcal{P}$ are projective in $\mathcal{A}$.*
2. *Any object of $\mathcal{A}$ is a quotient of an object of $\mathcal{P}$.*
3. *If $P$ and $Q$ are objects of $\mathcal{P}$, then so is $P \oplus Q$.*

*Then any functor $F$ from $\mathcal{P}$ to an abelian category $\mathcal{B}$ can be uniquely extended (up to isomorphism of functors) to a right exact functor from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. First uniqueness is almost obvious: let $F_1$ and $F_2$ be right exact functors from $\mathcal{A}$ to $\mathcal{B}$, and $\theta$ be an isomorphism from the restriction of $F_1$ to $\mathcal{P}$ to the restriction of $\mathcal{P}$. In particular, for any object $P$ of $\mathcal{P}$, there is an isomorphism $\theta_P$ from $F_1(P)$ to $F_2(P)$.

Now uniqueness will follow from the following

Proposition 2.15. *Let $\mathcal{A}$ be an abelian category, and $\mathcal{P}$ be a full subcategory of $\mathcal{A}$ satisfying the hypothesis of theorem 2.14. Let $F_1$ and $F_2$ be (non-necessarily additive) right exact functors from $\mathcal{A}$ to an abelian category $\mathcal{B}$. Then if $\theta$ is a natural transformation from the restriction of $F_1$ to $\mathcal{P}$ to the restriction of $F_2$ to $\mathcal{P}$, there exists a unique natural transformation $\tilde{\theta}$ from $F_1$ to $F_2$ which coincides with $\theta$ on $\mathcal{P}$.*

Proof. For any object $M$ of $\mathcal{A}$, choose a short exact sequence

$$Q \overset{\varphi}{\to} P \overset{\psi}{\to} M \to 0 \tag{2.5}$$

with $P$ and $Q$ in $\mathcal{P}$. Such a sequence exists by condition 2). Since $F_1$ and $F_2$ are right exact, the rows of the following commutative diagram are exact

$$\begin{array}{ccccccc}
F_1(Q \oplus P) & \xrightarrow{\Delta F_1(\varphi)} & F_1(P) & \xrightarrow{F_1(\psi)} & F_1(M) & \to 0 \\
\downarrow \theta_{Q\oplus P} & & \downarrow \theta_P & & & \\
F_2(Q \oplus P) & \xrightarrow{\Delta F_2(\varphi)} & F_2(P) & \xrightarrow{F_2(\psi)} & F_2(M) & \to 0
\end{array}$$

Then there is a unique morphism $\tilde{\theta}_M : F_1(M) \to F_2(M)$ completing this diagram into a commutative one. This morphism does not depend on the choice of the resolution 2.5: indeed, if

$$Q' \stackrel{\varphi'}{\to} P' \stackrel{\psi'}{\to} M \to 0$$

is another resolution of $M$ by objects of $\mathcal{P}$, then as $\psi'$ is an epimorphism and as $P$ is projective, there is a morphism $a : P \to P'$ such that $\psi' \circ a = \psi$. Thus if $\tilde{\theta}'_M : F_1(M) \to F_2(M)$ is built using the second resolution, I have

$$\tilde{\theta}'_M F_1(\psi) = \tilde{\theta}'_M F_1(\psi')F_1(a) = F_2(\psi')\theta_{P'}F_1(a) = \dots$$

$$\dots = F_2(\psi')F_2(a)\theta_P = F_2(\psi)\theta_P = \tilde{\theta}_M F_1(\psi)$$

So $\tilde{\theta}_M = \tilde{\theta}'_M$ since $F_1(\psi)$ is an epimorphism. Moreover if $M = P$ is already in $\mathcal{P}$, I can choose the resolution $P \stackrel{0}{\to} P \stackrel{1}{\to} M \to 0$ of $M$, and then

$$\tilde{\theta}_M = \tilde{\theta}_M F_1(1) = F_2(1)\theta_P = \theta_P$$

Now it is clear that the maps $\tilde{\theta}_M$ give a well defined natural transformation extending $\theta$, and that this extension is unique. □

To complete the proof of the theorem, I have to prove the existence of an extension $F'$ of $F$. For any object of $\mathcal{A}$, I choose an exact sequence (2.5). Since $F'$ must be right exact, and coincide with $F$ on $\mathcal{P}$, the sequence

$$F(Q \oplus P) \xrightarrow{\Delta F(\varphi)} F(P) \xrightarrow{F'(\psi)} F'(M) \to 0$$

must be exact. So I can define $F'(M)$ as the cokernel of $\Delta F(\varphi)$. Of course, I must make this definition functorial with respect to $M$.

I will show that if $f : M \to M'$ is a morphism in $\mathcal{A}$, then there is a well defined morphism $F'_f : F'(M) \to F'(M')$, which is moreover functorial with respect to $f$: this follows from standard arguments on projective resolutions.

If $Q' \stackrel{\varphi'}{\to} P' \stackrel{\psi'}{\to} M' \to 0$ is any exact sequence with $P'$ and $Q'$ in $\mathcal{P}$, then there is a map $a : P \to P'$ such that $\psi' \circ a = f \circ \psi$, because $P$ is projective and $\psi'$ is an epimorphism. Now $\psi' \circ a \circ \varphi = 0$, so as $\mathrm{Ker}\psi' = \mathrm{Im}\varphi'$, and as $Q$ is projective, there is a map $b : Q \to Q'$ such that $\varphi' \circ b = a \circ \varphi$.

Now I have the following diagram with exact rows

$$\begin{array}{ccccccc}
F(Q \oplus P) & \xrightarrow{\Delta F(\varphi)} & F(P) & \xrightarrow{F'(\psi)} & F'(M) & \to 0 \\
{\scriptstyle F\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}} \downarrow & & {\scriptstyle F(a)} \downarrow & & & \\
F(Q' \oplus P') & \xrightarrow{\Delta F(\varphi')} & F(P') & \xrightarrow{F'(\psi')} & F'(M') & \to 0
\end{array}$$

This diagram is commutative since

$$\Delta F(\varphi')F\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} = \Big(F(\varphi', 1) - F(0, 1)\Big)F\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} = F(\varphi' \circ b, a) - F(0, a) = \dots$$

$$\dots = F(a \circ \varphi, a) - F(0, a) = F(a)\Big(F(\varphi, 1) - F(0, 1)\Big) = F(a) \circ \Delta F(\varphi)$$

It follows that there is a morphism $F'_f : F'(M) \to F'(M')$ such that $F'_f \circ F'(\psi) = F'(\psi') \circ F(a)$.

This morphism does not depend on the choice of $a$ and $b$, since if $a' : P \to P'$ is another map such that $\psi' \circ a' = f \circ \psi$, then $\psi' \circ (a - a') = 0$, so there is a map $c : P \to Q'$ such that $a - a' = \varphi' \circ c$. Now $F\binom{c}{a'}$ is a map from $F(P)$ to $F(Q' \oplus P')$, and

$$\Delta F(\varphi')F\begin{pmatrix} c \\ a' \end{pmatrix} = \Big(F(\varphi', 1) - F(0, 1)\Big)F\begin{pmatrix} c \\ a' \end{pmatrix} = F(\varphi' \circ c + a') - F(a') = F(a) - F(a')$$

This proves first that $F'_f$ is well defined. So if $F = 1$, as I can take $a = 1$ and $b = 1$, I have $F'_1 = 1$. A similar argument shows that $F'_{f \circ g} = F'_f \circ F'_g$, so the correspondence $M \mapsto F'(M)$ and $f \mapsto F'_f$ is a functor from $\mathcal{A}$ to $\mathcal{B}$.

Moreover if $P$ is in $\mathcal{P}$, then I can choose the following exact sequence for $P$

$$P \xrightarrow{0} P \xrightarrow{1} P \to 0$$

The associated sequence is

$$F(P \oplus P) \xrightarrow{\Delta F(0)} F(P) \xrightarrow{F'(1)} F'(P) \to 0$$

As $\Delta F(0) = 0$, I have $F'(P) \simeq F(P)$. Moreover, if $f : P \to P'$ is a morphism in $\mathcal{P}$, then as the diagram

$$\begin{array}{ccccccc} P & \xrightarrow{0} & P & \xrightarrow{1} & P & \to 0 \\ f\downarrow & & f\downarrow & & f\downarrow & \\ P' & \xrightarrow{0} & P' & \xrightarrow{1} & P' & \to 0 \end{array}$$

is commutative, the following commutative diagram

$$\begin{array}{ccccccc} F(P \oplus P) & \xrightarrow{\Delta F(0)} & F(P) & \xrightarrow{1} & F(P) & \to 0 \\ F\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}\Big\downarrow & & F(f)\Big\downarrow & & F'(f)\Big\downarrow & \\ F(P' \oplus P') & \xrightarrow{\Delta F(0)} & F(P') & \xrightarrow{1} & F(P') & \to 0 \end{array}$$

shows that $F'(f) = F(f)$, and this induces an isomorphism between $F$ and the restriction of $F'$ to $\mathcal{P}$. Finally, the diagram

$$\begin{array}{ccccccc} F(P) & \xrightarrow{0} & F(P) & \xrightarrow{1} & F(P) & \to 0 \\ F\binom{0}{1}\Big\downarrow & & 1\Big\downarrow & & F'_\psi\Big\downarrow & \\ F(Q \oplus P) & \xrightarrow{\Delta F(\varphi)} & F(P) & \xrightarrow{F'(\psi)} & F'(M) & \to 0 \end{array}$$

shows that $F'_\psi = F'(\psi)$.

It remains to check that the functor $F'$ is right exact: denote by

$$Q_M \xrightarrow{\varphi_M} P_M \xrightarrow{\psi_M} M \to 0$$

the chosen resolution by objects of $\mathcal{P}$ for the object $M$ of $\mathcal{A}$. Suppose moreover as before that if $M = P$ is in $\mathcal{P}$, then this sequence is

$$P \xrightarrow{0} P \xrightarrow{1} P \to 0$$

so that $F'(P)$ can be identified with $F(P)$.

Now let $M$ be any object of $\mathcal{A}$, and consider first an exact sequence in $\mathcal{A}$

$$Q \overset{\varphi}{\to} P \overset{\psi}{\to} M \to 0$$

with $P$ and $Q$ in $\mathcal{P}$. By the above arguments there are maps $a : P \to P_M$, $b : Q \to Q_M$, $a' : P_M \to P$ and $b' : Q_M \to Q$ and a commutative diagram

$$\begin{array}{ccccccc} Q & \xrightarrow{\varphi} & P & \xrightarrow{\psi} & M & \to 0 \\ b \downarrow \uparrow b' & & a \downarrow \uparrow a' & & 1 \downarrow \uparrow 1 & \\ Q_M & \xrightarrow{\varphi_M} & P_M & \xrightarrow{\psi_M} & M & \to 0 \end{array}$$

As $\psi \circ a' \circ a = \psi_M \circ a = \psi$, there is a map $c : P_M \to Q$ such that $1 - a' \circ a = \varphi \circ c$. Similarly, there is a map $c' : P \to Q_M$ such that $1 - a \circ a' = \varphi_M \circ c'$. Now there is a commutative diagram

$$(2.6) \qquad \begin{array}{ccccccc} F(Q \oplus P) & \xrightarrow{\Delta F(\varphi)} & F(P) & \xrightarrow{F'(\psi)} & F'(M) & \to 0 \\ F\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \downarrow & & F(a) \downarrow & & 1 \downarrow & \\ F(Q_M \oplus P_M) & \xrightarrow{\Delta F(\varphi_M)} & F(P_M) & \xrightarrow{F'(\psi_M)} & F'(M) & \to 0 \end{array}$$

in which the bottom line is exact by construction of $F'(M)$. Now $F\binom{c}{a' \circ a}$ is a map from $F(P_M)$ to $F(Q \oplus P)$, and

$$\Delta F(\varphi) \circ F\begin{pmatrix} c \\ a' \circ a \end{pmatrix} = \Big(F(\varphi, 1) - F(0, 1)\Big) \circ F\begin{pmatrix} c \\ a' \circ a \end{pmatrix} = \dots$$

$$\dots = F(\varphi \circ c + a' \circ a) - F(a' \circ a) = F(1) - F(a') \circ F(a) = 1 - F(a') \circ F(a)$$

Similarly, the map $F\binom{c'}{a \circ a'} : F(P) \to F(Q_M \oplus P_M)$ is such that

$$\Delta F(\varphi_M) \circ F\begin{pmatrix} c' \\ a \circ a' \end{pmatrix} = 1 - F(a) \circ F(a')$$

This shows that $F(a)$ and $F(a')$ induce mutual inverse isomorphisms between the cokernel of $\Delta F(\varphi)$ and the cokernel of $\Delta F(\varphi_M)$, equal to $F'(M)$ by definition. In other words, the top line in 2.6 is exact.

In particular, if $\psi : P \to M$ is an epimorphism from an object of $\mathcal{P}$ to $M$, then $F'(\psi)$ is also an epimorphism.

Now let

$$0 \to M \overset{\varphi}{\to} N \overset{\psi}{\to} L \to 0$$

be an arbitrary short exact sequence in $\mathcal{A}$. It is well-known (see [**Wei94**] Horseshoe lemma 2.2.8) that it is possible to find a resolution

$$Q'_N \overset{\varphi'_N}{\to} P'_N \overset{\psi'_N}{\to} N \to 0$$

by objects of $\mathcal{P}$, such that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & Q_M & \to & Q'_N & \xrightarrow{\gamma} & Q_L & \to 0 \\ & \downarrow & & \varphi'_N \downarrow & & \downarrow & \\ 0 \to & P_M & \xrightarrow{\alpha} & P'_N & \xrightarrow{\beta} & P_L & \to 0 \\ & \downarrow & & \psi'_N \downarrow & & \downarrow & \\ 0 \to & M & \to & N & \to & L & \to 0 \end{array}$$

with exact rows and columns (note that this resolution needs not be equal to the prescribed resolution of $N$).

Then it is easy to check that the following diagram is commutative

$$\begin{array}{ccccccc} & & F(Q'_N \oplus P'_N) & \xrightarrow{F\begin{pmatrix}\gamma & 0\\ 0 & \beta\end{pmatrix}} & F(Q_L \oplus P_L) & \to 0 \\ & & \Delta F(\varphi'_N)\downarrow & & \Delta F(\varphi_L)\downarrow & \\ F(P_M \oplus P'_N) & \xrightarrow{\Delta F(\alpha)} & F(P'_N) & \xrightarrow{F(\beta)} & F(P_L) & \to 0 \\ F'\begin{pmatrix}\psi_M & 0\\ 0 & \psi'_N\end{pmatrix}\downarrow & & F'(\psi'_N)\downarrow & & F'(\psi_L)\downarrow & \\ F'(M \oplus N) & \xrightarrow{\Delta F'(\varphi)} & F'(N) & \xrightarrow{F'(\psi)} & F'(L) & \to 0 \\ \downarrow & & \downarrow & & \downarrow & \\ 0 & & 0 & & 0 & \end{array}$$

Moreover, its columns are exact by the above remarks, as well as its two top lines. Now the bottom line is also exact: the map $F'(\psi) \circ F'(\psi'_N) = F(\psi'_L) \circ F(\beta)$ is an epimorphism, because $F(\psi'_L)$ and $F(\beta)$ are. Hence $F'(\psi)$ is also an epimorphism. And if $\theta : F'(N) \to B$ is any map in $\mathcal{B}$ such that $\theta \circ \Delta F'(\varphi) = 0$, then

$$\theta \circ \Delta F'(\varphi) \circ F' \begin{pmatrix} \psi_M & 0 \\ 0 & \psi'_N \end{pmatrix} = \theta \circ F'(\psi'_N) \circ \Delta F(\alpha) = 0$$

As the middle row is exact, there is a map $\mu : F(P_L) \to B$ such that

$$\theta \circ F'(\psi'_N) = \mu \circ F(\beta)$$

Now

$$0 = \theta \circ F'(\psi'_N) \circ \Delta F(\varphi'_N) = \mu \circ F(\beta) \circ \Delta F(\varphi'_N) = \mu \circ \Delta F(\varphi_L) \circ F \begin{pmatrix} \gamma & 0 \\ 0 & \beta \end{pmatrix}$$

As the top line is exact, I have

$$\mu \circ \Delta F(\varphi_L) = 0$$

and as the right column is exact, there is a map $\lambda : F'(L) \to B$ such that $\mu = \lambda \circ F'(\psi_L)$. Thus

$$\theta \circ F'(\psi'_N) = \lambda \circ F'(\psi_L) \circ F(\beta) = \lambda \circ F'(\psi) \circ F'(\psi'_N)$$

As $F'(\psi'_N)$ is an epimorphism, I have $\theta = \lambda \circ F'(\psi)$, so the bottom line of the above diagram is exact, and the functor $F'$ is right exact. This completes the proof of the theorem. $\square$

## 3. Permutation Mackey functors

**3.1. Mackey functors.** Let $R$ be a commutative ring, and $G$ be a finite group. I will use Dress definition of Mackey functors for $G$ over $R$ (see [**Dre73**]):

DEFINITION 3.1. *A Mackey functor $M$ for $G$ over $R$ is a bivariant functor from the category $G$-**set** of finite $G$-sets to the category $R$-**Mod** of (left) $R$-modules, satisfying the following conditions:*

- *If $X$ and $Y$ are finite $G$-sets, and $i_X$ and $i_Y$ are the respective inclusions of $X$ and $Y$ into their disjoint union $X \sqcup Y$, then the following morphisms*

$$M(X) \oplus M(Y) \xrightarrow{\big(M_*(i_X), M_*(i_Y)\big)} M(X \sqcup Y) \xrightarrow{\begin{pmatrix} M^*(i_X) \\ M^*(i_Y) \end{pmatrix}} M(X) \oplus M(Y)$$

  *are mutual inverse isomorphisms.*
- *If*

$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ Z & \xrightarrow[d]{} & T \end{array}$$

  *is a cartesian square of finite $G$-sets, then $M_*(b)M^*(a) = M^*(d)M_*(c)$*

I will mostly consider the case $R = \mathbb{Z}$ of Mackey functors over the integers, called simply Mackey functors. The Mackey functors for $G$, and natural transformation of bivariant functors, form an abelian category, denoted by $Mack(G)$.

If $M$ is a Mackey functor for $G$, and $H$ is a subgroup of $G$, then $M(G/H)$ is also denoted $M(H)$. If $K \subseteq H$ are subgroups of $G$, then $p_K^H$ is the projection $G/K \to G/H$ defined by $p_K^H(xK) = xH$. The transfer $t_K^H$ is the map $M_*(p_K^H)$, and the restriction $r_K^H$ is the map $M^*(p_K^H)$.

**3.2. Burnside functors.** I will denote by $b_G$ or $b$ the Burnside Mackey functor for the group $G$. Its value $b(H)$ for a subgroup $H$ of $G$ is just the Grothendieck group of finite $H$-sets, for relations given by decomposition in disjoint union.

More generally, its value $b(X)$ on a finite $G$-set $X$ is the Grothendieck group of the category $G$-$\mathbf{set}{\downarrow}_X$ of finite sets over $X$ (see [**Bou97**]). If $f : X \to Y$ is a morphism in $G$-**set**, then $b_*(f) : b(X) \to b(Y)$ is defined by composition, and $b^*(f) : b(Y) \to b(X)$ by pull-back.

If $M$ is a Mackey functor for the group $G$, and $X$ is a finite $G$-set, the Mackey functor $M_X$ is defined (see [**Web91**], [**TW95**], [**Bou97**]) on a finite $G$-set $Y$ by $M_X(Y) = M(Y \times X)$, and for a morphism $f : Y \to Z$ in $G$-**set** by $M_{X,*}(f) = M_*(f \times Id_X)$ and $M_X^*(f) = M^*(f \times Id_X)$. This construction is functorial with respect to $X$: if $f : X \to X'$ is a morphism in $G$-**set**, then there are obvious morphisms $M_f : M_X \to M_{X'}$ and $M^f : M_{X'} \to M_X$.

More generally, if $X$ is any $G$-set (finite or not), and if $X = \sqcup_{\omega \in G\backslash X} \omega$ is its decomposition in (finite) $G$-orbits, I will set

$$M_X = \bigoplus_{\omega \in G\backslash X} M_\omega$$

For a finite $G$-set $X$, these two definitions of $M_X$ coincide, in that there is a canonical isomorphism between them: if $i_\omega$ is the inclusion of $\omega$ into $X$, then the sequence $(M_{i_\omega})_{\omega \in G\backslash X}$ is an isomorphism from $\oplus_{\omega \in G\backslash X} M_\omega$ to $M_X$.

Note that with this definition, for any $G$-set $X$ and any finite $G$-set $Y$, there is a natural isomorphism

$$\bigoplus_{(y,x)\in[G\backslash(Y\times X)]} M(G_{y,x}) \to M_X(Y) = \bigoplus_{\omega\in G\backslash X} M(Y\times\omega) \tag{3.7}$$

mapping the element $v \in M(G_{y,x})$ to the element $M_*(m_{y,x})(v) \in M(Y \times Gx)$, where $m_{y,x}$ is the map from $G/G_{y,x} \to Y \times Gx$ given by

$$m_{y,x}(gG_{y,x}) = (gy, gx)$$

To avoid the choice of a system of representatives $[G\backslash(Y \times X)]$, one can also view the left hand side module of (3.7) as

$$\Big(\bigoplus_{(y,x)\in Y\times X} M(G_{y,x})\Big)^G$$

Using this isomorphism, if $\psi : Y \to Y'$ is a morphism of finite $G$-sets, then the map $M_{X,*}(\psi)$ becomes the map

$$\Big(\bigoplus_{(y,x)\in Y\times X} M(G_{y,x})\Big)^G \to \Big(\bigoplus_{(y',x)\in Y'\times X} M(G_{y',x})\Big)^G$$

sending $(v_{y,x})_{(y,x)\in Y\times X}$ to $(v'_{y',x})_{(y',x)\in Y'\times X}$ defined by

$$v'_{y',x} = \sum_{y\in G_{y',x}\backslash\psi^{-1}(y')} t_{G_{y,x}}^{G_{y',x}} m_{y,x}$$

Similarly, the map $M_X^*(\psi)$ gives the map

$$\Big(\bigoplus_{(y',x)\in Y'\times X} M(G_{y',x})\Big)^G \to \Big(\bigoplus_{(y,x)\in Y\times X} M(G_{y,x})\Big)^G$$

sending $(v'_{y',x})_{(y',x)\in Y'\times X}$ to $(v_{y,x})_{(y,x)\in Y\times X}$ defined by

$$v_{y,x} = r_{G_{y,x}}^{G_{\psi(y),x}} v'_{\psi(y),x}$$

### 3.3. Permutation functors.

Definition 3.2. *A permutation Mackey functor is a Mackey functor isomorphic to $b_X$, for some $G$-set $X$. I denote by $PMack(G)$ the full subcategory of $Mack(G)$ formed by permutation Mackey functors.*

Note that if $X$ is any $G$-set, and $Y$ is a finite $G$-set, then $b_X(Y)$ is the Grothendieck group of the category of finite $G$-sets over $Y \times X$.

For a finite $G$-set $X$, and any Mackey functor $M$, it is easy to check (see [**Bou97**]) that

$$\mathrm{Hom}_{Mack(G)}(b_X, M) \simeq M(X) \tag{3.8}$$

This isomorphism is as follows: if $m \in M(X)$, then for any finite $G$-set $Y$, the map associated to $m$ from $b_X(Y) = b(Y \times X) \to M(Y)$ is defined by

$$\Big(Y \xleftarrow{\;a\;} Z \xrightarrow{\;b\;} X\Big) \mapsto M_*(a)M^*(b)(m) \in M(Y)$$

In particular, the functor $b_X$ is projective. Hence for any $G$-set $X$ (finite or not), the functor $b_X$ is projective, as a direct sum of projectives. If $M$ is any Mackey functor, then

$$\mathrm{Hom}_{Mack(G)}(b_X, M) \simeq \prod_{\omega\in G\backslash X} M(\omega)$$

**3.4. Lefschetz invariants.** Let $X$ be a $G$-set (finite or not). There is another possible interpretation of $b_X$, using Lefschetz invariants. If $Y$ is a finite $G$-poset, recall (see **[Thé87]**) that the Lefschetz invariant of $Y$ is the element of $b(G)$ defined by

$$\Lambda_Y^G = -\sum_{s\in G\backslash Sd(Y)} (-1)^{|s|} G/G_s$$

where $Sd(Y)$ is the set of totally ordered non-empty subsets of $Y$, and $|s|$ is the cardinality of $s$ and $G_s$ its stabilizer in $G$. If $H$ is a subgroup of $G$, then the (algebraic) number of fixed points of $H$ on $\Lambda_Y^G$ is the Euler-Poincaré characteristic of the set of fixed points of $H$ on $Y$

$$|(\Lambda_Y^G)^H| = \chi(Y^H)$$

So two $G$-posets $Y$ and $Z$ have the same Lefschetz invariant if and only if for any subgroup $H$ of $G$, the Euler-Poincaré characteristic $\chi(Y^H)$ and $\chi(Z^H)$ are equal.

Any $G$-set $X$ can be viewed as a $G$-poset with the discrete ordering ($x \leq y \Leftrightarrow x = y$): when no other poset structure is given, the discrete one will be understood. Similarly, all maps between $G$-posets will be $G$-equivariant maps, compatible with the (given or understood) poset structures.

Definition 3.3. *If $X$ and $Y$ are $G$-sets, and $(\Delta, f)$ is a (finite or not) $G$-poset over $Y\times X$, I denote by $f_Y$ (resp. $f_X$) the composition of $f$ with the projection onto $Y$ (resp. onto $X$).*

*I denote by $G$-$\mathbf{poset}\!\downarrow_{Y,X}$ the category of $G$-posets $(Z, f)$ over $Y\times X$ such that for any $y\in Y$, the fibre $f_Y^{-1}(y)$ is finite. Such a poset is said to have finite fibres over $Y$. Note that this implies in particular that $f^{-1}(y,x)$ is finite for any $(y,x)\in Y\times X$, and that $Z$ is finite if $Y$ is.*

*I say that two objects $(\Delta, f)$ and $(\Delta', f')$ of $G$-$\mathbf{poset}\!\downarrow_{Y,X}$ are equivalent (notation $(\Delta, f)\sim(\Delta', f')$), if*

$$\forall (y,x)\in Y\times X,\ \Lambda^{G_{y,x}}_{f^{-1}(y,x)} = \Lambda^{G_{y,x}}_{f'^{-1}(y,x)}$$

*I denote by $h_G(Y,X)$ the set of equivalence classes of objects of $G$-$\mathbf{poset}\!\downarrow_{Y,X}$, modulo this equivalence relation.*

Lemma 3.4. *Let $X$ be a $G$-set*

1. *Let $Y$ be a finite $G$-set. Then the correspondence which maps the finite $G$-poset $(Z, f)$ over $Y\times X$ to*

$$\Big(\Lambda^{G_{y,x}}_{f^{-1}(y,x)}\Big)_{(y,x)\in Y\times X} \in \Big(\bigoplus_{(y,x)\in Y\times X} b(G_{y,x})\Big)^G \simeq b_X(Y)$$

*induces a one to one correspondence $\theta_{Y,X}$ between $h_G(Y,X)$ and $b_X(Y)$.*

2. *If $(Z, f)$ and $(Z', f')$ are finite $G$-posets over $Y\times X$, then*

$$\theta_{Y,X}(Z\sqcup Z', f\sqcup f') = \theta_{Y,X}(Z,f) + \theta_{Y,X}(Z',f')$$

3. *If $\psi : Y \to Y'$ is a morphism of finite $G$-sets, and $(Z,f)$ is a finite $G$-poset over $Y \times X$, then*

$$b_{X,*}(\psi)\theta_{Y,X}(Z,f) = \theta_{Y',X}\Big(Z, (\psi \times Id_X) \circ f\Big)$$

*If $(Z',f')$ is a finite $G$-poset over $Y' \times X$, let $(Z, f_Y)$ be the $G$-set defined by the pull-back diagram*

$$\begin{array}{ccc} Z & \xrightarrow{\;a\;} & Z' \\ {\scriptstyle f_Y}\downarrow & & \downarrow{\scriptstyle f'_{Y'}} \\ Y & \xrightarrow[\psi]{} & Y' \end{array}$$

*viewed as a sub-$G$-poset of $Y \times Z'$. Then setting $f = (f_Y, f'_X \circ a) : Z \to Y \times X$ turns $(Z,f)$ into a finite $G$-poset over $Y \times X$, and*

$$b_X^* \theta_{Y',X}(Z',f') = \theta_{Y,X}(Z,f)$$

Proof. The correspondence $\theta$ is clearly well-defined and injective, by the very definition of equivalence of posets over $Y \times X$, since it is clear that for $(y,x) \in Y \times X$ and $g \in G$, I have

$$\Lambda^{G_{gy,gx}}_{f^{-1}(gy,gx)} = {}^g(\Lambda^{G_{y,x}}_{f^{-1}(y,x)})$$

Conversely, if $(\beta_{y,x}) \in \Big(\oplus_{(y,x)\in(Y\times X)} b(G_{y,x})\Big)^G$, then for each $(y,x) \in [G\backslash(Y \times X)]$, it is possible to find a finite $G_{y,x}$-poset $Z_{y,x}$ with Lefschetz invariant equal to $\beta_{y,x}$ (see Lemme 2 of [**Bou92**]). Of course, if $\beta_{y,x} = 0$, I take $Z_{y,x} = \emptyset$. Now define

$$Z = \sqcup_{(y,x)\in[G\backslash(Y\times X)]}\mathrm{Ind}_{G_{y,x}}^G Z_{y,x}$$

This is a finite $G$-poset. Let $f : Z \to Y \times X$ mapping the element $(g,z)$ of $\mathrm{Ind}_{G_{y,x}}^G Z_{y,x}$ to $(gy,gx)$. Then $(Z,f)$ is a finite $G$-poset over $Y \times X$, and moreover for $(y,x) \in [G\backslash(Y \times X)]$, it is clear that

$$f^{-1}(y,x) = \{1\} \times Z_{y,x} \subseteq \mathrm{Ind}_{G_{y,x}}^G Z_{y,x}$$

It is isomorphic to $Z_{y,x}$, so its Lefschetz invariant is $\beta_{y,x}$, and the first assertion follows.

The second assertion is clear, since the Lefschetz invariant of a disjoint union $E \sqcup E'$ is the sum of the Lefschetz invariant of $E$ and $E'$.

The third assertion follows from the fact that

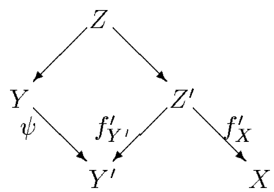
$$(\psi \circ f_Y, f_X)^{-1}(y',x) = \sqcup_{y\in\psi^{-1}(y')}(f_Y, f_X)^{-1}(y,x)$$

so that

$$\Lambda^{G_{y',x}}_{f'^{-1}(y',x)} = \sum_{y \in G_{y',x}\backslash\psi^{-1}(y')} \mathrm{Ind}_{G_{y,x}}^{G_{y',x}} \Lambda^{G_{y,x}}_{f^{-1}(y,x)}$$

Moreover the transfer for the Burnside functor is just induction.

Similarly, if $(Z', f')$ is a finite $G$-poset over $Y'$, let $Z$ be the pull-back of $Y$ and $Z'$ over $Y'$:




Let $f$ be the map from $Z$ to $Y \times X$ defined by this diagram. Now if $(y,x) \in Y \times X$, then

$$f^{-1}(y,x) = f'^{-1}\Big(\psi(y), x\Big)$$

so

$$\Lambda_{f^{-1}(y,x)}^{G_{y,x}} = \mathrm{Res}_{G_{y,x}}^{G_{\psi(y),x}} \Lambda_{f'^{-1}(\psi(y),x)}^{G_{\psi(y),x}}$$

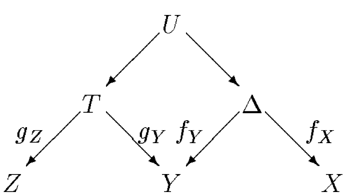
This completes the proof of the lemma. $\square$

Now I have a nice interpretation of $b_X$, and even a little more: observe that in the case of two finite $G$-sets $X$ and $Y$, isomorphism (3.8) gives

$$\mathrm{Hom}_{Mack(G)}(b_Y, b_X) \simeq b(Y \times X)$$

(see also [**Bou97**]). When $Y$ is infinite, this is no longer true. The correct formulation is the following:

PROPOSITION 3.5. *Let $G$ be a finite group, and $X$, $Y$ be any $G$-sets. For any object $(\Delta, f)$ of $G$-**poset**$\downarrow_{Y,X}$ and any finite $G$-set $Z$, let $b_{\Delta,f,Z}$ be the map sending the finite $G$-poset $(T,g)$ over $Z \times Y$ to the $G$-poset $(U,h)$ over $Z \times X$ defined by the pullback-diagram*




*Then the map $b_{\Delta,f,Z}$ passes down to equivalence classes of finite $G$-posets over $Z \times Y$, and this defines a morphism of Mackey functors $b_{\Delta,f}$ from $b_Y$ to $b_X$. Moreover this induces a one to one correspondence*

$$\theta_{Y,X} : h_G(Y,X) \simeq \mathrm{Hom}_{Mack(G)}(b_Y, b_X)$$

*such that $\theta_{Y,X}(\Delta \sqcup \Delta', f \sqcup f') = \theta_{Y,X}(\Delta, f) + \theta_{Y,X}(\Delta', f')$.*

PROOF. First it is clear that the pull-back $U$ is finite, since $g_Y(T)$ is, and $f$ has finite fibres over $Y$. It is also clear that $b_{\Delta,f,Z}$ passes down to equivalence classes of finite $G$-posets over $Z \times Y$: with the notation of the proposition, if $(z,x) \in Z \times X$, then

$$h^{-1}(z,x) = \sqcup_{y \in Y} g^{-1}(z,y) \times f^{-1}(y,x)$$

So

$$\Lambda_{h^{-1}(z,x)}^{G_{z,x}} = \sum_{y \in G_{z,x} \backslash Y} \mathrm{Ind}_{G_{z,y,x}}^{G_{z,x}} \Big(\mathrm{Res}_{G_{z,y,x}}^{G_{z,y}} \Lambda_{g^{-1}(z,y)}^{G_{z,y}}\Big) \Big(\mathrm{Res}_{G_{z,y,x}}^{G_{y,x}} \Lambda_{f^{-1}(y,x)}^{G_{y,x}}\Big)$$

So this only depends on the equivalence class of $(T, g)$ and $(\Delta, f)$.

Moreover, the maps $b_{\Delta,f,Z}$ define a morphism of Mackey functors: if $\psi : Z \to Z'$ is a morphism of finite $G$-sets, an element of $b_Y(Z)$ can be represented by the equivalence class of a finite $G$-poset $(T, g)$ over $Z \times Y$. Then $b_{Y,*}(\psi)(T, g)$ is the class of $(T, g')$, where $g' = (\psi \times Id_Y) \circ g$. But $b_{\Delta,f,Z}(T, g) = (U, h)$, and

$$b_{X,*}(\psi)(U, h) = \Big(U, (\psi \times Id_X) \circ h\Big)$$

On the other hand

$$b_{\Delta,f,Z'}\Big(T, (\psi \times Id_Y) \circ g\Big) = (U, h')$$

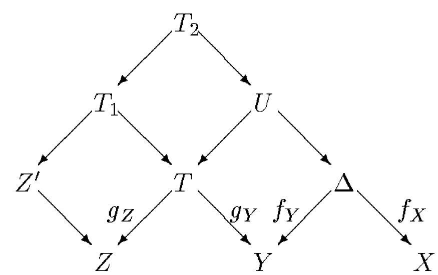
where $h' = (\psi \times Id_X) \circ h$. This proves that

$$b_{X,*} \circ b_{\Delta,f,Z} = b_{\Delta,f,Z'} \circ b_{Y,*}$$

Similarly, if now $\psi : Z' \to Z$ is a morphism of finite $G$-sets, then

$$b_Y^* \circ b_{\Delta,f,Z'} = b_{\Delta,f,Z} \circ b_X^*$$

This is because in the following diagram




if $(T, g)$ is a finite $G$-poset over $Z \times X$, if $T_1$ is the pull-back of $Z'$ and $T$ over $Z$, and $T_2$ the pull-back of $T_1$ and $U$ over $T$, then $T_2$ is also the pull-back of $Z'$ and $U$ over $Z$ (this last pull-back involves the composition of two cartesian squares, which is cartesian).

The last assertion of the proposition follows from the fact that if $Y$ is a finite $G$-set, then by (3.8)

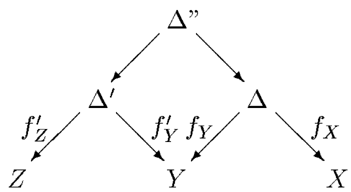
$$\mathrm{Hom}_{Mack(G)}(b_Y, b_X) = \bigoplus_{\omega \in G\backslash X} \mathrm{Hom}_{Mack(G)}(b_Y, b_\omega) \simeq \bigoplus_{\omega \in G\backslash X} b(Y \times \omega) \simeq b_X(Y)$$

Thus if $Y$ is any $G$-set

$$\mathrm{Hom}_{Mack(G)}(b_Y, b_X) = \prod_{\omega' \in G\backslash Y} b_X(\omega') \simeq \prod_{\omega' \in G\backslash Y} h_G(\omega', X)$$

Now a sequence $(T_{\omega'}) \in \prod_{\omega' \in G\backslash Y} h_G(\omega', X)$ defines a $G$-poset $T = \sqcup_{\omega' \in Y\backslash Y} T_{\omega'}$, with finite fibres over $Y$. The equivalence class of $T$ depends only on the sequence $(T_{\omega'})$. Conversely, the equivalence class of the $G$-poset $(T, f)$ over $Y \times X$, with finite fibres over $Y$, defines the sequence $T_{\omega'} = f^{-1}(\omega' \times X)$. This completes the proof of the proposition. □

Corollary 3.6. *Let $X$, $Y$ and $Z$ be $G$-sets. If $f : b_Y \to b_X$ is the morphism of Mackey functors defined by the class of the object $(\Delta, f)$ of $G$-**poset**$\downarrow_{Y,X}$, and $g : b_Z \to b_Y$ is the morphism defined by the class of the object $(\Delta', f')$ of $G$-**poset**$\downarrow_{Z,Y}$,* *then the morphism* $f \circ g : b_Z \to b_X$ *is defined by the class of the object* $\Delta$" *of* $G$-**poset**$\downarrow_{Z,X}$ *defined by the pull-back diagram*




Proof. This is clear from the proposition. Note that $\Delta$" has finite fibres over $Z$, if $\Delta$ has finite fibres over $Y$ and $\Delta'$ has finite fibres over $Z$. □

**3.5. Resolutions.** The subcategory of permutation Mackey functors is "big enough":

Proposition 3.7. *For any Mackey functor* $M$ *for the group* $G$, *there exists a* $G$*-set* $X$ *and an epimorphism* $\varphi : b_X \to M$.

Proof. Let $E$ be a set of subgroups of $G$, and for each $H \in E$, let $S_H$ be a non-empty subset of $M(H)$. Let

$$X = \sqcup_{H\in E}(G/H) \times S_H$$

where $G$ acts trivially on $S_H$. There is a natural morphism $\theta$ from $b_X$ to $M$: indeed the orbits of $G$ on $X$ are the sets $(G/H) \times \{s\}$, for $H \in E$ and $s \in S_H$. This gives

$$\mathrm{Hom}_{Mack(G)}(b_X, M) \simeq \prod_{\substack{H\in E\\ s\in S_H}} M(G/H)$$

The right hand side contains the element $(s)_{H\in E, s\in S_H}$, which corresponds to a morphism $\theta$ from $b_X$ to $M$. The element $s \in M(G/H)$ defines the morphism from $b_{G/H}$ to $M$ mapping the (discrete) set $(T, f)$ over $Z \times (G/H)$ to the element $M_*(f_Z)M^*(f_{G/H})(s)$. In particular, if $Z = T = G/H$, and $f$ is the diagonal inclusion, then this element is just $s$ itself. In other words, if $I = \mathrm{Im}\,\theta$, then $I(H)$ contains $S_H$, so it contains the sub-Mackey functor $J$ of $M$ generated by the union of the $S_H$, for $H \in E$ (this is the intersection of all the subfunctors $N$ of $M$ such that $N(H) \supseteq S_H$ for all $H \in E$ (see [**TW90**] Proposition 2.1)).

The image $I$ of $\theta$ is actually equal to $J$: indeed, for any subgroup $K$ of $G$, the value of $J$ at $K$ is

$$J(K) = < t_L^K({}^x r_{K^x}^H s) \mid H \in E,\ s \in S_H,\ x \in G,\ L \subseteq K \cap {}^x H >$$

But the element $t_L^K({}^x r_{K^x}^H s)$ is the image under $\theta_K$ of the element $(G/L, f)$ of $b_{G/H}(K)$, where $f(gL) = (gK, gxH)$.

This proves in particular that the morphism $\theta$ is an epimorphism if and only if the set $\sqcup_{H\in E} S_H$ generates $M$ as a Mackey functor. The proposition follows, taking for $E$ the set of all subgroups of $G$, and $S_H = M(H)$ for all $H$. □

## 4. Tensor induction for Mackey functors

**4.1. Bisets.** Let $G$ and $H$ be finite groups. A set $U$ is called a *biset* (more precisely an $H$-set-$G$) if the group $H \times G^{op}$ acts on $U$. Equivalently, the group $H$ acts on the left on $U$, and the group $G$ acts on the right, and those two actions commute.

If $U$ is an $H$-set-$G$, I denote by $U^{op}$ the $G$-set-$H$ which is equal to $U$ as a set, and double action is given for $g \in G$, $h \in H$, and $u \in U$ by

$$g.u.h \text{ (in } U^{op}) = h^{-1}ug^{-1} \text{ (in } U)$$

So let $U$ be a finite $H$-set-$G$, and $Z$ be a $G$-set. Let $\mathrm{Hom}_G(U^{op}, Z)$ be the set of morphism of $G$-sets from $U^{op}$ to $Z$. This is an $H$-set: if $h \in H$ and $\varphi \in \mathrm{Hom}_G(U^{op}, Z)$, then $h\varphi$ is the morphism of $G$-sets from $U^{op}$ to $Z$ defined by

$$(h\varphi)(u) = \varphi(h^{-1}u) \ \forall u \in U, \ h \in H$$

If now $Z$ is a $G$-poset, then $\mathrm{Hom}_G(U^{op}, Z)$ is an $H$-poset:

$$\varphi \leq \psi \text{ in } \mathrm{Hom}_G(U^{op}, Z) \ \Leftrightarrow \ \forall u \in U, \ \varphi(u) \leq \psi(u) \text{ in } Z$$

PROPOSITION 4.1. *The correspondence* $X \mapsto \mathrm{Hom}_G(U^{op}, X)$ *induces a functor* $T_U$ *from* $PMack(G)$ *to* $PMack(H)$*, mapping the object* $b_X$ *to* $b_{\mathrm{Hom}_G(U^{op},X)}$*, and the morphism* $\varphi : b_Y \to b_X$ *defined by the class of the object* $(\Delta, f)$ *of* $G$-$\mathbf{poset}{\downarrow}_{Y,X}$ *to the morphism defined by the class of the* $H$*-poset*

$$T_U(\Delta, f) = \Big(\mathrm{Hom}_G(U^{op}, \Delta), \mathrm{Hom}_G(U^{op}, f)\Big)$$

*over*

$$\mathrm{Hom}_G(U^{op}, Y \times X) \simeq \mathrm{Hom}_G(U^{op}, Y) \times \mathrm{Hom}_G(U^{op}, X)$$

PROOF. First I must check that if $(\Delta, f)$ is a $G$-poset over $Y \times X$, with finite fibres over $Y$, then

$$(D, F) = T_U(\Delta, f) = \Big(\mathrm{Hom}_G(U^{op}, \Delta), \mathrm{Hom}_G(U^{op}, f)\Big)$$

is an $H$-set over $\mathrm{Hom}_G(U^{op}, Y \times X)$, identified with

$$\mathrm{Hom}_G(U^{op}, Y) \times \mathrm{Hom}_G(U^{op}, X)$$

which has finite fibres over $\mathrm{Hom}_G(U^{op}, Y)$. Fix $\varphi \in \mathrm{Hom}_G(U^{op}, Y)$, and look for $\alpha \in \mathrm{Hom}_G(U^{op}, \Delta)$ such that $f_Y \circ \alpha = \varphi$: then for all $u \in U$, the element $\alpha(u)$ has to be in $f_Y^{-1}\varphi(u)$, which is a finite set by hypothesis. As $U$ is finite, there is only a finite number of possibles choices, hence a finite number of possibles $\alpha$'s.

Next I must show that the class of $T_U(\Delta, f)$ depends only on the class of $(\Delta, f)$. Fix $\varphi \in \mathrm{Hom}_G(U^{op}, Y)$ and $\psi \in \mathrm{Hom}_G(U^{op}, X)$, and a subgroup $K$ of $H_{\varphi,\psi}$. Then an element $\alpha$ of $\mathrm{Hom}_G(U^{op}, \Delta)^K$ such that $f \circ \alpha = (\varphi, \psi)$ is defined by the following conditions

$$\forall u \in U, \ \alpha(u) \in f^{-1}\Big(\varphi(u), \psi(u)\Big) \qquad \forall u \in U, \ \forall g \in G, \ \forall k \in K, \ \ \alpha(kug^{-1}) = g\alpha(u)$$

So $\alpha$ defines a sequence of elements $\alpha(u)$ in $f^{-1}\Big(\varphi(u), \psi(u)\Big)$, for $u$ in a system $[K\backslash U/G]$ of representatives of orbits of $K \times G^{op}$ on $U$. The element $\alpha(u)$ must be invariant by the subgroup of $G$ defined by

$$G_{K,u} = \{g \in G \mid \exists k \in K, \ kug^{-1} = u\}$$

Conversely, if I choose elements $a_u \in \Big(f^{-1}\Big(\varphi(u), \psi(u)\Big)\Big)^{G_{K,u}}$, for $u \in [K\backslash U/G]$, then I can define $\alpha(v)$ for any $v \in U$ by setting $\alpha(v) = ga_u$, if $v = kug^{-1}$, with $u \in [K\backslash U/G]$, $g \in G$, and $k \in K$.

In other words, there is a bijection

$$\Big(F^{-1}(\varphi, \psi)\Big)^K \simeq \prod_{u \in [K\backslash U/G]} \Big(f^{-1}\Big(\varphi(u), \psi(u)\Big)\Big)^{G_{K,u}}$$

This is clearly an isomorphism of posets, so

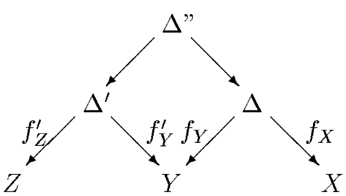
$$\chi\left(\left(F^{-1}(\varphi,\psi)\right)^K\right)=\prod_{u\in[K\backslash U/G]}\chi\left(\left(f^{-1}\left(\varphi(u),\psi(u)\right)\right)^{G_{K,u}}\right)$$

In particular, this depends only on the Lefschetz invariants $\Lambda_{f^{-1}(\varphi(u),\psi(u))}^{G_{\varphi(u),\psi(u)}}$ (note that $G_{K,u}$ is a subgroup of $G_{\varphi(u),\psi(u)}$ if $K$ is a subgroup of $H_{\varphi,\psi}$). Since these Euler-characteristics define the Lefschetz invariant $\Lambda_{F^{-1}(\varphi,\psi)}^{H_{\varphi,\psi}}$, the equivalence class of $(D,F)$ depends only on the class of $(\Delta,f)$.

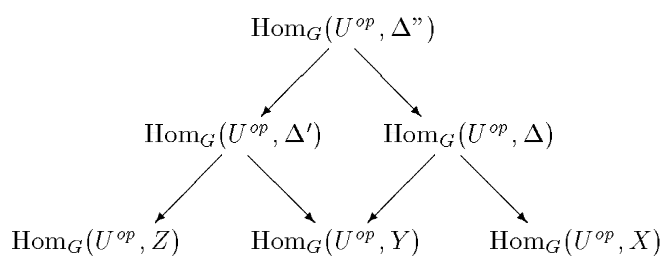
Finally I have to check functoriality: first the identity morphism of $b_X$ is associated to the class $(X,d_X)$ of the (discrete) set $X$ over $X\times X$, where $d_X(x)=(x,x)$. Clearly

$$T_U(X,d_X)=\left(\mathrm{Hom}_G(U^{op},X),d_{\mathrm{Hom}_G(U^{op},X)}\right)$$

so $T_U$ maps the identity to the identity. Moreover, if




is a composition of morphisms $b_Z\to b_Y\to b_X$, then taking images by the functor $\mathrm{Hom}_G(U^{op},-)$ gives the diagram




and the middle square in this diagram is cartesian, by definition. This proves that $T_U$ commutes with composition of morphisms, and this completes the proof of the proposition. □

**4.2. Tensor induction.** Now I can give the definition of tensor induction for Mackey functors:

DEFINITION 4.2. *Let $G$ and $H$ be finite groups, and $U$ be a finite $H$-set-$G$. I call tensor induction associated to $U$ the right exact functor from $Mack(G)$ to $Mack(H)$ extending the functor $T_U : PMack(G)\to Mack(H)$. This extension is again denoted by $T_U$.*

Recall that $T_U$ is constructed as follows: for a Mackey functor $M$ for $G$, choose an exact sequence

$$b_Y\xrightarrow{\ \varphi\ }b_X\xrightarrow{\ \psi\ }M\to 0$$

Then $T_U(M)$ is defined as the cokernel of the map $\Delta T_U(\varphi)$, so the following sequence is exact

$$T_U(b_Y \oplus b_X) \xrightarrow{\Delta T_U(\varphi)} T_U(b_X) \xrightarrow{T_U(\psi)} T_U(M) \to 0$$

Moreover

$$T_U(b_Y \oplus b_X) = T_U(b_{Y\sqcup X}) = b_{\mathrm{Hom}_G(U^{op}, Y\sqcup X)}$$

Suppose that $\varphi$ is defined by the $G$-poset $(D, f)$ over $Y \times X$. Then $\Delta T_U(D, f)$ is the morphism from $b_{\mathrm{Hom}_G(U^{op}, Y\sqcup X)}$ to $b_{\mathrm{Hom}_G(U^{op}, X)}$ defined by the difference

$$\begin{array}{ccc} & \mathrm{Hom}_G(U^{op}, D\sqcup X) & \\ \swarrow & & \searrow \\ \mathrm{Hom}_G(U^{op}, Y\sqcup X) & & \mathrm{Hom}_G(U^{op}, X) \end{array} \quad - \quad \begin{array}{ccc} & \mathrm{Hom}_G(U^{op}, \emptyset\sqcup X) & \\ \swarrow & & \searrow \\ \mathrm{Hom}_G(U^{op}, Y\sqcup X) & & \mathrm{Hom}_G(U^{op}, X) \end{array}$$

The left hand term is the image of the poset

$$\begin{array}{ccc} & D\sqcup X & \\ {\scriptstyle a}\swarrow & & \searrow{\scriptstyle b} \\ Y\sqcup X & & X \end{array}$$

where $a$ is $f_Y \sqcup Id_X$, and $b$ is $f_X$ on $D$ and identity on $X$. The other one is obtained from it by replacing $D$ by $\emptyset$.

Define $\mathrm{Hom}_G(U^{op}, D; X)$ as the set of $G$-morphisms $\alpha$ from $U$ to $D \sqcup X$ such that $\mathrm{Im}\,\alpha \not\subseteq X$. Then $T_U(M)$ is also the cokernel of the map

$$b_{\mathrm{Hom}_G(U,Y;X)} \to b_{\mathrm{Hom}_G(U,X)}$$

defined by the following poset

$$\begin{array}{ccc} & \mathrm{Hom}_G(U^{op}, D; X) & \\ \swarrow & & \searrow \\ \mathrm{Hom}_G(U^{op}, Y; X) & & \mathrm{Hom}_G(U^{op}, X) \end{array}$$

4.2.1. *Examples. The case $U = \emptyset$.* Let $U = \emptyset$. Then for any $G$-set $X$, the $H$-set $\mathrm{Hom}_G(U, X)$ is a one element set. It follows that the functor $T_\emptyset$ is the constant functor, equal to $b$.

*The case $H = G = U$.* Suppose $H = G$, and that $U$ is the set $G$, with double action given by left and right multiplication. Then for any $G$-set $X$

$$\mathrm{Hom}_G(G, X) \simeq X$$

It follows that the functor $T_G$ is the identity functor.

*Projectivity.* As the functor $T_U$ maps permutation functors to permutation functors, it follows that it maps projectives to projectives: if $P$ is any projective Mackey functor for $G$, then $P$ is a direct summand of some permutation functor $b_X$. Then $T_U(P)$ is a direct summand of $T_U(b_X) = b_{\mathrm{Hom}_G(U^{op},X)}$, hence it is projective.

**4.3. Composition.** The tensor induction $T_U$ is functorial with respect to $U$:

PROPOSITION 4.3. *Let $G$, $H$ and $K$ be finite groups. If $U$ is a finite $H$-set-$G$ and $V$ is a finite $K$-set-$H$, then there is an isomorphism of functors*

$$T_V \circ T_U \simeq T_{V\times_H U}$$

PROOF. Recall that $V\times_H U$ is the quotient $V\times U$, viewed as a $K$-set-$G$, by the action of $H$ given by $(v,u)h=(vh,h^{-1}u)$.

Now observe that $T_{V\times_H U}$ is right exact by definition, and that $T_V\circ T_U$ is right exact by proposition 2.9. Moreover, for any $G$-set $X$

$$(T_V\circ T_U)(b_X)=T_V(b_{\mathrm{Hom}_G(U^{op},X)})=b_{\mathrm{Hom}_H(V^{op},\mathrm{Hom}_G(U^{op},X))} \tag{4.9}$$

But there is an adjunction

$$\mathrm{Hom}_H\Big(V^{op},\mathrm{Hom}_G(U^{op},X)\Big)\simeq \mathrm{Hom}_G(U^{op}\times_H V^{op},X)$$

and moreover $U^{op}\times_H V^{op}\simeq (V\times_H U)^{op}$. Equation 4.9 is easily seen to be functorial with respect to $X$: in other words, the restrictions of $T_{V\times_H U}$ and $T_V\circ T_U$ to $PMack(G)$ are isomorphic. As they are both right exact, proposition follows from theorem 2.14. □

**4.4. Tensor product.** Recall from [**Bou97**] chapter 1 that if $M$ and $N$ are Mackey functors, then their tensor product is the Mackey functor $M\hat{\otimes}N$ defined on the finite $G$-set $X$ by

$$(M\hat{\otimes}N)(X)=\Big(\bigoplus_{Y\overset{\varphi}{\to}X} M(Y)\otimes N(Y)\Big)/\mathcal{J}$$

where $(Y,\varphi)$ is a (finite) $G$-set over $X$, and $\mathcal{J}$ is the submodule generated by the elements

$$M_*(f)(m)\otimes n'-m\otimes N^*(f)(n')\qquad \forall m\in M(Y),\ n'\in N(Y')$$

and the elements

$$M^*(f)(m')\otimes n-m\otimes N_*(f)(n)\qquad \forall m'\in M(Y'),\ n\in N(Y)$$

whenever $f:(Y,\varphi)\to(Y',\varphi')$ is a morphism of $G$-sets over $X$, which means that $\varphi'\circ f=\varphi$. If $m\in M(Y)$ and $n\in N(Y)$ for a $G$-set $(Y,\varphi)$ over $X$, I denote by $[m\otimes n]_{(Y,\varphi)}$ the image of $m\otimes n$ in $(M\hat{\otimes}N)(X)$.

If $\theta:X\to X'$ is a morphism of $G$-sets, then the image of $[m\otimes n]_{(Y,\varphi)}$ by $(M\hat{\otimes}N)_*(\theta)$ is $[m\otimes n]_{(Y,\theta\varphi)}$. If $(Y',\varphi')$ is a $G$-set over $X'$, if $m'\in M(Y')$ and $n'\in N(Y')$, then the image of $[m'\otimes n']_{(Y',\varphi')}$ by $(M\hat{\otimes}N)^*(f)$ is equal to $[M^*(a)(m)\otimes N^*(a)(n)]_{(Y,\varphi)}$, where $(Y,\varphi)$ is the pull-back of $(Y',\varphi')$ along $\theta$

$$\begin{array}{ccc} Y & \xrightarrow{a} & Y' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ X & \xrightarrow[\theta]{} & X' \end{array}$$

Also recall from [**Bou97**] Lemma 7.2.3 that if $X$ and $Y$ are finite $G$-sets, there are natural isomorphisms

$$(M\hat{\otimes}N)_{X\times Y}\simeq M_X\hat{\otimes}N_Y$$

Now this isomorphism clearly holds also for infinite sets $X$ and $Y$, because

$$M_X \hat{\otimes} N_Y \simeq \Big(\bigoplus_{\omega\in G\backslash X} M_\omega\Big)\hat{\otimes}\Big(\bigoplus_{\omega'\in G\backslash Y} N_{\omega'}\Big) \simeq \dots$$

$$\dots \simeq \bigoplus_{\substack{\omega\in G\backslash X\\ \omega'\in G\backslash Y}} M_\omega \hat{\otimes} N_{\omega'} \simeq \bigoplus_{\substack{\omega\in G\backslash X\\ \omega'\in G\backslash Y}} (M\hat{\otimes} N)_{\omega\times\omega'} \simeq (M\hat{\otimes} N)_{\sqcup\omega\times\omega'} \simeq \dots$$

$$\dots \simeq (M\hat{\otimes} N)_{X\times Y}$$

The following proposition states this isomorphism precisely, when $M$ and $N$ are both equal to the Burnside functor (note that $b\hat{\otimes} b \simeq b$ by Proposition 2.4.5 of [**Bou97**]):

Proposition 4.4. *Let $X$ and $Y$ be $G$-sets, and $Z$ be a finite $G$-set. If $(T,\varphi)$ is a finite $G$-set over $Z$, if $(U,f)$ is a finite $G$-poset over $T\times X$, and $(V,g)$ is a finite $G$-poset over $T\times Y$, denote by $U._TV$ the fibre product of $U$ and $V$ over $T$, viewed as a sub-$G$-poset of $U\times V$. Then $U._TV$ is a finite $G$-poset over $Z\times X\times Y$, and the correspondence*

$$(U,f)\otimes(V,g)\in b_X(T)\otimes b_Y(T)\mapsto U._TV\in b_{X\times Y}(Z)$$

*induces an isomorphism $(b_X\hat{\otimes} b_Y)(Z)\to b_{X\times Y}(Z)$, which is part of an isomorphism of Mackey functors*

$$b_X\hat{\otimes} b_Y \to b_{X\times Y}$$

Proof. First I have to specify the morphism $h: U._TV \to Z\times X\times Y$: if $(u,v)\in U._TV$, that is $(u,v)\in U\times V$ and $f_T(u)=g_T(v)$, then

$$h(u,v)=\Big(\varphi\circ f_T(u), f_X(u), g_Y(v)\Big)$$

Now if $(z,x,y)\in Z\times X\times Y$, then

$$h^{-1}(z,x,y)=\sqcup_{t\in\varphi^{-1}(z)} f^{-1}(t,x)\times g^{-1}(t,y)$$

which gives of course

$$\Lambda^{G_{z,x,y}}_{h^{-1}(z,x,y)}=\sum_{t\in G_{z,x,y}\backslash\varphi^{-1}(z)} \mathrm{Ind}_{G_{t,x,y}}^{G_{z,x,y}}\Big(\mathrm{Res}^{G_{t,x}}_{G_{t,x,y}}\Lambda^{G_{t,x}}_{f^{-1}(t,x)}\Big)\Big(\mathrm{Res}^{G_{t,y}}_{G_{t,x,y}}\Lambda^{G_{t,y}}_{g^{-1}(t,y)}\Big)$$

So the equivalence class of $(U._TV,h)$ depends only on the classes of $(U,f)$ and $(V,g)$.

Next I have to check that the submodule of relations $\mathcal{J}$ is mapped to zero in $b_{X\times Y}(Z)$: so let $\theta:(T,\varphi)\to(T',\varphi')$ be a morphism of $G$-sets over $Z$. Let $(U,f)$ be a finite $G$-poset over $T\times X$, which gives a $G$-poset $(U',f')$ over $T'$ by composition with $\theta\times Id_X$ (note that $U=U'$). Let $(V',g')$ be a finite $G$-poset over $T'$, which gives a $G$-set $(V,g)$ over $T$ by pull-back along $\theta$. I must check that the posets $U._TV$ and $U'._{T'}V'$ over $Z\times X\times Y$ are equivalent.

They are in fact isomorphic: an element of $U._TV$ is an element in $U\times V$, or a triple $(u,t,v')$ in $U\times T\times V'$, such that

$$\theta(t)=g'_{T'}(v') \qquad\qquad f_T(u)=t$$

This element is mapped to $\Big(\varphi f_T(u), f_X(u), g'_Y(v')\Big)\in Z\times X\times Y$.

On the other hand, an element of $U'._{T'}V'$ is an element $(u,v')\in U\times V'$, such that $\theta f_T(u)=g'_{T'}(v')$. It is mapped to $\Big(\varphi'\theta f_T(u), f_X(u), g'_Y(v')\Big)$. Now the map

$$(u,v')\in U'._{T'}V'\mapsto (u,f_T(u),v')\in U._TV$$

is clearly an isomorphism of posets.

So I have a well-defined map from $\Theta_Z : (b_X \hat{\otimes} b_Y)(Z) \to b_{X\times Y}(Z)$. This is an isomorphism: indeed, let $\Phi_Z$ be the map from $b_{X\times Y}(Z)$ to $(b_X \hat{\otimes} b_Y)(Z)$ which sends the finite $G$-poset $(W,f)$ over $Z\times X\times Y$ to

$$[(W, Id_W \times f_X)\otimes (W, Id_W \times f_Y)]_{(W,f_Z)}$$

where $Id_W \times f_X : W \to W\times X$ is defined by $(Id_W \times f_X)(w) = (w, f_X(w))$, and the maps $f_Z$, $f_X$ and $f_Y$ are obtained from $f$ by composition with the projections on $Z$, $X$ and $Y$.

It is clear that $\Theta_Z \circ \Phi_Z$ is the identity, since the fibre product $W._W W$ is equal to $W$. Conversely, if $(T,\varphi)$ is a finite $G$-set over $Z$, if $(U,f)$ is a finite $G$-poset over $T\times X$, and $(V,g)$ is a finite $G$-poset over $T\times Y$, then

$$\Phi_Z\circ\Theta_Z\Big([(U,f)\otimes(V,g)]_{(T,\varphi)}\Big)=[(U._TV, Id_{U._TV}\times h_X)\otimes(U._TV, Id_{U._TV}\times h_Y)]_{(U._TV,h_Z)}$$

But as $(U,f) = b_{X,*}(f_T)(U, Id_U\times f_X)$, up to an element of $\mathcal{J}$, I have

$$[(U,f)\otimes (V,g)]_{(T,\varphi)} = [(U, Id_U\times f_X)\otimes b_Y^*(f_T)(V,g)]_{(U,\varphi f_T)}$$

But $b_Y^*(f_T)(V,g) = (U._TV, k)$, where $k : U._TV \to U\times Y$ is the map defined by $k(u,v) = (u, g_Y(v))$. Now again $(U._TV, k) = b_{Y,*}(k_U)(U._TV, Id_{U._TV}\times k_Y)$, so up to $\mathcal{J}$, I have

$$[(U,f)\otimes(V,g)]_{(T,\varphi)} = [b_X^*(k_U)(U, Id_U\times f_X)\otimes(U._TV, Id_{U._TV}\times k_Y)]_{(U._TV,h_Z)} = \dots$$

$$\dots = [(U._TV, Id_{U._TV}\times h_X)\otimes (U._TV, Id_{U._TV}\times h_Y)]_{(U._TV,h_Z)}$$

So $\Phi_Z\circ\Theta_Z$ is also the identity. To complete the proof of the proposition, it suffices to check that the maps $\Phi_Z$ define a morphism of Mackey functors, which is clear. So the maps $\Theta_Z$ define also a morphism of Mackey functors, and $\Theta$ and $\Phi$ are inverse isomorphisms. □

Corollary 4.5. *Let $G$ and $H$ be finite groups, and $U$ be a finite $H$-set-$G$. If $M$ and $M'$ are Mackey functors for $G$ then*

$$T_U(M\hat{\otimes} M') \simeq T_U(M)\hat{\otimes} T_U(M')$$

*as Mackey functors for $H$, and this isomorphism is functorial in $M$ and $M'$.*

Proof. I will prove that there is an isomorphism of functors from $Mack(G)\times Mack(G)$ to $Mack(H)$

$$\Big([M,M']\mapsto T_U(M\hat{\otimes} M')\Big) \simeq \Big([M,M']\mapsto T_U(M)\hat{\otimes} T_U(M')\Big)$$

Indeed, the previous proposition states an isomorphism of functors between the restriction of those two functors to the subcategory $\mathcal{P} = PMack(G)\times PMack(G)$. But the objects of $\mathcal{P}$ are projective in $\mathcal{C} = Mack(G)\times Mack(G)$, and any object of $\mathcal{C}$ is the quotient of some object of $\mathcal{P}$. Moreover $\mathcal{P}$ is closed under direct sums.

The left hand side functor above is right exact, because it is composed of $T_U$, which is right exact, and of the tensor product, which is right exact (as any tensor product over a ring, which it actually is. See **[Bou97]** Chapter 1). The right hand side is composed of $T_U\times T_U$, which is right exact, and of the tensor product. So both sides are right exact, and isomorphic when restricted to $\mathcal{P}$. Hence they are isomorphic, by theorem 2.14. □

**4.5. Disjoint unions.** The tensor induction $T_U$ also behaves well with respect to disjoints unions of bisets:

PROPOSITION 4.6. *Let $G$ and $H$ be finite groups. If $U$ and $U'$ are finite $H$-sets-$G$, then for any Mackey functor $M$ for $G$*

$$T_{U\sqcup U'}(M)\simeq T_U(M)\hat{\otimes}T_{U'}(M)$$

*and this isomorphism is functorial in $M$.*

PROOF. Here again, both sides are right exact functors in $M$. So it suffices to check that their restrictions to $PMack(G)$ are isomorphic.

But for any $G$-set $X$

$$\mathrm{Hom}_G\Big((U\sqcup U')^{op},X\Big)=\mathrm{Hom}_G(U^{op}\sqcup U'^{op},X)\simeq \mathrm{Hom}_G(U^{op},X)\times\mathrm{Hom}_G(U'^{op},X)$$

This gives an isomorphism

$$T_{U\sqcup U'}(b_X)\simeq b_{\mathrm{Hom}_G(U^{op},X)\times\mathrm{Hom}_G(U'^{op},X)}\simeq b_{\mathrm{Hom}_G(U^{op},X)}\hat{\otimes}\, b_{\mathrm{Hom}_G(U'^{op},X)}=\ldots$$
$$\ldots=T_U(b_X)\hat{\otimes}T_{U'}(b_X)$$

which is easily seen to be functorial in $X$. So the proposition follows from theorem 2.14. □

**4.6. Direct sums.** One can think of $T_U(M)$ as a sort of $U$-th power of $M$. Here is the corresponding "binomial identity":

PROPOSITION 4.7. *Let $G$ and $H$ be finite groups, and $U$ be a finite $H$-set-$G$. For any Mackey functors $M$ and $M'$ for $G$*

$$T_U(M\oplus M')\simeq \bigoplus_{\substack{V\subseteq U\\ V\ G-\text{invariant}\\ V\ \text{mod}.H}}\mathrm{Ind}_{H_V}^H T_V(M)\hat{\otimes}T_{U-V}(M')$$

*(where $H_V=\{h\in H\mid hV=V\}$), and this isomorphism is functorial in $M$ and $M'$.*

PROOF. Both sides define functors from $Mack(G)\times Mack(G)$ to $Mack(H)$. The left hand side is clearly right exact, since it is composed of $T_U$ and of the direct sum, which is (additive!) exact. The right hand side is a direct sum of induced of tensor products of tensor inductions. As induction is an exact functor, the right hand side is also right exact. So again, it suffices to check that both sides are isomorphic when restricted to $PMack(G)\times PMack(G)$.

But if $X$ and $X'$ are $G$-sets, then there is a bijection

$$\mathrm{Hom}_G(U^{op},X\sqcup X')\simeq \bigsqcup_{\substack{V\subseteq U\\ V\ G-\text{invariant}}}\mathrm{Hom}_G(V^{op},X)\times\mathrm{Hom}_G((U-V)^{op},X')$$

Indeed any $\varphi:U\to X\sqcup X'$ is determined by $V=\varphi^{-1}(X)$, which is a sub-$G$-set of $U$, and its restrictions to $V$ and $U-V$, which are $G$-morphisms from $V$ to $X$ and $U-V$ to $X'$ respectively.

Now keeping track of the action of $H$ gives the isomorphism of $H$-sets

(4.10)

$$\mathrm{Hom}_G(U^{op},X\sqcup X')\simeq \bigsqcup_{\substack{V\subseteq U\\ V\ G-\text{invariant}\\ V\ \text{mod}.H}}\mathrm{Ind}_{H_V}^H\mathrm{Hom}_G(V^{op},X)\times\mathrm{Hom}_G((U-V)^{op},X')$$

But if $K$ is a subgroup of $G$, and $Z$ is a $K$-set, it is easy to see that

$$b_{\mathrm{Ind}_K^G Z} \simeq \mathrm{Ind}_K^G b_Z$$

This follows by taking direct sums from the case of a finite set $Z$. In that case for any $G$-set $Y$

$$b_{\mathrm{Ind}_K^G Z}(Y) = b(Y \times \mathrm{Ind}_K^G Z) = b\Big(\mathrm{Ind}_K^G(\mathrm{Res}_K^G Y.Z)\Big) = \dots$$

$$\dots = (\mathrm{Res}_K^G b)(\mathrm{Res}_K^G Y.Z) = b_Z(\mathrm{Res}_K^G Y) = (\mathrm{Ind}_K^G) b_Z(Y)$$

Now equation 4.10 gives the following isomorphism

$$b_{\mathrm{Ind}_G(U^{op}, X \sqcup X')} = \bigoplus_{\substack{V \subseteq U \\ V\ G-\text{invariant} \\ V \bmod . H}} \mathrm{Ind}_{H_V}^H b_{\mathrm{Hom}_G(V^{op},X)\times \mathrm{Hom}_G((U-V)^{op},X')} \simeq \cdots$$

$$\dots \simeq \bigoplus_{\substack{V \subseteq U \\ V\ G-\text{invariant} \\ V \bmod . H}} \mathrm{Ind}_{H_V}^H \Big( b_{\mathrm{Hom}_G(V^{op},X)} \hat{\otimes}\, b_{\mathrm{Hom}_G((U-V)^{op},X')}\Big)$$

This isomorphism is functorial in $X$ and $X'$, and this completes the proof of the proposition. □

## 5. Relations with the functors $\mathcal{L}_U$

**5.1. Construction of $\mathcal{L}_U$.** If $G$, $H$, and $K$ are groups, if $U$ is an $H$-set-$G$ and $V$ is a $K$-set-$H$, then in [**Bou96a**], I defined the product $V \circ_H U$ by

$$V \circ_H U = \{(v,u) \in V \times U \mid \forall h \in H,\ vh = v \Rightarrow \exists g \in G,\ hu = ug\}/H$$

This is a (generally strict) sub-$K$-set-$G$ of $V \times_H U$.

In particular, if $U$ is finite, the product $X \mapsto U \circ_G X$ is a functor from $G$-**set** to $H$-**set**, and those functors are exactly those preserving disjoint unions and cartesian squares ([**Bou96a**] Théorème 1). By composition, they give functors $M \mapsto M \circ U$ from $Mack(H)$ to $Mack(G)$.

Those functors between Mackey categories have left and right adjoints, described in [**Bou97**] Chapter 9. I denote by $\mathcal{L}_U$ the left adjoint of the functor $M \mapsto M \circ U$. So $\mathcal{L}_U$ is a functor from $Mack(G)$ to $Mack(H)$. Let me recall some notations and definitions:

Definition 5.1. *If $X$ is a finite $H$-set, let $\mathcal{D}_U(X)$ be the following category:*

- *The objects of $\mathcal{D}_U(X)$ are the finite $H$-sets over $X \times (U/G)$. If $(Y, f)$ is such an object, I denote by $f_X$ the $X$-component of $f$, and by $f_U$ its $U/G$-component. I denote by $U._fY$ or $U.Y$ the fibre product of $U$ and $Y$ over $U/G$. It is an $H$-set-$G$, with action given by $h(u,y)g = (hug, hy)$. To simplify the notations, I view $H\backslash U.Y$ as a left $G$-set (it should really be denoted $(H\backslash U.Y)^{op}$).*
- *A morphism $\alpha : (Y,f) \to (Y',f')$ in $\mathcal{D}_U(X)$ is a morphism of $H$-sets over $X \times (U/G)$, such that moreover the morphism $U.\alpha : U.Y \to U.Y'$ is injective on each $H$-orbit, that is*

$$\forall (u,y) \in U.Y,\ \forall h \in H,\ hu = u,\ \alpha(hy) = \alpha(y) \Rightarrow hy = y$$

*If $(Y,f)$ is an object of $\mathcal{D}_U(X)$, then a $G$-set $(Z,a)$ over $H\backslash U.Y$ is said to be $\nu$-disjoint if the following condition holds*

$$\forall (z,u,y)\in Z\times U\times Y,\ a(z)=H(u,y)\Rightarrow (u,z)\notin U\circ_G Z$$

*(this last condition means that there exists an $g\in G$ with $ug=u$ but $gz\neq z$).*

*If moreover $M$ is a Mackey functor for $G$, I set*

$$\mathcal{Q}_U(M)(Y,f)=M(H\backslash U.Y)/\sum_{(Z,a)}M_*(a)M(Z)$$

*where the sum runs through the $G$-sets $(Z,a)$ over $H\backslash U.Y$ which are $\nu$-disjoint.*

*Then $\mathcal{Q}_U(M)$ is a functor on $\mathcal{D}_U(X)$, with values in abelian groups. I set*

$$\mathcal{L}_U(M)(X)=\varinjlim_{(Y,f)\in\mathcal{D}_U(X)}\mathcal{Q}_U(M)(Y,f)$$

*If $(Y,f)$ is an object of $\mathcal{D}_U(X)$ and if $m\in M(H\backslash U.Y)$, I denote by $m_{(Y,f)}$ the image of $m$ in $\mathcal{L}_U(M)(X)$.*

The correspondence $X\mapsto \mathcal{L}_U(M)(X)$ can be turned into a Mackey functor for $H$, denoted by $\mathcal{L}_U(M)$.

Morever, if $\theta: M\to M'$ is a morphism of Mackey functors for $G$, then set

$$\mathcal{L}_U(\theta)(m_{(Y,f)})=\theta_{H\backslash U.Y}(m)_{(Y,f)}$$

This gives a well defined morphism $\mathcal{L}_U(\theta)$ of Mackey functors for $H$ from $\mathcal{L}_U(M)$ to $\mathcal{L}_U(M')$. More precisely:

THEOREM 5.2. *([**Bou97**] Theorem 9.5.2) Let $G$ and $H$ be finite groups, and $U$ be a finite $G$-set-$H$. The correspondence*

$$M\mapsto \mathcal{L}_U(M)$$

$$\theta\in \mathrm{Hom}_{Mack(G)}(M,M')\mapsto \mathcal{L}_U(\theta)\in \mathrm{Hom}_{Mack(H)}\Big(\mathcal{L}_U(M),\mathcal{L}_U(M')\Big)$$

*is a functor from $Mack(G)$ to $Mack(H)$, which is left adjoint to the functor $N\mapsto N\circ U$.*

COROLLARY 5.3. *The functor $\mathcal{L}_U$ is right exact.*

PROOF. This is because it is additive and has a right adjoint: see for example [**Wei94**] Theorem 2.6.1. □

### 5.2. Examples.

5.2.1. *Induction and restriction.* Let $H$ be a subgroup of $G$. If $U=G$, viewed as an $H$-set-$G$, then the functor $N\mapsto N\circ U$ is the induction functor $\mathrm{Ind}_H^G$, and the functor $\mathcal{L}_U$ is the restriction functor $\mathrm{Res}_H^G$. If $U=G$ is viewed as a $G$-set-$H$, then the functor $N\mapsto N\circ U$ is the restriction functor $\mathrm{Res}_H^G$, and the functor $\mathcal{L}_U$ is the induction functor $\mathrm{Ind}_H^G$.

5.2.2. *Inflation.* Let $N$ be a normal subgroup of $G$, and $H=G/N$. If $U=H$ is viewed as an $H$-set-$G$, then the functor $N\mapsto N\circ U$ is the inflation functor $\mathrm{Inf}_H^G$ (see [**TW90**],[**TW95**]). The functor $\mathcal{L}_U$ will be denoted by $M\mapsto M^N$ (it is denoted $M^+$ in [**TW95**]).

5.2.3. *Coinflation.* Let $N$ be a normal subgroup of $G$, and $H = G/N$. If $U = H$ is viewed as a $G$-set-$H$, then the functor $N \circ U$ will be denoted by $\rho_H^G$ (it is the functor $\beta^!$ of [**TW95**] section 5). The functor $\mathcal{L}_U$ will be denoted by $\iota_H^G$ (its existence is mentioned in [**TW95**] and it is called $\beta_!$).

The functor $\iota_H^G$ can be described as follows (see [**Bou97**] 9.9.3): if $X$ is a $G$-set, let $\mathcal{D}_H^G(X)$ be the following category:

- The objects of $\mathcal{D}_H^G(X)$ are the $G$-sets over $X$.
- A morphism $\alpha : (Y,f) \to (Y',f')$ in $\mathcal{D}_H^G(X)$ is a morphism of $G$-sets over $X$, which is moreover injective when restricted to every orbit of $N$ on $Y$.

Then if $M$ is a Mackey functor for $H$, the correspondence $(Y,f) \mapsto M(N\backslash Y)$ is a functor on $\mathcal{D}_H^G(X)$, and

$$\iota_H^G(M)(X) = \varinjlim_{(Y,f)\in\mathcal{D}_H^G(X)} M(N\backslash Y)$$

### 5.3. The case $U/G = \bullet$.

LEMMA 5.4. *Let $G$ and $H$ be finite groups, and $U$ be a finite $H$-set-$G$. Then for any $G$-set $X$*

$$\mathcal{L}_U(b_X) = b_{U\circ_G X}$$

PROOF. This follows from the case of a finite $G$-set $X$, which is a remark at the end of Chapter 9 of [**Bou97**]: indeed for any Mackey functor $N$ for $H$

$$\mathrm{Hom}_{Mack(H)}(\mathcal{L}_U(b_X),N) = \mathrm{Hom}_{Mack(G)}(b_X, N\circ U) = (N\circ U)(X) = \dots$$
$$\dots = N(U\circ_G X) = \mathrm{Hom}_{Mack(H)}(b_{U\circ_G X},N)$$

The lemma follows, because all those isomorphisms are natural in $N$. □

PROPOSITION 5.5. *Let $G$ and $H$ be finite groups, and $U$ be a finite $H$-set-$G$. The following conditions are equivalent:*

1. *The functor $T_U$ is additive.*
2. *The functors $T_U$ and $\mathcal{L}_U$ are isomorphic.*
3. *The group $G$ is transitive on $U$.*

PROOF. First if $T_U$ is additive, then for any $G$-sets $X$ and $Y$

$$T_U(b_{X\sqcup Y}) \simeq b_{\mathrm{Hom}_G(U^{op},X\sqcup Y)} \simeq T_U(b_X\oplus b_Y) \simeq T_U(b_X)\oplus T_U(b_Y) \simeq \dots$$
$$\dots \simeq b_{\mathrm{Hom}_G(U^{op},X)}\oplus b_{\mathrm{Hom}_G(U^{op},Y)} \simeq b_{\mathrm{Hom}_G(U^{op},X)\sqcup\mathrm{Hom}_G(U^{op},Y)}$$

As the rank of the evaluation at $\{1\}$ of $b_X$ is the cardinality of $X$, this implies in particular, if $X$ and $Y$ are finite trivial $G$-sets, that

$$(|X|+|Y|)^{|U/G|} = |X|^{|U/G|} + |Y|^{|U/G|}$$

which forces $|U/G| = 1$. So 3) holds.

Obviously 2) implies 1), since the functors $\mathcal{L}_U$ are always additive.

Now if 3) holds, I claim that $\mathrm{Hom}_G(U^{op},X) \simeq U\circ_G X$ for any $G$-set $X$: choose $u$ in $U$, take $\varphi\in\mathrm{Hom}_G(U^{op},X)$, and consider the couple

$$\theta(\varphi) = \Big(u,\varphi(u)\Big) \in U\times_G X$$

This does not depend on the choice of $u$, since by 3) any other choice is in the $G$-orbit of $u$, say equal to $ug$, for $g\in G$, and since

$$\Big(ug,\varphi(ug)\Big) = \Big(ug,g^{-1}\varphi(u)\Big) = \Big(u,\varphi(u)\Big)$$

Moreover if $g \in G$ fixes $u$ on the right, then

$$g\varphi(u) = \varphi(ug^{-1}) = \varphi(u)$$

This proves that $\theta$ is a map from $\mathrm{Hom}_G(U^{op}, X)$ to $U \circ_G X$. It is a morphism of $H$-sets, since for $h \in H$

$$\theta(h\varphi) = \Big(u, (h\varphi)(u)\Big) = \Big(u, \varphi(h^{-1}u)\Big)$$

Choose $g \in G$ such that $h^{-1}u = ug^{-1}$ (or $ug = hu$), which is possible by 3). Then

$$\Big(u, \varphi(h^{-1}u)\Big) = \Big(u, \varphi(ug^{-1})\Big) = \Big(u, g\varphi(u)\Big) = \Big(ug, \varphi(u)\Big) = \Big(hu, \varphi(u)\Big) = h\theta(\varphi)$$

Conversely, if $(u, x) \in U \circ_G X$, define a map $\theta'(u, x) : U \to X$ by

$$\theta'(u, x)(u') = g^{-1}x \quad \text{if} \quad g \in G,\ u' = ug$$

This map is well-defined by 3), and because $(u, x)$ is in $U \circ_G X$. It is a map of $H$-sets: if $h \in H$, then choose $g_0 \in G$ with $hu = ug_0$. Then

$$\Big(h\theta'(u, x)\Big)(u') = \theta'(u, x)(h^{-1}u') = \theta'(u, x)(h^{-1}ug) = \theta'(u, x)(ug_0^{-1}g) = g^{-1}g_0x$$

whereas $\theta'(hu, x)(u')$ is given by

$$\theta'(hu, x)(u') = \theta'(ug_0, x)(u') = g^{-1}g_0x$$

since $u' = (ug_0)g_0^{-1}g$.

Now $\theta'\theta(\varphi) = \theta'\Big(u, \varphi(u)\Big)$. But for $u' \in U$

$$\theta'\Big(u, \varphi(u)\Big)(u') = g^{-1}\varphi(u) \quad \text{if} \quad g \in G,\ u' = ug$$

So $\theta'\Big(u, \varphi(u)\Big)(u') = \varphi(ug) = \varphi(u')$, and the map $\theta' \circ \theta$ is the identity.

Conversely

$$\theta\theta'(u, x) = \Big(u, \theta'(u, x)(u)\Big) = (u, x)$$

So $\theta$ and $\theta'$ are mutually inverse isomorphisms.

Now for any $G$-set

$$T_U(b_X) = b_{\mathrm{Hom}_G(U^{op}, X)} \simeq b_{U \circ_G X} \simeq \mathcal{L}_U(b_X)$$

This is natural in $X$. As $T_U$ and $\mathcal{L}_U$ are right exact, they are isomorphic by theorem 2.14. So 2) holds, and this completes the proof of the proposition. □

Corollary 5.6. *Let $G$ and $H$ be finite groups, and $U$ be a finite $H$-set-$G$. Then for any Mackey functor $M$ for $G$*

1. *If $K$ is a subgroup of $H$*

$$\mathrm{Res}_K^H T_U(M) \simeq T_{\mathrm{Res}_{K \times G^{op}}^{H \times G^{op}} U}(M)$$

2. *If $N$ is a normal subgroup of $H$, then*

$$\Big(T_U(M)\Big)^N \simeq T_{N \backslash U}(M)$$

3. *If $N$ is a normal subgroup of a group $K$, and $H = K/N$, then*

$$\iota_H^K T_U(M) \simeq T_{\mathrm{Inf}_{H \times G^{op}}^{K \times G^{op}} U}(M)$$

*Moreover, these isomorphisms are natural in $M$.*

Proof. This is straightforward, using proposition 4.3, with $V = H$, viewed as a $K$-set-$H$ in case 1), and $V = H$ viewed as a $K$-set-$H$ in case 2), and $V = H$ viewed as an $H$-set-$K$ in case 3). □

**5.4. The products $T_V \circ \mathcal{L}_U$.** The previous corollary gives examples of composition $\mathcal{L}_V \circ T_U$. The next proposition computes the composition $T_V \circ \mathcal{L}_U$:

Proposition 5.7. *Let $G$, $H$ and $K$ be finite groups. Let $U$ be a finite $H$-set-$G$, and $V$ be a finite $K$-set-$H$. If $f : V^{op} \to U/G$ is a morphism of $H$-sets, let $V._{f,H}U$ denote the quotient set of the fibre product of $V$ and $U$ over $U/G$, by the right action of $H$ given by $(v,u)h = (vh, h^{-1}u)$. If $K_f$ is the stabilizer of $f$ in $K$, then $V._{f,H}U$ is a finite $K_f$-set-$G$ by $k.(v,u).g = (kv, ug)$. Then for any Mackey functor $M$ for $G$*

$$T_V \circ \mathcal{L}_U(M) \simeq \bigoplus_{f \in K\backslash \mathrm{Hom}_H(V^{op}, U/G)} \mathrm{Ind}_{K_f}^K T_{V._{f,H}U}(M)$$

*and this isomorphism is natural in $M$.*

Proof. Both sides are right exact functors. So it suffices to check that their restrictions to $PMack(G)$ coincide.

So let $M = b_X$, for some $G$-set $X$. The left hand side is

$$T_V \circ \mathcal{L}_U(b_X) \simeq T_V(b_{U \circ_G X}) = b_{\mathrm{Hom}_H(V^{op}, U \circ_G X)}$$

Now as there is always a (unique) map $p_X$ from $X$ to the trivial $G$-set $\bullet$, there is a natural map $\upsilon_X = U \circ_G p_X$ from $U \circ_G X$ to $U \circ_G \bullet \simeq U/G$, given by $(u,x) \mapsto uG$. So the functor $X \mapsto (U \circ_G X, \upsilon_X)$ is a functor from the category $G$-**Set** of (arbitrary) $G$-sets, to the category $H$-$\mathbf{Set}\downarrow_{U/G}$ of (arbitrary) $H$-sets over $U/G$. Conversely, if $(Y, f)$ is an $H$-set over $U/G$, then the set $H\backslash U._f Y$ is a $G$-set, and moreover:

Proposition 5.8. *Let $G$ and $H$ be finite groups, and $U$ be a finite $H$-set-$G$. The functor $(Y, f) \mapsto H\backslash U._f Y$ from $H$-$\mathbf{Set}\downarrow_{U/G}$ to $G$-**Set** is left adjoint to the functor $X \mapsto (U \circ_G X, \upsilon_X)$.*

Proof. The corresponding statement for finite sets is proposition 9.1.1 of [**Bou97**]. The proof extends verbatim to arbitrary sets: the unit of this adjunction is denoted by $\nu$ (it it the reason for the word $\nu$-disjoint in definition 5.1), and defined for an $H$-set $(Y, f)$ over $U/G$ by the morphism

$$\nu_{(Y,f)} : Y \to U \circ_G (H\backslash U._f Y)$$

given by $\nu_{(Y,f)}(y) = \Big(u, H(u,y)\Big)$ if $f(y) = uG$.

The counit of this adjunction is denoted by $\eta$. It is defined for a $G$-set $Z$ by the map

$$\eta_Z : H\backslash U._{\upsilon_Z}(U \circ_G Z) \to Z$$

defined by $\eta_Z\Big(H(u', (u,z)\Big) = h^{-1}z$ if $h \in H$ is such that $u' = uh$. □

This result means that there is a one to one correspondence between the set of morphisms of $H$-sets $\varphi$ from $V^{op}$ to $U \circ_G X$, such that $\upsilon \circ \varphi$ is a given morphism $f$ from $V^{op}$ to $U/G$, and the set of morphisms of $G$-sets from $H\backslash U._f V$ to $X$. In other words

$$\mathrm{Hom}_H(V^{op}, U \circ_G X) = \bigsqcup_{f \in \mathrm{Hom}_H(V^{op}, U/G)} \mathrm{Hom}_G(H\backslash U._f V, X)$$

Keeping track of the action of $K$ gives

$$\mathrm{Hom}_H(V^{op}, U\circ_G X) = \bigsqcup_{f\in K\backslash \mathrm{Hom}_H(V^{op},U/G)} \mathrm{Ind}_{K_f}^K \mathrm{Hom}_G(H\backslash U._f V, X)$$

Thus

$$b_{\mathrm{Hom}_H(V^{op},U\circ_G X)} \simeq \bigoplus_{f\in K\backslash \mathrm{Hom}_H(V^{op},U/G)} \mathrm{Ind}_{K_f}^K b_{\mathrm{Hom}_G(H\backslash U._f V,X)}$$

This isomorphism is natural in $X$. This completes the proof, since with the notations of the proposition 5.7, the map $(u,v)\mapsto (v,u)$ induces an isomorphism

$$H\backslash U._f V \simeq V._{f,H} U$$

of $K_f$-sets-$G$. □

PROPOSITION 5.9. *Let $G$ and $H$ be finite groups, and $U$ be a finite $H$-set-$G$. Let $X$ be a $G$-set and $M$ be a Mackey functor for $G$. Then there is an isomorphism*

$$T_U(M_X) \simeq T_U(M)_{\mathrm{Hom}_G(U^{op},X)}$$

*natural in $M$.*

PROOF. Here again its enough to check the isomorphism for permutation functors. But if $Y$ is a $G$-set, it is clear that $(b_Y)_X$ is isomorphic to $b_{Y\times X}$. As

$$\mathrm{Hom}_G(U^{op}, Y\times X) \simeq \mathrm{Hom}_G(U^{op}, Y)\times \mathrm{Hom}_G(U^{op}, X)$$

it follows that

$$T_U\Big((b_Y)_X\Big) \simeq \Big(b_{\mathrm{Hom}_G(U^{op},Y)}\Big)_{\mathrm{Hom}_G(U^{op},X)}$$

so the result holds for permutation functors, and the proposition follows.i □

REMARK 5.10. This proposition is also a consequence of proposition 5.7 in the case of a finite $G$-set $X$: let $\tilde{X} = G\times X$, viewed as a $G$-set-$G$ by $g.(g',x).g'' = (gg'g'', g.x)$. Then one can prove that the functor $M\mapsto M_X$ is isomorphic to the functor $\mathcal{L}_{\tilde{X}}$: this is because it is self adjoint, and moreover for any $G$-set $Y$, one has $\tilde{X}\circ_G Y \simeq X\times Y$.

## 6. Direct product of Mackey functors

**6.1. Definition.** There is a reasonable definition of the direct product of Mackey functors:

DEFINITION 6.1. *Let $G$ and $H$ be finite groups. If $M$ is a Mackey functor for $G$, and $N$ a Mackey functor for $H$, I denote by $M\boxtimes N$, and I call direct product of $M$ and $N$, the Mackey functor for $G\times H$ defined by*

$$M\boxtimes N = \iota_G^{G\times H}(M)\,\hat{\otimes}\,\iota_H^{G\times H}(N)$$

**6.2. Examples.**

6.2.1. *Direct product of permutation functors.* The direct product of two permutation Mackey functors is a permutation Mackey functor:

LEMMA 6.2. *Let $G$ and $H$ be finite groups. If $X$ is a $G$-set and $Y$ is an $H$-set, then*

$$b_X \boxtimes b_Y \simeq b_{\mathrm{Inf}_G^{G\times H} X\times \mathrm{Inf}_H^{G\times H} Y}$$

PROOF. This follows from the fact that

$$\iota_G^{G\times H}(b_X) = b_{\mathrm{Inf}_H^{G\times H} X}$$

by lemma 5.4, and from the isomorphism

$$b_{\mathrm{Inf}_H^{G\times H} X} \hat{\otimes} b_{\mathrm{Inf}_H^{G\times H} Y} \simeq b_{\mathrm{Inf}_H^{G\times H} X\times \mathrm{Inf}_H^{G\times H} Y}$$

6.2.2. *Extension of coefficients.* As a special case of direct product of Mackey functors, I will look at the case $G = \{1\}$: the category $Mack(\{1\})$ is equivalent to the category of abelian groups, because a Mackey functor for the trivial group is completely determined by its value on the trivial set, which is just an abelian group. Conversely, any abelian group $A$ defines a Mackey functor for the trivial group, still denoted by $A$, which value $A(X)$ on the finite set $X$ is equal to the group $A^{(X)}$ of functions from $X$ to $A$. If $f : X \to Y$ is a map of finite sets, then $A^*(f) : A(Y) \to A(X)$ is the composition with $f$, and $A_*(f) : A(X) \to A(Y)$ is defined by

$$A_*(f)(\alpha)(y) = \sum_{x\in f^{-1}(y)} \alpha(x)$$

PROPOSITION 6.3. *Let $G$ be a finite group. If $M$ is a Mackey functor for $G$, and $A$ is an abelian group, then after identification of $\{1\}\times G$ with $G$, for any finite $G$-set $X$*

$$(A \boxtimes M)(X) = A\otimes_{\mathbb{Z}} M(X)$$

*and for any map of finite $G$-sets $f : X \to Y$*

$$(A \boxtimes M)_*(f) = A\otimes_{\mathbb{Z}} M_*(f) \qquad\qquad (A \boxtimes M)^*(f) = A\otimes_{\mathbb{Z}} M^*(f)$$

In other words, the functor $A \boxtimes M$ is just the functor $M$, "with coefficients in $A$".

PROOF. First consider the case $A = \mathbb{Z}$. This is just the Burnside functor for the trivial group. But $\iota_{\{1\}}^G(b) = b$ by lemma 5.4, and I have

$$\mathbb{Z} \boxtimes M = b \hat{\otimes} \iota_G^G(M) = b \hat{\otimes} M \simeq M$$

More generally, if $S$ is any set, then the free abelian group $\mathbb{Z}^{(S)}$ is the functor $b_S$ for the trivial group. So

$$\iota_{\{1\}}^G(\mathbb{Z}^{(S)}) = b_{\mathrm{Inf}_{\{1\}}^G S} \hat{\otimes} M \simeq \bigoplus_{s\in S} M$$

So its value on any $G$-set $X$ is $\mathbb{Z}^{(S)} \otimes M(X)$. Now for a fixed Mackey functor $M$, the functor $A \mapsto A \boxtimes M$ is clearly additive and right exact. Choosing a resolution of $A$ by free $\mathbb{Z}$-modules

$$\mathbb{Z}^{(T)} \to \mathbb{Z}^{(S)} \to A \to 0$$

leads for any $G$-set $X$ to the exact sequence

$$(\mathbb{Z}^{(T)} \boxtimes M)(X) \to (\mathbb{Z}^{(S)} \boxtimes M)(X) \to (A \boxtimes M)(X) \to 0$$

This sequence is isomorphic to

$$\mathbb{Z}^{(T)} \otimes M(X) \to \mathbb{Z}^{(S)} \otimes M(X) \to (A \boxtimes M)(X) \to 0$$

This proves that $(A \boxtimes M)(X) \simeq A \otimes M(X)$, and this isomorphism is natural in $M$ and $A$. The proposition follows. □

**6.3. Associativity.** The direct product of Mackey functors is associative:

PROPOSITION 6.4. *Let $G$, $H$, and $K$ be finite groups. If $M$ is a Mackey functor for $G$, if $N$ is a Mackey functor for $H$ and $P$ is a Mackey functor for $P$, then after identification of $(G \times H) \times K$ with $G \times (H \times K)$, there is an isomorphism*

$$(M \boxtimes N) \boxtimes P \simeq M \boxtimes (N \boxtimes P)$$

*which is natural in $M$, $N$ and $P$.*

PROOF. Indeed, by definition

$$(M \boxtimes N) \boxtimes P = \iota_{G\times H}^{G\times H\times K}\left(\iota_G^{G\times H}(M) \hat{\otimes} \iota_H^{G\times H}(N)\right) \hat{\otimes} \iota_K^{G\times H\times K}(P)$$

Let $U = G \times H$, viewed as a $(G \times H \times K)$-set-$(G \times H)$. Then

$$\iota_{G\times H}^{G\times H\times K}\left(\iota_G^{G\times H}(M) \hat{\otimes} \iota_H^{G\times H}(N)\right) = T_U\left(\iota_G^{G\times H}(M) \hat{\otimes} \iota_H^{G\times H}(N)\right)$$

By proposition 4.4, this is also

$$T_U\left(\iota_G^{G\times H}(M)\right) \hat{\otimes} T_U\left(\iota_H^{G\times H}(N)\right)$$

Let $V = G$, wiewed as a $(G \times H)$-set-$G$. Then

$$T_U\left(\iota_G^{G\times H}(M)\right) = T_U \circ T_V(M) = T_{U\circ_{G\times H}V}(M)$$

But $U \circ_{G\times H} V$ is the set $G$, viewed as a $(G \times H \times K)$-set-$G$. Thus

$$T_{U\circ_{G\times H}V}(M) = \iota_G^{G\times H\times K}(M)$$

A similar argument shows that

$$T_U\left(\iota_H^{G\times H}(N)\right) = \iota_H^{G\times H\times K}(N)$$

Finally

$$(M \boxtimes N) \boxtimes P \simeq \left(\iota_G^{G\times H\times K}(M) \hat{\otimes} \iota_H^{G\times H\times K}(N)\right) \hat{\otimes} \iota_K^{G\times H\times K}(N)$$

On the other hand

$$M \boxtimes (N \boxtimes P) \simeq \iota_G^{G\times H\times K}(M) \hat{\otimes} \left(\iota_H^{G\times H\times K}(N) \hat{\otimes} \iota_K^{G\times H\times K}(N)\right)$$

Now proposition follows from the associativity of tensor product (see [**Bou97**] Proposition 1.9.1), and from the naturality of the above isomorphisms. □

**6.4. Tensor induction of direct products.**

PROPOSITION 6.5. *Let $G$, $H$, and $K$ be finite groups, and $U$ be a finite $K$-set-$(G \times H)$. Then $U/H$ is a finite $K$-set-$G$ and $U/G$ is a finite $K$-set-$H$. If $M$ is a Mackey functor for $G$ and $N$ is a Mackey functor for $H$, then*

$$T_U(M \boxtimes N) \simeq T_{U/H}(M) \hat{\otimes} T_{U/G}(N)$$

*and this isomorphism is natural in $M$ and $N$.*

PROOF. Let $V = G$, viewed as a $(G \times H)$-set-$G$, and $W = H$, viewed as a $(G \times H)$-set-$H$. Then by proposition 4.4

$$T_U(M \boxtimes N) = T_U\Big(T_V(M) \hat{\otimes} T_W(N)\Big) = \Big(T_U \circ T_V(M)\Big) \hat{\otimes} \Big(T_U \circ T_W(N)\Big)$$

But moreover

$$T_U \circ T_V(M) \simeq T_{U \times_{G \times H} V}(M)$$

and as $U \times_{G \times H} V \simeq U/H$ as $K$-set-$H$, this gives

$$T_U \circ T_V(M) \simeq T_{U/H}(M)$$

A similar argument shows that

$$T_U \circ T_W(N) \simeq T_{U/G}(N)$$

and the proposition follows. □

**6.5. Tensor product from direct product.** The tensor product of Mackey functors can be recovered from the direct product, by diagonal restriction:

PROPOSITION 6.6. *Let $G$ be a finite group, identified with the diagonal subgroup $\Delta(G)$ of $G \times G$. Then if $M$ and $N$ are Mackey functors for $G$*

$$\mathrm{Res}_{\Delta(G)}^{G \times G}(M \boxtimes N) \simeq M \hat{\otimes} N$$

*and this isomorphism is natural in $M$ and $N$.*

PROOF. Indeed the restriction to $\Delta(G)$ can be viewed as the tensor induction $T_U$ for the set $U = G \times G$, viewed as a $\Delta(G)$-set-$(G \times G)$. Now by proposition 6.5

$$T_U(M \boxtimes N) = T_{U/G}(M) \hat{\otimes} T_{U/G}(N)$$

Here the first $U/G$ is relative to the second factor of $G \times G$, and the second is relative to the first factor. Anyway, those two $G$-sets-$G$ are isomorphic to $G$, with its left and right action by multiplication. So $T_{U/G}$ is the identity functor, and the proposition follows. □

**6.6. Identification of the direct product.** The direct product of Mackey functors can be computed as follows:

PROPOSITION 6.7. *Let $G$ and $H$ be finite groups, and $X$ be a finite $(G \times H)$-set. If $M$ is a Mackey functor for $G$, and $N$ is a Mackey functor for $H$, then*

$$(M \boxtimes N)(X) \simeq \Big( \bigoplus_{Y \overset{\varphi}{\to} X} M(H\backslash Y) \otimes N(G\backslash Y)\Big)/\mathcal{J} \tag{6.11}$$

*where $(Y, \varphi)$ is a finite $(G \times H)$-set over $X$, and $\mathcal{J}$ is the submodule generated by*

$$M_*(H\backslash f)(m) \otimes n' - m \otimes N^*(G\backslash f)(n') \qquad \forall m \in M(H\backslash Y),\ n' \in N(G\backslash Y')$$

*whenever $f : (Y,\varphi) \to (Y',\varphi')$ is a morphism of $(G\times H)$-sets over $X$, which is injective when restricted to every $H$-orbit, and by the elements*

$$M^*(H\backslash f)(m')\otimes n - m'\otimes N_*(G\backslash f)(n) \qquad \forall m'\in M(H\backslash Y'),\ n\in N(G\backslash Y)$$

*whenever $f : (Y,\varphi) \to (Y',\varphi')$ is a morphism of $(G\times H)$-sets over $X$, which is injective when restricted to every $G$-orbit*

PROOF. Let $K = G\times H$. Recall that for a $K$-set $Y$, the value of $\iota_G^K(M)(Y)$ is

$$\iota_G^K(M)(Y) = \varinjlim_{(Z,f)\in\mathcal{D}_G^K(Y)} M(H\backslash Z)$$

where $\mathcal{D}_G^K(Y)$ is the category with $K$-sets over $Y$ as objects, the morphisms being the morphisms of $K$-sets over $Y$ which are injective when restricted to each orbit of $H$. If $m\in M(H\backslash Z)$, denote by $m_{(Z,f)}$ its image in $\iota_G^K(Y)$.

Let

$$(M\boxtimes' N)(X) = \Big(\bigoplus_{Y\overset{\varphi}{\to}X} M(H\backslash Y)\otimes N(G\backslash Y)\Big)/\mathcal{J}$$

If $(Y,\varphi)$ is a finite $K$-set over $X$, if $m\in M(H\backslash Y)$, and $n\in N(G\backslash Y)$, I denote by $[m\boxtimes n]_{(Y,\varphi)}$ the image of $m\otimes n$ in $(M\boxtimes' N)(X)$.

As

$$(M\boxtimes N)(X) = (\iota_G^K(M)\hat{\otimes}\iota_H^K(N))(X) = \Big(\bigoplus_{Y\overset{\varphi}{\to}X}\iota_G^K(M)(H\backslash Y)\otimes\iota_H^K(N)(G\backslash Y)\Big)/\mathcal{K}$$

for a suitable submodule $\mathcal{K}$, I can try to define a morphism $\Phi$ from $(M\boxtimes' N)(X)$ to $(M\boxtimes N)(X)$ by

$$\Phi\Big([m\boxtimes n]_{(Y,\varphi)}\Big) = [m_{(Y,Id_Y)}\otimes n_{(Y,Id_Y)}]_{(Y,\varphi)}$$

This is possible if $\mathcal{J}$ is mapped to zero. So let $f : (Y,\varphi)\mapsto(Y',\varphi')$ be a morphism of $K$-sets over $X$, which is injective on every $H$-orbit. Let $m\in M(H\backslash Y)$ and $n'\in N(G\backslash Y')$. Now

$$\Phi\Big([m\boxtimes N^*(G\backslash f)(n')]_{(Y,\varphi)}\Big) = [m_{(Y,Id_Y)}\otimes N^*(G\backslash f)(n')_{(Y,Id_Y)}]_{(Y,\varphi)}$$

But $N^*(G\backslash f)(n')_{(Y,Id_Y)} = \iota_H^K(N)^*(f)(n'_{(Y',Id_{Y'})})$, so in $(M\boxtimes N)(X)$, I have

$$\Phi\Big([m\boxtimes N^*(G\backslash f)(n')]_{(Y,\varphi)}\Big) = [\iota_G^K(M)_*(f)(m_{(Y,Id_Y)})\boxtimes n'_{(Y',Id'_Y)}]_{(Y',\varphi')}$$

As $\iota_G^K(M)_*(f)(m_{(Y,Id_Y)}) = m_{(Y,f)}$ and as $f$ is supposed to be injective on every $H$-orbit, it is a morphism in $\mathcal{D}_G^K(X)$ from $(Y,f)$ to $(Y',Id_{Y'})$. Thus in $\iota_G^K(Y')$, I have

$$m_{(Y,f)} = M_*(H\backslash f)(m)_{(Y',Id_{Y'})}$$

This gives

$$\Phi\Big([m\boxtimes N^*(G\backslash f)(n')]_{(Y,\varphi)}\Big) = [M_*(H\backslash f)(m)_{(Y',Id_{Y'})}\otimes n'_{(Y',Id_{Y'})}]_{(Y',\varphi')} = \ldots$$

$$\ldots = \Phi\Big([M_*(H\backslash f)(m)\boxtimes n']_{(Y',\varphi')}\Big)$$

A similar argument obtained by reversing the roles of $M$ and $N$ shows that $\Phi$ is well-defined.

I will now define a map $\Theta$ from $(M \boxtimes N)(X)$ to $(M \boxtimes' N)(X)$: let $(Y, \varphi)$ be a $K$-set over $X$. If $(Z, f)$ is an object of $\mathcal{D}_G^K(Y)$, if $m \in M(H\backslash Z)$, if $(T, g)$ is an object of $\mathcal{D}_H^K(Y)$ and $N \in N(G\backslash T)$, I build the fibre product $Z.T$

$$\begin{array}{ccc} Z.T & \xrightarrow{\pi_Z} & Z \\ {\scriptstyle \pi_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow[g]{} & Y \end{array}$$

and then I set

$$\Theta([m_{(Z,f)} \otimes n_{(T,g)}]_{(Y,\varphi)}) = [M^*(H\backslash\pi_Z)(m) \boxtimes N^*(G\backslash\pi_T)(n)]_{(Z.T,\varphi\circ f\circ\pi_Z)}$$

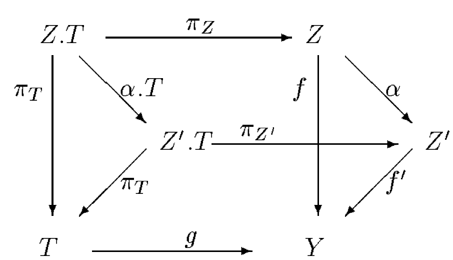
This map is also well-defined: I must check that $\Theta$ is compatible with the inductive limits in the definition of $\iota$, and that $\mathcal{K}$ is mapped to zero.

Suppose first that I replace $m_{(Z,f)}$ by $M_*(H\backslash\alpha)(m)_{(Z',f')}$, for some morphism $\alpha : (Z, f) \to (Z', f')$ in $\mathcal{D}_G^K(Y)$. Then

$$\Theta\Big([M_*(H\backslash\alpha)(m)_{(Z',f')} \otimes n_{(T,g)}]_{(Y,\varphi)}\Big) = \dots$$

$$\dots = [M^*(H\backslash\pi_{Z'})M_*(H\backslash\alpha)(m) \boxtimes N^*(G\backslash\pi_T)(n)]_{(Z'.T,\varphi\circ f'\circ\pi_{Z'})}$$

There is a commutative diagram




Now it is clear that the square

$$\begin{array}{ccc} Z.T & \xrightarrow{\pi_Z} & Z \\ {\scriptstyle \alpha.T}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Z'.T & \xrightarrow[\pi_{Z'}]{} & Z' \end{array}$$

is cartesian, so the square

$$\begin{array}{ccc} H\backslash Z.T & \xrightarrow{H\backslash\pi_Z} & H\backslash Z \\ {\scriptstyle H\backslash\alpha.T}\downarrow & & \downarrow{\scriptstyle H\backslash\alpha} \\ H\backslash Z'.T & \xrightarrow[H\backslash\pi_{H\backslash Z'}]{} & H\backslash Z' \end{array}$$

is also cartesian by Lemma 9.3.3 of [**Bou97**], because $\alpha$ is injective on the $H$-orbits. Now it follows that

$$M^*(H\backslash\pi_{Z'})M_*(H\backslash\alpha) = M_*(H\backslash\alpha.T)M^*(H\backslash\pi_Z)$$

Moreover, the morphism $\alpha.T : Z.T \to Z'.T$ is injective on every $H$-orbit, because $\alpha$ is. Thus in $(M \boxtimes' N)(X)$, I have

$$[M_*(H\backslash\alpha.T)M^*(H\backslash\pi_Z) \boxtimes N^*(G\backslash\pi_T)(n)]_{(Z'.T,\varphi\circ f'\circ\pi_{Z'})} = \dots$$
$$\dots = [M^*(H\backslash\pi_Z) \boxtimes N^*(G\backslash\alpha.T)N^*(G\backslash\pi_T)(n)]_{(Z.T,\varphi\circ f\circ\pi_Z)}$$

But

$$N^*(G\backslash\alpha.T)N^*(G\backslash\pi_T)(n) = N^*(G\backslash\pi_T)(n)$$

This proves that

$$\Theta\Big([M_*(H\backslash\alpha)(m)_{(Z',f')} \otimes n_{(T,g)}]_{(Y,\varphi)}\Big) = \Theta\Big([m_{(Z,f)} \otimes n_{(T,g)}]_{(Y,\varphi)}\Big)$$

and a similar argument shows that $\Theta$ is compatible with the relations coming from the inductive limits.

Now suppose that $\alpha : (Y,\varphi) \to (Y',\varphi')$ is a morphism of $K$-sets over $X$. Let $(Z,f)$ be an object of $\mathcal{D}_G^K(Y)$ and $m \in M(H\backslash Z)$. Let $(T',g')$ be an object in $\mathcal{D}_H^K(Y')$, and $n' \in N(G\backslash T)$. In $(M \boxtimes N)(X)$, I have

$$(6.12)\quad [m_{(Z,f)} \otimes \iota_H^K(N)^*(\alpha)(n'_{(T',g')})]_{(Y,\varphi)} = [\iota_G^K(M)_*(\alpha)(m_{(Z,f)}) \otimes n'_{(T',g')}]_{(Y',\varphi')}$$

I have to check that both sides have the same image in $(M \boxtimes' N)(X)$. But

$$\iota_G^K(M)_*(\alpha)(m_{(Z,f)}) = m_{(Z,\alpha\circ f)}$$

Moreover, if the square

$$\begin{array}{ccc} T & \xrightarrow{a} & T' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Y & \xrightarrow[\alpha]{} & Y' \end{array}$$

is cartesian, then

$$\iota_H^K(N)^*(\alpha)(n'_{(T',g')}) = N^*(G\backslash a)(n')_{(T,g)}$$

The image by $\Theta$ of the right hand side of (6.12) is

$$(6.13)\qquad [M^*(H\backslash\pi_Z)(m) \boxtimes N^*(G\backslash\pi_T)N^*(G\backslash a)(n')]_{(Z.T,\varphi\circ g\circ\pi_T)}$$

Now the square

$$\begin{array}{ccc} Z.T & \xrightarrow{a\circ\pi_T} & T' \\ {\scriptstyle \pi_Z}\downarrow & & \downarrow{\scriptstyle g'} \\ Z & \xrightarrow[\alpha\circ f]{} & Y' \end{array}$$

is cartesian, because it is composed of two cartesian squares. So the image by $\Theta$ of the left hand side of (6.12) is

$$[M^*(H\backslash\pi_Z)(m) \boxtimes N^*(G\backslash(a\circ\pi_T))(n')]_{(Z.T,\varphi'\circ g'\circ a\circ\pi_T)}$$

This is equal to (6.13). So $\Theta$ is well-defined.

It is clear that $\Theta\circ\Phi$ is the identity: indeed, if $(Y,\varphi)$ is a $K$-set over $X$, if $m \in M(H\backslash Y)$ and $n \in N(G\backslash Y)$, then

$$\Theta\circ\Phi([m\boxtimes n]_{(Y,\varphi)}) = \Theta([m_{(Y,Id)} \otimes n_{(Y,Id)}]_{(Y,\varphi)}) = [m\boxtimes n]_{(Y,\varphi)}$$

since $Y.Y = Y$. Moreover, if $(Z,f)$ and $(T,g)$ are any $K$-sets over $Y$, if $m \in M(H\backslash Z)$ and $n \in N(H\backslash T)$, then in $(M \boxtimes N)(X)$

$$[m_{(Z,f)} \otimes n_{(T,g)}]_{(Y,\varphi)} = [\iota_G^K(M)_*(f)(m_{(Z,Id)}) \otimes n_{(T,g)}]_{(Y,\varphi)} = \dots$$

$$\ldots = [m_{(Z,Id)} \otimes \iota_H^K(N)^*(f)(n_{(T,g)})]_{(Z,\varphi\circ f)} = \ldots$$

$$\ldots = [m_{(Z,Id)} \otimes N^*(G\backslash\pi_T)(n)_{(Z.T,\pi_Z)}]_{(Z,\varphi\circ f)} = \ldots$$

$$\ldots = [m_{(Z,Id)} \otimes \iota_H^K(N)_*(\pi_Z)\Big(N^*(G\backslash\pi_T)(n)_{(Z.T,Id_{Z.T})}\Big)]_{(Z,\varphi\circ f)} = \ldots$$

$$\ldots = [M^*(H\backslash\pi_Z)(m)_{(Z.T,Id_{Z.T})} \otimes N^*(G\backslash\pi_T)(n)_{(Z.T,Id_{Z.T})}]_{(Y,\varphi)}$$

Thus $\Phi\circ\Theta$ is also the identity, and this proves the proposition. □

From now on, I will identify $(M \boxtimes N)(X)$ with the right hand side of (6.11).

Corollary 6.8. *If $\theta : X \to X'$ is a morphism of $(G\times H)$-sets, then*

$$(M \boxtimes N)_*(\theta)\Big([m \boxtimes n]_{(Y,\varphi)}\Big) = [m \boxtimes n]_{(Y,\theta\circ\varphi)}$$

*If $(Y',\varphi')$ is a $(G\times H)$-set over $X'$, if $m' \in M(H\backslash Y')$ and $n' \in N(G\backslash Y')$, then*

$$(M \boxtimes N)^*\Big([m' \boxtimes n']_{(Y',\varphi')}\Big) = [M^*(H\backslash a)(m') \boxtimes N^*(G\backslash a)(n')]_{(Y,\varphi)}$$

*where $Y$, $\varphi$, and $a$ are defined by the cartesian square*

$$\begin{array}{ccc} Y & \xrightarrow{a} & Y' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ X & \xrightarrow[\theta]{} & X' \end{array}$$

Proof. This follows from straightforward translations, using the isomorphisms $\Phi$ and $\Theta$ of the proposition. □

Remark 6.9. Those identifications give an explicit form of the isomorphisms of proposition 6.3

$$(A \boxtimes M)(X) \simeq A \otimes M(X)$$

Just map $a\otimes m \in A\otimes M(X)$ to $[A^*(p_X)(a) \boxtimes m]_{(X,Id_X)} \in (A\boxtimes M)(X)$, where $p_X : X \to \bullet$ is the only possible map. Conversely, map $[\alpha\otimes m]_{(Y,\varphi)}$ to

$$\sum_{y\in G\backslash Y} \alpha(y)\otimes M_*(\varphi)M_*(i_y)M^*(i_y)(m)$$

where $i_y : G/G_y \to Y$ is defined by $i_y(gG_y) = gy$.

Remark 6.10. Recall from [**Bou96b**] Section 3 the following notations: if $L$ is a subgroup of $G\times H$, denote by $G' = p_1(L)$ (resp. $H' = p_2(L)$) the projection of $L$ on $G$ (resp. on $H$). Let

$$k_1(L) = \{g \in G \mid (g,1) \in L\} \qquad\qquad k_2(L) = \{h\in H \mid (1,h)\in L\}$$

Then $k_j(L) \trianglelefteq p_j(L)$ for $j = 1, 2$, and the quotients $K = p_1(L)/k_1(L)$ and $p_2(L)/k_2(L)$ are isomorphic. Let $s : G' \to K$ be the canonical surjection. Then there exists a surjection $t : H' \to K$ such that

$$L = \{(g,h) \in G'\times H' \mid s(g) = t(h)\}$$

The previous identification of $M\boxtimes N$ gives the following evaluation at the subgroup $L$ of $G\times H$:

$$(M\boxtimes N)(L) = \Big(\bigoplus_{K\subseteq L} M\Big(p_1(K)\Big)\otimes N\Big(p_2(K)\Big)\Big)/\mathcal{J}$$

where $\mathcal{J}$ is the submodule generated by

$$t_{p_1(K')}^{p_1(K)} m \otimes n - m \otimes r_{p_2(K')}^{p_2(K)} n \quad \text{for} \quad K' \subseteq K,\ k_2(K') = k_2(K)$$
$$r_{p_1(K')}^{p_1(K)} m \otimes n - m \otimes t_{p_2(K')}^{p_2(K)} n \quad \text{for} \quad K' \subseteq K,\ k_1(K') = k_1(K)$$
$$xm \otimes yn - m \otimes n \quad \text{for} \quad (x, y) \in L$$

**6.7. Wreath products.** Let $G$ be a group, and $n$ be a positive integer. I denote by $G \wr S_n$ the wreath product of $G$ with the symmetric group $S_n$: it is the semi-direct product of $G^n$ and $S_n$, for the action of $S_n$ by permutations of the factors of $G^n$. There is a more intrinsic presentation of those wreath products:

PROPOSITION 6.11. *Let $G$ be a group, and $E$ be a free right $G$-set. If $n = |E/G|$ is finite, then $\mathrm{Aut}_G(E) \simeq G \wr S_n$.*

PROOF. Choose a system of representatives $\Omega$ of $E/G$. Then as a right $G$-set

$$E \simeq \Omega \times G$$

where $G$ acts trivially on $\Omega$. Now if $\theta \in \mathrm{Aut}_G(\Omega \times G)$

$$\forall \omega \in \Omega,\ \forall g \in G,\ \theta(\omega, g) = \theta(\omega, 1)g$$

so there exists a map $\gamma : G \to \Omega$ and a permutation $\sigma$ of $\Omega$, such that

$$\theta(\omega, g) = \Big(\sigma(\omega), \gamma(\omega)g\Big)$$

Conversely, if $\gamma \in G^{\Omega}$ and $\sigma$ is a bijection of $\Omega$, this formula defines a $G$-automorphism of $\Omega \times G$. If $|\Omega| = n$, this states a bijection between $\mathrm{Aut}_G(\Omega \times G)$ and $G \wr S_n$, which is easily seen to be a group isomorphism. □

DEFINITION 6.12. *If $G$ is a finite group, and $n$ is a non-negative integer, define $M^{\boxtimes n}$ as the Mackey functor for $G^n$ equal to the Burnside functor for the trivial group if $n = 0$, and to $M^{n-1} \boxtimes M$ if $n \geq 1$.*

Note in particular that $M^{\boxtimes 1} \simeq M$.

PROPOSITION 6.13. *Let $G$ be a finite group, and $E$ be a finite free right $G$-set. Then if $|E/G| = n$, and if $H = \mathrm{Aut}_G(E)$, the group $H$ is isomorphic to $G \wr S_n$, and the set $E$ is a finite $H$-set-$G$. Then for any Mackey functor $M$ for $G$*

$$\mathrm{Res}_{G^n}^{H} T_E(M) \simeq M^{\boxtimes n}$$

PROOF. It follows from corollary 5.6 that

$$\mathrm{Res}_{G^n} T_E(M) = T_{\mathrm{Res}_{G^n \times G^{op}} E}(M)$$

Now the restriction of $E$ to $G^n \times G^{op}$ is the disjoint union of $n$-copies of $G$: more precisely, if $G(i)$ is the $i^{th}$-factor of $G^n$

$$\mathrm{Res}_{G^n \times G^{op}} E = \bigsqcup_{i=1}^{n} \mathrm{Inf}_{G(i) \times G^{op}}^{G^n \times G^{op}} G$$

It follows that

$$\mathrm{Res}_{G^n} T_E(M) \simeq \hat{\bigotimes_{i=1}^{n}} T_{\mathrm{Inf}_{G(i) \times G^{op}}^{G^n \times G^{op}} G}(M)$$

But moreover

$$T_{\mathrm{Inf}_{G(i) \times G^{op}}^{G^n \times G^{op}} G}(M) = \iota_{G(i)}^{G^n}(M)$$

An easy induction argument shows that

$$M^{\boxtimes n} = \iota_{G(1)}^{G^n}(M)\,\hat{\otimes}\ldots\hat{\otimes}\,\iota_{G(n)}^{G^n}(M)$$

and this completes the proof of the proposition. □

REMARK 6.14. Suppose that $H$ is a subgroup of $G$, of index $n$. Let $E = G$ viewed as a free right $H$-set. Then $|E/H| = n$. Moreover, the group $G$ acts on the left on $E$ by multiplication, and this action commutes with the right action of $H$. This gives a morphism $G \to \mathrm{Aut}_H(E) \simeq H \wr S_n$: this is the Frobenius morphism.

In that case, the functor $T_E$ certainly deserves the name of tensor induction, and could be denoted by $\mathrm{Ten}_H^G$.

## 7. Tensor induction for Green functors

**7.1. Green functors.** Recall the following definition (see [**Bou97**] Chapter 2)

DEFINITION 7.1. *Let $R$ be a commutative ring. A Green functor $A$ (over $R$) for the group $G$ is a Mackey functor (over $R$) endowed for any $G$-sets $X$ and $Y$ of bilinear maps*

$$A(X) \times A(Y) \to A(X \times Y)$$

*denoted by $(a, b) \mapsto a \times b$ which are bifunctorial, associative, and unitary, in the following sense:*

- *(Bifunctoriality) If $f : X \to X'$ and $g : Y \to Y'$ are morphisms of $G$-sets, then the squares*

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ {\scriptstyle A_*(f)\times A_*(g)}\downarrow & & \downarrow{\scriptstyle A_*(f\times g)} \\ A(X') \times A(Y') & \xrightarrow[\times]{} & A(X' \times Y') \end{array}$$

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ {\scriptstyle A^*(f)\times A^*(g)}\uparrow & & \uparrow{\scriptstyle A^*(f\times g)} \\ A(X') \times A(Y') & \xrightarrow[\times]{} & A(X' \times Y') \end{array}$$

  *are commutative.*
- *(Associativity) If $X$, $Y$ and $Z$ are $G$-sets, then the square*

$$\begin{array}{ccc} A(X) \times A(Y) \times A(Z) & \xrightarrow{Id_{A(X)}\times(\times)} & A(X) \times A(Y \times Z) \\ {\scriptstyle (\times)\times Id_{A(Z)}}\downarrow & & \downarrow{\scriptstyle \times} \\ A(X \times Y) \times A(Z) & \xrightarrow[\times]{} & A(X \times Y \times Z) \end{array}$$

  *is commutative, up to identifications $(X\times Y)\times Z \simeq X\times Y\times Z \simeq X\times(Y\times Z)$.*
- *(Unitarity) If $\bullet$ denotes the $G$-set with one element, there exists an element $\varepsilon \in A(\bullet)$ such that for any $G$-set $X$ and for any $a \in A(X)$*

$$A_*(p_X)(a \times \varepsilon) = a = A_*(q_X)(\varepsilon \times a)$$

  *denoting by $p_X$ (resp. $q_X$) the (bijective) projection from $X \times \bullet$ (resp. from $\bullet \times X$) to $X$.*

*If $A$ and $B$ are Green functors for the group $G$, a morphism $f$ (of Green functors) from $A$ to $B$ is a morphism of Mackey functors such that for any $G$-sets $X$ and $Y$, the square*

$$\begin{array}{ccc} A(X)\times A(Y) & \xrightarrow{\times} & A(X\times Y) \\ {\scriptstyle f_X\times f_Y}\downarrow & & \downarrow{\scriptstyle f_{X\times Y}} \\ B(X)\times B(Y) & \xrightarrow[\times]{} & B(X\times Y) \end{array}$$

*is commutative.*
*If moreover $f_\bullet : A(\bullet)\to B(\bullet)$ maps the unit of $A$ to the unit of $B$, then I will say that $f$ is unitary.*

Equivalently, a Green functor $A$ (over $\mathbb{Z}$) can be defined as a Mackey functor equipped with morphisms

$$\mu : A\hat{\otimes} A\to A \qquad \varepsilon : b\to A$$

satisfying the usual conditions of associativity and unitarity: the following diagrams are commutative

$$(7.14)\qquad \begin{array}{ccc} A\hat{\otimes}A\hat{\otimes}A & \xrightarrow{1\hat{\otimes}\mu} & A\hat{\otimes}A \\ {\scriptstyle \mu\hat{\otimes}1}\downarrow & & \downarrow{\scriptstyle \mu} \\ A\hat{\otimes}A & \xrightarrow[\mu]{} & A \end{array} \qquad \begin{array}{ccccc} b\hat{\otimes}A & \xrightarrow{\varepsilon\hat{\otimes}1} & A\hat{\otimes}A & \xleftarrow{1\hat{\otimes}\varepsilon} & A\hat{\otimes}b \\ & {\scriptstyle\eta}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle\eta} & \\ & & A & & \end{array}$$

Here $\eta$ is any of the canonical isomorphisms $b\hat{\otimes}A\simeq A\simeq A\hat{\otimes}b$.

Similarly, if $A$ is a Green functor for $G$, an $A$-module $M$ is a Mackey functor equipped with a morphism $\mu : A\hat{\otimes}M\to M$, such that the following diagrams are commutative

$$(7.15)\qquad \begin{array}{ccc} A\hat{\otimes}A\hat{\otimes}M & \xrightarrow{1\hat{\otimes}\mu} & A\hat{\otimes}M \\ {\scriptstyle \mu\hat{\otimes}1}\downarrow & & \downarrow{\scriptstyle \mu} \\ A\hat{\otimes}M & \xrightarrow[\mu]{} & M \end{array} \qquad \begin{array}{ccc} b\hat{\otimes}M & \xrightarrow{\varepsilon\hat{\otimes}1} & A\hat{\otimes}M \\ & {\scriptstyle\eta}\searrow & \downarrow{\scriptstyle\mu} \\ & & M \end{array}$$

I will denote by $A$-**Mod** the category of $A$-modules.

**7.2. Tensor induction.** Tensor induction maps Green functors to Green functors:

PROPOSITION 7.2. *Let $G$ and $H$ be finite groups, and $U$ be a finite $H$-set-$G$.*

1. *If $A$ is a Green functor for $G$, then $T_U(A)$ is a Green functor for $H$.*
2. *If $M$ is an $A$-module, then $T_U(M)$ is a $T_U(A)$-module.*
3. *The correspondence $M\mapsto T_U(M)$ is a (generally non-additive) right exact functor from $A$-**Mod** to $T_U(A)$-**Mod**.*

PROOF. This is clear, since $T_U$ commute with tensor products, and maps the Burnside functor to the Burnside functor. So the image by $T_U$ of the diagrams

(7.14) give commutative diagrams

$$\begin{array}{ccc}
T_U(A)\,\hat{\otimes}\,T_U(A)\,\hat{\otimes}\,T_U(A) & \xrightarrow{1\,\hat{\otimes}\,T_U(\mu)} & T_U(A)\,\hat{\otimes}\,T_U(A)\\
{\scriptstyle T_U(\mu)\,\hat{\otimes}\,1}\downarrow & & \downarrow{\scriptstyle T_U(\mu)}\\
T_U(A)\,\hat{\otimes}\,T_U(A) & \xrightarrow[T_U(\mu)]{} & T_U(A)
\end{array}$$

$$\begin{array}{ccccc}
b\,\hat{\otimes}\,T_U(A) & \xrightarrow{T_U(\varepsilon)\,\hat{\otimes}\,1} & T_U(A)\,\hat{\otimes}\,T_U(A) & \xleftarrow{1\,\hat{\otimes}\,T_U(\varepsilon)} & T_U(A)\,\hat{\otimes}\,b\\
 & \eta\searrow & \downarrow{\scriptstyle T_U(\mu)} & \swarrow\eta & \\
 & & T_U(A) & &
\end{array}$$

So $T_U(A)$ is a Green functor.

Similarly, if $M$ is an $A$-module, then the images of the diagrams (7.15) give a structure of $T_U(A)$ module on $T_U(M)$. By definition, the correspondence $M \mapsto T_U(M)$ is a right exact functor. □

REMARK 7.3. There is a similar result for the functors $\mathcal{L}_U$ (see [**Bou97**] Proposition 10.3.2). In the case $U/G = \bullet$, I have seen that $T_U = \mathcal{L}_U$. One can check that the two structures of Green functor coincide in this case.

REMARK 7.4. If $G$ and $H$ are finite groups, if $A$ is a Green functor for $G$ and $B$ is a Green functor for $H$, then $A \boxtimes B$ is a Green functor for $G \times H$: this follows from the fact that $\iota_G^{G\times H}(A)$ and $\iota_H^{G\times H}(B)$ are, and that the tensor product of Green functors is a Green functor (see [**Bou97**] Proposition 6.3.1).

**7.3. Examples.**

7.3.1. *Extension of coefficients.* Let $G$ and $H$ be finite groups. If $R$ is a commutative ring, then a Mackey functor over $R$ for $G$ is just a module over the Green functor

$$R \boxtimes b \simeq \iota_{\{1\}}^G(R)\,\hat{\otimes}\,b \simeq \iota_{\{1\}}^G(R)$$

If $U$ is a finite $G$-set-$H$, then $T_U(M)$ is a module over the Green functor

$$T_U(R \boxtimes b) \simeq T_{U/G}(R)\,\hat{\otimes}\,T_{U/\{1\}}(b) \simeq T_{U/G}(R)$$

This functor $T_{U/G}(R)$ depends only on $R$ and $U/G$, which is just a finite $H$-set. More generally, if $A$ is an abelian group, and $V$ is an $H$-set, the functor $T_V(A)$ is a Mackey functor for $H$, that may be quite complicated, even if $A$ is cyclic:

PROPOSITION 7.5. *Let $n$ be an positive integer, and $\Omega$ be a set of cardinality $n$. If $H$ is a finite group, and $V$ is a finite $H$-set, then $T_V(\mathbf{Z}/n\mathbf{Z})$ is the quotient of the Burnside functor $b$ for $H$ by the subfunctor generated by the elements $\Omega^W = \mathrm{Hom}_{\{1\}}(W,\Omega) \in b(H_W)$, where $W$ is a non-empty subset of $V$.*

PROOF. Let $0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0$ be the standard resolution. As Mackey functors over the trivial group, it can be viewed as

$$0 \to b_\bullet \xrightarrow{n} b_\bullet \to \mathbf{Z}/n\mathbf{Z} \to 0$$

The $\{1\}$-poset over $\bullet\times\bullet$ defining the morphism $n : b_\bullet \to b_\bullet$ is just the discrete set $\Omega$ with $n$ elements. Now $T_V(\mathbf{Z}/n\mathbf{Z})$ is defined by the following exact sequence

$$b_{\mathrm{Hom}_{\{1\}}(V,\bullet;\bullet)} \to b_{\mathrm{Hom}_{\{1\}}(V,\bullet)} \to T_V(\mathbf{Z}/n\mathbf{Z}) \to 0$$

This is also

$$b_{\mathrm{Hom}_{\{1\}}(V,\bullet;\bullet)} \to b \to T_V(\mathbf{Z}/n\mathbf{Z}) \to 0$$

Evaluation at the trivial $H$-set gives

$$b\Big(\mathrm{Hom}_{\{1\}}(V,\bullet;\bullet)\Big) \to b(\bullet) \to T_V(\mathbf{Z}/n\mathbf{Z})(\bullet) \to 0$$

Now an element $\varphi$ of $\mathrm{Hom}_{\{1\}}(V,\bullet;\bullet)$ is completely determined by the preimage $W$ of the first point, which must be non-empty. If $K$ is a subgroup of $H$, then $\varphi$ in invariant by $K$ if and only if $W$ is. So the $H$-sets $H/K \to \mathrm{Hom}_{\{1\}}(V,\bullet;\bullet)$ are in one to one correspondence with the $K$-invariant non-empty subsets of $V$, and they generate $b\Big(\mathrm{Hom}_{\{1\}}(V,\bullet;\bullet)\Big)$ as $K$ runs through the subgroups of $H$. To find the image of these elements in $b(\bullet)$, I have to build pull-backs

$$\begin{array}{ccc} T & \longrightarrow & \mathrm{Hom}_{\{1\}}(V,\Omega;\bullet) \\ \big\downarrow & & \big\downarrow a \\ H/K & \underset{\varphi}{\longrightarrow} & \mathrm{Hom}_{\{1\}}(V,\bullet;\bullet) \end{array}$$

Now if $\varphi$ is associated to the subset $W$ of $V$, the pull-back $T$ is isomorphic to

$$T \simeq \mathrm{Ind}_K^H a^{-1}\Big(\varphi(K)\Big)$$

Moreover $a^{-1}\Big(\varphi(K)\Big)$ is the set of maps $\beta : V \to \Omega \sqcup \bullet$, such that $\beta(W) \subseteq \Omega$, and $\beta(V-W) = \bullet$. So it is isomorphic to $\mathrm{Hom}_{\{1\}}(W,\Omega)$. Thus

$$\mathrm{Ind}_K^H a^{-1}\Big(\varphi(K)\Big) = \mathrm{Ind}_K^H \mathrm{Res}_K^{H_W} \Omega^W$$

Now the proposition follows, since for any subgroup $L$ of $H$, the module

$$T_V(L) = (\mathrm{Res}_L^H T_V)(\bullet) = T_{\mathrm{Res}_L^H V}(\bullet)$$

can be computed by the previous procedure. □

EXAMPLE 7.6. Suppose $H = \mathbb{Z}/p\mathbb{Z}$, for a prime $p$, and $V = H$, acted on by multiplication. Then if $\emptyset \neq W \subseteq V$, either $H_W$ is trivial, or $H_W = H$ and $W = V$. This proves that $T_V(\mathbb{Z}/n\mathbb{Z})(H)$ is the quotient of $b(H)$, generated by $H/H$ and $H/\{1\}$, by the submodule generated by the elements

$$nH/H + \frac{n^p - n}{p} H/1 \quad \text{and} \quad nH/1$$

whereas $T_V(\mathbb{Z}/n\mathbb{Z})(1)$ is the quotient of $b(1) \simeq \mathbb{Z}$ by $n\{1\}/\{1\}$. So the evaluation of $T_V(\mathbb{Z}/n\mathbb{Z})$ at the trivial subgroup is always $\mathbb{Z}/n\mathbb{Z}$. Its evaluation at $H$ is the quotient of $\mathbb{Z}^2$ by the submodule generated by $(n,d)$ and $(0,n)$, where $d$ is the g.c.d of $n$ and $\frac{n^p-n}{p}$, equal to $n$ if $n$ is prime to $p$, and to $n/p$ if $p \mid n$. So if $n$ is prime to $p$, the functor $T_V$ is just $\mathbb{Z}/n\mathbb{Z} \boxtimes b$. But if $p \mid n$, then

$$T_V(\mathbb{Z}/n\mathbb{Z})(H) \simeq \mathbb{Z}/(n/p)\mathbb{Z} \times \mathbb{Z}/(np)\mathbb{Z}$$

This example shows that $T_U(R)$ need not be a Mackey functor over $R$, even if $R = \mathbb{Z}/p\mathbb{Z}$ for a prime $p$. However, there is a simple case:

Corollary 7.7. *Let $n$ be a positive integer, and $H$ be a finite group. If $V$ is a (non-empty) finite $H$-set, then*

1. *If $n$ is prime to the order of $H$, then*
$$T_V(\mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z} \boxtimes b$$
2. *Let $h$ be the least integer such that if $H_1 \subset H_2 \ldots \subset H_h = H$ is a strictly increasing sequence of subgroups of $G$, then $H_1$ is of order prime to $n$. Then $T_V(\mathbb{Z}/n\mathbb{Z})$ is a Mackey functor over $\mathbb{Z}/n^h\mathbb{Z}$.*

Proof. The first assertion is proved by induction on the order of $H$. If $H$ is trivial, this is true, because $T_U(\mathbb{Z}/n\mathbb{Z})(1)$ is a tensor product of copies of $\mathbb{Z}/n\mathbb{Z}$, thus isomorphic to $\mathbb{Z}/n\mathbb{Z}$. This is because the functor $T_\bullet$ is the identity functor for the trivial group. Note next that $\Omega^{W_1 \sqcup W_2} = \Omega^{W_1} \times \Omega^{W_2}$ in $b(H)$. Now if $W$ is any orbit of $H$ on $V$, then $\Omega^W$ can be written as

$$\Omega^W = nH/H + \sum_f H/H_f \tag{7.16}$$

where the $H_f$ are proper subgroups of $H$, equal to stabilizers of *non-constant* functions $f$ from $W$ to $\Omega$. Moreover, the cyclic group $C = \mathbb{Z}/n\mathbb{Z}$ acts on $\Omega^W$, and this action commutes with the action of $H$. The stabilizer of $f \in \Omega^W$ in $C \times H$ is the direct product $D \times K$ of its projections on $C$ and $H$, because $C$ and $H$ have relatively prime orders. Now $D$ stabilizes $f$, so $D = 1$.

Moreover, if $c \in C$ and $f \in \Omega^W$, then the stabilizer of $c.f$ in $H$ is equal to the stabilizer of $f$, and $c.f$ cannot be in the orbit of $f$ under $H$ if $c \neq 1$. This proves that the coefficient of $H/H_f$ in the sum (7.16) is a multiple of $n$, so $T_V(\mathbb{Z}/n\mathbb{Z})(H)$ is the quotient of $b(H)$ by a submodule $I$ contained in $nb(H)$. Moreover, by induction hypothesis, the element $nH_f/H_f$ is zero in $T_V(\mathbb{Z}/n\mathbb{Z})(H_f)$. As

$$nH/H_f = \mathrm{Ind}_{H_f}^H(nH_f/H_f)$$

this element is zero in $T_V(\mathbb{Z}/n\mathbb{Z})(H)$. So $nH/H = 0$ in $T_V(\mathbb{Z}/n\mathbb{Z})(H)$, and $nb(H) \subseteq I$. So

$$T_V(\mathbb{Z}/n\mathbb{Z})(H) = b(H)/nb(H) \simeq (\mathbb{Z}/n\mathbb{Z} \boxtimes b)(H)$$

This proves the first assertion.

The second assertion follows by induction on $h$: the case $h = 1$ is the first assertion. All I have to show then is that $n^h T_V(\mathbb{Z}/n\mathbb{Z})(H) = 0$. But this is clear, since multiplying equation (7.16) by $n^{h-1}$ gives

$$n^h H/H + \sum_f n^{h-1} H/H_f = 0 \quad \text{in} \quad T_V(H)$$

By induction hypothesis, for each $f$ in this sum, I have $n^{h-1}H_f/H_f = 0$ in $T_V(H_f)$, so $n^h H/H = 0$, as required. □

7.3.2. *Direct product of Green functors.* If $A$ and $B$ are Green functors for the group $G$, then $A \hat{\otimes} B$ is also a Green functor for $G$ (see **[Bou97]** 6.3). The product on $A \hat{\otimes} B$ follows from associativity and commutativity of tensor product of Mackey functors, which give the following morphisms

$$(A \hat{\otimes} B) \hat{\otimes} (A \hat{\otimes} B) \xrightarrow{\simeq} A \hat{\otimes} B \hat{\otimes} A \hat{\otimes} B \xrightarrow{\simeq} A \hat{\otimes} A \hat{\otimes} B \hat{\otimes} B \to A \hat{\otimes} B$$

Similarly, the unit of $A \hat{\otimes} B$ follows from the isomorphism $b \hat{\otimes} b \simeq b$

$$b \simeq b \hat{\otimes} b \xrightarrow{\varepsilon_A \hat{\otimes} \varepsilon_B} A \hat{\otimes} B$$

The following lemma deals with similar morphisms between directs products:

LEMMA 7.8. *Let $G$ and $H$ be finite groups.*

1. *If $A$ and $B$ are Mackey functors for $G$, if $C$ and $D$ are Mackey functors for $H$, then there are isomorphisms*
$$(A\hat{\otimes}B)\boxtimes(C\hat{\otimes}D)\simeq(A\boxtimes C)\hat{\otimes}(B\boxtimes D)$$
*of Mackey functors for $G\times H$, which are natural in $A$, $B$, $C$, and $D$.*
2. *There is an isomorphism*
$$b\boxtimes b\simeq b$$
*of Mackey functors for $G\times H$.*

PROOF. The first assertion follows from associativity and commutativity of tensor product, and of corollary 4.5, since

$$(A\hat{\otimes}B)\boxtimes(C\hat{\otimes}D)=\iota_G^{G\times H}(A\hat{\otimes}B)\hat{\otimes}\iota_H^{G\times H}(C\hat{\otimes}D)\simeq\ldots$$
$$\ldots\simeq\iota_G^{G\times H}(A)\hat{\otimes}\iota_G^{G\times H}(B)\hat{\otimes}\iota_H^{G\times H}(C)\hat{\otimes}\iota_H^{G\times H}(D)\simeq\ldots$$
$$\ldots\simeq\iota_G^{G\times H}(A)\hat{\otimes}\iota_H^{G\times H}(C)\hat{\otimes}\iota_G^{G\times H}(B)\hat{\otimes}\iota_H^{G\times H}(D)\simeq(A\boxtimes C)\hat{\otimes}(B\boxtimes D)$$

The second assertion is also clear, since
$$b\boxtimes b=\iota_G^{G\times H}(b)\hat{\otimes}\iota_H^{G\times H}(b)\simeq b\hat{\otimes}b\simeq b$$
Note that in $b\boxtimes b\simeq b$, the first $b$ is the Burnside functor for $G$, the second is the Burnside functor for $H$, and the third is the Burnside functor for $G\times H$. □

Now if $A$ is a Green functor for $G$, and $H$ is a Green functor for $H$, then the morphisms
$$(A\boxtimes B)\hat{\otimes}(A\boxtimes B)\xrightarrow{\simeq}(A\hat{\otimes}A)\boxtimes(B\hat{\otimes}B)\to A\boxtimes B$$
$$b\simeq b\boxtimes b\xrightarrow{\varepsilon_A\boxtimes\varepsilon_B}A\boxtimes B$$
turn $A\boxtimes B$ into a Green functor for $G\times H$. Similarly, if $M$ is an $A$-module and $N$ is a $B$-module, then $M\boxtimes N$ has a natural structure of $A\boxtimes B$-module. Moreover, the isomorphisms of corollary 4.5 and proposition 6.5 are isomorphisms of Green functors:

PROPOSITION 7.9. *Let $G$, $H$, and $K$ be finite groups.*

1. *If $A$ is a Green functor for $G$, if $B$ is a Green functor for $H$, and $U$ is a finite $K$-set-$(G\times H)$, then there is an isomorphism of Green functors*
$$T_U(A\boxtimes B)\simeq T_{U/H}(A)\hat{\otimes}T_{U/G}(B)$$
2. *If $A$ and $B$ are Green functors for $G$, and if $U$ is a finite $H$-set-$G$, then there there is an isomorphism of Green functors*
$$T_U(A\hat{\otimes}B)\simeq T_U(A)\hat{\otimes}T_U(B)$$
3. *If $A$ is a Green functor for $G$, and $U$ and $U'$ are finite $H$-sets-$G$, then there there is an isomorphism of Green functors*
$$T_{U\sqcup U'}(A)\simeq T_U(A)\hat{\otimes}T_{U'}(A)$$
4. *If $A$ is a Green functor for $G$, and if $U$ is a finite $H$-set-$G$, then there is an isomorphism of Green functors*
$$T_U(A^{op})\simeq T_U(A)^{op}$$

PROOF. The two first assertions follows from the definition of the Green functor structure on $T_U(A \boxtimes B)$ and $T_U(A \hat{\otimes} B)$. The third one follows from the the following diagram

$$\begin{array}{ccc}
T_{U\sqcup U'}(A)\hat{\otimes} T_{U\sqcup U'}(A) & \xrightarrow{\simeq} & T_{U\sqcup U'}(A\hat{\otimes} A) \\
\downarrow & & \\
T_U(A)\hat{\otimes} T_{U'}(A)\hat{\otimes} T_U(A)\hat{\otimes} T_{U'}(A) & & \\
\downarrow & & \Big\downarrow T_{U\sqcup U'}(p) \\
T_U(A)\hat{\otimes} T_U(A)\hat{\otimes} T_{U'}(A)\hat{\otimes} T_{U'}(A) & & \\
\downarrow & & \\
T_U(A\hat{\otimes} A)\hat{\otimes} T_{U'}(A\hat{\otimes} A) & & \\
\downarrow & & \\
T_U(A)\sqcup T_{U'}(A) & \xrightarrow[\simeq]{} & T_{U\sqcup U'}(A)
\end{array}$$

where $p : A\hat{\otimes} A \to A$ is the product of $A$. This diagram is commutative, because for any $G$-sets $X$ and $Y$, the following diagram is commutative

$$\begin{array}{ccc}
b_{\mathrm{Hom}_(U\sqcup U',X)}\hat{\otimes} b_{\mathrm{Hom}_G(U\sqcup U',X)} & \xrightarrow{\simeq} & b_{\mathrm{Hom}_G(U\sqcup U',X\times Y)} \\
\downarrow & & \\
b_{\mathrm{Hom}_G(U,X)}\hat{\otimes} b_{\mathrm{Hom}_G(U',X)}\hat{\otimes} b_{\mathrm{Hom}_G(U,Y)}\hat{\otimes} b_{\mathrm{Hom}_G(U',Y)} & & \Big\downarrow Id \\
\downarrow & & \\
b_{\mathrm{Hom}_G(U,X)}\hat{\otimes} b_{\mathrm{Hom}_G(U,Y)}\hat{\otimes} b_{\mathrm{Hom}_G(U',X)}\hat{\otimes} b_{\mathrm{Hom}_G(U',Y)} & & \\
\downarrow & & \\
b_{\mathrm{Hom}_G(U,X\times Y)}\hat{\otimes} b_{\mathrm{Hom}_G(U',X\times Y)} & \xrightarrow[\simeq]{} & b_{\mathrm{Hom}_G(U\sqcup U',X\times Y)}
\end{array}$$

For last assertion, recall that the product on $A^{op}$ can be defined by

$$A\hat{\otimes} A \xrightarrow{\sigma} A\hat{\otimes} A \to A$$

where $\sigma$ denotes the natural isomorphism expressing the commutativity of the tensor product. Now the assertion follows from the following commutative diagram, where $M$ and $N$ are Mackey functors for $G$

$$\begin{array}{ccc}
T_U(M\hat{\otimes} N) & \xrightarrow{\simeq} & T_U(M)\hat{\otimes} T_U(N) \\
{\scriptstyle T_U(\sigma)}\Big\downarrow & & \Big\downarrow{\scriptstyle \sigma} \\
T_U(N\hat{\otimes} M) & \xrightarrow[\simeq]{} & T_U(N)\hat{\otimes} T_U(M)
\end{array}$$

To check the commutativity of this diagram, it suffices to suppose $M = b_X$ and $N = b_Y$, for some $G$-sets $X$ and $Y$. In that case, it follows from the commutativity of the following square

$$\begin{array}{ccc}
\mathrm{Hom}_G(U^{op}, X\times Y) & \xrightarrow{\simeq} & \mathrm{Hom}_G(U^{op}, X)\times \mathrm{Hom}_G(U^{op}, Y) \\
{\scriptstyle \mathrm{Hom}_G(U^{op},s)}\Big\downarrow & & \Big\downarrow{\scriptstyle s} \\
\mathrm{Hom}_G(U^{op}, Y\times X & \xrightarrow[\simeq]{} & \mathrm{Hom}_G(U^{op}, Y)\times \mathrm{Hom}_G(U^{op}, Y)
\end{array}$$

where $s$ is the bijection exchanging the factors of a product of sets. $\square$

## 8. Cohomological tensor induction

**8.1. Cohomological Mackey functors.** Let $G$ be a finite group, and $R$ be a commutative ring. A Mackey functor $M$ for $G$ over $R$ is called *cohomological* if for any subgroups $K \subseteq H$ of $G$, the composition $t_K^H r_K^H$ is multiplication by the index $[H:K]$. This is equivalent to say that $M$ is a module over the Green functor $FP_R$ (see [**TW95**] Proposition 16.3). The functor $FP_V$ is defined more generally for an $RG$-module $V$ and a finite $G$-set $X$ by

$$FP_V(X) = \mathrm{Hom}_{RG}(RX, V)$$

where $RX$ is the permutation $RG$-module associated to $X$ (see [**Bou97**] 4.5.2). Clearly $FP_R \simeq R \times FP_{\mathbb{Z}}$, so a Mackey functor over $R$ is cohomological if and only if it is cohomological as a Mackey functor (over $\mathbb{Z}$).

Lemma 8.1. *Let $G$ be a finite group, and $M$ be a Mackey functor for $G$. Then $M$ admits a biggest cohomological quotient $M^{coh}$, given by*

$$M^{coh} = FP_{\mathbb{Z}} \hat{\otimes} M$$

Proof. This follows from the alternative description of the tensor product of Mackey functors (see [**Bou97**] Proposition 1.5.1): if $M$ and $N$ are Mackey functors for $G$, if $H$ is a subgroup of $G$, then

$$(N \hat{\otimes} M)(H) \simeq \Big( \bigoplus_{K \subseteq H} N(K) \otimes M(K) \Big) / \mathcal{J}$$

where $\mathcal{J}$ is the submodule generated by the elements

$$t_L^K n \otimes m - n \otimes r_L^K m \quad \text{for} \quad L \subseteq K \subseteq H,\ n \in N(L),\ m \in M(K)$$

$$r_L^K n \otimes m - n \otimes t_L^K m \quad \text{for} \quad L \subseteq K \subseteq H,\ n \in N(K),\ m \in M(L)$$

$$hn \otimes hm - n \otimes m \quad \text{for} \quad K \subseteq H,\ n \in N(K),\ m \in M(K),\ h \in H$$

Moreover, if $K$ is a subgroup of $G$, then $FP_{\mathbb{Z}}(K) \simeq \mathbb{Z}$. Restriction maps are identity, and transfers are multiplication by the index. Thus

$$(FP_{\mathbb{Z}} \hat{\otimes} M)(H) \simeq \Big( \bigoplus_{K \subseteq H} \mathbb{Z} \otimes M(K) \Big) / \mathcal{J}$$

where $\mathcal{J}$ is generated by

$$1 \otimes m - 1 \otimes t_L^K m \quad \text{for} \quad L \subseteq K \subseteq H,\ m \in M(L)$$

$$1 \otimes r_L^K m - [K:L] 1 \otimes m \quad \text{for} \quad L \subseteq K \subseteq H,\ m \in M(K)$$

$$1 \otimes m - 1 \otimes hm \quad \text{for} \quad K \subseteq H,\ m \in M(K),\ h \in H$$

Now the element $1 \otimes m$, for $K \subseteq H$, and $m \in M(K)$, is equal to $1 \otimes t_K^H(m)$ in $(FP_{\mathbb{Z}} \otimes M)(H)$. This proves that $(FP_{\mathbb{Z}} \otimes M)(H)$ is the quotient of $M(H)$ by the submodule generated by the elements $t_K^H r_K^H m - [H:K]m$, for $K \subseteq H$ and $m \in M(K)$. Thus $FP_{\mathbb{Z}} \hat{\otimes} M$ is a cohomological quotient of $M$. Moreover, it is clear that any morphism from $M$ to a cohomological functor must factor through $FP_{\mathbb{Z}} \hat{\otimes} M$, and the lemma follows. □

Lemma 8.2. *Let $G$ be a finite group, and $X$ be a $G$-set. Then*

$$(b_X)^{coh} \simeq FP_{\mathbb{Z}X}$$

PROOF. This follows from the previous lemma, and from the isomorphisms

$$(b_X)^{coh} = FP_{\mathbb{Z}} \hat{\otimes} b_X \simeq (FP_{\mathbb{Z}} \hat{\otimes} b)_X \simeq (FP_{\mathbb{Z}})_X$$

Moreover the isomorphism $(FP_{\mathbb{Z}})_X \simeq FP_{\mathbb{Z}}$ follows from the case of a finite $G$-set $X$: in that case indeed, for a finite $G$-set $Y$

$$(FP_{\mathbb{Z}})_X(Y) = FP_{\mathbb{Z}}(Y\times X) = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}(Y\times X), \mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}Y\otimes\mathbb{Z}X, \mathbb{Z}) \simeq \dots$$

$$\dots \simeq \mathrm{Hom}_{\mathbb{Z}G}\Big(\mathbb{Z}Y, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}X, \mathbb{Z})\Big) \simeq \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}Y, \mathbb{Z}X) = FP_{\mathbb{Z}X}(Y)$$

since $\mathbb{Z}X$ is self-dual. Those isomorphisms are moreover functorial in $Y$. □

**8.2. Tensor induction of cohomological Mackey functors.** The image of cohomological Mackey functors by tensor induction is often zero:

PROPOSITION 8.3. *Let $G$ and $H$ be finite groups, and $U$ be a finite $H$-set-$G$. The following conditions are equivalent:*

1. *There exists a cohomological Mackey functor $M$ such that $T_U(M)$ is non-zero.*
2. *The functor $T_U(FP_{\mathbb{Z}})$ is non-zero.*
3. *There exists a prime number $p$ such that for any $u \in U$, the stabilizer $G_u$ of $u$ in $G$ is a $p$-group.*

PROOF. If $M$ is cohomological, then $M$ is a module over $FP_{\mathbb{Z}}$. Thus $T_U(M)$ is a module over $T_U(FP_{\mathbb{Z}})$. So if $T_U(FP_{\mathbb{Z}}) = 0$, then $T_U(M)$ is a module over the zero Green functor, so it is zero. Thus 1) implies 2).

Now suppose 2) holds. Decompose $U$ as a disjoint union of transitive bisets $U_i$, for $i \in I$. Then

$$T_U(FP_{\mathbb{Z}}) \simeq \underset{i\in I}{\otimes} T_{U_i}(FP_{\mathbb{Z}})$$

Each $U_i$ is a transitive biset, isomorphic to $(H \times G)/L_i$, for some subgroup $L_i$ of $H \times G$. Choose an index $i$, and denote by $H' = p_1(L_i)$ (resp. $G' = p_2(L_i)$) the projection of $L_i$ on $H$ (resp. on $G$). Let

$$k_1(L_i) = \{h \in H \mid (h,1) \in L_i\} \qquad N = k_2(L_i) = \{g \in G \mid (1,g) \in L_i\}$$

Let $K = p_2(L_i)/k_2(L_i)$, and $s : G' \to K$ be the canonical surjection. Then there exists a surjection $t : H' \to K$ such that

$$L_i = \{(h,g) \in H' \times G' \mid t(h) = s(g)\}$$

Moreover, if

$$K_1 = \{\Big(h, t(h) \mid h \in H'\Big)\} \subseteq H \times K \qquad K_2 = \{\Big(s(g), g\Big) \mid g \in G'\} \subseteq K \times G$$

then (see [**Bou96b**] Lemme 3)

$$U_i = (G \times H)/L_i \simeq (H \times K)/K_1 \times_K (K \times G)/K_2$$

Thus the functor $T_{U_i}$ factors as

$$T_{U_i} = T_{(H\times K)/K_1} \circ T_{(K\times G)/K_2}$$

Moreover $G$ is transitive on $(K \times G)/K_2$ (since the projection of $K_2$ on $K$ is $K$ itself), so

$$T_{(K\times G)/K_2} = \mathcal{L}_{(K\times G)/K_2}$$

Thus for any Mackey functor $M$ for $G$

$$T_{(K\times G)/K_2}(M) = \left(\mathrm{Res}_{G'}^{G} M\right)^N$$

Note that the stabilizer in $G$ of the element $u = (h, g)L_i$ of $U_i$ is equal to ${}^gN$. Now if $M = FP_{\mathbb{Z}}$, then $\mathrm{Res}_{G'}^{G} M = FP_{\mathbb{Z}}$. Moreover, for any subgroup $K' = S/N$ of $K = G'/N$

$$(FP_{\mathbb{Z}})^N(K') = FP_{\mathbb{Z}}(S)/\mathcal{J}$$

where $\mathcal{J}$ is generated by the submodules $t_T^S FP_{\mathbb{Z}}(T)$, for subgroups $T$ of $S$ not containing $N$. In other words

$$\left(\mathrm{Res}_{G'}^{G} FP_{\mathbb{Z}}\right)^N(S/N) = \mathbb{Z}/gcd\{[S:T] \mid N \not\subseteq T \subset S\}\mathbb{Z}$$

Now if $N$ is not a $p$-group for some $p$, then the Sylow subgroups $T$ of $S$ (for various primes), do not contain $N$, and have relatively prime index in $S$. So if $N$ is not a $p$-group for some prime $p$, then

$$\left(\mathrm{Res}_{G'}^{G} FP_{\mathbb{Z}}\right)^N = 0$$

and it follows that $T_{U_i}(FP_{\mathbb{Z}}) = 0$ in that case. Moreover, if $N$ is a non-trivial $p$-group, then the $q$-Sylow subgroups of $S$ for $q \neq p$ do not contain $N$, and the greatest common divisor of their indexes in $S$ is a power of $p$ (dividing the order of $S$). In particular, the functor $\left(\mathrm{Res}_{G'}^{G} FP_{\mathbb{Z}}\right)^N$ is a Mackey functor over $\mathbb{Z}/p^a\mathbb{Z}$, where $p^a$ is the $p$-part of the order of $G$. Thus if $N$ is a non-trivial $p$-group, corollary 7.7 shows that $T_{U_i}(FP_{\mathbb{Z}})$ is a Mackey functor over $\mathbb{Z}/p^m\mathbb{Z}$, for some power $p^m$ of $p$.

But the tensor product of a Mackey functor over $\mathbb{Z}/p^m\mathbb{Z}$ and a Mackey functor over $\mathbb{Z}/q^l\mathbb{Z}$ is zero if $p$ and $q$ are relatively prime (since the tensor product of abelian groups of relatively prime orders is zero). So if 2) holds, then there exists a prime $p$ such that for any $u \in U$, the stabilizer of $u$ in $G$ is either trivial, or a non-trivial $p$-group. So 3) holds.

I will admit for a while that 3) implies 1): this will be a consequence of proposition 8.11. □

**8.3. Cohomological tensor induction.** The fixed point functor $FP_R$ plays for cohomological Mackey functors over $R$ the role of the Burnside functor for Mackey functors. One can try to translate in the category $Comack_R(G)$ of cohomological Mackey functors over $R$ the definitions of tensor induction for Mackey functors, in order to define a "cohomological tensor induction".

Recall the following theorem:

THEOREM 8.4. *(Thévenaz-Webb* **[TW95]** *Theorem 16.5). Let $G$ be a finite group and $R$ be a commutative ring.*

1. *A Mackey functor $M$ for $G$ over $R$ is cohomological if and only if it is isomorphic to a quotient of a fixed point functor $FP_V$, where $V$ is an $RG$-module, which can be chosen to be a permutation module.*
2. *A cohomological Mackey functor is projective in $Comack_R(G)$ if and only if it is isomorphic to a fixed point functor $FP_V$, where $V$ is a direct summand of a permutation module for $G$.*

I will also use the following:

LEMMA 8.5. *Let $G$ be a finite group. If $X$ and $Y$ are $G$-sets, then*

$$\mathrm{Hom}_{Comack_R(G)}(FP_{RX}, FP_{RY}) \simeq \mathrm{Hom}_{RG}(RX, RY)$$

PROOF. This is a direct consequence of the adjunction property of the functors $FP_V$ (see [**TW90**] 6.1): the functor $V \mapsto FP_V$ is right adjoint to the functor of evaluation at $\{1\}$. Moreover $FP_{RX}(1) \simeq RX$. □

DEFINITION 8.6. *A permutation cohomological Mackey functor for $G$ over $R$ is a Mackey functor isomorphic to $FP_{RX}$, for some $G$-set $X$. The full subcategory of $Comack_R(G)$ consisting of permutation cohomological Mackey functors will be denoted by $PComack_R(G)$.*

Thus the category $PComack_R(G)$ can be described as follows: the objects of $PComack_R(G)$ are the $G$-sets. A morphism in $PComack_R(G)$ from $Y$ to $X$ is a morphism of $RG$-modules from $RY$ to $RX$, or equivalently a matrix $m(x, y)$ of coefficients in $R$, indexed by $X \times Y$, which is $G$-invariant (that is $m(gx, gy) = m(x, y)$ for all $g \in G$ and $(x, y) \in X \times Y$), such that for all $y \in Y$, the coefficient $m(x, y)$ is zero except for a finite number of $x \in X$. The composition of morphisms is given by matrix multiplication.

The previous theorem shows that the subcategory $\mathcal{P} = PComack_R(G)$ of $\mathcal{C} = Comack_R(G)$ satisfies the hypothesis of theorem 2.14: it is formed of projective objects, any object of $\mathcal{C}$ is a quotient of some object of $\mathcal{P}$, and $\mathcal{P}$ is closed by direct sums. So any functor from $\mathcal{P}$ to an abelian category can be uniquely extended to a right exact functor defined on $\mathcal{C}$.

DEFINITION 8.7. *(see [**Bou96a**] 3.1) Let $R$ be a commutative ring, and $G$ and $H$ be finite groups. If $U$ is an $H$-set-$G$, I will say that $U$ is free-$R$ if for any $u \in U$, the prime factors of the order of the stabilizer $G_u$ of $u$ in $G$ are equal to zero in $R$.*

Since two distinct primes cannot vanish in $R$ if $R$ is non-zero, there are only two cases left if $U$ is free-$R$:

- Either $R$ has prime characteristic $p > 0$, and all the groups $G_u$ are $p$-groups (the set $U$ will be called free-$p$ in that case).
- Or $R$ has characteristic zero or non-prime, and all the groups $G_u$ are trivial (so the set $U$ is right free).

PROPOSITION 8.8. *Let $R$ be a commutative ring. Let $G$ and $H$ be finite groups, and $U$ be a finite free-$R$ $H$-set-$G$. If $m(x, y)$ is a matrix defining a morphism in $Comack_R(G)$ from $Y$ to $X$, if $\varphi : U^{op} \to X$ and $\psi : U^{op} \to Y$ are morphisms of $G$-sets, define*

$$\tilde{m}(\varphi, \psi) = \prod_{u \in U/G} m\Big(\varphi(u), \psi(u)\Big)$$

*(this does not depend of the choice of representatives of $U/G$). Then the correspondence*

$$X \mapsto \mathrm{Hom}_G(U^{op}, X) \qquad\qquad m \mapsto \tilde{m}$$

*is a functor $T_U^{R\,coh}$ from $PComack_R(G)$ to $PComack_R(H)$.*

PROOF. First I must check that $\tilde{m}$ defines a morphism in $Comack_R(H)$ from $\mathrm{Hom}_G(U^{op}, Y)$ to $\mathrm{Hom}_G(U^{op}, X)$: for a given $\psi$, the coefficient

$$\tilde{m}(\varphi, \psi) = \prod_{u \in U/G} m\Big(\varphi(u), \psi(u)\Big)$$

is non-zero if and only if for any $u \in U$, the coefficient $m\Big(\varphi(u), \psi(u)\Big)$ is non-zero. So for each $u$, there is only a finite number of possible values for $\varphi(u)$. As $U$ is finite, there is a finite number of $\varphi$ such that $\tilde{m}(\varphi, \psi) \neq 0$.

Next I have to check that $\tilde{m}$ is $H$-invariant. But this is clear, since for $h \in H$

$$\tilde{m}(h\varphi, h\psi) = \prod_{u \in U/G} m\Big(\varphi(h^{-1}u), \psi(h^{-1}u)\Big) = \tilde{m}(\varphi, \psi)$$

since the image by $h^{-1}$ of a system of representatives of $U/G$ is another system of representatives.

Now of course, if $m$ represents the identity morphism, then $\tilde{m}(\varphi, \psi)$ is non-zero if and only if $\varphi(u) = \psi(u)$ for all $u \in U$, or equivalently if $\varphi = \psi$. So $\tilde{m}$ is the identity morphism.

Finally, let $Z$ be another $G$-set, and $p$ be a matrix representing a morphism in $Comack_R(G)$ from $Z$ to $Y$. The product matrix $m.p$ is defined by

$$(m.p)(x, z) = \sum_{y \in Y} m(x, y)p(y, z)$$

Let $\theta \in \mathrm{Hom}_G(U^{op}, Z)$, and $\varphi \in \mathrm{Hom}_G(U^{op}, X)$. Then

$$\widetilde{m.p}(\varphi, \theta) = \prod_{u \in U/G} \left( \sum_{y \in Y} m\Big(\varphi(u), y\Big) p\Big(y, \theta(u)\Big) \right) \tag{8.17}$$

Now for a given $u \in U/G$

$$\sum_{y \in Y} m\Big(\varphi(u), y\Big) p\Big(y, \theta(u)\Big) = \sum_{\substack{y \in G_u \backslash Y \\ g \in G_u/G_{u,y}}} m\Big(\varphi(u), gy\Big) p\Big(gy, \theta(u)\Big) = \dots$$

$$\dots = \sum_{\substack{y \in G_u \backslash Y \\ g \in G_u/G_{u,y}}} m\Big(\varphi(ug), y\Big) p\Big(y, \theta(ug)\Big) = \sum_{y \in G_u \backslash Y} [G_u : G_{u,y}] m\Big(\varphi(u), y\Big) p\Big(y, \theta(u)\Big)$$

As $U$ is free-$R$, the coefficient $[G_u : G_{u,y}]$ is zero unless $G_{u,y} = G_u$, or equivalently $y \in Y^{G_u}$.

Now expanding the product in equation (8.17) is equivalent to choosing for each $u \in U/G$ an element $y_u \in Y^{G_u}$. This in turn is equivalent to defining a $G$-morphism $\psi$ from $U^{op}$ to $Y$ (by $\psi(u') = gy_u$ if $u \in U/G$, $g \in G$, and $u' = ug^{-1}$). This gives finally

$$\widetilde{m.p}(\varphi, \theta) = \sum_{\psi \in \mathrm{Hom}_G(U^{op}, Y)} \prod_{u \in U/G} m\Big(\theta(u), \psi(u)\Big) p\Big(\psi(u), \varphi(u)\Big) = \dots$$

$$\dots = \sum_{\psi \in \mathrm{Hom}_G(U^{op}, Y)} \tilde{m}(\theta, \psi)\tilde{p}(\psi, \varphi)$$

This proves that $m \mapsto \tilde{m}$ is multiplicative, and the proposition follows. □

DEFINITION 8.9. *Let $R$ be a commutative ring, and $G$ and $H$ be finite groups. If $U$ is a finite free-$R$ $H$-set-$G$, I will call cohomological tensor induction the unique right exact functor extending the functor $T_U^{R\,coh}$ from $PComack_R(G)$ to $Comack_R(H)$. This extension will still be denoted by $T_U^{R\,coh}$.*

Recall that if $M$ is a cohomological Mackey functor for $G$ over $R$, then $T_U^{R\,coh}(M)$ is obtained by choosing a resolution of $M$

$$FP_{RY} \xrightarrow{\varphi} FP_{RX} \xrightarrow{\psi} M \to 0$$

by permutation cohomological Mackey functors. Then $T_U^{R\,coh}(M)$ is defined by the exact sequence

$$T_U^{R\,coh}(FP_{RY} \oplus FP_{RX}) \xrightarrow{\Delta T_U^{R\,coh}(\varphi)} T_U^{R\,coh}(FP_{RX}) \longrightarrow T_U^{R\,coh}(M) \to 0$$

**8.4. Extension of coefficients.** A cohomological Mackey functor $M$ for $G$ over $R$ is a module over $FP_R$. If $f : R \to R'$ is a morphism of commutative rings, then the induced morphism $FP_R \to FP_{R'}$ turns $FP_{R'}$ into a right $FP_R$-module. I can then consider the functor $R' \boxtimes_R M$ defined by

$$R' \boxtimes_R M = FP_{R'} \mathop{\hat{\otimes}}_{FP_R} M$$

(see [**Bou97**] 6.6 for the definition of this tensor product over $FP_R$). It is a module over $FP_{R'}$, or equivalently a cohomological Mackey functor over $R'$. Its evaluation on a $G$-set $X$ is simply given by tensoring with $R'$

$$(R' \boxtimes_R M)(X) = R' \otimes_R M(X)$$

If $M$ admits a resolution

$$FP_{RY} \xrightarrow{\varphi} FP_{RX} \xrightarrow{\psi} M \to 0$$

then the sequence

$$R' \boxtimes_R FP_{RY} \xrightarrow{R' \boxtimes_R \varphi} R' \boxtimes_R FP_{RX} \xrightarrow{R' \boxtimes_R \psi} R' \boxtimes_R M \to 0$$

is still exact, because the functor $R' \otimes_R -$ is right exact. But clearly, for any $G$-set $X$

$$R' \boxtimes_R FP_{RX} = FP_{R'X}$$

So I have a resolution of $R'M$ by permutation cohomological Mackey functors over $R'$.

Now if $H$ is another finite group, and $U$ is a finite free-$R$ $H$-set-$G$, then $U$ is also free-$R'$, since the morphism $f$ is unitary. By definition $T_U^{R'\,coh}(R' \boxtimes_R M)$ is the cokernel of

$$T_U^{R'\,coh}(FP_{R'Y} \oplus FP_{R'X}) \xrightarrow{\Delta T_U^{R'\,coh}(\varphi)} T_U^{R'\,coh}(FP_{R'X}) \tag{8.18}$$

But moreover

$$T_U^{R'\,coh}(FP_{R'X}) = FP_{R'\mathrm{Hom}_G(U^{op},X)} = \ldots$$

$$\ldots = R' \boxtimes_R FP_{R\mathrm{Hom}_G(U^{op},X)} = R' \boxtimes_R T_U^{R\,coh}(FP_{RX})$$

So $T_U^{R'\,coh}(R' \boxtimes_R M)$ is also the cokernel of

$$R' \boxtimes_R T_U^{R\,coh}(FP_{RY} \oplus FP_{RX}) \xrightarrow{R' \boxtimes_R \Delta T_U^{R\,coh}(\varphi)} R' \boxtimes_R T_U^{R\,coh}(FP_{RX})$$

And this proves the following

LEMMA 8.10. *Let $f : R \to R'$ be a unitary morphism of commutative rings. If $G$ and $H$ are finite groups, if $U$ is a finite free-$R$ $H$-set-$G$, then $U$ is also free-$R'$, and for any cohomological Mackey functor $M$ for $G$ over $R$, there is an isomorphism*

$$T_U^{R'\,coh}(R' \boxtimes_R M) \simeq R' \boxtimes_R T_U^{R\,coh}(M)$$

*which is natural in $M$.*

Thus cohomological tensor induction commutes with extension of coefficients. In particular, if $p$ is a prime, if $U$ is free-$p$, then for any cohomological Mackey functor $M$ over $\mathbb{F}_p$

$$T_U^{R\,coh}(R \boxtimes_{\mathbb{F}_p} M) \simeq R \boxtimes_{\mathbb{F}_p} T_U^{\mathbb{F}_p\,coh}(M)$$

**8.5. Comparison.** If $M$ is a cohomological Mackey functor for $G$ over a commutative ring $R$, and if $U$ is finite free-$R$ $H$-set-$G$, I can build $T_U^{R\,coh}(M)$. But as $M$ is also a Mackey functor, I can build $T_U(M)$. Those two constructions are different, but closely related:

PROPOSITION 8.11. *Let $G$ and $H$ be finite groups, and $R$ be a commutative ring. If $U$ is a finite free-$R$ $H$-set-$G$, then for any Mackey functor $M$ for $G$, the functor $R \boxtimes M^{coh}$ is a cohomological Mackey functor over $R$, and there are isomorphisms*

$$T_U^{R\,coh}(R \boxtimes M^{coh}) \simeq R \boxtimes T_U(M)^{coh} \simeq FP_R \hat{\otimes} T_U(M) \tag{8.19}$$

*which are moreover natural in $M$.*

*In particular, if $M$ is a cohomological Mackey functor, and if $U$ is right free, then*

$$T_U^{\mathbb{Z}\,coh}(M) = T_U(M)^{coh} \tag{8.20}$$

*Similarly, if $p$ is a prime, if $U$ is free-$p$, then for any cohomological Mackey functor $M$ over $\mathbb{F}_p$*

$$T_U^{\mathbb{F}_p coh}(M) \simeq \mathbb{F}_p \boxtimes T_U(M)^{coh} \simeq FP_{\mathbb{F}_p} \hat{\otimes} T_U(M) \tag{8.21}$$

PROOF. The right hand side isomorphism in (8.19) is clear, since

$$R \boxtimes T_U(M)^{coh} \simeq R \boxtimes \Big(FP_{\mathbb{Z}} \hat{\otimes} T_U(M)\Big) \simeq FP_R \hat{\otimes} T_U(M)$$

Now both sides of the left hand side isomorphism in (8.19) are functors from $Mack(G)$ to $Comack_R(H)$, which are clearly right exact, since they are composed of right exact functors. So it suffices to check that their restrictions to $PMack(G)$ are isomorphic. But if $M = b_X$, for some $G$-set $X$, then $M^{coh} = FP_{\mathbb{Z}X}$, and

$$R \boxtimes M^{coh} = FP_{RX}$$

It follows that

$$T_U^{coh}(R \boxtimes M^{coh}) \simeq FP_{R\mathrm{Hom}_G(U^{op},X)}$$

On the other hand, the functor $T_U(M)$ is $b_{\mathrm{Hom}_G(U^{op},X)}$, thus

$$R \boxtimes T_U(M)^{coh} \simeq R \boxtimes FP_{\mathbb{Z}\mathrm{Hom}_G(U^{op},X)} \simeq FP_{R\mathrm{Hom}_G(U^{op},X)}$$

So the functors $M \mapsto T_U^{coh}(R \boxtimes M^{coh})$ and $M \mapsto R \boxtimes T_U(M)^{coh}$ coincide on objects of $PMack(G)$. To see that the restrictions of those functors to $PMack(G)$ are isomorphic, I have to look at their actions on morphisms.

Let me first give this action for the functor $Q : M \mapsto M^{coh}$ from $Mack(G)$ to $Comack_{\mathbb{Z}}(G)$. Lemma 8.2 shows that it maps $Pmack(G)$ to $PComack_{\mathbb{Z}}(G)$. Now if $X$ and $Y$ are $G$-sets, and if $a$ is a morphism in $PMack(G)$ from $Y$ to $X$, represented by a poset $(\Delta, f)$ over $Y \times X$ with finite fibres over $Y$, I want to describe the morphism $Q(a)$. This is a $G$ invariant matrix indexed by $X \times Y$, with coefficients in $\mathbb{Z}$, which is obtained by evaluation at the trivial subgroup of the morphism $FP_{\mathbb{Z}Y} \to FP_{\mathbb{Z}X}$ deduced from $a$. Since for any Mackey functor $M$, the evaluations of $M$ and $M^{coh}$ at $\{1\}$ coincide, the morphism $Q(a)$ is also the evaluation of $a$ at $\{1\}$, and its expression will follow from the explicit description of the isomorphism

$$b_X(1) \simeq \mathbb{Z}X$$

But an element of $b_X(1) \simeq h_G(G/1, X)$ is an equivalence class of finite $G$-posets over $(G/1) \times X$. If $(Z, f)$ is such a poset, then as the stabilizer $G_{g,x}$ of any element $(g, x)$ of $G \times X$ is trivial, I have $b(G_{g,x}) \simeq \mathbb{Z}$, and the Lefchetz invariant $\Lambda^{G_{g,x}}_{f^{-1}(g,x)}$ corresponds to $\chi\Big(f^{-1}(g, x)\Big)$ under this isomorphism. Now the isomorphism of lemma 3.4 shows that the map

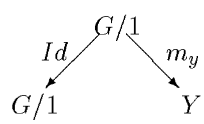
$$q(Z, f) = \sum_{x \in X} \chi\Big(f^{-1}(1, x)\Big) x$$

induces the required isomorphism $h_G(G/1, X) \simeq b_X(1) \simeq \mathbb{Z}X$. The inverse isomorphism maps $x_0 \in X$ to the poset $(Z, f)$ defined by

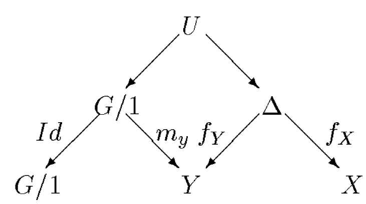
$$Z = G/1 (= G) \qquad \forall g \in G,\ f(g) = (g, gx_0) \in (G/1) \times X$$

were $G$ is given the discrete ordering (clearly $f^{-1}(1, x)$ is empty if $x \neq x_0$, and $f^{-1}(1, x_0) = \{1\}$ is a singleton).

Now if $y \in Y$ is a basis element of $\mathbb{Z}Y$, it can be viewed as the element




of $b_Y(G/1)$, where $m_y(g) = gy$ for $g \in G$. Its image by $a$ in $b_X(1)$ is given by pull-back




Denote by $\varphi$ the map $U \to (G/1) \times X$ in this diagram. I can write

$$Q(a)(y) = \sum_{x \in X} \chi\Big(\varphi^{-1}(1, x)\Big) x$$

But moreover

$$\varphi^{-1}(1, x) = \{(g, \delta) \in G \times \Delta \mid g = 1,\ gy = f_Y(\delta),\ f_X(\delta) = x\} \simeq f^{-1}(y, x)$$

In other words

$$Q(a)(y) = \sum_{x\in X} \chi\Big(f^{-1}(y,x)\Big)x$$

and the $(x, y)$ entry in the matrix $Q(a)$ is $\chi\Big(f^{-1}(y,x)\Big)$.

It follows from the definition of the functor $T_U^{R\,coh}$ that the morphism

$$T_U^{R\,coh}\Big(R\boxtimes Q(a)\Big) : T_U^{R\,coh}\Big(R\boxtimes (b_Y)^{coh}\Big) \to T_U^{R\,coh}\Big(R\boxtimes (b_X)^{coh}\Big)$$

is defined by the following matrix of coefficients in $R$

$$\alpha(\varphi,\psi) = \prod_{u\in U/G} \Big(\chi\Big(f^{-1}(\psi(u),\varphi(u))\Big)\Big)$$

for $\varphi \in \mathrm{Hom}_G(U^{op}, X)$ and $\psi \in \mathrm{Hom}_G(U^{op}, Y)$.

On the other hand, let $F$ denote the morphism

$$\mathrm{Hom}_G(U^{op}, f) : \mathrm{Hom}_G(U^{op}, \Delta) \to \mathrm{Hom}_G(U^{op}, Y) \times \mathrm{Hom}_G(U^{op}, X)$$

It represents a morphism from $T_U(b_Y)$ to $T_U(b_X)$. The associated morphism from $R\boxtimes T_U(b_Y)^{coh}$ to $R\boxtimes T_U(b_X)^{coh}$ is given by the matrix

$$\beta(\varphi,\psi) = \chi\Big(F^{-1}(\psi,\varphi)\Big)$$

But I have shown in the proof of proposition 4.1 that for any subgroup $K$ of $H_{\psi,\varphi}$

$$\chi\Big(F^{-1}(\psi,\varphi)^K\Big) = \prod_{u\in[K\backslash U/G]} \chi\left(\Big(f^{-1}\Big(\psi(u),\varphi(u)\Big)\Big)^{G_{K,u}}\right)$$

where

$$G_{K,u} = \{g\in G \mid \exists k\in K,\ ku = ug\}$$

In the case $K = \{1\}$, I have $G_{K,u} = G_u$, which is a $p$-group if $R$ is of prime characteristic $p$, and trivial otherwise. In both cases, I have

$$\chi\left(\Big(f^{-1}\Big(\psi(u),\varphi(u)\Big)\Big)^{G_u}\right) = \chi\Big(f^{-1}\Big(\psi(u),\varphi(u)\Big)\Big) \qquad \text{(in } R\text{)}$$

Hence the matrices $\alpha$ and $\beta$ are equal. This shows that the restrictions of the functors $M \mapsto T_U^{coh}(R\boxtimes M^{coh})$ and $M \mapsto R\boxtimes T_U(M)^{coh}$ to $PMack(G)$ are isomorphic. Since they are both right exact, those two functors are isomorphic, and this completes the proof of proposition 8.19.

Thus if $M$ is cohomological and $U$ is right free, then $M^{coh} = M$ and (8.20) follows from (8.19) for $R = \mathbb{Z}$.

Now if $M$ is a cohomological Mackey functor over $\mathbb{F}_p$, then $M^{coh} = M$, because $M$ is cohomological, and $\mathbb{F}_p\boxtimes M = M$ since $M$ is a functor over $\mathbb{F}_p$. So (8.21) follows from (8.19) in the case $R = \mathbb{F}_p$. This completes the proof of proposition 8.11. □

As a consequence, I can now prove the missing implication 3) ⇒ 1) in proposition 8.3

Corollary 8.12. *Let $G$ and $H$ be finite groups. If $p$ is a prime, and $U$ is a finite free-$p$ $H$-set-$G$, then $FP_{\mathbb{F}_p}$ is a cohomological Mackey functor, and $T_U(FP_{\mathbb{F}_p}) \neq 0$.*

PROOF. Indeed

$$FP_{\mathbb{F}_p}\hat{\otimes}T_U(FP_{\mathbb{F}_p})\simeq T_U^{\mathbb{F}_p\,coh}(FP_{\mathbb{F}_p})=FP_{\mathbb{F}_p}\neq 0$$

so in particular $T_U(FP_{\mathbb{F}_p})\neq 0$. □

**8.6. Composition and graded bisets.** In this section, I will explain how to compose those functors $T_U^{R\,coh}$. Since cohomological tensor induction seems quite compatible with tensor induction of Mackey functors, one could expect a formula similar to proposition 4.3. However, one must be very careful about the meaning of such a formula.

The context is the following: there are three finite groups $G$, $H$, and $K$, a finite biset $U$, wich is an $H$-set-$G$, and a finite biset $V$, which is a $K$-set-$H$. The problem is to compare the functors $T_V^{R\,coh}\circ T_U^{R\,coh}$ and $T_{V\times_H U}^{R\,coh}$. Let $X$ be a $G$-set. Then

$$T_V^{R\,coh}\circ T_U^{R\,coh}(FP_{RX})=T_V^{R\,coh}(FP_{R\mathrm{Hom}_G(U^{op},X)})=FP_{R\mathrm{Hom}_H(V^{op},\mathrm{Hom}_G(U^{op},X))}$$

As usual, the canonical isomorphism

$$\mathrm{Hom}_H\Big(V^{op},\mathrm{Hom}_G(U^{op},X)\Big)\simeq\mathrm{Hom}_G\Big((V\times U)^{op},X\Big) \tag{8.22}$$

shows that $T_V^{R\,coh}\circ T_U^{R\,coh}(FP_{RX})$ and $T_{V\times_H U}^{R\,coh}(FP_{RX})$ are isomorphic. So the restrictions of $T_V^{R\,coh}\circ T_U^{R\,coh}$ and $T_{V\times_H U}^{R\,coh}$ to $PComack_R(G)$ are isomorphic on objects. However, they are *not* isomorphic as functors, which means that the previous isomorphism is badly behaved with respect to morphisms.

To see this, let $m:Y\to X$ be a morphism in $PComack_R(G)$, i.e. a $G$-invariant matrix indexed by $X\times Y$, such that for any $y\in Y$, there is only a finite number of $x\in X$ for which $m(x,y)\neq 0$. Using isomorphism 8.22, it follows from the definitions that for $\varphi\in\mathrm{Hom}_G\Big((V\times_H U)^{op},X\Big)$ and $\psi\in\mathrm{Hom}_G\Big((V\times_H U)^{op},Y\Big)$, I have

$$T_V^{R\,coh}\circ T_U^{R\,coh}(m)(\varphi,\psi)=\prod_{v\in V/H}\prod_{u\in U/G}m\Big(\varphi(v,u),\psi(v,u)\Big) \tag{8.23}$$

On the other hand

$$T_{V\times_H U}^{R\,coh}(m)(\varphi,\psi)=\prod_{(v,u)\in(V\times_H U)/G}m\Big(\varphi(v,u),\psi(v,u)\Big) \tag{8.24}$$

As the sets $V/H\times U/G$ and $(V\times_H U)/G$ are not isomorphic in general, it follows that the expressions in 8.23 and 8.24 are not equal. To see how much they differ, fix sets $[U/G]$ and $[V/H]$ of orbits representatives, and consider the map

$$\theta:[V/H]\times[U/G]\to(V\times_H U)/G\,,\qquad \theta(v_0,u_0)=(v_0,u_0)G\,.$$

This map is surjective, since if $(v,u)\in V\times_H U$, there exist $v_0\in[V/H]$ (unique) and $h\in H$ such that $v=v_0h$. Now in $V\times_H U$, I have $(v,u)=(v_0,hu)$, and there exist $u_0\in[U/G]$ (unique) and $g\in G$ such that $hu=u_0g$. Then clearly

$$\theta(v_0,u_0)=(v_0,u_0)G=(v_0,u_0g)G=(v_0,hu)G=(v_0h,u)G=(v,u)G\,.$$

Now two pairs $(v_0,u_0)$ and $(v_1,u_1)$ have the same image under $\theta$ if and only if there exist $h\in H$ and $g\in G$ such that

$$v_1=v_0h\qquad\text{and}\qquad hu_1=u_0g\,.$$

The first equality gives $v_1=v_0=v_0h$, since $v_1$ and $v_0$ are in the set of representatives $[V/H]$. The second one gives $u_1=h^{-1}u_0g$. In other words $u_1G$ is in the left orbit of $u_0G$ under the right stabilizer $H_{v_0}$ of $v_0$. So the inverse image $\theta^{-1}\big((v,u)G\big)$ has cardinality $[H_v : H_{v,uG}]$, where

$$H_v = \{h \in H \mid vh = v\} \qquad H_{v,uG} = \{h \in H \mid vh = v,\ \exists g \in G,\ hu = ug\}$$

and I can rewrite equation 8.23 as

(8.25)

$$(T_V^{R\,coh} \circ T_U^{R\,coh})(m)(\varphi\psi) = \prod_{(v,u)\in[(V\times_H U)/G]} \Big(m\big(\varphi(v,u),\psi(v,u)\big)\Big)^{[H_v:H_{v,uG}]}$$

In particular, this shows that the functors $T_V^{R\,coh} \circ T_U^{R\,coh}$ and $T_{V\times_H U}^{R\,coh}$ are isomorphic if and only if elevation at the power $[H_v : H_{v,uG}]$ is the identity endomorphism of $R$, for any $(v,u) \in V \times_H U$. This will be the case in any of the following situations:

- The group $H$ acts freely on $V$, i.e. $H_v = \{1\}$ for all $v \in V$. This will be true by hypothesis if $R$ has characteristic 0 or non-prime.
- The group $G$ acts transitively on $U$, since in that case, the set $U/G$ is a single point.
- The ground ring is $R = \mathbb{F}_p$.

In general, one can pass from 8.24 to 8.25 by elevating each term $m\Big(\varphi(u),\psi(u)\Big)$ to the power $[H_v : H_{v,uG}]$. This is always equal to 1 if $R$ has characteristic 0 or non-prime, and a power of $p$ if $R$ has prime characteristic $p$. In any case, elevation at the power $[H_v : H_{v,uG}]$ is an endomorphism of the ring $R$.

So this suggests to use such ring endomorphisms to define functors between categories of cohomological Mackey functors, and leads to the following definition:

DEFINITION 8.13. *An* End$(R)$*-graded $H$-set-$G$, or graded biset for short, is a couple $(U,a)$ where $U$ is a finite free-$R$ $H$-set-$G$, and $a$ is a function from $H\backslash U/G$ to* End$(R)$.

*If $(U',a')$, then the disjoint union of $(U,a)$ and $(U',a')$ is the graded biset defined by*

$$(U,a) \sqcup (U',a') = (U \sqcup U', a \sqcup a')$$

*where $a \sqcup a'$ is the obvious function $H\backslash(U \sqcup U')/G \to$* End$(R)$.

*If $K$ is another group, and $(V,b)$ is an* End$(R)$*-graded $K$-set-$H$, then the product $(V,b) \times_H (U,a)$ is the* End$(R)$*-graded $K$-set-$G$ defined by*

$$(V,b) \times_H (U,a) = (V \times_H U, b \times_H a)$$

*where $b \times_H a$ is the function from $K\backslash(V \times_H U)/G$ to* End$(R)$ *defined by*

$$(b \times_H a)(v,u) = b(v) \circ a(u) \circ \pi_{v,u}$$

*where $\pi_{v,u}$ is elevation at the power $[H_v : H_{v,uG}]$.*

DEFINITION 8.14. *If $(U,a)$ is an* End$(R)$*-graded $H$-set-$G$, and if $X$ is a $G$-set, let*

$$T_{(U,a)}(X) = \mathrm{Hom}_G(U^{op}, X)$$

*If $m$ is a morphism in $PComack_R(G)$ from $Y$ to $X$, let*

$$T_{(U,a)}(m)(\varphi,\psi) = \prod_{u\in U/G} a(u)\Big[m\Big(\varphi(u),\psi(u)\Big)\Big]$$

*for $\varphi \in \mathrm{Hom}_G(U^{op},X)$ and $\psi \in \mathrm{Hom}_G(U^{op},Y)$.*

Clearly, if I take for $a$ the constant function equal to the identity endomorphism of $R$, then the functor $T_{(U,a)}$ is just the previous functor $T_U^{R\,coh}$.

Lemma 8.15. *The correspondence $T_{(U,a)}$ is a functor from $PComack_R(G)$ to $PComack_R(H)$.*

Proof. Let $X$, $Y$ and $Z$ be $G$-sets, and let $m : Y \to X$ and $n : Z \to Y$ be morphisms in $PComack_R(G)$. Then by definition, for $\varphi \in \mathrm{Hom}_G(U^{op}, X)$ and $\theta \in \mathrm{Hom}_G(U^{op}, Z)$

$$T_{(U,a)}(m \circ n)(\varphi, \theta) = \prod_{u \in U/G} a(u)\left[(m \circ n)\Big(\varphi(u), \theta(u)\Big)\right] \tag{8.26}$$

Moreover, for $u \in U$

$$(m \circ n)\Big(\varphi(u), \theta(u)\Big) = \sum_{y \in Y} m\Big(\varphi(u), y\Big) n\Big(y, \theta(u)\Big)$$

This can also be written as

$$(m \circ n)\Big(\varphi(u), \theta(u)\Big) = \sum_{\substack{y \in G_u\backslash Y\\ g \in G_u/G_{u,y}}} m\Big(\varphi(u), gy\Big) n\Big(gy, \theta(u)\Big) = \dots$$

$$\dots = \sum_{y \in G_u\backslash Y} [G_u : G_{u,y}] m\Big(\varphi(u), y\Big) n\Big(y, \theta(u)\Big) = \sum_{y \in Y^{G_u}} m\Big(\varphi(u), y\Big) n\Big(y, \theta(u)\Big)$$

Now expanding the product in 8.26 is equivalent to choosing an element $y \in Y^{G_u}$ for all $u \in U/G$. This in turn is equivalent to choosing a $G$-morphism from $U^{op}$ to $Y$. Finally, this gives

$$T_{(U,a)}(m \circ n)\Big(\varphi, \theta\Big) = \sum_{\psi \in \mathrm{Hom}_G(U^{op}, Y)} \prod_{u \in U/G} a(u)\left[m\Big(\varphi(u), \psi(u)\Big) n\Big(\psi(u), \theta(u)\Big)\right] = \dots$$

$$\dots = \sum_{\psi \in \mathrm{Hom}_G(U^{op}, Y)} \prod_{u \in U/G} a(u)\left[m\Big(\varphi(u), \psi(u)\Big)\right] \prod_{u \in U/G} a(u)\left[n\Big(\varphi(u), \psi(u)\Big)\right] = \dots$$

$$\dots = \sum_{\psi \in \mathrm{Hom}_G(U^{op}, Y)} T_{(U,a)}(m)(\varphi, \psi) T_{(U,a)}(n)(\psi, \theta)$$

It proves that $T_{(U,a)}$ is multiplicative on morphisms. Moreover, it is clear that if $m$ is the identity morphism, then

$$T_{(U,a)}(\varphi, \psi) = \prod_{u \in U/G} a(u)(\delta_{\varphi(u), \psi(u)})$$

where $\delta$ is a Kronecker symbol. This is non-zero if and only if $\varphi = \psi$, and in this case it is equal to 1.

It follows that $T_{(U,a)}$ maps the identity morphism of $X$ to the identity morphism of $T_{(U,a)}(X)$. This completes the proof of the lemma. □

Notation 8.16. I will denote by $T_{(U,a)}^{R\,coh}$ the unique extension of the functor $T_{(U,a)}$ to a right exact functor from $Comack_R(G)$ to $Comack_R(H)$. This will be called *the cohomological tensor induction* associated to the graded biset $(U, a)$.

### 8.7. Properties of cohomological tensor induction.

8.7.1. *Tensor product over $R$.* Before stating properties of cohomological tensor induction associated to a graded biset, I have to extend the notion of tensor product to cohomological Mackey functors over $R$: first note that cohomological Mackey functors over $R$ are also Mackey functors over $R$, or $(R\boxtimes b)$-modules. Since $R\boxtimes b$ is a commutative Green functor, any left $(R\boxtimes b)$-module is also a right $(R\boxtimes b)$-module. Thus if $M$ and $N$ are Mackey functors over $R$, I can define

$$M\hat{\otimes}_R N = M \mathop{\hat{\otimes}}_{R\boxtimes b} N$$

(see [**Bou97**] 6.6 for the definition of the tensor product in the right hand side). With this definition, it is easy to check that for any subgroup $H$ of $G$

$$(M \mathop{\hat{\otimes}}_R N)(H) \simeq \Big(\mathop{\oplus}_{K\subseteq H} M(K) \mathop{\hat{\otimes}}_R N(K)\Big)/\mathcal{J}$$

where $\mathcal{J}$ is the $R$-submodule generated by

$$t_L^K n\otimes m - n\otimes r_L^K m \quad \text{for} \quad L\subseteq K\subseteq H,\ n\in N(L),\ m\in M(K)$$

$$r_L^K n\otimes m - n\otimes t_L^K m \quad \text{for} \quad L\subseteq K\subseteq H,\ n\in N(K),\ m\in M(L)$$

$$hn\otimes hm - n\otimes m \quad \text{for} \quad K\subseteq H,\ n\in N(K),\ m\in M(K),\ h\in H$$

In other words, to compute $M\hat{\otimes}_R N$, replace $\otimes$ by $\otimes_R$, and "submodule" by $R$-submodule in the formulae for $M\hat{\otimes} N$.

Lemma 8.17. *Let $G$ be a finite group, and $R$ be a commutative ring. If $M$ and $N$ are cohomological Mackey functors for $G$ over $R$, then so is $M\hat{\otimes}_R N$.*

Proof. Indeed $M\hat{\otimes}_R N$ is a Mackey functor over $FP_R\hat{\otimes}_R FP_R$. Now the lemma follows from the following isomorphism of Green functors

$$FP_R \mathop{\hat{\otimes}}_R FP_R \simeq FP_R$$

which in turn follows from the previous identification of $M\hat{\otimes}_R N(H)$: indeed, if $H$ is a subgroup of $G$, then

$$(FP_R \mathop{\hat{\otimes}}_R FP_R)(H) = \Big(\mathop{\oplus}_{K\subseteq H} R \mathop{\hat{\otimes}}_R R\Big)/\mathcal{J}$$

where $\mathcal{J}$ is the $R$-submodule generated by

$$[K:L][1\otimes 1]_L - [1\otimes 1]_K \quad \text{for} \quad L\subseteq K\subseteq H$$

$$[1\otimes 1]_{{}^hK} - [1\otimes 1]_K \quad \text{for} \quad K\subseteq H,\ h\in H$$

So for any $L\subseteq H$, I have $[1\otimes 1]_L = [H:L][1\otimes 1]_H$, and this proves that $(FP_R\hat{\otimes}_R FP_R)(H)\simeq R$. □

Remark 8.18. The corresponding statement for direct product is false: if $G$ and $H$ are finite groups, if $M$ is a cohomological Mackey functor for $G$ over $R$ and $N$ is a cohomological Mackey functor for $H$ over $R$, then in general $M\boxtimes N$ is *not* a cohomological Mackey functor: this is because the functor $\iota_G^{G\times H}$ does not preserve cohomological Mackey functors.

8.7.2. *Properties.*

THEOREM 8.19. *Let $R$ be a commutative ring.*

1. *If $G$, $H$ and $K$ are finite groups, if $(U,a)$ is an $\mathrm{End}(R)$-graded $H$-set-$G$ and $(V,b)$ is an $\mathrm{End}(R)$-graded $K$-set-$H$, then there is an isomorphism of functors*
$$T^{R\,coh}_{(V,b)}\circ T^{R\,coh}_{(U,a)}\simeq T^{R\,coh}_{(V,b)\times_H(U,a)}$$
2. *If $G$ and $H$ are finite groups, if $M$ and $N$ are cohomological Mackey functors for $G$ over $R$, and if $(U,a)$ is an $\mathrm{End}(R)$-graded $H$-set-$G$, then*
$$T^{R\,coh}_{(U,a)}(M\hat{\otimes}_R N)\simeq T^{R\,coh}_{(U,a)}(M)\hat{\otimes}_R T^{R\,coh}_{(U,a)}(N)$$
3. *If $G$ and $H$ are finite groups, if $(U,a)$ and $(U',a')$ are $\mathrm{End}(R)$-graded $H$-sets-$G$, and $M$ is a cohomological Mackey functor for $G$ over $R$, then*
$$T^{R\,coh}_{(U,a)\sqcup(U',a')}(M)\simeq T^{R\,coh}_{(U,a)}(M)\hat{\otimes}_R T^{R\,coh}_{(U',a')}(M)$$
4. *If $G$ and $H$ are finite groups, if $(U,a)$ is an $\mathrm{End}(R)$-graded $H$-set-$G$, and if $M$ and $M'$ are cohomological Mackey functors for $G$, then*
$$T^{R\,coh}_{(U,a)}(M\oplus M')=\bigoplus_{\substack{V\subseteq U\\ V\text{ invariant by }G\\ V\text{ mod. }H}}\mathrm{Ind}_{H_V}^H T^{R\,coh}_{(V,a_{|V})}(M)\hat{\otimes}_R T^{R\,coh}_{(U-V,a_{|(U-V)})}(M')$$
5. *If $G$ and $H$ are finite groups, if $(U,a)$ is an $\mathrm{End}(R)$-graded $H$-set-$G$, if $X$ is a $G$-set and $M$ a cohomological Mackey functor for $G$, then $M_X$ is a cohomological Mackey functor for $G$, and*
$$T^{R\,coh}_{(U,a)}(M_X)\simeq T^{R\,coh}_{(U,a)}(M)_{\mathrm{Hom}_G(U^{op},X)}$$

*Moreover, all these isomorphisms are natural.*

PROOF. Assertion 1) should be clear from the discussion of composition of functors $T_U^{R\,coh}$. The other assertions state isomorphisms between right exact functors, so it is enough to check these isomorphisms on the restrictions to the corresponding subcategories of permutation cohomological Mackey functors, and this is easy: for instance, assertion 2) is a consequence of the natural isomorphism
$$FP_{RX}\hat{\otimes}FP_{RY}\simeq FP_{R(X\times Y)}$$
and of the bijection $\mathrm{Hom}_G(U^{op},X\times Y)\simeq\mathrm{Hom}_G(U^{op},X)\times\mathrm{Hom}_G(U^{op},Y)$. □

8.7.3. *Cohomological Green functors.* A cohomogical Green functor $A$ for $G$ over $R$ is a Green functor over $R$, which is cohomological as a Mackey functor. Equivalently, it is a Green functor over $R$, such that the unit morphism $\varepsilon: b\to A$ factors through $b\to FP_{\mathbb{Z}}$. This is also equivalent to say that $A$ is a Green functor, with a (unitary) morphism of Green functors $FP_R\to A$. In particular, any $A$-module is a cohomological Mackey functor for $G$ over $R$.

LEMMA 8.20. *Let $R$ be a commutative ring. Let $G$ and $H$ be finite groups, and $(U,a)$ be an $\mathrm{End}(R)$-graded $H$-set-$G$. If $A$ is a cohomological Green functor for $G$ over $R$, then $T^{R\,coh}_{(U,a)}(A)$ is a cohomological Green functor for $H$ over $R$. If $M$ is an $A$-module, then $T^{R\,coh}_{(U,a)}(M)$ is a $T^{R\,coh}_{(U,a)}(A)$-module, and the correspondence $M\mapsto T^{R\,coh}_{(U,a)}(M)$ is a functor from $A$-**Mod** to $T^{R\,coh}_{(U,a)}(A)$-**Mod**.*

PROOF. The images by $T_{(U,a)}^{R\,coh}$ of the commutative diagrams

$$\begin{array}{ccc} A\hat{\otimes}_R A\hat{\otimes}_R A & \xrightarrow{1\hat{\otimes}_R\mu} & A\hat{\otimes}_R A \\ {\scriptstyle \mu\hat{\otimes}_R 1}\downarrow & & \downarrow{\scriptstyle \mu} \\ A\hat{\otimes}_R A & \xrightarrow[\mu]{} & A \end{array} \qquad \begin{array}{ccccc} FP_R\hat{\otimes}_R A & \xrightarrow{\varepsilon\hat{\otimes}_R 1} & A\hat{\otimes}_R A & \xleftarrow{1\hat{\otimes}_R\varepsilon} & A\hat{\otimes}_R FP_R \\ & {\scriptstyle\eta}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle\eta} & \\ & & A & & \end{array}$$

give a Green functor structure on $T_{(U,a)}^{R\,coh}(A)$, and a unitary morphism of Green functors $T_{(U,a)}^{R\,coh}(FP_R) = FP_R \to T_{(U,a)}^{R\,coh}(A)$, so $T_{(U,a)}^{R\,coh}(A)$ is a cohomological Green functor. The other assertions of the lemma are trivial. □

PROPOSITION 8.21. *Let $R$ be a commutative ring, and $G$ and $H$ be finite groups.*

1. *If $A$ and $B$ are cohomological Green functor for $G$ over $R$, and if $(U,a)$ is an* End$(R)$*-graded $H$-set-$G$, then there is an isomorphism of cohomological Green functors*
$$T_{(U,a)}^{R\,coh}(A\hat{\underset{R}{\otimes}}B) \simeq T_{(U,a)}^{R\,coh}(A)\hat{\underset{R}{\otimes}}T_{(U,a)}^{R\,coh}(B)$$
2. *If $A$ is a cohomological Green functor for $G$ over $R$, and $(U,a)$ and $(U',a')$ are* End$(R)$*-graded $H$-sets-$G$, then there is an isomorphism of Green functors*
$$T_{(U,a)\sqcup(U',a')}^{R\,coh}(A) \simeq T_{(U,a)}^{R\,coh}(A)\hat{\underset{R}{\otimes}}T_{(U',a')}^{R\,coh}(A)$$
3. *If $A$ is a cohomological Green functor for $G$ over $R$, and $(U,a)$ is an* End$(R)$*-graded $H$-set-$G$ then there is an isomorphism of Green functors*
$$T_{(U,a)}^{R\,coh}(A^{op}) \simeq T_{(U,a)}^{R\,coh}(A)^{op}$$

PROOF. The proof of these assertions is similar to the proof of the corresponding assertions of proposition 7.9. □

## 9. Tensor induction for $p$-permutation modules

**9.1. Definition.** Let $k$ be a field of characteristic $p$. If $G$ and $H$ are finite groups, and if $(U,a)$ is an End$(k)$-graded $H$-set-$G$, I have built the functor $T_{(U,a)}^{k\,coh}$ from $Comack_k(G)$ to $Comack_k(H)$. This functor maps permutation cohomogical Mackey functors for $G$ over $k$ to permutation cohomological Mackey functors for $H$. Hence it maps projective cohomological Mackey functors for $G$ to projective cohomological Mackey functors for $H$.

But a projective cohomological Mackey functor $M$ for $G$ over $k$ is isomorphic to $FP_V$, where $V$ is a direct summand of a permutation $kG$-module, or $p$-permutation $kG$-module (see [**Bro85**]): this is a $kG$-module admitting an $S$-invariant $k$-basis, for some $p$-Sylow subgroup $S$ of $G$. Note that the module $V$ is the evaluation of $FP_V$ at $\{1\}$. So if $V$ is a $p$-permutation $kG$-module, then $T_{(U,a)}^{k\,coh}(FP_V)$ is isomorphic to $FP_W$, for some $p$-permutation $kG$-module $W$. This leads to the following:

DEFINITION 9.1. *Let $kG$-$p$-**Mod** denote the full subcategory of the category of $kG$-modules formed by $p$-permutation $kG$-modules (note that this is generally not an abelian category).*

*The composed functor*

$$V \mapsto T^{k\,coh}_{(U,a)}(FP_V)(1)$$

*from* $kG$-$p$-**Mod** *to* $kH$-$p$-**Mod** *will be called tensor induction for* $p$*-permutation modules, and denoted by* $T^{k\,per}_{(U,a)}$.

*If* $a$ *is the constant function equal to the identity endomorphism of* $k$*, then* $T^{k\,per}_{(U,a)}$ *will simply be denoted* $T^{k\,per}_{U}$.

**9.2. Properties.** Evaluation at the trivial subgroup of the isomorphisms of theorem 8.19 gives the following

PROPOSITION 9.2. *Let* $p$ *be a prime number and* $k$ *be a field of characteristic* $p$.

1. *If* $G$*,* $H$*, and* $K$ *are finite groups, if* $(U,a)$ *is an* $\mathrm{End}(k)$*-graded* $H$*-set-*$G$ *and* $(V,b)$ *is an* $\mathrm{End}(k)$*-graded* $K$*-set-*$H$*, then*
$$T^{k\,per}_{(V,b)} \circ T^{k\,per}_{(U,a)} = T^{k\,per}_{(V,b)\times_H(U,a)}$$
2. *If* $G$ *and* $H$ *are finite groups, if* $M$ *and* $N$ *are* $p$*-permutation* $kG$*-modules, and if* $U$ *is an* $\mathrm{End}(k)$*-graded* $H$*-set-*$G$*, then*
$$T^{k\,per}_{(U,a)}(M\otimes_k N) \simeq T^{k\,per}_{(U,a)}(M)\otimes_k T^{k\,per}_{(U,a)}(N)$$
3. *If* $G$ *and* $H$ *are finite groups, if* $(U,a)$ *and* $(U',a')$ *are* $\mathrm{End}(k)$*-graded* $H$*-sets-*$G$*, and* $M$ *is a* $p$*-permutation* $kG$*-module, then*
$$T^{k\,per}_{(U,a)\sqcup(U',a')}(M) \simeq T^{k\,per}_{(U,a)}(M)\otimes_k T^{k\,per}_{(U',a')}(M)$$
4. *If* $G$ *and* $H$ *are finite groups, if* $(U,a)$ *is an* $\mathrm{End}(k)$*-graded* $H$*-set-*$G$*, and if* $M$ *and* $M'$ *are* $p$*-permutation* $kG$*-modules, then*
$$T^{k\,per}_{(U,a)}(M\oplus M') = \bigoplus_{\substack{V\subseteq U\\ V \text{ invariant by } G\\ V \bmod H}} \mathrm{Ind}_{H_V}^{H} T^{k\,per}_{(V,a_{|V})}(M)\otimes_k T^{k\,per}_{(U-V,a_{|U-V})}(M')$$

*Moreover, all these isomorphisms are natural.*

**9.3. $p$-permutation algebras.**

DEFINITION 9.3. *A* $p$*-permutation* $kG$*-algebra is a* $k$*-algebra, with an action of* $G$*, admitting an* $S$*-invariant* $k$*-basis, for some* $p$*-Sylow subgroup* $S$ *of* $G$.

PROPOSITION 9.4. *Let* $p$ *be a prime number and* $k$ *be a ring of characteristic* $p$*. Let* $G$ *and* $H$ *be finite groups.*

1. *If* $(U,a)$ *is an* $\mathrm{End}(k)$*-graded* $H$*-set-*$G$*, and* $A$ *is a* $p$*-permutation* $kG$*-algebra, then* $T^{k\,per}_{(U,a)}(A)$ *is a* $p$*-permutation* $kH$*-algebra.*
2. *If* $(U,a)$ *is an* $\mathrm{End}(k)$*-graded* $H$*-set-*$G$*, and* $A$ *and* $B$ *are* $p$*-permutation* $kG$*-algebras, then there is an isomorphism of* $p$*-permutation* $k$*-algebras*
$$T^{k\,per}_{(U,a)}(A\otimes_k B) \simeq T^{k\,per}_{(U,a)}(A)\otimes_k T^{k\,per}_{(U,a)}(B)$$
3. *If* $(U,a)$ *and* $(U',a')$ *are* $\mathrm{End}(k)$*-graded* $H$*-sets-*$G$*, and* $A$ *is a* $p$*-permutation* $kG$*-algebra, then there is an isomorphism of* $p$*-permutation* $kH$*-algebras*
$$T^{k\,per}_{(U,a)\sqcup(U',a')}(A) \simeq T^{k\,per}_{(U,a)}(A)\otimes_k T^{k\,per}_{(U',a')}(A)$$
4. *If* $(U,a)$ *is an* $\mathrm{End}(k)$*-graded* $H$*-set-*$G$*, and* $A$ *is a* $p$*-permutation* $kG$*-algebra, then*
$$T^{k\,per}_{(U,a)}(A^{op}) \simeq T^{k\,per}_{(U,a)}(A)^{op}$$

**Proof:** Since $A$ is a $kG$-algebra, then $FP_A$ is a cohomological Green functor for $G$ over $k$, so $T_{(U,a)}^{k\,coh}(FP_A)$ is a cohomological Green functor for $H$ over $k$. But as $A$ is a $p$-permutation $kG$-module, the functor $T_{(U,a)}^{k\,coh}(FP_A)$ is isomorphic to $FP_B$, for $B = T_U^{k\,per}(A)$. Now evaluation at the trivial subgroup of the product for the Green functor $FP_B$ gives a $p$-permutation $kH$-algebra structure on $B$. The first assertion follows. The other ones are evaluations at the trivial subgroup of the corresponding assertions of proposition 8.21. □

.

**9.4. Identification.** Let $p$ be a prime number, and $k$ be a field of characteristic $p$. Let $G$ and $H$ be finite groups, and $U$ be a finite free-$p$ $H$-set-$G$ (for simplicity, I will not handle the case of a graded biset in this section). It is possible to give an explicit description of the functor $T_U^{k\,per}$.

NOTATION 9.5. If $M$ is a $kG$-module, denote by $\{M\}$ the underlying $G$-set. Consider the $H$-set $\mathrm{Hom}_G(U^{op}, \{M\})$ as a set of functions from $U$ to $M$. If $x \in k$ and $f \in \mathrm{Hom}_G(U^{op}, \{M\})$, define $xf \in \mathrm{Hom}_G(U^{op}, \{M\})$ by

$$(xf)(u) = xf(u)\ \forall u \in U$$

If $\lambda \in \mathrm{Hom}_G(U^{op}, \{k\})$, set

$$\pi(\lambda) = \prod_{u \in U/G} \lambda(u)$$

This does not depend on the choice of $U/G$. Define $\lambda * f \in \mathrm{Hom}_G(U^{op}, \{M\})$ by

$$(\lambda * f)(u) = \lambda(u) f(u)\ \forall u \in U$$

If $f$ and $f'$ are elements of $\mathrm{Hom}_G(U^{op}, \{M\})$, let $<f + f'> \in \mathrm{Hom}_G(U^{op}, \{M\})$ denote the sum of $f$ and $f'$, defined by

$$<f + f'>(u) = f(u) + f'(u)\ \forall u \in U$$

If $V \subseteq U$ is a $G$-subset of $U$, define the element $[f, f']_V$ of $\mathrm{Hom}_G(U^{op}, \{M\})$ by

$$[f, f']_V(u) = \begin{cases} f(u) & \text{if } u \in V \\ f'(u) & \text{if } u \in U - V \end{cases}$$

If $P$ is a $p$-subgroup of $G$, define the Brauer quotient $M[P]$ by

$$M[P] = M^P / (\sum_{Q \subset P} Tr_Q^P M^Q)$$

Let $Br_P$ denote the natural projection $M^P \to M[P]$. If $f \in \mathrm{Hom}_G(U, \{M\})$, then in particular for any $u \in U$, the element $f(u)$ is in $M^{G_u}$, so I can consider $Br_{G_u}\Big(f(u)\Big)$. Recall finally that if $X$ is a $G$-set, then for any $p$-group $P$, the module $(kX)[P]$ is isomorphic to $k(X^P)$.

DEFINITION 9.6. *If $M$ is a $kG$-module, define*

$$t_U(M) = k\mathrm{Hom}_G(U^{op}, \{M\})/\mathcal{J}$$

*where $\mathcal{J}$ is the subspace generated by the elements*

$$(\lambda * f) - \pi(\lambda) f \quad \textit{for} \quad \lambda \in \mathrm{Hom}_G(U^{op}, \{k\}),\ f \in \mathrm{Hom}_G(U^{op}, \{M\})$$

$$<f + f'> - \sum_{\substack{V \subseteq U \\ V\ G\text{-invariant}}} [f, f']_V \quad \textit{for} \quad f, f' \in \mathrm{Hom}_G(U^{op}, \{M\})$$

*and by all the elements* $f \in \mathrm{Hom}_G(U^{op}, \{M\})$ *such that there exists* $u \in U$ *with* $Br_{G_u}\Big(f(u)\Big) = 0$.

LEMMA 9.7. *The correspondence* $M \mapsto t_U(M)$ *is a functor from* $kG$-**Mod** *to* $kH$-**Mod**.

PROOF. If $\varphi : M \to M'$ is a morphism of $kG$-modules, then $\varphi$ induces a morphism $\{\varphi\}$ of the underlying $G$-sets, hence a morphism

$$\Phi = \mathrm{Hom}_G(U^{op}, \{\varphi\}) : \mathrm{Hom}_G(U^{op}, \{M\}) \to \mathrm{Hom}_G(U^{op}, \{M'\})$$

Now if $\lambda \in \mathrm{Hom}_G(U^{op}, \{k\})$ and $f \in \mathrm{Hom}_G(U^{op}, \{M\})$, then for any $u \in U$

$$\Phi(\lambda * f)(u) = \varphi\Big(\lambda(u)f(u)\Big) = \lambda(u)\varphi\Big(f(u)\Big)$$

so $\Phi(\lambda * f) = \lambda * \Phi(f)$. Similarly $\Phi(xf) = x\Phi(f)$ if $x \in k$. Thus

$$\Phi\Big(\lambda * f - \pi(\lambda)f\Big) = \lambda * \Phi(f) - \pi(\lambda)\Phi(f)$$

Similarly

$$\Phi(<f + f'>)(u) = \varphi\Big(f(u) + f'(u)\Big) = \varphi\Big(f(u)\Big) + \varphi\Big(f'(u)\Big)$$

so $\Phi(<f + f'>) =< \Phi(f) + \Phi(f') >$. And if $V$ is a $G$-subset of $U$, then

$$\Phi([f, f']_V)(u) = \varphi\Big([f, f'](u)\Big) = \begin{cases} \varphi\Big(f(u)\Big) & \text{if } u \in V \\ \varphi\Big(f'(u)\Big) & \text{if } u \in U - V \end{cases}$$

Thus $\Phi([f, f']_V) = [\Phi(f), \Phi(f')]_V$.

Finally, if there exists $u \in U$ such that $Br_{G_u}\Big(f(u)\Big) = 0$, then as $\varphi$ commutes with $Br_{G_u}$

$$Br_{G_u}\Big(\Phi(f)(u)\Big) = \varphi Br_{G_u}\Big(f(u)\Big) = 0$$

This proves that $\Phi$ passes down to a quotient map $t_U(\varphi) : t_U(M) \to t_U(M')$, and the lemma follows. □

PROPOSITION 9.8. *Let* $p$ *be a prime number, and* $k$ *be a field of characteristic* $p$. *Let* $G$ *and* $H$ *be finite groups, and* $U$ *be a finite free-*$p$ $H$*-set-*$G$. *Then the functor* $T_U^{k\,per}$ *is isomorphic to the restriction of* $t_U$ *to* $kG$-$p$-**Mod**.

PROOF. Suppose that $M = kX$, for some $G$-set $X$. Then by definition

$$T_U^{k\,per}(M) \simeq \Big(FP_{k\mathrm{Hom}_G(U^{op}, X)}\Big)(1) \simeq k\mathrm{Hom}_G(U^{op}, X)$$

Now $\mathrm{Hom}_G(U^{op}, X) \simeq \prod_{u \in U/G} X^{G_u}$, and as $kX^{G_u} \simeq (kX)[G_u]$, I have

$$T_U^{k\,per}(M) \simeq \bigotimes_{u \in U/G} (kX)[G_u]$$

where $\otimes = \otimes_k$. Now the proposition follows from the following:

LEMMA 9.9. *Let* $p$ *be a prime number, and* $k$ *be a field of characteristic* $p$. *Let* $G$ *and* $H$ *be finite groups, and* $U$ *be a finite free-*$p$ $H$*-set-*$G$. *Then for any* $kG$*-module* $M$, *there is an isomorphism of vector spaces*

$$t_U(M) \simeq \bigotimes_{u \in U/G} M[G_u]$$

*(where* $\otimes = \otimes_k$*) which is natural in* $M$.

PROOF. There is a natural map

$$t_U(M) = k\mathrm{Hom}_G(U^{op}, \{M\})/\mathcal{J} \to \underset{u\in U/G}{\otimes} M[G_u]$$

sending $f \in \mathrm{Hom}_G(U^{op}, \{M\})$ to $\otimes_{u\in U/G} Br_{G_u}\Big(f(u)\Big)$. Conversely, I can map $\otimes_{u\in U/G} Br_{G_u}(m_u)$ to the image of the element $f \in \mathrm{Hom}_G(U^{op}, \{M\})$ defined by $f(u') = gm_u$, if $u' = ug^{-1}$ for $u \in U/G$ and $g \in G$. Those maps are well-defined, and inverse to each other. □

So the restrictions of $T_U^{k\,per}$ and $t_U$ to the category of permutation modules are isomorphic.

Now if $M$ is any $p$-permutation $kG$-module, there is an exact sequence

$$FP_{kY} \overset{\varphi}{\to} FP_{kX} \overset{\psi}{\to} FP_M \to 0 \tag{9.27}$$

for suitable $G$-sets $X$ and $Y$. By definition, the sequence

$$T_U^{k\,coh}(FP_{k(Y\sqcup X)}) \xrightarrow{\Delta T_U^{k\,coh}(\varphi)} T_U^{k\,coh}(FP_{kX}) \to T_U^{k\,coh}(FP_M) \to 0$$

is exact, and evaluation at the trivial subgroup gives an exact sequence

$$T_U^{k\,per}(k(Y\sqcup X)) \xrightarrow{\Delta T_U^{k\,per}(\varphi)} T_U^{k\,per}(kX) \to T_U^{k\,per}(M) \to 0$$

The left part of this sequence is isomorphic to

$$t_U\Big(k(Y\sqcup X)\Big) \xrightarrow{\Delta t_U(\varphi)} t_U(kX)$$

So it suffices to prove that the sequence

$$t_U\Big(k(Y\sqcup X)\Big) \xrightarrow{\Delta t_U(\varphi)} t_U(kX) \to t_U(M) \to 0 \tag{9.28}$$

is exact if $M$ is a $p$-permutation module. And this sequence is exact if and only if it is an exact sequence of $k$-vector spaces.

But if $M$ is a $p$-permutation $kG$-module, the morphism $\psi$ in the sequence (9.27) is a split epimorphism, since $FP_M$ is a projective cohomological Mackey functor. Thus for any $u \in U$, the sequence

$$(kY)[G_u] \xrightarrow{\varphi[G_u]} (kX)[G_u] \xrightarrow{\psi[G_u]} M[G_u] \to 0$$

is exact. Hence the direct product of those sequences for $u \in S = U/G$

$$\Big((kY)[G_u]\Big)_{u\in S} \xrightarrow{(\varphi[G_u])_{u\in S}} \Big((kX)[G_u]\Big)_{u\in S} \xrightarrow{(\psi[G_u])_{u\in S}} \Big(M[G_u]\Big)_{u\in S} \to 0 \tag{9.29}$$

is exact. Now up to the isomorphism of the lemma, the sequence (9.28) becomes

$$\underset{u\in U/G}{\otimes}\Big((kY)[G_u] \oplus (kX)[G_u]\Big) \to \underset{u\in U/G}{\otimes}(kX)[G_u] \to \underset{u\in U/G}{\otimes} M[G_u] \to 0$$

It is exact, since the sequence (9.29) is exact, and the functor

$$(L_u)_{u\in U/G} \mapsto \underset{u\in U/G}{\otimes} L_u$$

is right exact. So the sequence (9.28) is exact, and this completes the proof of the proposition. □

REMARK 9.10. For $u \in U$, let

$$H_{(u)} = \{h \in H \mid \exists g \in G,\ hug = u\}$$

Then keeping track of the action of $H$ in lemma 9.9 gives the isomorphims of $kH$-modules

$$t_U(M) \simeq \bigotimes_{u \in H\backslash U/G} \mathrm{ten}_{H_{(u)}}^{H} M[G_u]$$

where $\mathrm{ten}_{H_{(u)}}^{H}$ is the ordinary tensor induction for $kH$-modules (described in the next section). The module $M[G_u]$ is a $kH_{(u)}$-module via the following action

$$h.m = g.m \quad \text{if} \quad h \in H,\ g \in G,\ hug^{-1} = u$$

## 10. Tensor induction for modules

**10.1. Definition.** Let $R$ be a commutative ring. If $G$ is a finite group, let $RG$-**FMod** denote the full subcategory of $RG$-**Mod** consisting of free $RG$-modules. Since a free $RG$-module is always the permutation module associated to a free $G$-set, this category can also be seen as the category of free permutation modules: its objects are the free $G$-sets, and a morphism from $Y$ to $X$ is a $G$-invariant matrix $m(x, y)$, indexed by $X \times Y$, with coefficients in $R$, and such that for any $y \in Y$, the coefficient $m(x, y)$ is zero except for a finite number of $x$. Composition of morphisms is given by matrix multiplication.

Let $G$ and $H$ be finite groups, and $U$ be a finite right free $H$-set-$G$. Then if $X$ is a free $G$-set, the $H$-set $\mathrm{Hom}_G(U^{op}, X)$ need not be free in general. But I can consider the $RH$-module

$$t_U(X) = R\mathrm{Hom}_G(U^{op}, X)$$

If $m : Y \to X$ is a morphism in $RG$-**FMod**, then for $\varphi \in \mathrm{Hom}_G(U^{op}, X)$ and $\psi \in \mathrm{Hom}_G(U^{op}, Y)$, define

$$\tilde{m}(\varphi, \psi) = \prod_{u \in U/G} m\Big(\varphi(u), \psi(u)\Big)$$

This matrix defines a morphism of $RH$-modules from $t_U(Y)$ to $t_U(X)$: for a given $\psi$, there is only a finite number of $\varphi$ such that $\tilde{m}(\varphi, \psi)$ is non-zero, and moreover $\tilde{m}$ is $H$-invariant (see the proof of proposition 8.8). Actually:

LEMMA 10.1. *Let $R$ be a commutative ring, and $G$ and $H$ be finite groups. Let $U$ be a finite right free $H$-set-$G$. Then the correspondence*

$$X \mapsto t_U(X) = R\mathrm{Hom}_G(U^{op}, X) \qquad\qquad m \mapsto \tilde{m}$$

*is a functor $t_U$ from $RG$-**FMod** to $RH$-**Mod**.*

PROOF. Clearly if $m$ is the identity matrix, so is $\tilde{m}$. I have only to check that the correspondence $m \mapsto \tilde{m}$ is multiplicative.

But if $Z$ is another free $G$-set, and $p$ is a morphism from $Z$ to $Y$, then the morphism $m \circ p : Z \to X$ is represented by the product matrix $m.p$ defined for $x \in X$ and $z \in Z$ by

$$(m.p)(x, z) = \sum_{y \in Y} m(x, y)p(y, z)$$

Now if $\theta \in \mathrm{Hom}_G(U^{op}, Z)$ and $\varphi \in \mathrm{Hom}_G(U^{op}, X)$

$$\widetilde{m.p}(\varphi,\theta) = \prod_{u\in U/G}\left(\sum_{y\in Y} m\Big(\varphi(u),y\Big)p\Big(y,\theta(u)\Big)\right)$$

Now expanding this product is equivalent to choosing a sequence $(y_u)_{u\in U/G}$ of elements of $Y$. But as $U^{op}$ is a free $G$-set, this is equivalent to choosing a $G$-morphism $\psi$ from $U^{op}$ to $Y$. Thus

$$\widetilde{m.p}(\varphi,\theta) = \sum_{\psi\in \mathrm{Hom}_G(U^{op},Y)}\ \prod_{u\in U/G} m\Big(\varphi(u),\psi(u)\Big)p\Big(\psi(u),\theta(u)\Big) = \dots$$

$$\dots = \sum_{\psi\in \mathrm{Hom}_G(U^{op},Y)} \tilde{m}(\varphi,\psi)\tilde{m}(\psi,\theta)$$

So $m \mapsto \tilde{m}$ is multiplicative, and the lemma follows. □

Note that the subcategory $\mathcal{P} = RG$-**FMod** of $RG$-**Mod** satisfies the hypothesis of theorem 2.14: its objects are projective, it is closed under directs sums, and any $RG$-module is a quotient of a free $RG$-module. This leads to the following:

Definition 10.2. *Let $R$ be a commutative ring. If $G$ and $H$ are finite groups, and $U$ is a finite right free $H$-set-$G$, I call tensor induction associated to $U$ the unique right exact functor from $RG$-**Mod** to $RH$-**Mod** extending the functor $t_U$ from $RG$-**FMod** to $RH$-**Mod**. This extension is still denoted by $t_U$.*

**10.2. Identification.** The module $t_U(M)$ can be computed as follows:

Proposition 10.3. *Let $R$ be a commutative ring. Let $G$ and $H$ be finite groups, and $U$ be a finite right free $H$-set-$G$. Then for any $RG$-module $M$, there is an isomorphism*

$$t_U(M) \simeq R\mathrm{Hom}_G(U^{op}, \{M\})/\mathcal{J}$$

*where $\mathcal{J}$ is the $R$-submodule generated by the elements*

$$(\lambda * f) - \pi(\lambda)f \quad \text{for} \quad \lambda \in \mathrm{Hom}_G(U^{op}, \{k\}),\ f \in \mathrm{Hom}_G(U^{op}, \{M\})$$

$$< f+f' > - \sum_{\substack{V\subseteq U\\ V\ G\text{-invariant}}} [f,f']_V \quad \text{for} \quad f,f' \in \mathrm{Hom}_G(U^{op}, \{M\})$$

*This isomorphism is moreover natural in $M$.*

Remark 10.4. This shows that the notation $t_U$ is coherent with the definition 9.6 given for $p$-permutation $kG$-modules, when $k$ is a field of characteristic $p$: indeed, if $U$ is right free, then $G_u = \{1\}$ for all $u \in U$, so all morphisms $Br_{G_u}$ are equal to identity, and their kernels are zero. Moreover, if $f(u) = 0$ for some $u$, then setting $\lambda(v) = 0$ on the orbit of $u$, and $\lambda(v) = 1$ elsewhere, I have $f = \lambda * f$ and $\pi(\lambda) = 0$, so $f$ is in the submodule $\mathcal{J}$ of the proposition.

Proof. It is easy to see that if $M$ is an $RG$-module, then

$$R\mathrm{Hom}_G(U^{op}, \{M\})/\mathcal{J} \simeq \bigotimes_{u\in U/G} M$$

like in lemma 9.9. This proves that the functor

$$M \mapsto R\mathrm{Hom}_G(U^{op}, \{M\})/\mathcal{J}$$

is right exact. Now if $M$ is a permutation module $RX$, the previous isomorphism gives

$$R\mathrm{Hom}_G(U^{op}, \{M\})/\mathcal{J} \simeq \bigotimes_{u\in U/G} RX \simeq R\mathrm{Hom}_G(U^{op}, X) = t_U(M)$$

So both functors in the proposition are right exact functors, which are isomorphic when restricted to $\mathcal{P} = RG$-**FMod**. The proposition follow from theorem 2.14. □

REMARK 10.5. This proposition proves that if $M = RX$ is *any* permutation module (not only for a free $X$), then

$$t_U(M) \simeq R\mathrm{Hom}_G(U^{op}, X)$$

### 10.3. Properties.

THEOREM 10.6. *Let $R$ be a commutative ring.*

1. *If $G$, $H$, and $K$ are finite groups, if $U$ is a finite right free $H$-set-$G$ and $V$ is a finite right free $K$-set-$H$, then $V \times_H U$ is a finite right free $K$-set-$G$ and*
$$t_V \circ t_U = t_{V\times_H U}$$
2. *If $G$, $H$ and $K$ are finite groups, if $U$ is a finite right free $K$-set-$(G \times H)$, if $M$ is an $RG$-module and $N$ is an $RH$-module, then*
$$t_U(M \boxtimes_R N) = t_{U/H}(M) \underset{R}{\otimes} t_{U/G}(N)$$
*where $M \boxtimes_R N$ denotes the tensor product $M \otimes_R N$, viewed as an $R(G\times H)$-module.*
3. *If $G$ and $H$ are finite groups, if $M$ and $N$ are $RG$-modules, and if $U$ is a finite right free $H$-set-$G$, then*
$$t_U(M \otimes_R N) \simeq t_U(M) \otimes_R t_U(N)$$
4. *If $G$ and $H$ are finite groups, if $U$ and $U'$ are finite right free $H$-sets-$G$, and $M$ is an $RG$-module, then*
$$t_{U\sqcup U'}(M) \simeq t_U(M) \otimes_R t_{U'}(M)$$
5. *If $G$ and $H$ are finite groups, if $U$ is a finite right free $H$-set-$G$, and if $M$ and $M'$ are $RG$-modules, then*
$$t_U(M \oplus M') = \bigoplus_{\substack{V\subseteq U \\ V \text{ invariant by } G \\ V \text{ mod. } H}} \mathrm{Ind}_{H_V}^H t_V(M) \otimes_R t_{U-V}(M')$$
6. *If $G$, $H$, and $K$ are finite groups, if $U$ is a finite right free $H$-set-$G$, and $V$ is a finite right free $K$-set-$H$, then for any $RG$-module $M$*
$$t_V(RU \otimes_{RG} M) \simeq \bigoplus_{f\in K\backslash \mathrm{Hom}_H(V^{op}, U/G)} \mathrm{Ind}_{K_f}^K t_{V\cdot_{f,H} U}(M)$$

*Moreover, all these isomorphisms are natural.*

PROOF. All those assertions state isomorphisms between right exact functors. So it suffices to check that the restrictions to $RG$-**FMod** are isomorphic, and this is straightforward: let me just give the details for the last one, where the notations refer to proposition 5.7. If $X$ is a $G$-set, then

$$RU \underset{RG}{\otimes} RX \simeq R(U \times_G X) \simeq R(U \circ_G X)$$

since $U$ is right free. Now I have seen in proposition 5.7 that

$$\mathrm{Hom}_H(V^{op}, U \circ_G X) = \bigsqcup_{f \in K\backslash \mathrm{Hom}_H(V^{op}, U/G)} \mathrm{Ind}_{K_f}^K \mathrm{Hom}_G(H\backslash U._f V, X)$$

It follows that

$$t_V(RU \underset{RG}{\otimes} M) \simeq \bigoplus_{f \in K\backslash \mathrm{Hom}_H(V^{op}, U/G)} \mathrm{Ind}_{K_f}^K R\mathrm{Hom}_G(H\backslash U._f V, X)$$

Moreover $H\backslash U._f V$ and $V._{f,H} U$ are isomorphic as $K_f$-sets-$G$. This proves the desired isomorphism for free permutation modules, and the assertion follows from right exactness. □

**10.4. Comparison.** Tensor induction for Mackey functors and for modules are compatible:

Proposition 10.7. *Let $G$ and $H$ be finite groups.*

1. *If $U$ is a finite right free $H$-set-$G$, then for any Mackey functor $M$ for $G$, there is an isomorphim of $\mathbb{Z}G$-modules*

$$T_U(M)(1) \simeq t_U\Big(M(1)\Big)$$

*where $t_U$ denotes tensor induction for $\mathbb{Z}G$-modules.*

2. *If $R$ is a commutative ring, and $U$ is a finite right free $H$-set-$G$, then for any cohomological Mackey functor $M$ for $G$ over $R$, there is an isomorphism of $RG$-modules*

$$T_U^{R\,coh}(M)(1) \simeq t_U\Big(M(1)\Big)$$

*where $t_U$ denotes tensor induction for $RG$-modules.*

*Those isomorphisms are moreover natural in $M$.*

Proof. Both assertions state isomorphisms between right exact functors (from from $Mack(G)$ to $\mathbb{Z}G$-**Mod** for the first one, and from $Comack_R(G)$ to $RG$-**Mod** for the second). So it suffices to check the isomorphism of the restrictions to $PMack(G)$ for the first assertion, and $PComack_R(G)$ for the second.

But if $M = b_X$ for some $G$-set $X$, then

$$T_U(M)(1) = (b_{\mathrm{Hom}_G(U^{op}, X)})(1) \simeq \mathbb{Z}\mathrm{Hom}_G(U^{op}, X) \simeq t_U(\mathbb{Z}X)$$

Similarly, if $M = FP_{RX}$ for some $G$-set $X$, then

$$T_U^{R^{coh}}(M)(1) = \Big(FP_{R\mathrm{Hom}_G(U^{op}, X)})(1) = R\mathrm{Hom}_G(U^{op}, X) = t_U(X)$$

Now proposition follows from theorem 2.14. □

**10.5. Examples.**

10.5.1. *Restriction.* Let $H$ be a subgroup of $G$, and $U = G$, viewed as an $H$-set-$G$. Then if $X$ is a free $G$-set, it is clear that $\mathrm{Hom}_G(U^{op}, X)$ is isomorphic to the restriction of $X$ to $H$. Since

$$\mathrm{Res}_H^G RX = R(\mathrm{Res}_H^G X)$$

the functors $\mathrm{Res}_H^G$ and $t_U$ are isomorphic when restricted to $RG$-**FMod**. Since they are both right exact, they are isomorphic, and $t_U$ is just the restriction functor in that case.

10.5.2. *Inflation.* Let $N$ be a normal subgroup of $G$ and $H = G/N$. Let $U = H$, viewed as a right free $G$-set-$H$. If $X$ is an $H$-set, then $\mathrm{Hom}_H(U^{op}, X)$ is isomorphic to $\mathrm{Inf}_H^G X$. Thus

$$t_U(X) = R\mathrm{Inf}_H^G X = \mathrm{Inf}_H^G(RX)$$

Here again, the functors $t_U$ and $\mathrm{Inf}_H^G$ are right exact functors, with isomorphic restrictions to $RG$-**FMod**. So the functor $t_U$ is just the inflation functor in that case.

10.5.3. *Ordinary tensor induction.* Let $H$ be a subgroup of $G$, and $U = G$, viewed as a $G$-set-$H$. Then for any $H$-set $X$

$$t_U(X) = R\mathrm{Hom}_H(U^{op}, X) \simeq R(X^{[G:H]}) = (RX)^{\otimes[G:H]}$$

So the functor $t_U$ is isomorphic to the (ordinary) tensor induction functor $\mathrm{ten}_H^G$ (see [**Ben91**] 3.15).

In particular, theorem 10.6 5) and 6) gives explicit formulas for $\mathrm{ten}_H^G(M \oplus N)$, and $\mathrm{ten}_H^G(\mathrm{Ind}_K^H M)$ (compare with [**Ben91**] Proposition 3.15.2), whereas Mackey formula for tensor induction, transitivity of tensor induction, formulas for composition of tensor induction with restriction and inflation follow from theorem 10.6 1).

## References

[ACPW98] A. Adem, J. Carlson, S. Priddy, and P. Webb (eds.), *Group representations: Cohomology, Group Actions and Topology*, vol. 63, AMS Proceedings of Symposia in pure mathematics, Janvier 1998.

[Ben91] D.J. Benson, *Representations and cohomology I*, Cambridge studies in advanced mathematics, vol. 30, Cambridge University Press, 1991.

[Bou92] Serge Bouc, *Exponentielle et modules de Steinberg*, J. of Algebra **150** (1992), no. 1, 118–157.

[Bou96a] Serge Bouc, *Construction de foncteurs entre catégories de G-ensembles*, J. of Algebra **183** (1996), no. 0239, 737–825.

[Bou96b] Serge Bouc, *Foncteurs d'ensembles munis d'une double action*, J. of Algebra **183** (1996), no. 0238, 664–736.

[Bou97] Serge Bouc, *Green functors and G-sets*, Lecture Notes in Mathematics, vol. 1671, Springer, October 1997.

[Bro85] Michel Broué, *On Scott modules and p-permutation modules: an approach through the Brauer morphism*, Proc.AMS **93** (1985), no. 3, 401–408.

[BT98] Serge Bouc and Jacques Thévenaz, *The group of endo-permutation modules*, Preprint, June 1998.

[Dre73] A.W.M Dress, Contributions to the theory of induced representations, Lecture Notes in Mathematics, vol. 342, 183–240, Lecture Notes in Mathematics, Springer-Verlag, 1973, pp. 183–240.

[Thé87] Jacques Thévenaz, *Permutation representation arising from simplicial complexes*, J.Combin. Theory **46** (1987), 122–155, Ser.A.

[TW90] Jacques Thévenaz and Peter Webb, *Simple Mackey functors*, Proceedings of the 2nd International group theory conference Bressanone 1989, Rend. Circ. Mat. Palermo, vol. 23, 1990, Serie II, pp. 299–319.

[TW95] Jacques Thévenaz and Peter Webb, *The structure of Mackey functors*, Trans. Amer. Math. Soc. **347** (1995), no. 6, 1865–1961.

[Web91] Peter Webb, *A split exact sequence for Mackey functors*, Comment. Math. Helv. **66** (1991), 34–69.

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge studies in advanced mathematics, vol. 38, Cambridge University Press, 1994.

S. Bouc, Equipe des groupes finis, UFR de Mathématiques, Université Paris 7-Denis Diderot, 2 place Jussieu, 75251. Paris Cedex 05, France

*E-mail address*: `bouc@math.jussieu.fr`