Resolutions of Mackey functors[1]

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Abstract. I will build some standard resolutions for Mackey functors which are projective relative to p-subgroups. Those resolutions are closely related to the poset of p-subgroups. They lead to generalizations of known results on cohomology. They give a way to compute the Cartan matrix for Mackey functors, in terms of p-permutation modules, and to precise the structure of projective Mackey functors. They also provide results on complexes of projective Mackey functors and complexes of p-permutation modules.

1. Introduction

I will build some standard resolutions for Mackey functors which are projective relative to p-subgroups. Those resolutions are closely related to the poset of p-subgroups. They lead to generalizations of known results on cohomology. They give a way to compute the Cartan matrix for Mackey functors, in terms of p-permutation modules, and to precise the structure of projective Mackey functors. They also provide results on complexes of projective Mackey functors and complexes of p-permutation modules.

2. Notation

2.1. The Mackey algebra. Most of the results I will need can be found in J.Thévenaz and P.Webb's paper ([**6**]) on the structure of Mackey functors : among other possible definitions, a Mackey functor is a module over the Mackey algebra. A possible definition of this algebra is the following :

Definition 2.1. Let G be a finite group, and R be a commutative ring. The *Mackey algebra* $\mu_R(G)$ is the algebra generated by the elements t_H^K, r_H^K, and $c_{g,H}$, where H and K are subgroups of G such that $H \subseteq K$, and g is an element of G, subject to the relations :

$$t_K^L t_H^K = t_H^L \text{ for all } H \subseteq K \subseteq L$$

$$r_H^K r_K^L = r_H^L \text{ for all } H \subseteq K \subseteq L$$

$$c_{g,{}^hK} c_{h,K} = c_{gh,K} \text{ for all } g, h, K$$

$$t_H^H = r_H^H = c_{h,H} \text{ for all } h, H \text{ such that } h \in H$$

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[1]This is a rough translation of my paper *Résolutions de foncteurs de Mackey* (in french) AMS PSPM 63 (1998) 31-83. A few comments have been added in footnotes (date : 2008-11-24)

$$c_{g,K}t_H^K = t_{^gH}^{^gK}c_{g,H} \text{ for all } g,h,H,K$$
$$c_{g,K}r_K^H = r_{^gK}^{^gH}c_{g,H} \text{ for all } g,h,H,K$$
$$\sum_H t_H^H = \sum_H r_H^H = 1$$
$$r_K^H t_L^H = \sum_{x\in K\backslash H/L} t_{K\cap {^xL}}^K r_{K\cap {^xL}}^{^xL} c_{x,L} \text{ for all } K\subseteq H\supseteq L \quad \text{(Mackey axiom)}$$

any other product of r_H^K, t_H^K and $c_{g,H}$ being equal to zero.

Setting $c_x = \sum_H c_{x,H}$ for $x\in G$, it follows that $c_{x,H} = c_x t_H^H$, and that the map $x\mapsto c_x$ endows $\mu_R(G)$ with a structure of interior G-algebra (in Puig's sense [**3**]).

Proposition (3.4) of [**6**] proves that the $c_{x,1}$, hence also the c_x, are linearly independent, which allows for an identification of x and c_x. With this notation, Proposition (3.2) of [**6**] proves that $\mu_R(G)$ admits a basis over R, consisting of the elements $t_{^xL}^H x r_L^K$, where H and K are subgroups of G, where $x\in K\backslash G/H$, and L is a subgroup of $H\cap K^x$ up to conjugation by $H\cap K^x$.

If H is a subgroup of G, the defining relations of $\mu_R(G)$ allow to define a morphism from the Burnside ring $b_R(H)$ of H with coefficients in R, to the algebra $\mu_R(G)$, sending the element H/K to the element $t_K^H r_K^H$. I will denote by $X\mapsto [X]$ this morphism.

On the other hand, there is a natural morphism z from the Burnside ring of G to the center of the Mackey algebra ([**6**], Proposition (9.2)), defined by

$$z(X) = \sum_{K\subseteq G} [\mathrm{Res}_K^G X]$$

which allows the use of the idempotents of the Burnside ring to split the Mackey algebra into smaller pieces.

In the remainder of this paper, I will only consider the "characteristic p" case : I will assume that any prime number different from p is invertible in the ring R. Under these conditions (cf [**6**] Section 9-10), the algebra $\mu_R(G)$ is Morita-equivalent to a direct product of algebras indexed by the p-perfect subgroups of G (i.e. the subgroups with no non trivial quotient p-group).

More precisely, the algebra $\mu_R(G)$ is Morita-equivalent to the direct product over the p-perfect subgroups H of G (up to G-conjugation) of the algebras I will denote by $\mu_R^1(N_G(H)/H)$ (the category of $\mu_R^1(G)$-modules is denoted by $Mack_R(G,1)$ in [**6**]) : the algebra $\mu_R^1(G)$ is the piece of $\mu_R(G)$ corresponding to the idempotent f_1^G of the Burnside ring of G. Denoting by $\underline{s}_p(G)$ the set of p-subgroups of G, the idempotent f_1^G of the Burnside ring $b_{\mathbb{Q}}(G)$ with rational coefficients is defined par

$$f_1^G = \sum_{P\in \underline{s}_p(G)/G} e_P^G$$

with

$$e_P^G = \frac{1}{|N_G(P)|}\sum_{Q\subseteq P} \tilde{\chi}]Q,P[|Q|G/Q$$

where $\tilde{\chi}]Q,P[$ is the reduced Euler-Poincaré characteristic of the set of p-subgroups strictly containing Q and strictly contained in P. An easy computation shows then that

$$f_1^G = -\sum_{P\in \underline{s}_p(G)/G} \frac{\tilde{\chi}(s_p(N_G(P)/P))}{|N_G(P)/P|}G/P$$

with $s_p(G) = \underline{s}_p(G) - \{1\}$.

This expression shows that f_1^G is p-integral, hence f_1^G in $b_R(G)$ (it is indeed well known that $\tilde{\chi}(s_p(G))$ is divisible by the p-part of the order of G, since e.g. it is the degree of a projective character of G (cf. [**4**])).

With this notation, the algebra $\mu_R^1(G)$ identifies to $z(f_1^G)\mu_R(G)$. Another way of seeing this algebra up to Morita equivalence is given by the following lemma :

Lemma 2.2. *Let $e = \sum_{P\in \underline{s}_p(G)} t_P^P$. Then $e\mu_R(G)e$ is a subalgebra of $\mu_R^1(G)$, and the inclusion of $e\mu_R(G)e$ in $\mu_R^1(G)$ is a Morita equivalence.*

First I have to check that $e\mu_R(G)e$ is a subalgebra of $\mu_R^1(G)$: it suffices to check that $z(f_1^G)e = e$, which will follow from the equality $z(f_1^G)t_P^P = t_P^P$ for all p-subgroup P of G. But it follows from Section 9 of [**6**] that

$$z(f_1^G)t_P^P = [\mathrm{Res}_P^G(f_1^G)]t_P^P = [\mathrm{Res}_P^G(f_1^G)P/P] = [\mathrm{Res}_P^G(f_1^G)]$$

and the result will follow, if I know that $\mathrm{Res}_P^G f_1^G = f_1^P$, since the expression of f_1^P shows that $f_1^P = P/P$ if P is a p-group. This in turn follows from the

Lemma 2.3. *Let H be a subgroup of G. Then $\mathrm{Res}_H^G f_1^G = f_1^H$*

The computation can be done inside $b_{\mathbb{Q}}(G)$: the idempotents e_L^G are such that for all X of $b_{\mathbb{Q}}(G)$

$$X = \sum_{\substack{L\in G\\ L mod. G}} |X^L| e_L^G$$

Then

$$\mathrm{Res}_H^G e_P^G = \sum_{\substack{L\subseteq H\\ L mod. H}} \mathrm{Res}_H^G e_P^G . e_L^H = \sum_{\substack{L\subseteq H\\ L mod. H}} |(\mathrm{Res}_H^G e_P^G)^L| e_L^H$$

and since $|(\mathrm{Res}_H^G e_P^G)^L|$ is non zero only if L is a conjugate of P in G, in which case it is equal to 1, I see that $\mathrm{Res}_H^G e_P^G$ is the sum of the e_L^H, where L runs through the conjugates of P contained in H, up to H-conjugation. Then $\mathrm{Res}_H^G f_1^G$ is the sum over the p-subgroups L of H, modulo H, of the e_L^H. Lemma 2.3, and the first assertion of Lemma 2.2. The second one follows then from

Lemma 2.4. [2] *Let A be a ring (with identity element), and e be an idempotent of A. The following assertions are equivalent :*
1) The inclusion of eAe in A is a Morita equivalence.
2) The two sided ideal of A generated by e is equal to A (i.e. $AeA = A$).

The inclusion of eAe in A defines a functor of restriction r from the category A-mod of left A-modules to eAe-mod, by $r(M) = eM$. This functor has a left adjoint i, defined by $i(M) = A \otimes_{eAe} M$. Then $ri(M) = eA \otimes_{eAe} M = eAe \otimes_{eAe} M = M$ and $ir(M)$ identifies with AeM. Hence if 1) holds, then in particular $AeA = A$, hence 2) holds. Conversely, if 2) holds, then $ir(M) = AeM = AeAM = AM = M$, hence 1) holds.

To complete the proof of Lemma 2.2, it remains to check that the identity element of $\mu_R^1(G)$, i.e. $z(f_1^G)$, lies in the two sided ideal of $\mu_R^1(G)$ generated by e. It suffices to show that $t_P^H r_P^H$ lies in this ideal, for any subgroup H of G and any p-subgroup P of H. But this is clear, since $t_P^H r_P^H = t_P^H e r_P^H$. Lemma 2.2 follows.

[2]This lemma is true, but the proof given here is not quite complete

Remark 2.5. The $e\mu_R(G)e$-modules are actually very natural objects : they are exactly the Mackey functors "defined only over p-subgroups ". Whereas the $\mu_R^1(G)$-modules are the Mackey functors which are projective relative to p-subgroups ([**6**] Theorem 9.7). The algebra $\mu_R^1(G)$ identifies with the subalgebra of $\mu_R(G)$ generated by the $x \in G$, and the r_Q^H and t_Q^H, where Q is a p-subgroup of G and H an arbitrary subgroup of G.

On the other hand, Lemma 2.2 has the following consequence :

Corollary 2.6. *Let S be a Sylow p-subgroup of G, and M_* be a complex of Mackey functors in $Mack_R(G,1)$. Then M_* is acyclic (resp. split acyclic) if and only if its restriction to S is.*

Indeed, a Morita equivalence maps an exact sequence (resp. a split exact sequence) to an exact sequence (resp. to a split exact sequence).

2.2. Another algebra. I will use also here another algebra, that I denote by $r\mu_R(G)$, defined similarly by considering only p-subgroups of G, and "forgetting" the generators t_H^K and the Mackey axiom.

More precisely, let A be a free R-module on the set of triples (x,Q,P), where x is an element of G, and P,Q are p-subgroups of G such that $Q \subseteq P$. I turn A into an algebra by defining the multiplication of the basis elements by

$$(x,Q,P)(y,R,S) = \delta_{P^y,R}(xy,Q^y,S)$$

It is easy to see that A is an associative algebra. Moreover, let

$$e = \sum_{P \in \underline{s}_p(G)} (1,P,P)$$

then

$$e.(y,Q,R) = \sum_{P \in \underline{s}_p(G)} \delta_{P^y,Q}(y,P^y,R) = (y,Q,R)$$

and

$$(y,Q,R).e = \sum_{P \in \underline{s}_p(G)} \delta_{R,P}(y,Q,P) = (y,Q,R)$$

hence e is an identity element of A.

Let then I the R-submodule of A generated by the elements of the form $u_{x,z,Q,P} = (xz,Q^z,P) - (x,Q,P)$, where $z \in P$. Then I is a two sided ideal of A. Indeed

$$\begin{aligned}(y,R,S)u_{x,z,Q,P} &= \delta_{S^{xz},Q^z}(yxz,R^{xz},P) - \delta_{S^x,Q}(yx,R^x,P)\\ &= \delta_{S^x,Q}((yxz,R^{xz},P) - (yx,R^x,P)) = \delta_{S^x,Q}u_{yx,z,R^x,P}\end{aligned}$$

Similarly

$$\begin{aligned}u_{x,z,Q,P}(y,R,S) &= \delta_{P^y,R}(xzy,Q^{zy},S) - \delta_{P^y,R}(xy,Q^y,S)\\ &= \delta_{P^y,R}u_{xy,z^y,Q^y,S}\end{aligned}$$

which makes sense, since if $P^y = R$, then $z^y \in R \subseteq S$.

I will denote by $r\mu_R(G)$ the quotient algebra A/I. It is easy to find a basis of $r\mu_R(G)$:

Lemma 2.7. *The elements (x,Q,P), for $Q \subseteq P \in \underline{s}_p(G)$, and $x \in G/P$, form a basis of $r\mu_R(G)$.*

Lemma 2.7 is just a reformulation of the following :

LEMMA 2.8. *Let A be an R-algebra, admitting a basis B over R. Let E be an equivalence relation over B such that the elements $x-y$, with xEy, generate a two sided ideal I of A. Then A/I admits a basis over R, consisting of the images in A/I of the elements of B/E.*

Indeed, it is clear that the images of B/E generate A/I. since all the elements of the equivalence class of $x \in B$ are congruent to x modulo I. And if $\sum_{x\in B/E} r_x x \in I$, with $r_x \in R$, then there exists scalars $s_{x,y} \in R$ such that

$$\sum_{x\in B/E} r_x x = \sum_{xEy} s_{x,y}(x-y)$$

Then for all x

$$r_x = \sum_{xEy} s_{x,y} - \sum_{yEx} s_{y,x}$$

Summing this relation for x in the equivalence class of z, I can suppose that r_z alone is non zero, hence

$$r_z = \sum_{xEz}(\sum_{xEy} s_{x,y} - \sum_{yEx} s_{y,x}) = \sum_{x,yEz} s_{x,y} - \sum_{x,yEz} s_{x,y} = 0$$

since the relation E is symmetric and transitive. The above lemmas follow.

I can then define an application f from $r\mu_R(G)$ to $e\mu_R^1(G)e$ by sending (x,Q,P) to xr_Q^P. It is easy to check that the map f is a morphism of algebras (with identity elements).

LEMMA 2.9. *The morphism f is injective.*

(In the sequel, I will identify $r\mu_R(G)$ with its image in $\mu_R(G)$.)
To show this, I must check that the elements xr_Q^P, for $x \in G/P$ and $Q \subseteq P$, are linearly independent in $\mu_R(G)$: but if

$$\sum_{x\in G/P,Q,P} r_{x,Q,P}\, x r_Q^P = 0 \quad \text{in } \mu_R(G)$$

then multiplying this relation on the left by t_R^R and on the right by r_P^P, for given subgroups R and P, it follows that

$$\sum_{\substack{x\in G/P,Q\\ {}^xQ=R}} r_{x,Q,P}\, t_R^R x r_Q^P = 0$$

i.e.

$$\sum_{\substack{x\in G/P\\ R^x\subseteq P}} r_{x,R^x,P}\, t_R^R x r_{R^x}^P = 0$$

Then using the above basis of $\mu_R(G)$, it follows that, for any double coset Rx_0P and any subgroup Q_0 of $R^{x_0}\cap P$

$$\sum_{\substack{x\in Rx_0P/P\\ R^x=Q_0 mod. R^{x_0}\cap P}} r_{x,R^x,P} = 0$$

i.e.

$$\sum_{\substack{y\in R/(R\cap {}^{x_0}P)\\ R^{yx_0}=Q_0 mod. R^{x_0}\cap P}} r_{yx_0,R^{yx_0},P}=0$$

or equivalently

$$\sum_{\substack{y\in R/(R\cap {}^{x_0}P)\\ R^{x_0}=Q_0 mod. R^{x_0}\cap P}} r_{yx_0,R^{x_0},P}=0$$

In particular taking $Q_0=R^{x_0}\cap P$, I conclude that $r_{x_0,R^{x_0}\cap P,P}=0$, and as this relation is true for all x_0,R,P, the above linear independence claim holds, and the injectivity of f follows.

It is also natural to consider here the algebra $t\mu_R(G)$ defined similarly to $r\mu_R(G)$ by "forgetting" the generators r_H^K. This algebra identifies with the subalgebra of $\mu_R(G)$ generated by the elements t_Q^Px. It also identifies to the opposite algebra of $r\mu_R(G)$: the map sending t_Q^P to r_Q^P and $c_{x,Q}$ to $c_{x^{-1},Q}$ defines indeed an anti-isomorphism from $t\mu_R(G)$ to $r\mu_R(G)$. The results I will prove here for $r\mu_R(G)$ will have a "dual" version for $t\mu_R(G)$, and the previous argument will allow for a single proof for both.

The algebra homomorphism from $r\mu_R(G)$ to $\mu_R^1(G)$ yields a forgetful functor from the category of Mackey functors which are projective relative to p-subgroups to the category of $r\mu_R(G)$-modules. I will denote this functor by $\mathcal{R}$: if M is a Mackey functor, then $\mathcal{R}(M)$ is the functor M, for which I only consider evaluations at p-subgroups, and restrictions and conjugation by elements of G.

The functor $\mathcal{R}$ admits a left adjoint : let N be a $r\mu_R(G)$-module. I set $N(P)=r_P^PN$. If H is a subgroup of G, then H acts on $\oplus_{P\in\underline{s}_p(H)}N(P)$, and I set

$$\mathcal{I}(N)(H)=H_0(H,\oplus_{P\in\underline{s}_p(H)}N(P))$$

that I will also denote by $(\oplus_{P\in\underline{s}_p(H)}N(P))_H$.

If P is a p-subgroup of H, and v is an element of $\oplus_{P\in\underline{s}_p(H)}N(P)$, I denote by $\pi_H(v)$ its image in $\mathcal{I}(N)(H)$.

If $H\subseteq K$ are subgroups of G, and if $v\in\oplus_{P\in\underline{s}_p(H)}N(P)$, I set

$$t_H^K(\pi_H(v))=\pi_K(v)$$

which makes sense since $H\subseteq K$.

Similarly, if Q is a p-subgroup of K, and $v\in N(Q)$, I set

$$r_H^K(\pi_K(v))=\sum_{x\in H\backslash K/Q}\pi_H(r_{H\cap {}^xQ}^{{}^xQ}xv)$$

It is easy to see that the map r_H^K is well defined.

Finally in the same situation, if $x\in G$, I set

$$c_{x,K}(\pi_K(v))=\pi_{{}^xK}(xv)$$

With this notation :

Lemma 2.10. *The correspondence sending N to $\mathcal{I}(N)$ defines a functor from the category of $r\mu_R(G)$-modules in $Mack_R(G,1)$, and this functor is left adjoint to the functor $\mathcal{R}$.*

I must check that $\mathcal{I}(N)$ is a Mackey functor, which is moreover in $Mack_R(G,1)$, and then check the adjunction property. For the first property, the only non obvious points are the transitivity of restrictions and the Mackey axiom. So let $H \subseteq K \subseteq L$ be subgroups of G, and $x = \pi_L(n) \in \mathcal{I}(N)(L)$, with $P \in \underline{s}_p(L)$ and $n \in N(P)$.

Then

$$r_K^L(x) = \sum_{l \in K\backslash L/P} \pi_K(r_{K\cap {}^lP}^{{}^lP} ln)$$

Thus

$$\begin{aligned} r_H^K r_K^L(x) &= \sum_{k \in H\backslash K/K\cap {}^lP} \sum_{l \in K\backslash L/P} \pi_H(r_{H\cap {}^k(K\cap {}^lP)}^{{}^k(K\cap {}^lP)} k r_{K\cap {}^lP}^{{}^lP} ln) \\ &= \sum_{k \in H\backslash K/K\cap {}^lP} \sum_{l \in K\backslash L/P} \pi_H(r_{H\cap {}^{kl}P}^{{}^{kl}P} kln) \end{aligned}$$

But the map $(k,l) \mapsto kl$ is a bijection from the set of pairs (k,l) such that $l \in K\backslash L/P$ and $k \in H\backslash K/K \cap {}^lP$ to the set $H\backslash L/P$. Thus indeed $r_H^K r_K^L(x) = r_H^L(x)$, and restrictions are transitive.

For the Mackey axiom, let $H \subseteq K \supseteq L$ be subgroups of G, and $x = \pi_L(n) \in \mathcal{I}(N)(L)$, with $P \in \underline{s}_p(L)$. Then $t_L^K(x) = \pi_K(n)$ hence

$$r_H^K t_L^K(x) = \sum_{k \in H\backslash K/P} \pi_H(r_{H\cap {}^kP}^{{}^kP} kn)$$

On the other hand

$$\begin{aligned} \sum_{y \in H\backslash K/L} t_{H\cap {}^yL}^H r_{H\cap {}^yL}^{{}^yL} yx &= \sum_{y \in H\backslash K/L} t_{H\cap {}^yL}^H r_{H\cap {}^yL}^{{}^yL} \pi_{{}^yL}(yn) \\ = \sum_{y \in H\backslash K/L} t_{H\cap {}^yL}^H & \sum_{z \in H\cap {}^yL\backslash {}^yL/{}^yP} \pi_{H\cap {}^yL}(r_{H\cap {}^yL\cap {}^{zy}P}^{{}^{zy}P} zyn) \\ &= \sum_{y \in H\backslash K/L} \sum_{z \in H\cap {}^yL\backslash {}^yL/{}^yP} \pi_H(r_{H\cap {}^{zy}P}^{{}^{zy}P} zyn) \end{aligned}$$

Replacing z by z^y in this sum, I get

$$r_H^K t_L^K(x) = \sum_{y \in H\backslash K/L} \sum_{z \in H^y\cap L\backslash L/P} \pi_H(r_{H\cap {}^{yz}P}^{{}^{yz}P} yzn)$$

and as above, the map $(y,z) \mapsto yz$ induces a bijection from the set of pairs (y,z) such that $y \in H\backslash K/L$ and $z \in H^y \cap L\backslash L/P$ to the set $H\backslash K/P$. Then indeed

$$r_H^K t_L^K(x) = \sum_{y \in H\backslash K/L} t_{H\cap {}^yL}^H r_{H\cap {}^yL}^{{}^yL} yx$$

and the Mackey axiom holds. Thus $\mathcal{I}(N)$ is indeed a Mackey functor.

To check that it is in $Mack_R(G,1)$, I must check that $z(f_1^G)$ acts trivially on it. So let H be a subgroup of G, and $x = \pi_H(n) \in \mathcal{I}(H)(N)$, with $n \in N(P)$. As $x = t_P^H(\pi_P(n))$, I have $z(f_1^G)x = [f_1^H]t_P^H(\pi_P(n))$. Then if I know that $[f_1^H]t_P^H = t_P^H[\mathrm{Res}_P^H f_1^H]$, the proof will be complete, since $\mathrm{Res}_P^H f_1^H = f_1^P = P/P$. But this follows from

Lemma 2.11. *Let $K \subseteq H$ be subgroups of G, and $X \in b(H)$. Then*

$$[X]t_K^H = t_K^H[\mathrm{Res}_K^H X] \quad \textit{in} \quad \mu_R(G)$$

Indeed if $X=H/L$, then $[X]=t_L^Hr_L^H$ and

$$[X]t_K^H=t_L^Hr_L^Ht_K^H=\sum_{x\in L\backslash H/K}t_{L\cap {}^xK}^H r_{L\cap {}^xK}^{{}^xK}x=\sum_{x\in L\backslash H/K}xt_{L^x\cap K}^Hr_{L^x\cap K}^K$$
$$=\sum_{x\in L\backslash H/K}t_{L^x\cap K}^Hr_{L^x\cap K}^K$$

since $x\in H$, which can be written as

$$[X]t_K^H=t_K^H\sum_{x\in L\backslash H/K}t_{L^x\cap K}^Kr_{L^x\cap K}^K=t_K^H[\mathrm{Res}_K^H H/L]$$

proving Lemma2.11.

It remains to check the adjunction property : so let α be a morphism from $\mathcal{I}(N)$ to a Mackey functor M. Such a morphism is characterized by specifying, for all subgroup H of G, of a morphism α_H of $\mathcal{I}(N)(H)$ in $M(H)$, such that

$$\alpha_Ht_K^H=t_K^H\alpha_K\ ,\ \alpha_Kr_K^H=r_K^H\alpha_H\ ,\ c_{x,H}\alpha_H=\alpha_{{}^xH}c_{x,H}$$

for all $K\subseteq H$ and all $x\in G$.

Then if P is a p-subgroup of G, I define a map β_P from $N(P)$ to $M(P)$ by $\beta_P=\alpha_P\pi_P$. If Q is a subgroup of P, and if $n\in N(P)$, then

$$r_Q^P(\pi_P(n))=\sum_{x\in Q\backslash P/P}\pi_Q(r_{Q\cap {}^xP}^{{}^xP}xn)=\pi_Q(r_Q^P(n))$$

hence

$$r_Q^P\beta_P(n)=r_Q^P\alpha_P\pi_P(n)=\alpha_Qr_Q^P\pi_P(n)=\alpha_Q\pi_Qr_Q^P(n)=\beta_Qr_Q^P(n)$$

Similarly

$$c_{x,P}\beta_P(n)=c_{x,P}\alpha_P\pi_P(n)=\alpha_{{}^xP}c_{x,P}\pi_P(n)=\alpha_{{}^xP}\pi_{{}^xP}c_{x,P}(n)=\beta_{{}^xP}c_{x,P}(n)$$

and β defines a morphism from N to $\mathcal{R}(M)$.

Conversely, if β is such a morphism, then for any p-subgroup P of G, I have a morphism β_P of $N(P)$ in $M(P)$, such that

$$r_Q^P\beta_P=\beta_Qr_Q^P\ \ c_{x,P}\beta_P=\beta_{{}^xP}c_{x,P}$$

Then if H is a subgroup of G, I define a map α_H from $\mathcal{I}(N)(H)$ to $M(H)$ by $\alpha_H(\pi_H(n))=t_P^H\beta_P(n)$, if P is a p-subgroup of H, and if $n\in N(P)$. If K is a subgroup of H, if Q is a subgroup of P, and if $n\in N(Q)$, then

$$\alpha_Ht_K^H(\pi_K(n))=\alpha_H(\pi_H(n))=t_Q^H\beta_Q(n)$$

whereas

$$t_K^H\alpha_K(\pi_K(n))=t_K^Ht_Q^K\beta_Q(n)$$

which proves that $\alpha_Ht_K^H=t_K^H\alpha_K$.

Similarly, if $n\in N(P)$, then

$$\alpha_Kr_K^H(\pi_H(n))=\alpha_K(\sum_{x\in K\backslash H/P}\pi_K(r_{K\cap {}^xP}^{{}^xP}xn))=\sum_{x\in K\backslash H/P}t_{K\cap {}^xP}^K\beta_{K\cap {}^xP}r_{K\cap {}^xP}^{{}^xP}xn$$
$$=\sum_{x\in K\backslash H/P}t_{K\cap {}^xP}^Kr_{K\cap {}^xP}^{{}^xP}\beta_{{}^xP}xn=\sum_{x\in K\backslash H/P}t_{K\cap {}^xP}^Kr_{K\cap {}^xP}^{{}^xP}x\beta_Pn$$
$$=r_K^Ht_P^H\beta_P(n)=r_K^H\alpha_H(\pi_H(n))$$

which proves that $\alpha_Kr_K^H=r_K^H\alpha_H$.

Finally

$$c_{x,H}\alpha_H\pi_H(n) = c_{x,H}t_P^H\beta_P(n) = t_{^xP}^{^xH}c_{x,P}\beta_P(n) = t_{^xP}^{^xH}\beta_{^xP}c_{x,P}(n) = \alpha_{^xH}c_{x,H}(\pi_H(n))$$

which proves that $c_{x,H}\alpha_H = \alpha_{^xH}c_{x,H}$, hence that α defines a morphism from $\mathcal{I}(N)$ to M.
These correspondences between $\mathrm{Hom}_{\mu_R(G)}(\mathcal{I}(N), M)$ and $\mathrm{Hom}_{r\mu_R(G)}(N, \mathcal{R}(M))$ are clearly mutual inverse bijections, and this proves the adjunction property, completing the proof of Lemma 2.10.

3. Examples of $r\mu_R(G)$-modules

Let I the R-submodule of $r\mu_R(G)$ generated by the xr_Q^P, where $x \in G$, and Q is a proper subgroup of P. Then :

Lemma 3.1. *The submodule I is a nilpotent two sided ideal of $r\mu_R(G)$, and the quotient $r\mu_R(G)/I$ is isomorphic to the direct product over the p-subgroups P of G up to G-conjugation, of the algebras $\mathrm{Ind}_{N_G(P)/P}^G RN_G(P)/P$.*

(about the notion of induced algebra, see [**3**])

Indeed, if Q is a proper subgroup of P, and if $A \subseteq B$ are p-subgroups of G, and if $c \in G$, then $cr_A^B xr_Q^P$ is equal to zero if $B^x \neq Q$, and to $cxr_{A^x}^P$ otherwise, and then A^x is a proper subgroup of P. Similarly, the product $xr_Q^P cr_A^B$ is equal to zero if $P^c \neq A$, and to $xcr_{Q^c}^B$ otherwise, and then Q^c is a proper subgroup of B.

It is clear moreover that I is nilpotent : indeed, if $x_1r_{Q_1}^{P_1}x_2r_{Q_2}^{P_2}\ldots x_nr_{Q_n}^{P_n} \neq 0$, then $P_1^{x_2} = Q_2$, $P_2^{x_3} = Q_3$, etc, and the sequence $P_1^{x_2\ldots x_n}, P_2^{x_3\ldots x_n}, \ldots, P_n$ is strictly decreasing. Its length is hence bounded by the p-valuation of the order of G.

Then if P is a p-subgroup of G, let $e_P = \sum_{Q=_GP} r_Q^Q$. The elements e_P are idempotents of $r\mu_R(G)$. Moreover e_P and e_R are orthogonal if P and R are not conjugate in G. Finally $\sum_{P mod. G} e_P = \sum_P r_P^P$ is the identity element of $r\mu_R(G)$.

On the other hand, if the product $r_P^P cr_A^B r_P^P$ is non zero, then $P^c = A \subseteq B = P$, hence the algebra $r_P^P r\mu_R(G) r_P^P$ has a basis consisting of the elements xr_P^P, for $x \in N_G(P)/P$ (which form a subset of the basis of $r\mu_R(G)$ of Lemma 2.7). Moreover, the map $x \in N_G(P)/P \mapsto xr_P^P$ defines a morphism of algebras α of $RN_G(P)/P$ in $r_P^P r\mu_R(G) r_P^P$, which is hence an isomorphism. It is then clear that the map from $\mathrm{Ind}_{N_G(P)/P}^G RN_G(P)/P$ to $e_P r\mu_R(G) e_P$, sending $g \otimes u \otimes h$ to $g\alpha(u)h$ is a isomorphism, and this completes the proof of Lemma 3.1.

But if H is a subgroup of G, and A is an interior H-algebra, the inclusion of A in $\mathrm{Ind}_H^G A$ is a Morita equivalence : this follows e.g. from Lemma 2.4. This remark, together with Lemma 3.1, yields a classification of the simple $r\mu_R(G)$-modules.

Indeed, let P be a p-subgroup of G and V be an $RN_G(P)/P$-module. I define an $r\mu_R(G)$-module $N_{P,V}$ by $N_{P,V} = \mathrm{Ind}_{N_G(P)/P}^G V$. The element cr_A^B acts by 0 on $N_{P,V}$ if A is a proper subgroup of B, or if B is not conjugate to P. And if $A = B = P^x$, then $cr_A^B.(y \otimes v) = 0$ if $xy \notin N_G(P)$, and $cr_A^B.(y \otimes v) = cx^{-1} \otimes xyv$ otherwise (note that this element does not depend on the choice of the element x such that $B = P^x$).

This defines a genuine $r\mu_R(G)$-module : I must essentially check that Q acts trivially on $N_{P,V}(Q)$ and that

$$cr_{P^x}^{P^x}(dr_{P^y}^{P^y}.(z \otimes v)) = (cr_{P^x}^{P^x}dr_{P^y}^{P^y}).(z \otimes v)$$

The left hand side is equal to 0 if $yz \notin N_G(P)$, and to $cr_{P^x}^{P^x}.(dy^{-1} \otimes yzv)$ otherwise. This is equal to zero if $xdy^{-1} \notin N_G(P)$, and to $cx^{-1} \otimes xdzv = cx^{-1} \otimes xdy^{-1}yzv = cdy^{-1} \otimes yzv$ otherwise. But the right hand side is equal to zero if $P^{xd} \neq P^y$, i.e. if $xdy^{-1} \notin N_G(P)$, and to $cdr_{P^y}^{P^y}.(z \otimes v)$ otherwise. This is equal to zero if $yz \notin N_G(P)$, and to $cdy^{-1} \otimes yzv$ otherwise. Hence the two sides are equal.

The definition of $N_{P,V}$ implies that $N_{P,V}(Q) = r_Q^Q N_{P,V}$ is equal to zero if Q is not conjugate to P in G. And if $Q = P^x$, then $N_{P,V}(Q)$ identifies with $x^{-1} \otimes V$. Then if $q \in Q$, and if $x^{-1} \otimes v \in N_{P,V}(Q)$, I have $q.(x^{-1} \otimes v) = qx^{-1} \otimes v = x^{-1}.{}^xq \otimes v = x^{-1} \otimes^x qv = x^{-1} \otimes v$ since ${}^xQ = P$ and since P acts trivially on V. Thus $N_{P,V}$ is a genuine $r\mu_R(G)$-module.

The previous remarks, together with Lemma 3.1, yield the following

Proposition 3.2. *The modules $N_{P,V}$, when P runs through a set of representatives of conjugacy classes of p-subgroups of G, and V through a set of representatives of isomorphism classes of simple $RN_G(P)/P$-modules, form a full set of representatives of isomorphism classes of simple $r\mu_R(G)$-modules.*

In order to describe the indecomposable projective $r\mu_R(G)$-modules, I need a notation : if A and B are subgroups of G, I denote by $T_G(A,B)$ the set of $x \in G$ such that $A^x \subseteq B$. Let then P be a p-subgroup of G, and V be a $RN_G(P)/P$-module. If Q is a p-subgroup of G, I define an R-submodule of $\mathrm{Ind}_{N_G(P)/P}^G V$, denoted by $L_{P,V}(Q)$, by

$$L_{P,V}(Q) = \oplus_{x \in T_G(Q,P)/N_G(P)} x \otimes V$$

and I set $L_{P,V} = \oplus_Q L_{P,V}(Q)$.

If $Q \supseteq R$, then the inclusion of $T_G(Q,P)/N_G(P)$ in $T_G(R,P)/N_G(P)$ induces a morphism r_R^Q from $L_{P,V}(Q)$ to $L_{P,V}(R)$. And if $x \in G$, the natural bijection $yN_G(P) \mapsto xyN_G(P)$ from $T_G(Q,P)/N_G(P)$ to $T_G({}^xQ,P)/N_G(P)$ induces a morphism, that I still denote by x, from $L_{P,V}(Q)$ to $L_{P,V}({}^xQ)$. Then

Proposition 3.3. *These definitions turn $L_{P,V}$ into an $r\mu_R(G)$-module, and the functor from $RN_G(P)/P$-Mod to $r\mu_R(G)$-Mod sending V to $L_{P,V}$ is left adjoint to the restriction functor induced by the inclusion $u \mapsto ur_P^P$ from $RN_G(P)/P$ to $r\mu_R(G)$.*

The action of cr_A^B on the element $x \otimes v$ of $L_{P,V}(Q)$ is defined by

$$cr_A^B.x \otimes v = \delta_{B,Q} cx \otimes v$$

where, when $B = Q$, the element $cx \otimes v$ is seen in $L_{P,V}({}^cA)$. In particular, if $q \in Q$, then $q.(x \otimes v) = qx \otimes v = xq^x \otimes v = x \otimes q^x v = x \otimes v$, since $Q^x \subseteq P$. Thus Q acts trivially on $L_{P,V}(Q)$.

Similarly,

$$dr_E^F.(cr_A^B.x \otimes v) = \delta_{F,{}^cA}\delta_{B,Q} dcx \otimes v$$

with $dcx \otimes v \in L_{P,V}({}^dE)$ if $B = Q$ and $F = {}^cA$.

Since $dr_E^F.cr_A^B = \delta_{F^c,A} dcr_{E^c}^B$, I have

$$(dr_E^F.cr_A^B).x \otimes v = \delta_{F^c,A}\delta_B^Q dcx \otimes v$$

with $dcx \otimes v \in L_{P,V}({}^{dc}(E^c))$ if $F^c = A$ and $B = Q$. Thus $L_{P,V}$ is indeed an $r\mu_R(G)$-module.

It is clear that the construction sending V to $L_{P,V}$ is functorial in V. If M is an $r\mu_R(G)$-module, a morphism α from $L_{P,V}$ to M is defined by specifying, for any p-subgroup Q of G, a morphism α_Q from $L_{P,V}(Q)$ to $M(Q)$, such that

$$\alpha_R r_R^Q = r_R^Q \alpha_Q \quad x\alpha_Q = \alpha_{^xQ} x$$

Since $L_{P,V}(P)$ identifies with V as an $RN_G(P)/P$-module, this yields a morphism $\beta = \alpha_P$ from V to $M(P)$. And since the element $x \otimes v$ of $L_{P,V}(Q)$ is equal to $r_Q^{^xP}(x \otimes v)$, I have

$$\alpha_Q(x \otimes v) = r_Q^{^xP} \alpha_{^xP}(x \otimes v) = r_Q^{^xP} x\alpha_P(1 \otimes v) = xr_{Q^x}^P \beta(v)$$

and β determines entirely α.

Conversely, if β is a morphism from V to $M(P)$, if Q is a p-subgroup of G, and if $x \otimes v \in L_{P,V}(Q)$, I set $\alpha_Q(x \otimes v) = xr_{Q^x}^P \beta(v)$, which makes sense since $Q^x \subseteq P$.

Then $r_R^Q \alpha_Q(x \otimes v) = r_R^Q xr_{Q^x}^P \beta(v) = r_{R^x}^P \beta(v) = \alpha_R r_R^Q(x \otimes v)$, hence $\alpha_R r_R^Q = r_R^Q \alpha_Q$.

On the other hand, $y\alpha_Q(x \otimes v) = yxr_{Q^x}^P \beta(v)$, whereas $\alpha_{^yQ}(y.(x \otimes v)) = \alpha_{^yQ}(yx \otimes v) = yxr_{(^yQ)^{yx}}^P \beta(v)$, which proves that α is a morphism of $r\mu_R(G)$-modules.

Finally it is clear that these correspondences between $\mathrm{Hom}_{r\mu_R(G)}(L_{P,V}, M)$ and $\mathrm{Hom}_{RN_G(P)/P}(V, M(P))$ are mutual inverse bijections. The proposition follows.

Corollary 3.4. *If E_V is a projective cover of the simple $RN_G(P)/P$-module V, then L_{P,E_V} is a projective cover of the simple $r\mu_R(G)$-module $N_{P,V}$.*

This follows indeed from the isomorphisms

$$\mathrm{Hom}_{r\mu_R(G)}(L_{P,E_V}, M) \cong \mathrm{Hom}_{RN_G(P)/P}(E_V, M(P))$$

and $N_{P,V}(P) \cong V$: if f is an essential morphism from E_V to V, then f defines a morphism $L(f)$ of L_{P,E_V} in $N_{P,V}$, which is also essential : if L_1 is a submodule of L_{P,E_V} which maps *onto* $N_{P,V}$ by $L(f)$, then $L_1(P)$ maps *onto* V by f, hence $L_1(P) = E_V$. And since $L_1(Q)$ is the sum for $x \in T_G(Q,P)$ of $xr_{Q^x}^P(L_1(P))$, it follows that $L_1(Q) = L_{P,V}(Q)$ for all Q, hence that $L_1 = L_{P,V}$.

Remark 3.5. The module $L_{P,E_V}(1)$ identifies with $X = \mathrm{Ind}_{N_G(P)/P}^G E_V$ as an RG-module. In the case where R is a field of characteristic p, it is easy to check that the module $L_{P,E_V}(Q)$ identifies with the module

$$X[Q] = X^Q / (\sum_{R \subset Q} Tr_R^Q(X^R))$$

The projective indecomposable $r\mu_R(G)$-modules are then in one to one correspondence with p-permutation RG-modules of the form $\mathrm{Ind}_{N_G(P)/P}^G E_V$, where P is a p-subgroup of G and E_V is an indecomposable projective $RN_G(P)/P$-module.

The restriction functor induced by the inclusion $u \mapsto ur_P^P$ from $RN_G(P)/P$ to $r\mu_R(G)$ also admits a right adjoint : if V is an $RN_G(P)/P$ module, and Q a p-subgroup of G, then Q acts on the right on $N_G(P)\backslash T_G(P,Q)$, hence on the left on $\oplus_{x \in N_G(P)\backslash T_G(P,Q)} x^{-1} \otimes V$, and I set

$$L_{P,V}^o(Q) = (\oplus_{x \in N_G(P)\backslash T_G(P,Q)} x^{-1} \otimes V)^Q$$

Then if $Q \subseteq R$, the inclusion of $N_G(P)\backslash T_G(P,Q)$ in $N_G(P)\backslash T_G(P,R)$ allows to define a morphism r_Q^R from $L_{P,V}^o(R)$ to $L_{P,V}^o(Q)$, by $r_Q^R(\oplus_x x^{-1} \otimes v_x) = \oplus_{P^x \subseteq Q} x^{-1} \otimes v_x$.

Similarly, the natural bijection $y \mapsto yx^{-1}$ from $N_G(P)\backslash T_G(P,Q)$ to $N_G(P)\backslash T_G(P,^x Q)$ allows to define a morphism denoted by x from $L^o_{P,V}(Q)$ to $L^o_{P,V}(^xQ)$, by setting $x. \oplus_y (y^{-1} \otimes v_y) = \oplus_y (yx^{-1})^{-1} \otimes v_y = \oplus_y xy^{-1} \otimes v_y$.

It is then clear that Q acts trivially on $L^o_{P,V}(Q)$. Moreover

$$\begin{aligned} dr^F_E(cr^B_A.(\oplus_x x^{-1} \otimes v_x)) &= \delta_{B,Q} dr^F_E(\oplus_{P^x \subseteq A} cx^{-1} \otimes v_x) \\ &= \delta_{B,Q}\delta_{F,^cA} \oplus_{P^x \subseteq A, P^{xc^{-1}} \subseteq E} dcx^{-1} \otimes v_x \end{aligned}$$

But if $F = {}^cA$, then $P^{xc^{-1}} \subseteq E$ implies that $P^x \subseteq E^c \subseteq F^c = A$, hence

$$dr^F_E(cr^B_A.(\oplus_x x^{-1} \otimes v_x)) = \delta_{B,Q}\delta_{F,^cA} \oplus_{P^{xc^{-1}} \subseteq E} dcx^{-1} \otimes v_x$$

On the other hand, since $dr^F_E cr^B_A = \delta_{F^c,A} dcr^B_{E^c}$, I also have

$$(dr^F_E cr^B_A) \oplus_x x^{-1} \otimes v_x = \delta_{B,Q}\delta_{F^c,A} \oplus_{P^x \subseteq E^c} dcx^{-1} \otimes v_x$$

and $L^o_{P,V} = \oplus_Q L^o_{P,V}(Q)$ is indeed an $r\mu_R(G)$-module.

This construction is clearly functorial in V. And if α is a morphism from an $r\mu_R(G)$-module M to $L^o_{P,V}$, then as $L^o_{P,V}(P) = 1 \otimes V$, this yields a morphism β from $M(P)$ to V, such that $\alpha_P = 1 \otimes \beta$. If Q is a p-subgroup of G, if $m \in M(Q)$, and if $\alpha_Q(m) = \oplus_{x \in N_G(P)\backslash T_G(P,Q)} x^{-1} \otimes v_x$, let $x \in N_G(P)\backslash T_G(P,Q)$. Since $r^Q_{P^x}\alpha_Q(m) = x^{-1} \otimes v_x$, I must hence have $x^{-1} \otimes v_x = r^Q_{P^x}\alpha_Q(m) = \alpha_{P^x} r^Q_{P^x}(m) = x^{-1}\alpha_P x r^Q_{P^x}(m)$, hence $v_x = \beta(xr^Q_{P^x}(m))$, and β determine entirely α.

Conversely, if β is given, I define α_Q from $M(Q)$ to $L^o_{P,V}(Q)$ by

$$\alpha_Q(m) = \oplus_{x \in N_G(P)\backslash T_G(P,Q)} x^{-1} \otimes \beta(xr^Q_{P^x}(m))$$

It is easy to see that α is a morphism from M to $L^o_{P,V}$. These two constructions are clearly inverse to each other. Thus

Proposition 3.6. *The functor which maps V to $L^o_{P,V}$ is right adjoint to the restriction functor from $r\mu_R(G)$-Mod to $RN_G(P)/P$-Mod.*

This allows of course for a description of injective $r\mu_R(G)$-modules.

4. Resolutions

4.1. Projectivity relative to a functor. [3] The following definition generalizes the notion of relative projectivity, when $\mathcal{R}$ is a functor of restriction to a subgroup.

Definition 4.1. Let C and D be categories, and $\mathcal{R}$ a functor from C to D. I will say that an object M of C is projective relative to $\mathcal{R}$ (or $\mathcal{R}$-projective) if for any diagram

$$\begin{array}{ccc} & & M \\ & & \downarrow \beta \\ & \alpha & \\ X & \rightarrow & Y \end{array}$$

such that the morphism $\mathcal{R}(\alpha)$ is a split epimorphism, there exists a morphism ϕ from M to X such that $\alpha\phi = \beta$.

[3] The content of this section is very close to the notion of *comonad* or *cotriple* in category theory. See e.g. C.A. Weibel. *An introduction to homological algebra*, Cambridge studies in advanced mathematics 38, Chapter 8.6

The following lemma is a straightforward consequence of the definitions :

Lemma 4.2. *a) If M is projective relative to $\mathcal{R}$, and if $f : M \to N$ is a split epimorphism, then N is projective relative to $\mathcal{R}$.*
b) If M_1 and M_2 are projective relative to $\mathcal{R}$, and if N is a coproduct of M_1 and M_2, then N is projective relative to $\mathcal{R}$

The following lemma is a formalization of known results in the classical cases of relative projectivity, when $\mathcal{R}$ has a left adjoint :

Lemma 4.3. *Let C and D be categories, and $\mathcal{R}$ be a functor of C in D, admitting a left adjoint $\mathcal{I}$. Let M be an object of C. The following conditions are equivalent :*
1. *The object M is projective relative to $\mathcal{R}$.*
2. *The counit morphism $\mathcal{I}\mathcal{R}(M) \to M$ is a split epimorphism.*
3. *There exists an object Y of D and a split epimorphism $\mathcal{I}(Y) \to M$.*

If X is an object of C, if Y is an object of D, and α (resp. β) is a morphism from Y to $\mathcal{R}(X)$ (resp. a morphism from $\mathcal{I}(Y)$ to X), I denote by α^* (resp. β_*) the morphism from $\mathcal{I}(Y)$ to X (resp. from Y to $\mathcal{R}(X)$) associated by adjunction. Then $(\alpha^*)_* = \alpha$ and $(\beta_*)^* = \beta$.

Since the isomorphism from $\mathrm{Hom}_C(\mathcal{I}(-), -)$ to $\mathrm{Hom}_D(-, \mathcal{R}(-))$ is an isomorphism of bifunctors, if $f \in \mathrm{Hom}_D(Y_1, Y_2)$, and if $\alpha \in \mathrm{Hom}_D(Y_2, \mathcal{R}(X))$, I have $(\alpha f)^* = \alpha^* \mathcal{I}(f)$. Similarly, if $g \in \mathrm{Hom}_C(X_1, X_2)$, and if $\beta \in \mathrm{Hom}_C(\mathcal{I}(Y), X_1)$, then $(g\beta)_* = \mathcal{R}(g)\beta_*$.

Then if 1) holds, the diagram

$$\begin{tikzcd} & M \arrow[d, "Id_M"] \\ \mathcal{I}\mathcal{R}(M) \arrow[r, "(Id_{\mathcal{R}(M)})^*"] & M \end{tikzcd}$$

shows that 2) holds, if I know that the morphism $u = \mathcal{R}((Id_{\mathcal{R}(M)})^*)$ is a split epimorphism. But if I set $v = (Id_{\mathcal{I}\mathcal{R}(M)})_*$, I have

$$uv = ((Id_{\mathcal{R}(M)})^* Id_{\mathcal{R}(M)})_* = ((Id_{\mathcal{R}(M)})^*)_* = Id_{\mathcal{R}(M)}$$

It is clear that 2) implies 3).
Conversely, if $u : \mathcal{I}(Y) \to M$ is a split epimorphism, then there exists $v : M \to \mathcal{I}(Y)$ such that $uv = Id_M$. On the other hand, the morphism $\mathcal{I}(u_*)$ is a morphism from $\mathcal{I}(Y)$ to $\mathcal{I}\mathcal{R}(M)$ and

$$(Id_{\mathcal{R}(M)})^* \mathcal{I}(u_*) = (Id_{\mathcal{R}(M)} u_*)^* = u$$

hence

$$(Id_{\mathcal{R}(M)})^* \mathcal{I}(u_*) v = uv = Id_M$$

which prove 2).
Finally if 2) holds, let $\tau : M \to \mathcal{I}\mathcal{R}(M)$ be such that $(Id_{\mathcal{R}(M)})^* \tau = Id_M$. If I have a diagram

$$\begin{tikzcd} & M \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"] & Y \end{tikzcd}$$

such that $\mathcal{R}(\alpha)$ is a split epimorphism, then there exists a morphism

$$\gamma : \mathcal{R}(Y) \to \mathcal{R}(X)$$

such that $\mathcal{R}(\alpha)\gamma = Id_{\mathcal{R}(Y)}$. Setting then

$$\phi = (Id_{\mathcal{R}(X)})^*\mathcal{I}(\gamma)\mathcal{I}\mathcal{R}(\beta)\tau$$

it follows that

$$\begin{aligned}\alpha\phi &= \alpha(Id_{\mathcal{R}(X)})^*\mathcal{I}(\gamma\mathcal{R}(\beta))\tau \\ &= (\mathcal{R}(\alpha)Id_{\mathcal{R}(X)})^*\mathcal{I}(\gamma\mathcal{R}(\beta))\tau \\ &= (\mathcal{R}(\alpha)\gamma\mathcal{R}(\beta))^*\tau = (\mathcal{R}(\beta))^*\tau \\ &= (\mathcal{R}(\beta)Id_{\mathcal{R}(M)})^*\tau = \beta(Id_{\mathcal{R}(M)})^*\tau = \beta\end{aligned}$$

which proves 1), and the equivalence of the three conditions.

Lemma 4.4. *Let C and D be categories, and $\mathcal{R}$ be a functor from C to D, admitting a left adjoint $\mathcal{I}$. Then $\mathcal{R}$ is faithful if and only if for any object M of C, the counit morphism from $\mathcal{I}\mathcal{R}(M)$ to M is an epimorphism.*

Indeed, the morphism from $\mathcal{I}\mathcal{R}(M)$ to M is an epimorphism if and only if for any object X, the morphism

$$\mathrm{Hom}(M,X) \to \mathrm{Hom}(\mathcal{I}\mathcal{R}(M),X)$$

is injective, i.e. by adjunction if and only if the morphism

$$\mathrm{Hom}(M,X) \to \mathrm{Hom}(\mathcal{R}(M),\mathcal{R}(X))$$

is injective.

Remark 4.5. It follows that if $\mathcal{R}$ is faithful, then any projective object is projective relative to $\mathcal{R}$.

I suppose now that C and D are abelian categories, that $\mathcal{R}$ is faithful and admits a left adjoint $\mathcal{I}$. I denote by $K(M)$ the kernel of the epimorphism from $\mathcal{I}\mathcal{R}(M)$ to M.

Then $K(M)$ is the quotient of $\mathcal{I}\mathcal{R}K(M)$ by $K(K(M)) = K^2(M)$, which itself is the quotient of $\mathcal{I}\mathcal{R}K^2(M)$ by $K(K^2(M)) = K^3(M)$, and I can build that way a resolution of M by objects of C of the form $\mathcal{I}(L)$, which are projective relative to $\mathcal{R}$. Actually :

Lemma 4.6. *Let C and D be abelian categories, and $\mathcal{R}$ be a faithful functor from C to D admitting a left adjoint $\mathcal{I}$. Then for any object M of C, there exists a resolution*

$$\ldots \to L_i \to L_{i-1} \to \ldots \to L_0 \to M \to 0$$

where the L_i's are projective relative to $\mathcal{R}$, such that the complex

$$\ldots \to \mathcal{R}(L_i) \to \mathcal{R}(L_{i-1}) \to \ldots \to \mathcal{R}(L_0) \to \mathcal{R}(M) \to 0$$

is exact and split, and such a resolution of M is unique to homotopy.

The existence of such a resolution follows from the previous argument, and from the fact that the morphism $\mathcal{R}\mathcal{I}\mathcal{R}(M) \to \mathcal{R}(M)$ is a split epimorphism (see the proof of Lemma 4.3).

A standard homological argument shows that if M and N are objects of C, if X_* is a resolution of M by $\mathcal{R}$-projective objects, and Y_* a resolution of N such that the complex $\mathcal{R}(Y_*) \to \mathcal{R}(N)$ is split, any homomorphism of M in N can be lifted to an homomorphism from X_* to Y_*, and that such a lift is unique up to homotopy. The uniqueness of the resolution follows.

REMARK 4.7. If M is projective, a resolution of M with the properties of Lemma 4.6 is split, because M is projective relative to $\mathcal{R}$.

4.2. Resolutions of Mackey functors. The functors $\mathcal{R}$ and $\mathcal{I}$ defined above between the categories $r\mu_R(G)$-Mod and $Mack_R(G,1)$ fulfill the assumptions of Lemma 4.6 (the functor $\mathcal{R}$ is faithful because it is the composition of a forgetful functor and an equivalence of categories).

Thus, any Mackey functor in $Mack_R(G,1)$ admits a resolution by $\mathcal{R}$-projective functors, whose image by $\mathcal{R}$ is split. This means that this resolution can be split by homomorphisms which commute with conjugations by G, and with restrictions (but dot not commute, in general, with transfers). Moreover such a resolution is unique up to homotopy. The additional fact here is that this resolution can be chosen to be *finite*.

Indeed, let M a Mackey functor. Then $K(M)(H)$ is the image in the quotient $(\oplus_{P\in\underline{s}_p(H)}M(P))_H$ of the set of sequences n_P such that $\sum_P t_P^H n_P = 0$. In particular, if H is a p-subgroup Q of G, this condition is equivalent to $n_Q = -\sum_{P\subset Q} t_P^Q n_P$. It follows that $\mathcal{R}K(M)(Q)$ identifies with

$$\mathcal{R}(M)_1(Q) = (\oplus_{P\subset Q}M(P))_Q$$

More generally, let M a $r\mu_R(G)$-module : I will set $M_0 = M$, and if i is a positive integer, I set

$$M_i(Q) = (\oplus_{P_0\subset P_1\ldots\subset P_{i-1}\subset Q}M(P_0))_Q$$

If $n \in M(P_0)$, I denote by $n_{P_0,P_1,\ldots,P_{i-1},Q}$ the corresponding element of $(\oplus_{P_0\subset P_1\ldots\subset P_{i-1}\subset Q}M(P_0))$, and $\pi_Q(n_{P_0,P_1,\ldots,P_{i-1},Q})$ its image in $M_i(Q)$. If $x \in G$, I set

$$x\pi_Q(n_{P_0,\ldots,P_{i-1},Q}) = \pi_{^xQ}((xn)_{^xP_0,\ldots,^xP_{i-1},^xQ})$$

and if S is a subgroup of Q, I set

$$r_S^Q\pi_Q(n_{P_0,\ldots,P_{i-1},Q}) = \sum_{\substack{x\in S\backslash Q/P_0\\ S\cap^x P_0\neq S\cap^x P_1\ldots\neq S\cap^x P_{i-1}\neq S}} \pi_S((xr^{P_0}_{S^x\cap P_0}n)_{S\cap^x P_0,\ldots,S\cap^x P_{i-1},S})$$

LEMMA 4.8. *The previous definitions turn M_i into an $r\mu_R(G)$-module.*

The only non-trivial point is the transitivity of restrictions. So let $T\subseteq R\subseteq Q$ be p-subgroups of G, and $s = (P_0,\ldots,P_{i-1},Q)$ be an increasing sequence of subgroups of Q. I will say that s is a proper sequence if s is strictly increasing. If A is a subgroup of G, I will denote by $A\cap s$ the sequence $(A\cap P_0,\ldots,A\cap P_{i-1},A\cap Q)$.

Then if $n \in M(P_0)$, I have

$$r_S^Q(\pi_Q(n_s)) = \sum_{\substack{x\in S\backslash Q/P_0\\ S\cap^x s\ proper}} \pi_S((xr^{P_0}_{S^x\cap P_0}n)_{S\cap^x s})$$

Thus

$$r_T^S r_S^Q(\pi_Q(n_s)) = \sum_{\substack{x\in S\backslash Q/P_0\\ S\cap^x s\ proper}} \sum_{\substack{y\in R\backslash S/S\cap^x P_0\\ R\cap^y(S\cap^x s)\ proper}} \pi_R((yr^{S\cap^x P_0}_{R^y\cap S\cap^x P_0}xr^{P_0}_{S^x\cap P_0}n)_{R\cap^y(S\cap^x s)})$$

i.e.

$$r_T^S r_S^Q(\pi_Q(n_s)) = \sum_{\substack{x\in S\backslash Q/P_0\\ S\cap^x s\ proper}} \sum_{\substack{y\in R\backslash S/S\cap^x P_0\\ R\cap^{yx} s\ proper}} \pi_R((yxr^{P_0}_{R^{yx}\cap P_0}n)_{R\cap^{yx}s})$$

Then the element $z = yx$ runs through a set of representatives of doubles cosets $R\backslash Q/P_0$ such that $R \cap {}^z s$ is proper. Thus

$$r_T^S r_S^Q(\pi_Q(n_s)) = \sum_{\substack{z\in R\backslash Q/P_0\\ R\cap^z s\ proper}} \pi_R((zr^{P_0}_{R^z\cap P_0}n)_{R\cap^z s}) = r_T^Q(\pi_Q(n_s))$$

which proves the lemma.

The previous construction associates to any $r\mu_R(G)$-module M some $r\mu_R(G)$-modules M_i, and moreover $M_i = 0$ if i is greater than (or equal to) the p-valuation of the order of G. If M is a Mackey functor in $Mack_R(G, 1)$, and i a non negative integer, I will denote by $\delta_i(M)$ the functor $\mathcal{I}(\mathcal{R}(M)_i)$, and I will build a resolution of M from these functors. The functor $\delta_i(M)$ admits the following simple description :

Lemma 4.9. *Let M in $Mack_R(G, 1)$, and H a subgroup of G. Then*

$$\delta_i(M)(H) = (\oplus_{P_0\subset\ldots\subset P_i\in\underline{s}_p(H)} M(P_0))_H$$

If $s = (P_0, \ldots, P_i)$ is a sequence of p-subgroups of H, and if $n \in M(P_0)$, I denote by n_s the corresponding element of $(\oplus_{P_0\subset\ldots\subset P_i\in\underline{s}_p(H)} M(P_0))$ if the sequence s is strictly increasing, and I set $n_s = 0$ otherwise. I denote by $\pi_H(n_s)$ the image of n_s in $(\oplus_{P_0\subset\ldots\subset P_i\in\underline{s}_p(H)} M(P))_H$. I can then define a linear map f from $\delta_i(M)(H)$ to $(\oplus_{P_0\subset\ldots\subset P_i\in\underline{s}_p(H)} M(P))_H$ by

$$f(\pi_H\pi_Q(n_s)) = \pi_H(n_{s\cup Q})$$

which makes sense since if $h \in H$ and $q \in Q$, then

$$f(\pi_H(h\pi_Q(q.n_s))) = f(\pi_H\pi_{^hQ}((hqn)_{^{hq}s})) = \pi_H((hqn)_{^{hq}s\cup^h Q}))$$

$$= \pi_H(h.(qn)_{^q s\cup Q}) = \pi_H(q.n_{s\cup Q}) = \pi_H(n_{s\cup Q})$$

Conversely, I define a linear map g from $(\oplus_{P_0\subset\ldots\subset P_i\in\underline{s}_p(H)} M(P))_H$ to $\delta_i(M)(H)$ by

$$g(\pi_H(n_s)) = \pi_H\pi_{sups}(n_{s-sups})$$

which makes sense because if $h \in H$, then

$$g(\pi_H(h.n_s)) = g(\pi_H((hn)_{^h s})) = \pi_H\pi_{^h sups}((hn)_{^h s-^h sups}))$$

$$= \pi_H(h.\pi_{sups}(n_{s-sups})) = \pi_H\pi_{sups}(n_{s-sups})$$

It is then clear that f and g are mutual inverse isomorphisms, which proves the lemma.

If K is a subgroup of H, if $s = (P_0, \ldots, P_i)$ is a sequence of p-subgroups of K, and if $n \in M(P_0)$, it follows from the definitions that

$$t_K^H(\pi_K(n_s)) = \pi_H(n_s)$$

Similarly, if $s = (P_0, \ldots, P_i)$ is a sequence of p-subgroups of H, and if $n \in M(P_0)$, then

$$r_K^H(\pi_H(n_s)) = \sum_{\substack{x\in K\backslash H/P_0\\ K\cap^x s\ proper}} \pi_K((xr^{P_0}_{K^x\cap P_0}n)_{K\cap^x s})$$

Hence I can identify $\delta_0(M)$ with $\mathcal{IR}(M)$. I will denote by d_0 the canonical morphism from $\delta_0(M)$ to M. If $i > 0$, I define a linear map d_i from $\delta_i(M)$ to $\delta_{i-1}(M)$ by

$$d_{i,H}(\pi_H(n_s)) = \pi_H((t_{P_0}^{P_1}n)_{s_0} + \sum_{j=1}^{i}(-1)^j n_{s_j})$$

where s_j denotes the sequence $s - \{P_j\}$. Then

THEOREM 4.10. *Let M in $Mack_R(G,1)$. The sequence*

$$0 \to \ldots \stackrel{d_{i+1}}{\to} \delta_i(M) \stackrel{d_i}{\to} \ldots \stackrel{d_1}{\to} \delta_0(M) \stackrel{d_0}{\to} M \to 0$$

is an exact complex of Mackey functors in $Mack_R(G,1)$, and the complex

$$0 \to \ldots \stackrel{\mathcal{R}(d_{i+1})}{\to} \mathcal{R}(\delta_i(M)) \stackrel{\mathcal{R}(d_i)}{\to} \ldots \stackrel{\mathcal{R}(d_1)}{\to} \mathcal{R}(\delta_0(M)) \stackrel{\mathcal{R}(d_0)}{\to} \mathcal{R}(M) \to 0$$

is exact and split.

First I must that the maps d_i are morphisms of Mackey functors. This is clear for d_0. If $i > 0$, if $H \subseteq K$ are subgroups of G, if $s = (P_0, \ldots, P_i)$ is an increasing sequence of p-subgroups of H, and if $n \in M(P_0)$, then

$$d_{i,K}(t_H^K \pi_H(n_s)) = d_{i,K}(\pi_K(n_s)) = \pi_K((t_{P_0}^{P_1}n)_{s_0} + \sum_{j=1}^{i}(-1)^j n_{s_j})$$

whereas

$$t_H^K d_{i,H}(\pi_H(N_s)) = t_H^K(\pi_H((t_{P_0}^{P_1}n)_{s_0} + \sum_{j=1}^{i}(-1)^j n_{s_j}) = \pi_K((t_{P_0}^{P_1}n)_{s_0} + \sum_{j=1}^{i}(-1)^j n_{s_j})$$

and the maps d_i commute with traces.

Similarly, if $s = (P_0, \ldots, P_i)$ is an increasing sequence of p-subgroups of K, and if $n \in M(P_0)$, then

$$d_{i,H}(r_H^K \pi_K(n_s)) = d_{i,H}(\sum_{x \in H\backslash K/P_0} \pi_H((xr_{H^x \cap P_0}^{P_0}n)_{H \cap {}^x s}))$$

$$= \sum_{x \in H\backslash K/P_0} \pi_H((t_{H \cap {}^x P_0}^{H \cap {}^x P_1} xr_{H^x \cap P_0}^{P_0}n)_{(H \cap {}^x s)_0} + \sum_{j=1}^{i}(-1)^j (xr_{H^x \cap P_0}^{P_0})_{(H \cap {}^x s)_j}) \qquad (1)$$

On the other hand

$$r_H^K d_{i,K}(\pi_K(n_s)) = r_H^K \pi_K((t_{P_0}^{P_1}n)_{s_0} + \sum_{j=1}^{i}(-1)^j n_{s_j})$$

But if $j > 0$, then the smallest element of s_j is P_0, and

$$r_H^K(\pi_K(n_{s_j})) = \sum_{x \in H\backslash K/P_0} \pi_H((xr_{H^x \cap P_0}^{P_0}n)_{H \cap {}^x s_j})$$

And these terms are equal to the corresponding terms of Equality (1), since for any m of $M(P_0)$, I have $m_{(H \cap {}^x s)_j} = m_{H \cap {}^x s_j}$. Similarly

$$r_H^K \pi_K((t_{P_0}^{P_1}n)_{s_0}) = \sum_{y \in H\backslash K/P_1} \pi_H(yr_{H^y \cap P_1}^{P_1}(t_{P_0}^{P_1}n)_{H \cap {}^y s_0})$$

$$= \pi_H \sum_{y\in H\backslash K/P_1} (y \sum_{z\in H^y\cap P_1\backslash P_1/P_0} t_{H^y\cap {}^zP_0}^{H^y\cap P_1} r_{H^y\cap {}^zP_0}^{{}^zP_0} zn)_{H\cap {}^ys_0})$$

$$= \sum_{y\in H\backslash K/P_1} \sum_{z\in H^y\cap P_1\backslash P_1/P_0} \pi_H(t_{H\cap {}^{yz}P_0}^{H\cap {}^yP_1} r_{H\cap {}^{yz}P_0}^{{}^{yz}P_0} yzn)_{H\cap {}^ys_0})$$

and as $z \in P_1 = \inf s_0$, I have ${}^yP_1 = {}^{yz}P_1$ and $H \cap {}^ys_0 = H \cap {}^{yz}P_0$, and I can sum over $x = yz$, which runs through the set $H\backslash K/P_0$. It follows that

$$r_H^K \pi_K((t_{P_0}^{P_1} n)_{s_0}) = \sum_{x\in H\backslash K/P_0} \pi_H(t_{H\cap {}^xP_0}^{H\cap {}^xP_1} r_{H\cap {}^xP_0}^{{}^xP_0} xn)_{H\cap {}^xs_0})$$

$$= \sum_{x\in H\backslash K/P_0} \pi_H(t_{H\cap {}^xP_0}^{H\cap {}^xP_1} x r_{H^x\cap P_0}^{P_0} n)_{H\cap {}^xs_0})$$

which is equal to the corresponding term of Equality (1), since $(H\cap {}^xs)_0 = H\cap {}^xs_0$, which proves that the maps d_i commute with restrictions.

Finally it is clear that the d_i's commute with conjugations by elements of G, hence they are indeed morphisms of Mackey functors.

A standard computation shows that $d_id_{i+1} = 0$ for all i, and the sequence in the statement of Theorem 4.10 is indeed a complex of Mackey functors (in $Mack_R(G, 1)$ by definition of functors $\delta_i(M)$).

To check that this complex is exact, it suffices to check that its image by $\mathcal{R}$ is acyclic, since the functor $\mathcal{R}$ is a forgetful functor. I will show that this complex is split and acyclic, as claimed in Theorem 4.10.

Let Q be a p-subgroup of G. If $s = (P_0, \ldots, P_i)$ is a sequence of subgroups of Q, and if $n \in M(P_0)$, I set $\alpha_{i,Q}(\pi_Q(n_s)) = \pi_Q(n_{s\cup Q})$. Similarly, if $n \in M(Q)$, I set $\alpha_{-1,Q}(n) = \pi_Q(n_{\{Q\}})$. This yields for any i a map from $\delta_i(M)(Q)$ to $\delta_{i+1}(M)(Q)$. This map is actually a morphism from $\mathcal{R}(\delta_i(M))$ to $\mathcal{R}(\delta_{i+1}(M))$: indeed, if S is a subgroup of Q, then

$$\alpha_{i,S} r_S^Q \pi_Q(n_s) = \alpha_i \sum_{x\in S\backslash Q/P_0} \pi_S((x r_{S^x\cap P_0}^{P_0} n)_{S\cap {}^xs}) = \sum_{x\in S\backslash Q/P_0} \pi_S((x r_{S^x\cap P_0}^{P_0} n)_{(S\cap {}^xs)\cup S})$$

But

$$r_S^Q \alpha_{i,Q} \pi_Q(n_s) = r_S^Q \pi_Q(n_{s\cup Q}) = \sum_{\substack{x\in S\backslash Q/P_0 \\ S\cap {}^x(s\cup Q)\ proper}} \pi_S((x r_{S^x\cap P_0}^{P_0} n)_{S\cap {}^x(s\cup Q)})$$

and since for any m of $M(S)$, I have $m_{(S\cap {}^xs)\cup S} = m_{S\cap {}^x(s\cup Q)}$ if $S \cap {}^x(s \cup Q)$ is proper, and $m_{(S\cap {}^xs)\cup S} = 0$ otherwise, the maps α_i commute with restrictions. It is clear on the other hand that they commute with conjugations, hence that this are morphisms from $\mathcal{R}(\delta_i(M))$ to $\mathcal{R}(\delta_{i+1}(M))$.

Finally

$$d_{i+1,Q}\alpha_{i,Q}\pi_Q(n_s) = d_{i+1,Q}\pi_Q(n_{s\cup Q}) = \pi_Q((t_{P_0}^{P_1} n)_{(s\cup Q)_0} + \sum_{j=1}^{i+1} (-1)^j n_{(s\cup Q)_j})$$

On the other hand

$$\alpha_{i-1,Q} d_{i,Q} \pi_Q(n_s) = \alpha_{i-1,Q}\pi_Q((t_{P_0}^{P_1} n)_{s_0} + \sum_{j=1}^{i} (-1)^j n_{s_j})$$

$$= \pi_Q((t_{P_0}^{P_1} n)_{s_0 \cup Q} + \sum_{j=1}^{i} (-1)^j n_{s_j \cup Q})$$

By difference, it follows that

$$d_{i+1,Q}\alpha_{i,Q}\pi_Q(n_s) - \alpha_{i-1,Q}d_{i,Q}\pi_Q(n_s) = (-1)^{i+1}\pi_Q(n_{(s\cup Q)_{i+1}}) = (-1)^{i+1}\pi_Q(n_s)$$

hence $d_{i+1}\alpha_i - \alpha_{i-1}d_i = (-1)^{i+1}Id$, and a suitable change of sign on the α_i's proves the last assertion of the theorem.

4.3. Dual results. I have showed above that the algebra $t\mu_R(G)$ identifies with the opposite algebra of $r\mu_R(G)$. The previous results have a translation in terms of this algebra, obtained by replacing restrictions by traces, traces by restrictions, elements of G by their inverses, coinvariants by invariants, and reversing the arrows.

I will denote by $\mathcal{T}$ the restriction functor associated to the inclusion from $t\mu_R(G)$ into $\mu_R(G)$. The functor $\mathcal{T}$ admits a right adjoint, that I denote by $\mathcal{J}$, defined par

$$\mathcal{J}(N)(H) = (\oplus_{P\in \underline{s}_p(H)} N(P))^H$$

when N is a $t\mu_R(G)$-module and H a subgroup of G.

If K is a subgroup of H, and if $\oplus_P n_P \in \mathcal{J}(N)(H)$, then $r_K^H(\oplus_P n_P) = \oplus_Q m_Q$, with $m_Q = n_Q$ if Q is a subgroup of K, and $m_Q = 0$ otherwise.

Similarly, if $\oplus_Q m_Q \in \mathcal{J}(N)(K)$, then $t_K^H(\oplus_Q m_Q) = \oplus_P n_P$, with

$$n_P = \sum_{x\in P\backslash H/K} t_{P\cap {}^xK}^P x n_{P^x\cap K}$$

If M is a Mackey functor in $Mack_R(G,1)$, I can define similarly the functors $\partial^i(M)$ by the formulas

$$\partial^i(M)(H) = (\oplus_{P_0\subset\ldots\subset P_i\in \underline{s}_p(H)} M(P_0))^H$$

With this notation $r_K^H(\oplus_s n_s) = \oplus_t m_t$, with $m_t = n_t$ if $\sup\, t \subseteq K$, and $m_t = 0$ otherwise. Similarly, $t_K^H(\oplus_t m_t) = \oplus_s n_s$, with

$$n_s = \sum_{x\in P_0\backslash H/K} t_{P_0\cap {}^xK}^{P_0} x m_{s^x\cap K} \quad \text{where} \quad P_0 = \inf s$$

The differential $\hat{d}^i$ from $\partial^i(M)$ to $\partial^{i+1}(M)$ is given by $\hat{d}^i(\oplus_s n_s) = \oplus_t m_t$ where

$$m_t = r_{P_0}^{P_1} n_{s_0} + \sum_{j=1}^{i} (-1)^j n_{s_j}$$

The dual result to Theorem 4.10 is then the following :

Theorem 4.11. *Let M in $Mack_R(G,1)$. The sequence*

$$0 \to M \overset{\hat{d}^{-1}}{\to} \partial^0(M) \overset{\hat{d}^0}{\to} \ldots \overset{\hat{d}^{i-1}}{\to} \partial^i(M) \overset{\hat{d}^i}{\to} \ldots \to 0$$

is an exact complex of Mackey functors in $Mack_R(G,1)$, and the complex

$$0 \to \mathcal{T}(M) \overset{\mathcal{T}(\hat{d}^{-1})}{\to} \mathcal{T}(\partial^0(M)) \overset{\mathcal{T}(\hat{d}^0)}{\to} \ldots \overset{\mathcal{T}(\hat{d}^{i-1})}{\to} \mathcal{T}(\partial^i(M)) \overset{\mathcal{T}(\hat{d}^i)}{\to} \ldots \to 0$$

is exact and split.

5. Applications

5.1. Steinberg Modules. Theorem 4.11 allows to extend to any Mackey functor in $Mack_R(G,1)$ a result of P.Webb expressing homology in terms of Steinberg modules (cf [**7**]) :

Proposition 5.1. *Let M be a Mackey functor in $Mack_R(G,1)$, and H be a subgroup of G. Then*

$$M(H) = -\sum_{P\in\underline{s}_p(H)/N_G(H)} \mathrm{Ind}_{N_G(H,P)/N_H(P)}^{N_G(H)/H}\mathrm{Hom}_{RN_H(P)/P}(St_p(N_H(P)/P), M(P))$$

in the Green ring of $RN_G(H)/H$-modules.

Remark 5.2. Webb's Theorem is the case where $H = G$ and $M(K) = \hat{H}^n(K, V)_p$, where V denotes a $\mathbf{Z}_pG$-module and $n \in \mathbf{Z}$: in this case, the term corresponding to $P = 1$ in the above sum is equal to zero. On the other hand, the fact that $\hat{H}^n(-, V)_p$ lies in $Mack_R(G,1)$ is proved in [**6**] (section [16]).

Recall that the Steinberg module over R of the group G at the prime number p is defined by

$$St_p(G) = -R - \sum_s (-1)^{|s|}\mathrm{Ind}_{N_G(s)}^G R$$

where the summation runs over a set of representatives of G-conjugacy classes of strictly increasing sequences s of non trivial p-subgroups of G, and $|s|$ denotes the cardinality of s. This sum is an element of the Green ring of finitely generated RG-modules. On the other hand, if A, B and C are RG-modules, then $\mathrm{Hom}_{RG}(A - B, C)$ is equal by definition to $\mathrm{Hom}_{RG}(A, C) - \mathrm{Hom}_{RG}(B, C)$.

Proposition 5.1 follows from the fact that, by Theorem 4.11, if H is a subgroup of G, the sequence

$$0 \to M(H) \to \partial^0(M)(H) \to \ldots \to \partial^i(M)(H) \to \ldots \to 0$$

is a split exact sequence of $RN_G(H)/H$-modules. Denoting by $\Delta_i(H)$ the set of sequences $P_0 \subset \ldots \subset P_i$ of p-subgroups of H, I have

$$M(H) = \sum_{i\geq 0}(-1)^i\partial^i(M)(H) = \sum_{i\geq 0}(-1)^i(\oplus_{s\in\Delta_i(H)}M(\inf s))^H$$

But $\oplus_{s\in\Delta_i(H)}M(\inf s)$ is an $RN_G(H)$-module, isomorphic to

$$\oplus_{s\in\Delta_i(H)/N_G(H)}\mathrm{Ind}_{N_G(H,s)}^{N_G(H)}M(\inf s)$$

Moreover, if H is a normal subgroup of the group G, if K is a subgroup of G, and L is a RK-module, the module $(\mathrm{Ind}_K^G L)^H$ identifies to to $\mathrm{Ind}_{HK/H}^{G/H} L^{H\cap K}$ as an RG/H-module. This remark shows that

$$\partial^i(M)(H) = \oplus_{s\in\Delta_i(H)/N_G(H)}\mathrm{Ind}_{HN_G(H,s)/H}^{N_G(H)/H}M(\inf s)^{N_H(s)}$$

On the other hand, the module $St_p(N_H(P)/P)$ is a $RN_G(H,P)$-module, isomorphic to

$$-\sum_i(-1)^i \sum_{\substack{s\in\Delta_i(H)/N_G(H,P)\\ \inf s = P}} \mathrm{Ind}_{N_G(H,s)}^{N_G(H,P)} R$$

hence the module $\mathrm{Hom}_R(St_p(N_H(P)/P), M(P))$ is isomorphic to

$$-\sum_i (-1)^i \sum_{\substack{s\in\Delta_i(H)/N_G(H,P)\\ \inf s=P}} \mathrm{Ind}_{N_G(H,s)}^{N_G(H,P)} \mathrm{Hom}_R(R, M(P))$$

Another application of the previous remark, for the group $N_G(H,P)$ and its normal subgroup $N_H(P)$, gives then
$\mathrm{Hom}_{RN_H(P)}(St_p(N_H(P)/P), M(P)) =$

$$-\sum_i (-1)^i \sum_{\substack{s\in\Delta_i(H)/N_G(H,P)\\ \inf s=P}} \mathrm{Ind}_{N_G(H,s)/N_H(s)}^{N_G(H,P)/N_H(P)} M(P)^{N_H(s)}$$

and the right hand side of the equality of the proposition becomes

$$\sum_{P\in \underline{s}_p(H)/N_G(H)} \sum_i (-1)^i \sum_{\substack{s\in\Delta_i(H)/N_G(H,P)\\ \inf s=P}} \mathrm{Ind}_{N_G(H,s)/N_H(s)}^{N_G(H)/H} M(P)^{N_H(s)}$$

i.e.

$$\sum_i (-1)^i \sum_{s\in\Delta_i(H)/N_G(H)} \mathrm{Ind}_{N_G(H,s)/N_H(s)}^{N_G(H)/H} M(\inf s)^{N_H(s)}$$

and this is equal to $\sum_i (-1)^i \partial^i(M)(H)$, which proves the proposition.

5.2. $\mathcal{T}$-injective functors and $\mathcal{R}$-projective functors.

5.2.1. *Residues.* The definitions and results of Section 3 about the projectivity relative to $\mathcal{R}$ can be dualized by reversing the arrows and replacing "left" by "right", and "projective" by "injective" (in other words, I will say that an object M is injective relative to the functor $\mathcal{T}$ if it is projective relative to the dual functor of $\mathcal{T}$ for the opposite categories).

For the next statement, I will need the following notation :
Let M a Mackey functor for the group G, and K be a subgroup of G. I will denote by $\overline{M}(K)$ the quotient

$$\overline{M}(K) = M(K)/\sum_{L\subset K} t_L^K M(L)$$

and $\underline{M}(K)$ the intersection

$$\underline{M}(K) = \bigcap_{L\subset K} \mathrm{Ker}\ r_L^K$$

These definitions of "residues" of M at K are dual to each other : if N is a Mackey functor for the group $N_G(K)/K$, I define (with Thévenaz and Webb cf.[**5**]) a functor from the category of $R(N_G(K)/K)$-Mackey functors to the category of $\mu_R(G)$-modules mapping N to the functor

$$F(N) = \mathrm{Ind}_{N_G(K)}^G \mathrm{Inf}_{N_G(K)/K}^{N_G(K)} N$$

Then the functor F a a left adjoint, defined by

$$M^K(L/K) = M(L)/\sum_{K\not\subset H\subset L} t_H^L M(H)$$

and a right adjoint, defined by

$$M_K(L/K) = \bigcap_{K\not\subset H\subset L} \mathrm{Ker}\ r_H^L$$

Then $\overline{M}(K)$ is the value at K/K of M^K, and $\underline{M}(K)$ is the value at K/K of M_K.

Let then X be an $r\mu_R(G)$-module, and P be a p-subgroup of G. It is easy to see that $\overline{\mathcal{I}(X)}(P)$ identifies with $X(P)$. If FP_V denotes the $N_G(P)/P$-Mackey functor of fixed points on V, defined by $FP_V(H) = V^H$, $r_K^H v = v$, and $t_K^H(w) = Tr_K^H(v)$, this implies that

$$\mathrm{Hom}(\mathcal{I}(X), \mathrm{Ind}_{N_G(P)}^G \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} FP_V) = \mathrm{Hom}(\mathcal{I}(X)^P, FP_V) =$$
$$\mathrm{Hom}_{RN_G(P)/P}(X(P), V)$$

the second equality following from the fact that that the functor sending V to FP_V is right adjoint to the evaluation functor at P/P (from the category of $RN_G(P)/P$-Mackey functors to the category of $RN_G(P)/P$-modules)(cf.[**5**] Proposition 6.1).

Since moreover

$$\mathrm{Hom}(\mathcal{I}(X), \mathrm{Ind}_{N_G(P)}^G \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} FP_V) = \mathrm{Hom}(X, \mathcal{R}(\mathrm{Ind}_{N_G(P)}^G \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} FP_V))$$

it follows that the functor mapping V to $\mathcal{R}(\mathrm{Ind}_{N_G(P)}^G \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} FP_V)$ is right adjoint to the functor sending X to $X(P)$ (from the category of $r\mu_R(RN_G(P)/P)$-modules to the category of $RN_G(P)/P$-modules), hence that it identifies with the functor sending V to $L_{P,V}^o$ (cf. Proposition 3.6).

The following proposition shows how to compute the values of an $\mathcal{R}$-projective or a $\mathcal{T}$-injective Mackey functor from its residues :

Proposition 5.3. *Let M be a Mackey functor in $Mack_R(G,1)$.*

(1) *If M is $\mathcal{R}$-projective (e.g. if M is projective), then*
$$M(H) = (\oplus_{P\in \underline{s}_p(H)} \overline{M}(P))_H$$
as an $RN_G(H)/H$-module.

(2) *If M is $\mathcal{T}$-injective (e.g. if M is injective), then*
$$M(H) = (\oplus_{P\in \underline{s}_p(H)} \underline{M}(P))^H$$
as an $RN_G(H)/H$-module.

These two results are dual to each other, hence it suffices to prover the second one, which will be a consequence of the following stronger (but more obscure) result :

Proposition 5.4. *Let M be a Mackey functor in $Mack_R(G,1)$. If M is $\mathcal{T}$-injective, then*

$$\mathcal{R}(M) = \bigoplus_{P\in \underline{s}_p(G)/G} L_{P,\underline{M}(P)}^o = \mathcal{R}(\bigoplus_{P\in \underline{s}_p(G)/G} \mathrm{Ind}_{N_G(P)}^G \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} FP_{\underline{M}(P)})$$

Proposition 5.3 follows, since

$$(\bigoplus_{P\in \underline{s}_p(G)/G} \mathrm{Ind}_{N_G(P)}^G \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} FP_{\underline{M}(P)})(H) =$$
$$\bigoplus_{P\in \underline{s}_p(G)/G}\ \bigoplus_{x\in N_G(P)\backslash G/H} (\mathrm{Inf}_{N_G(P)/P}^{N_G(P)} FP_{\underline{M}(P)})(N_G(P)\cap {}^xH)$$
$$= \bigoplus_{P\in \underline{s}_p(G)/G}\ \bigoplus_{x\in N_G(P)\backslash T_G(P,H)/H} FP_{\underline{M}(P)}(N_{^xH}(P)/P)$$

and using rational coefficients for this computation, this sum is equal to

$$\bigoplus_{P\in\underline{s}_p(G)}\frac{|N_G(P)|}{|G|}\bigoplus_{x\in T_G(P,H)}\frac{|N_{{}^xH}(P)|}{|N_G(P)||H|}\underline{M}(P)^{N_{{}^xH}(P)/P}=$$

$$\bigoplus_{P\in\underline{s}_p(G),x\in G,P^x\subseteq H}\frac{|N_H(P^x)|}{|G||H|}\underline{M}(P^x)^{N_H(P^x)/P^x}$$

$$=\bigoplus_{P\in\underline{s}_p(H)}\frac{|N_H(P)|}{|H|}\underline{M}(P)^{N_H(P)}=(\oplus_{P\in\underline{s}_p(H)}\underline{M}(P))^H$$

Proposition 5.4 shows that the identification of part 2 of Proposition 5.3 commutes with restrictions.

In order to prove Proposition 5.4, I will first characterize those $r\mu_R(G)$-modules which are isomorphic to direct sums of modules of the form L^o_{P,V_P}. First I notice that I can define $\underline{M}(P)$ when M is an $r\mu_R(G)$-module and P is a p-subgroup of G, since this definition uses only restriction maps. Then :

Lemma 5.5. *Let M be an $r\mu_R(G)$-module. The following conditions are equivalent :*

1. *There exists $N_G(P)/P$-modules V_P such that M is isomorphic to*
$$\bigoplus_{P\in\underline{s}_p(G)/G}L^o_{P,V_P}$$
2. *The module M is isomorphic to*
$$\bigoplus_{P\in\underline{s}_p(G)/G}L^o_{P,\underline{M}(P)}$$
3. *For any p-subgroup P of G, the inclusion of $\underline{M}(P)$ in $M(P)$ is a split injection of $RN_G(P)/P$-modules, and the quotient $M(P)/\underline{M}(P)$ is isomorphic to $(\lim\limits_{\longleftarrow Q\subset P}M(Q))^P$.*

I recall that $L^o_{P,V}$ is defined by

$$L^o_{P,V}(Q)=(\oplus_{x\in N_G(P)\backslash T_G(P,Q)}x^{-1}\otimes V)^Q$$

and

$$r^R_Q(\oplus_x x^{-1}\otimes v_x)=\oplus_{P^x\subseteq Q}x^{-1}\otimes v_x$$

In particular $L^o_{P,V}(R)$ is zero if P is not contained in R up to conjugation, hence $\underline{L}^o_{P,V}(R)=0$ in this case.

Otherwise, if $v=\oplus_{x\in N_G(P)\backslash T_G(P,R)}x^{-1}\otimes v\in\underline{L}_{P,V}(R)$, and if R is not conjugate to P, let $y\in T_G(P,R)$. Since $r^R_{P^y}(v)=y^{-1}\otimes v_y=0$, it follows that $v=0$. This proves that $\underline{L}^o_{P,V}(R)$ is zero if R is not conjugate to P, and equal to V otherwise, hence Assertion 1) implies Assertion 2).

It is clear conversely that Assertion 2) implies Assertion 1). The previous argument also shows that Assertion 1) implies the first part of Assertion 3), because if $M=L^o_{P,V}$, then the injection of $\underline{M}(R)$ in $M(R)$ is zero if R is not conjugate to P, and the identity map otherwise.

To prove the second part, I notice that the kernel of the natural morphism from $M(R)$ to $(\lim\limits_{\longleftarrow\, Q\subset R} M(Q))^R$ is equal to $\underline{M}(R)$. Hence it suffices to show that this morphism is surjective.

So let $M = L^o_{P,V}$. There is nothing to prove if P is not contained in R up to conjugation. Similarly, if $R = P$, then $(\lim\limits_{\longleftarrow\, Q\subset R} M(Q))^R = 0$, and $\underline{M}(R) = M(R)$ in this case. Hence I can suppose that R contains P strictly up to conjugation. Then if $v \in (\lim\limits_{\longleftarrow\, Q\subset R} M(Q))^R$, for any proper subgroup Q of R, I have an element v_Q of $M(Q)$, such that if $S \subseteq Q$, then $r^Q_S(v_Q) = v_S$. In particular, if $x \in T_G(P,R)$, then the element v_{P^x} can be written $x^{-1} \otimes v_x$. Since v is fixed by R, the element $w = \oplus_{x\in N_G(P)\backslash T_G(P,R)} x^{-1} \otimes v_x$ is in $M(R)$, and it is clear that $r^R_Q(w) = v_Q$ for any proper subgroup Q of R, which proves the second part of Assertion 3).

It remains to show that Assertion 3) implies Assertion 2). Let X be the module $\oplus_{P\in \underline{s}_p(G)/G} L^o_{P,\underline{M}(P)}$. I notice first that

$$\mathrm{Hom}(M, X) = \oplus_P \mathrm{Hom}_{RN_G(P)/P}(M(P), \underline{M}(P))$$

from the adjunction property of the functor $V \mapsto L^o_{P,V}$. Then by the first part of Assertion 3), I can choose, for any P modulo G, a section of the inclusion of $\underline{M}(P)$ in $M(P)$, and this choice determines a morphism ϕ from M to X.

This morphism is such that $\underline{\phi}(P)$ is the identity map for any P. Let then K be the kernel of ϕ, and Q be a minimal p-subgroup of G such that $K(Q) \neq 0$. Then by definition of Q,

$$K(Q) = \underline{K}(Q) = \mathrm{Ker}\ \underline{\phi}(Q) = 0$$

and this contradiction proves that ϕ is injective. Let Y the cokernel of ϕ. I postpone the proof of the following lemma :

Lemma 5.6. *The functor sending M to $\underline{M}(P)$ is left exact, and its first derived functor D_P is given by*

$$D_P(M) = (\lim_{\longleftarrow\, Q\subset P} M(Q))^P / (\text{Image of } \ M(P))$$

Then the exact sequence $0 \to M \xrightarrow{\phi} X \to Y \to 0$ yields for any P the exact sequence $0 \to \underline{M}(P) \xrightarrow{\phi(P)} \underline{X}(P) \to \underline{Y}(P) \to 0$, by the second part of Assertion 3). Since $\underline{\phi}(P)$ is the identity map, it follows that $\underline{Y}(P) = 0$ for all P.

Then if Q is minimal such that $Y(Q) \neq 0$, I have $\underline{Y}(Q) = Y(Q) = 0$, and this contradiction proves that Y is zero, hence that ϕ is surjective, and Assertion 2) of Lemma 5.5 follows.

It remains to prove Lemma 5.6. If $0 \to X \to Y \to Z \to 0$ is an exact sequence of $r\mu_R(G)$-modules, it is clear that $\underline{X}(P)$ is a submodule of $\underline{Y}(P)$. And if w is in the kernel of the map $\underline{Y}(P) \to \underline{Z}(P)$, then w is an element of $Y(P)$ which maps to 0 in $Z(P)$. Thus w is in $X(P) \cap \underline{Y}(P) = \underline{X}(P)$, which proves that the functor $M \mapsto \underline{M}(P)$ is left exact.

To complete the proof of Lemma 5.6, it remains to see that its first derived functor D_P is as claimed. I will denote by d_P the functor which maps M to the quotient of $(\lim\limits_{\longleftarrow\, Q\subset P} M(Q))^P$ by the image of $M(P)$.

I have already observed (in the proof of the implication 1)⇒ 3) of Lemma 5.5), that $d_P(M) = 0$ if M is of the form $L^o_{P,V}$. The adjunction property of the functor

$V \mapsto L^o_{P,V}$ shows that $L^o_{P,V}$ is injective if V is, and I have indeed $D_P(L^o_{P,V}) = d_P(L^o_{P,V}) = 0$ in this case.

Let then M be an arbitrary $r\mu_R(G)$-module. I choose, for any P modulo G, an injective $RN_G(P)/P$-module I_P containing $\underline{M}(P)$. The inclusion map from $\underline{M}(P)$ into I_P extends to a morphism from $M(P)$ to I_P, and I get this way a morphism from M to $X = \oplus_P L^o_{P,I_P}$, and X is an injective module. It is easy to see that this morphism is moreover injective.

Let Y the cokernel of this morphism. If I show that any exact sequence $0 \to M \to X \to Y \to 0$ yields an exact sequence

$$0 \to \underline{M}(P) \to \underline{X}(P) \to \underline{Y}(P) \to d_P(M) \to d_P(X)$$

I get then an exact sequence

$$0 \to \underline{M}(P) \to \underline{X}(P) \to \underline{Y}(P) \to d_P(M) \to 0$$

Since moreover, by definition of D_P, I have the exact sequence

$$0 \to \underline{M}(P) \to \underline{X}(P) \to \underline{Y}(P) \to D_P(M) \to 0$$

Lemma 5.6 will be proved.

So let $0 \to M \to X \xrightarrow{b} Y \to 0$ be an exact sequence, in which I consider M as a submodule of X, and y be an element of $\underline{Y}(P)$. Then there exists x in $X(P)$ such that $y = b(x)$. Moreover, if Q is a proper subgroup of P, then $r^P_Q x$ belongs to $M(Q)$.

The sequence $(r^P_Q x)_{Q\subset P}$ defines an element of $(\lim\limits_{\longleftarrow\, Q\subset P} M(Q))^P$, well defined up to an element of the image of $M(P)$, i.e. an element of $d_P(M)$. This element is zero if and only if there exists an element m of $M(P)$ such that $r^P_Q x = r^P_Q m$ for all subgroup proper Q of P, i.e. if y is in the image of $\underline{X}(P)$ (because then $y = b(x-m)$, and $x - m \in \underline{X}(P)$). Thus I have built an exact sequence

$$0 \to \underline{M}(P) \to \underline{X}(P) \to \underline{Y}(P) \xrightarrow{c} d_P(M)$$

The inclusion a from M to X yields a morphism $d_P(a)$ from $d_P(M)$ to $d_P(X)$. Let m be an element of $d_P(M)$. Then m is represented by a sequence $(m_Q)_{Q\subset P}$ of elements of $M(Q)$ such that $r^Q_S(m_Q) = m_S$ if $S \subseteq Q \subset P$, and $pm_Q = m_{{}^pQ}$ if $p \in P$. The element m is in the kernel of $d_P(a)$ if and only if there exists an element x of $X(P)$ such that $m_Q = r^P_Q x$ for all $Q \subset P$, hence if and only if m is in the image of $\underline{Y}(P)$ (the element m is then the image by c of $b(x)$).

Hence I have proved that the sequence

$$0 \to \underline{M}(P) \to \underline{X}(P) \to \underline{Y}(P) \xrightarrow{c} d_P(M) \xrightarrow{d_P(a)} d_P(X)$$

is exact, which proves Lemma 5.6, and completes the proof of Lemma 5.5.

It remains to see how Lemma 5.5 implies Proposition 5.4. So let M be a $\mathcal{T}$-injective Mackey functor in $Mack_R(G,1)$. I will show that $\mathcal{R}(M)$ fulfills Condition 3 of Lemma 5.5.

This condition being inherited by direct summands, I can suppose that M is of the form $\mathcal{J}(X)$. Let then P be a p-subgroup of G. By definition

$$\mathcal{J}(X)(P) = (\bigoplus_{Q\subseteq P} X(Q))^P$$

If R is a subgroup of P, and if $v = (v_Q) \in \mathcal{J}(X)(P)$, then the Q-component of $r^P_R(v)$ is equal to v_Q (for $Q \subseteq R$). Hence if v is in $\underline{\mathcal{J}(X)}(P)$, then v_Q is zero for any subgroup of a proper subgroup of P, i.e. for any proper subgroup of P. Hence the only non zero component of v is its P-component. But it is clear that $\mathcal{J}(X)(P)$ splits as

$$\mathcal{J}(X)(P) = X(P) \bigoplus \big(\bigoplus_{Q\subset P} X(Q)\big)^P$$

as an $N_G(P)/P$-module, which proves that the injection of $\underline{\mathcal{J}(X)}(P)$ in $\mathcal{J}(X)(P)$ is a split.

Similarly, if $w \in (\varprojlim_{Q\subset P} \mathcal{J}(X)(Q))^P$, then w is a sequence of elements w_Q of $\mathcal{J}(X)(Q)$ such that $r^Q_R(w_Q) = w_R$ if $Q \subset P$, and $x.w_Q = w_{{}^xQ}$ if $x \in P$. Each element w_Q is itself defined by a sequence $w_{R,Q}$ of elements of $X(R)$, for $R \subseteq Q$. Then if $R \subseteq S \subseteq Q \subset P$, I must have $r^Q_S(w_Q) = w_S$, hence $w_{R,S} = w_{R,Q}$. Setting $u_R = w_{R,R}$ for $R \subset P$, and $u_P = 0$, I define indeed a element $u = (u_R)$ of $\mathcal{J}(X)(P)$ (because w is invariant by P) such that $r^P_Q(u) = w_Q$ for $Q \subset P$. The natural morphism from $\mathcal{J}(X)(P)$ to $(\varprojlim_{Q\subset P} \mathcal{J}(X)(Q))^P$ is hence indeed surjective, which proves Condition 3 of Lemma 5.5, and Proposition 5.4.

5.2.2. *Homomorphisms.* Let X be an $r\mu_R(G)$-module, and Y be a $t\mu_R(G)$-module. I define a $t\mu_R(G)$-module $h(X,Y)$ by

$$h(X,Y)(P) = \mathrm{Hom}_R(X(P), Y(P))$$

If P is a subgroup of Q, and if $\phi \in h(X,Y)(P)$, I set

$$t^Q_P(\phi) = t^Q_P.\phi.r^Q_P$$

If x is a element of G, I define the conjugate of ϕ by x by

$$x(\phi) = x.\phi.x^{-1}$$

A similar (though more complicated) construction, exists for Mackey functors : if X and Y are G-Mackey functors, I define the Mackey functor $H(X,Y)$ by

$$H(X,Y)(K) = \mathrm{Hom}(\mathrm{Res}^G_K X, \mathrm{Res}^G_K Y)$$

Hence an element of $H(X,Y)(K)$ is a sequence of homomorphisms ϕ_L of $X(L)$ in $Y(L)$, for all subgroups L of K such that

$$\phi_L.t^L_M = t^L_M.\phi_M \qquad \phi_M.r^L_M = r^L_M.\phi_L \qquad \text{si } M \subseteq L \subseteq K$$
$$x.\phi_L = \phi_{{}^xL}.x \qquad \text{if } x \in K,\ L \subseteq K$$

The restriction of such an element $\phi = (\phi_L)_{L\subseteq K}$ to a subgroup M of K is defined by $r^K_M(\phi)_L = \phi_L$ for $L \subseteq M$. The trace of ϕ from K to a subgroup N of G containing K is defined by

$$t^N_K(\phi)_L = \sum_{x\in K\backslash N/L} t^L_{K^x\cap L}.x^{-1}.\phi_{K\cap {}^xL}.x.r^L_{K^x\cap L}$$

Finally, if $x \in G$, then the conjugate of ϕ by x is defined par

$$(x\phi)_L = x.\phi_{L^x}.x^{-1}$$

Its a rather tedious calculation to check that $H(X,Y)$ is indeed a Mackey functor. This construction plays the same role for Mackey functors as the functor $\mathrm{Hom}(-,-)$ plays for G-modules : for example, a Mackey functor X is projective relative to a subgroup K of G if and only if $H(X,X)(G) = t^G_K H(X,X)(K)$ (Higman's criterion).

On the other hand, it is easy to see that if X and Y are in $Mack_R(G,1)$, then so does $H(X,Y)$ (this follows from the fact that $\mathrm{Res}_K^G f_1^G = f_1^K$).

The previous two constructions are related by the following lemma :

Lemma 5.7. *Let X be an $r\mu_R(G)$-module, and Y be a $t\mu_R(G)$-module. Then*

$$H(\mathcal{I}(X),\mathcal{J}(Y)) \simeq \mathcal{J}(h(X,Y))$$

Corollary 5.8. *Let X and Y be Mackey functors in $Mack_R(G,1)$. If X is $\mathcal{R}$-projective and if Y is $\mathcal{T}$-injective, then $H(X,Y)$ is $\mathcal{T}$-injective, and in particular*

$$\mathrm{Hom}(X,Y) = \oplus_{P\in\underline{s}_p(G)/G}\mathrm{Hom}_{RN_G(P)/P}(\overline{X}(P),\underline{Y}(P))$$

Let K be a subgroup of G. There is an obvious restriction functor from the category of $r\mu_R(G)$-modules to the category of $r\mu_R(K)$-modules : if P is a p-subgroup of K, and X a $r\mu_R(G)$-module, then $\mathrm{Res}_K^G(X)(P) = X(P)$. It is clear moreover that this functor commutes with the functors $\mathcal{I}$ and $\mathcal{R}$: more precisely, if $\mathcal{I}_G$ (resp. $\mathcal{I}_K$) denotes the functor $\mathcal{I}$ for the group G (resp. for the group K), then $\mathrm{Res}_K^G\mathcal{I}_G = \mathcal{I}_K\mathrm{Res}_K^G$ (here the first Res denotes restriction for Mackey functors, the second one restriction for $r\mu_R(G)$-modules).

Similarly, there is a restriction functor, that I still denote by Res_K^G, from the category of $t\mu_R(G)$-modules to the category of $t\mu_R(K)$-modules, which commutes with functors $\mathcal{J}$ and $\mathcal{T}$.

With this notation

$$\mathrm{Hom}(\mathrm{Res}_K^G\mathcal{I}_G(X),\mathrm{Res}_K^G\mathcal{J}(Y)) = \mathrm{Hom}(\mathcal{I}_K(\mathrm{Res}_K^G X),\mathrm{Res}_K^G\mathcal{J}(Y)) =$$
$$\mathrm{Hom}(\mathrm{Res}_K^G X,\mathcal{R}_K\mathrm{Res}_K^G\mathcal{J}_G(Y)) = \mathrm{Hom}(\mathrm{Res}_K^G X,\mathcal{R}_K\mathcal{J}_K(\mathrm{Res}_K^G Y))$$

But $\mathcal{J}_K(\mathrm{Res}_K^G Y)$ is $\mathcal{T}_K$-injective, and $\underline{\mathcal{J}_K(\mathrm{Res}_K^G)(Y)}(P) = Y(P)$. Proposition 5.4 shows then that

$$\mathcal{R}_K\mathcal{J}_K(\mathrm{Res}_K^G Y) = \bigoplus_{P\in\underline{s}_p(K)/K} L^o_{P,Y(P)}$$

and then

$$\mathrm{Hom}(\mathrm{Res}_K^G\mathcal{I}_G(X),\mathrm{Res}_K^G\mathcal{J}(Y)) = \bigoplus_{P\in\underline{s}_p(K)/K}\mathrm{Hom}_{N_G(P)/P}(X(P),Y(P))$$

hence

$$\mathrm{Hom}(\mathrm{Res}_K^G\mathcal{I}_G(X),\mathrm{Res}_K^G\mathcal{J}(Y)) = (\bigoplus_{P\in\underline{s}_p(K)} h(X,Y)(P))^K$$

which is also $\mathcal{J}(h(X,Y))(K)$. It is easy to describe this isomorphism precisely : if for all p-subgroup P of K, I have a morphism ϕ_P from $X(P)$ to $Y(P)$ such that $x\phi_P x^{-1} = \phi_{^xP}$ if $x\in K$, and if L is a subgroup of K, I define a morphism ϕ_L from $(\oplus_{P\subseteq L}X(P))_L$ to $(\oplus_{Q\subseteq L}Y(Q))^L$ by

$$\phi_L(\pi_L(u))_Q = \sum_{x\in P\backslash L/Q} t^Q_{Q\cap {^xP}}.x.r^P_{Q^x\cap P}.\phi_P(u)$$

This identification allows to check that the resulting isomorphism is indeed an isomorphism of Mackey functors, which proves Lemma 5.7.

Then if X is $\mathcal{R}$-projective and if Y is $\mathcal{T}$-injective, the functor $H(X,Y)$ is a direct summand of a functor of the form $H(\mathcal{I}(A),\mathcal{J}(B))$, isomorphic to $\mathcal{J}(h(A,B))$, hence it is $\mathcal{T}$-injective. To prove the corollary, it remains then to apply Proposition 5.4 and the following lemma :

Lemma 5.9. *Let X and Y be G-Mackey functors, and K be a subgroup of G. Then $H(X,Y)_P$ identifies with $H(X^P,Y_P)$*

This lemma shows indeed that

$$\underline{H(X,Y)}(P) = H(X^P,Y_P)(P/P) = \mathrm{Hom}_R(\overline{X}(P),\underline{Y}(P))$$

hence that $(\oplus_{P\in \underline{s}_p(G)}\underline{H(X,Y)}(P))^G = \oplus_{P\in \underline{s}_p(G)/G}\mathrm{Hom}_{N_G(P)/P}(\overline{X}(P),\underline{Y}(P))$.

So it remains to prove Lemma 5.9. Let K be a subgroup of G. Then by definition, $H(X,Y)(K)$ is the group of homomorphisms (as K-Mackey functors) from $\mathrm{Res}^G_K X$ to $\mathrm{Res}^G_K Y$. An element of $H(X,Y)(K)$ is hence a sequence (ϕ_L) of homomorphisms from $X(L)$ to $Y(L)$, for $L\subseteq K$, which commute with restrictions, traces, and elements of K.

If P is a normal subgroup of K, such an element is in $H(X,Y)_P(K/P)$ if its restriction to any subgroup M of K not containing P is zero, i.e. if $\phi_M=0$ if $P\not\subseteq M$.

If L/P is a subgroup of K/P, then $r^L_M\phi_L = \phi_M r^L_M = 0$ if $P\not\subseteq M$, and the image of ϕ_L is contained in $Y_P(L/P)$. Similarly, since $\phi_L t^L_M = t^M_L\phi_M = 0$, the map ϕ_L factors through $X^P(L/P)$. I define that way for all subgroup L/P of K/P a morphism from $X^P(L/P)$ to $Y_P(L/P)$. It is clear that these morphisms commute with restrictions, traces and elements of K, hence I have defined a morphism of K/P-Mackey functors from X^P to Y_P.

This construction can be obviously reversed, and this proves Lemma 5.9.

5.3. Projective functors and Cartan matrix.

5.3.1. *Notation and recall.* I will suppose in this section that $\mathcal{R}$ is a field k, of characteristic $p>0$.

Under these conditions, the simple Mackey functors in $Mack_k(G,1)$ are indexed by the G-conjugacy classes of pairs (Q,V), where Q is a p-subgroup of G and V a simple $kN_G(Q)/Q$-module (cf.[**5**] Theorem 8.3 and [**6**] Theorem 9.7).
The simple functor $S^G_{Q,V}$ is defined for $Q\neq 1$ by

$$S^G_{Q,V} = \mathrm{Ind}^G_{N_G(Q)}\mathrm{Inf}^{N_G(Q)}_{N_G(Q)/Q} S^{N_G(Q)/Q}_{1,V}$$

and the functor $S^G_{1,V}$ is the unique minimal subfunctor of the fixed points functor FP_V. The value at a subgroup H is given by $S^G_{1,V}(H) = Tr^H_1(V)$.

Denoting by FQ_V the functor of coinvariants on V, whose value at H is $FQ_V(H) = V_H$, the functor $S^G_{1,V}$ is also the quotient of FQ_V by its unique maximal subfunctor.

I will denote by $P^G_{Q,V}$ the projective cover of the functor $S^G_{Q,V}$. Then $P^G_{Q,V}$ is in $Mack_k(G,1)$. Denoting by b_Q the Burnside functor for the group Q, the functor $P^G_{Q,V}$ is a direct summand of the functor $\mathrm{Ind}^G_Q b_Q$. The module $P_{Q,V}(1)$ is an indecomposable p-permutation kG-module (cf.[**6**] Theorem 12.7). Conversely, if W is a p-permutation kG-module, there exists a unique projective Mackey functor L in $Mack_k(G,1)$ such that $L(1)=W$.

Thus the projective Mackey functors in $Mack_k(G,1)$ are characterized by their value at the trivial subgroup of G. The main result of this section will be a formula to compute the Cartan matrix of $Mack(G,1)$ from the p-permutation modules.

If M is a Mackey functor, I will denote by M^* its dual, defined by $M^*(H) = M(H)^*$, by $t^K_H(\phi) = \phi\circ r^K_H$, by $r^K_H(\phi) = \phi\circ r^K_H$ and $x.\phi = \phi\circ x^{-1}$. If M and N are Mackey functors, then $\mathrm{Hom}(M,N)$ identifies with $\mathrm{Hom}(N^*,M^*)$, hence the dual of a projective functor is an injective functor, and the dual of an injective functor is a projective functor.

The dual of the functor $S_{Q,V}^G$ being the functor S_{Q,V^*}^G, the dual of the functor $P_{Q,V}^G$ is the injective hull I_{Q,V^*}^G of the functor S_{Q,V^*}^G.

If L is a projective Mackey functor, then the natural morphisms from $FQ_{L(1)}$ to L and from L to $FP_{L(1)}$ are respectively injective and surjective (cf.[**6**] Lemma 12.4). The same is true by duality if L is an injective functor, since the dual of the functor FP_V is the functor FQ_{V^*}.

Finally if K is a subgroup of G, the dual of the functor M^K is the functor $(M^*)_K$.

5.3.2. *Residues.* The Mackey functors in $Mack_R(G,1)$ which are moreover projective (resp. injective) are $\mathcal{R}$-projective (resp. $\mathcal{T}$-injective). I can apply Corollary 5.8 to these functors (and this argument also holds when R is an arbitrary ring). When R is a field k, the residues projective or injective Mackey functors can be easily computed :

Lemma 5.10. *Let M be a projective (resp. injective) functor in $Mack_k(G,1)$, and P a p-subgroup of G. Then $\overline{M}(P)$ (resp. $\underline{M}(P)$) is isomorphic to $M(1)[P]$.*

(for the notation $V[P]$, see Remark 3.5)

It suffices by duality of prove this lemma when M is projective : indeed, if M is injective, then M^* is projective and

$$\underline{M}(P) = M_P(P/P) = ((M^*)^*)_P(P/P) = ((M^*)^P)^*(P/P) = ((M^*)^P(P/P))^* =$$

$$(M^*(1)[P])^* = (M(1)[P]^*)^* = M(1)[P]$$

since if V is a p-permutation module, then $V^*[P] = (V[P])^*$.

So let M be a projective Mackey functor in $Mack_k(G,1)$, and Q be a p-subgroup of G. The natural morphism α from $M(Q)$ to $FP_{M(1)}(Q) = M(1)^Q$ is the map sending v to $r_1^Q v$. This morphism is surjective. On the other hand, if v is a relative trace $t_S^Q w$ from a proper subgroup S of Q, then

$$r_1^Q v = \sum_{x\in Q/S} t_1^1 x r_1^S w = Tr_S^Q(w)$$

and the morphism α passes down to quotients, giving a surjective morphism $\bar{\alpha}$ from $\overline{M}(Q)$ to $M(1)[Q]$. I must now show that this morphism is injective. This property being inherited by direct sums and direct summands, I can suppose that M is of the form $\mathrm{Ind}_S^G b_S$, where S is a p-subgroup of G.

In this case, the module $M(1)$ is isomorphic to $\mathrm{Ind}_S^G k$, and denoting by $\overline{T}_G(Q,S)$ the quotient $N_G(Q)\backslash G/S$, I have (cf.[**2**])

$$M(1)[Q] = (\mathrm{Ind}_S^G k)[Q] = \sum_{x\in \overline{T}_G(Q,S)} \mathrm{Ind}_{N_{^xS}(Q)/Q}^{N_G(Q)/Q} k$$

On the other hand, if L is a Mackey functor for the group $N_G(Q)/Q$, then

$$\mathrm{Hom}((\mathrm{Ind}_S^G b_S)^Q, L) = \mathrm{Hom}(\mathrm{Ind}_S^G b_S, \mathrm{Ind}_{N_G(Q)}^G \mathrm{Inf}_{N_G(Q)/Q}^{N_G(Q)} L)$$

or

$$\mathrm{Hom}((\mathrm{Ind}_S^G b_S)^Q, L) = \mathrm{Hom}(\mathrm{Res}_{N_G(Q)}^G \mathrm{Ind}_S^G b_S, \mathrm{Inf}_{N_G(Q)/Q}^{N_G(Q)} L)$$

Moreover

$$\mathrm{Res}^G_{N_G(Q)}\mathrm{Ind}^G_S b_S = \sum_{x\in N_G(Q)\backslash G/S} \mathrm{Ind}^{N_G(Q)}_{N_G(Q)\cap {}^xS} b_{N_G(Q)\cap {}^xS}$$

hence

$$\mathrm{Hom}((\mathrm{Ind}^G_S b_S)^Q, L) = \sum_{x\in N_G(Q)\backslash G/S} \mathrm{Hom}(b_{N_G(Q)\cap {}^xS}, \mathrm{Res}^{N_G(Q)}_{N_G(Q)\cap {}^xS}\mathrm{Inf}^{N_G(Q)}_{N_G(Q)/Q} L)$$

Since for any group S and any S-Mackey functor X, the group $\mathrm{Hom}(b_S, X)$ is isomorphic to $X(S)$, and since $(\mathrm{Inf}^{N_G(Q)}_{N_G(Q)/Q} L)(N_{{}^xS}(Q))$ is zero if Q is not contained in ${}^xS$, i.e. if $x \notin \overline{T}_G(Q,S)$, and equal to $L(N_{{}^xS}(Q)/Q)$ otherwise, I have

$$\mathrm{Hom}((\mathrm{Ind}^G_S b_S)^Q, L) = \sum_{x\in \overline{T}_G(Q,S)} L(N_{{}^xS}(Q)/Q)$$

Finally, for any L, I have

$$\mathrm{Hom}((\mathrm{Ind}^G_S b_S)^Q, L) = \sum_{x\in \overline{T}_G(Q,S)} \mathrm{Hom}(\mathrm{Ind}^{N_G(Q)/Q}_{N_{{}^xS}(Q)/Q} b_{N_{{}^xS}(Q)/Q}, L)$$

But the functor sending M to M^Q being left adjoint of an exact functor, it follows that M^Q is projective if M is. Then the functors $X = (\mathrm{Ind}^G_S b_S)^Q$ and $Y = \sum_{x\in \overline{T}_G(Q,S)} \mathrm{Ind}^{N_G(Q)/Q}_{N_{{}^xS}(Q)/Q} b_{N_{{}^xS}(Q)/Q}$ are both projective, and such that $\mathrm{Hom}(X, L) = \mathrm{Hom}(Y, L)$ for any L. Hence they are isomorphic.

So they have the same value at Q/Q, which gives

$$\overline{M}(Q) = \sum_{x\in \overline{T}_G(Q,S)} \mathrm{Ind}^{N_G(Q)/Q}_{N_{{}^xS}(Q)/Q} k = M(1)[Q]$$

and completes the proof of Lemma 5.10.

5.3.3. *Cartan matrix.* I suppose moreover here that the field k is big enough (i.e. that it is a splitting field for all the groups $N_G(Q)/Q$, for $Q \in \underline{s}_p(G)$).

Let Q (resp. L) be a p-subgroup of G, and V (resp. W) be a simple $kN_G(Q)/Q$-module (resp. a simple $kN_G(L)/L$-module). Lemma 5.10 and Corollary 5.8 show that

$$\mathrm{Hom}(P^G_{Q,V}, (P^G_{L,W})^*) = \oplus_{S\in \underline{s}_p(G)/G} \mathrm{Hom}_{kN_G(S)/S}(P^G_{Q,V}(1)[S], (P^G_{L,W})^*(1)[S])$$

On the other hand, in the Grothendieck group of Mackey functors

$$P^G_{L,W} = \sum_{H,M} C_{(L,W),(H,M)} S^G_{H,M}$$

where the sum runs over the indexing pairs of simple Mackey functors in $Mack_k(G,1)$, denoting by $C_{(L,W),(H,M)}$ the coefficient of the Cartan matrix corresponding to the simple functors $S^G_{L,W}$ and $S^G_{H,M}$. The dual of $S^G_{H,M}$ being S^G_{H,M^*}, it follows that

$$(P^G_{L,W})^* = \sum_{H,M} C_{(L,W),(H,M)} S^G_{H,M^*}$$

and since $\mathrm{Hom}(P^G_{Q,V}, S^G_{H,M^*})$ is zero if the pair (H, M^*) is not conjugate to the pair (Q,V), and one dimensional k (since k is big enough), it follows that

$$C_{(L,W),(Q,V^*)} = \sum_{S\in \underline{s}_p(G)/G} \dim_k \mathrm{Hom}_{kN_G(S)/S}(P^G_{Q,V}(1)[S], (P^G_{L,W})^*(1)[S])$$

But $(P^G_{L,W})^*(1) = (P^G_{L,W}(1))^*$, and $(P^G_{L,W}(1))^* = P^G_{L,W^*}(1)$ (because $P^G_{L,W}(1)$ is the Green correspondent of the projective cover of W by Theorem 12.7 of [**6**], and since the Green correspondence commutes with the duality). Changing V to V^* in the above equality gives

$$C_{(L,W),(Q,V)} = \sum_{S\in \underline{s}_p(G)/G} \dim_k \mathrm{Hom}_{kN_G(S)/S}((P^G_{Q,V}(1))^*[S], (P^G_{L,W}(1))^*[S])$$

i.e.

$$C_{(L,W),(Q,V)} = \sum_{S\in \underline{s}_p(G)/G} \dim_k \mathrm{Hom}_{kN_G(S)/S}(P^G_{L,W}(1)[S], P^G_{Q,V}(1)[S])$$

Finally, by linearity, this gives :

Proposition 5.11. *Let k be a "big enough" field. If M and N are projective Mackey functors in $Mack_k(G,1)$, then*

$$\dim_k \mathrm{Hom}(M,N) = \sum_{S\in \underline{s}_p(G)/G} \dim_k \mathrm{Hom}_{kN_G(S)/S}(M(1)[S], N(1)[S])$$

Corollary 5.12. *Let $pg(G)$ the Green ring of p-permutation kG-modules. The bilinear form on $pg(G)$ defined by*

$$< X, Y >_G = \sum_{S\in \underline{s}_p(G)/G} \dim_k \mathrm{Hom}_{kN_G(S)/S}(X[S], Y[S])$$

is symmetric, positive, and definite. Moreover, if H is a subgroup of G, then

$$< \mathrm{Ind}_H^G X, Y >_G = < X, \mathrm{Res}^G_H Y >_H$$

This corollary follows from the fact that the Cartan matrix of $Mack_k(G,1)$ is symmetric, positive, and definite, and that the corresponding bilinear form satisfies Frobenius reciprocity (moreover, if M is a kH-Mackey functor, then $(\mathrm{Ind}_H^G M)(1) = \mathrm{Ind}_H^G M(1)$).

5.3.4. *Another formula.* I suppose here that R is a field k of characteristic p. The notation about the generalized Steinberg modules was introduced in [**1**].

Proposition 5.13. *Let L be a projective Mackey functor in $Mack_k(G,1)$, and X be any Mackey functor for G over k. Then*

$$\dim_k \mathrm{Hom}(L,X) = \sum_{P\in \underline{s}_p(G)/G} \dim_k \mathrm{Hom}_{kN_G(P)/P}(St(N_G(P)/P, L(1)[P]), X(P))$$

To prove this proposition, I use the following ingredients :

1) If N is a normal subgroup of G, there exists a natural algebra homomorphism from $\mu_k(G/N)$ to $\mu_k(G)$, mapping the element $t^{K/N}_{\bar{x}L/N}\bar{x}r^{H/N}_{L/N}$ to $t^K_{^xL}xr^H_L$. If N is a normal p-subgroup of G, then this morphism maps the algebra $\mu^1_k(G/N)$ in the algebra $\mu^1_k(G)$: indeed, if P/N is a p-subgroup of G/N, then the image of $t^{H/N}_{P/N}$ (resp. $r^{H/N}_{P/N}$) is t^H_P (resp. r^H_P), and P is a p-subgroup of G.

2) This morphism induces an exact functor $\rho^G_{G/N}$ from the category of G-Mackey functors to the category of G/N-Mackey functors : if M is a G-Mackey functor, then the functor $\rho^G_{G/N}(M)$ is such that

$$\rho^G_{G/N}(M)(H/N) = M(H)$$

Moreover, if N is a normal p-subgroup of G, the functor $\rho^G_{G/N}$ maps $Mack_k(G,1)$ in $Mack_k(G/N,1)$.

The functor $\rho^G_{G/N}$ admits a left adjoint, denoted by $\iota^G_{G/N}$. Since the functor $\rho^G_{G/N}$ is exact, the functor $\iota^G_{G/N}(M)$ is projective if M is. If N is a normal p-subgroup of G, and if M is in $Mack_k(G/N,1)$, then $\iota^G_{G/N}(M)$ is in $Mack_k(G,1)$.
3) If M is a G/N-Mackey functor, then for any kG-module V

$$\mathrm{Hom}(\iota^G_{G/N}(M), FP_V) = \mathrm{Hom}(M, \rho^G_{G/N}(FP_V))$$

and it is easy to see that $\rho^G_{G/N}(FP_V)$ identifies with FP_{V^N}. Then

$$\mathrm{Hom}(\iota^G_{G/N}(M), FP_V) = \mathrm{Hom}_{kG/N}(M(N/N), V^N) = \mathrm{Hom}_{kG}(M(N/N), V)$$

On the other hand, since

$$\mathrm{Hom}(\iota^G_{G/N}(M), FP_V) = \mathrm{Hom}_G(\iota^G_{G/N}(M)(1), V)$$

it follows that $\iota^G_{G/N}(M)(1)$ is isomorphic as a kG-module to the module $M(N/N)$.
4) If $M = A - B$ is a virtual projective kG-module, I denote by FQ_M the Mackey virtual projective functor $FQ_A - FQ_B$, in the Green ring of projective Mackey functors in $Mack_k(G,1)$. If X and Y are virtual projective Mackey functors in $Mack_k(G,1)$ such that $X(1) = Y(1)$ in the ring of Green of kG-modules, then $X = Y$.
The previous considerations prove the following lemma :

Lemma 5.14. *Let L be a projective Mackey functor in $Mack_k(G,1)$. Then*

$$L = \sum_{P \in \underline{s}_p(G)/G} \mathrm{Ind}^G_{N_G(P)} \iota^{N_G(P)}_{N_G(P)/P}(FQ_{St(N_G(P)/P, L(1)[P])})$$

in the Green ring of projective functors in $Mack_k(G,1)$.

Indeed, the module $L(1)$ can be written (cf.[**1**])

$$L(1) = \sum_{P \in \underline{s}_p(G)/G} \mathrm{Ind}^G_{N_G(P)} St(N_G(P)/P, L(1)[P])$$

Then both sides of the equality of the lemma are virtual projective Mackey functors in $Mack_k(G,1)$, which have the same value at the trivial subgroup. Hence they are equal.

The proposition follows, applying the functor $\mathrm{Hom}(-,X)$ to both sides.

Remark 5.15. It is possible to compute effectively the functor $\iota^G_{G/N}$: let M be a kG/N-Mackey functor, and K be a subgroup of G. Let $\omega_N(K)$ the set of subgroups of K ordered by the relation

$$L \preceq L' \Leftrightarrow \begin{cases} L \subseteq L' \\ L \cap N = L' \cap N \end{cases}$$

I denote by

$$\varinjlim_{L \in \omega_N(K)} M(LN/N)$$

the quotient of $\oplus_{L \subseteq K} M(LN/N)$ by the submodule generated by elements of the form $t^{L'N/N}_{LN/N} m - m$, for $L \preceq L'$ and $m \in M(LN/N)$.

The group K acts on $\varinjlim_{L\in\omega_N(K)} M(LN/N)$, and then

$$\iota_H^G(M)(K) = \Big(\varinjlim_{L\in\omega_N(K)} M(LN/N)\Big)_K$$

is the largest quotient on which K acts trivially. If L is a subgroup of K, and m an element of $M(LN/N)$, I denote by m_L^K the image of m in $\iota_H^G(M)(K)$.

If $K \subseteq K'$, then $t_K^{K'}$ (resp. $r_K^{K'}$) is the map from $\iota_H^G(M)(K)$ to $\iota_H^G(M)(K')$ (resp. from $\iota_H^G(M)(K')$ to $\iota_H^G(M)(K)$) defined by

$$t_K^{K'}(m_L^K) = m_L^{K'} \qquad r_K^{K'}(m'^{K'}_{L'}) = \sum_{x\in K\backslash K'/L'} \Big({}^x r^{L'N/N}_{(K^x\cap The)N/N} m'\Big)^K_{K\cap {}^xL'}$$

Finally if $x \in G$, then ${}^x(m_L^K) = ({}^xm)^{{}^xK}_{{}^xL}$.

6. Projective functors and image of $\mathcal{I}$

I will try to see in this section under which conditions a projective Mackey functor in $Mack(G,1)$ lies in the image of $\mathcal{I}$. I will suppose that R is a complete local ring, with residue field of characteristic p.

6.1. Finite projective resolutions. I will state the following equivalence :

THEOREM 6.1. *Let X be an $r\mu_R(G)$-module. The following conditions are equivalent :*

1. *The functor $\mathcal{I}(X)$ is projective.*
2. *The module X has a finite projective resolution.*

The proof of this theorem requires a series of preliminary results.

LEMMA 6.2. *Let X and Y be projective functors in $Mack_R(G,1)$. The morphism $\phi \mapsto \phi(1)$ of evaluation at 1 from* $\mathrm{Hom}(X,Y)$ *to* $\mathrm{Hom}_{RG}(X(1),Y(1))$ *is surjective. Moreover ϕ is a split injective (resp. a split surjective) if and only if $\phi(1)$ is a split injective (resp. a split surjective).*

I have already recalled that the natural morphism from Y to $FP_{Y(1)}$ is surjective if Y is projective. Let then f in $\mathrm{Hom}_{RG}(X(1),Y(1))$. It yields by adjunction a morphism F of X in $FP_{Y(1)}$. Then the diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \xrightarrow{F} & FP_{Y(1)} \end{array}$$

can be completed to a commutative diagram by a morphism ϕ from X to Y, because the vertical arrow is surjective, and the functor X is projective. It is clear that $\phi(1)$ is equal to f, which proves the first assertion.

It is clear moreover that if ϕ is a split injection, then $\phi(1)$ is also split injective. Conversely, if $\phi(1)$ is a split injection, then there exists a morphism f from $Y(1)$ to $X(1)$ such that $f\phi(1) = Id_{X(1)}$. The morphism f can be lifted to a morphism $\tilde{f}$ from Y to X, and then the morphism $\psi = \tilde{f}\phi$ is an endomorphism of X such that $\psi(1) = Id$. Then some power of ψ is a Fitting element[4]. Its image I is then a

[4]This probably requires X to be finitely generated

direct summand of X, hence I is a projective Mackey functor in $Mack(G,1)$ such that $I(1) = X(1)$, which proves that I is isomorphic to X, hence that $\psi = \tilde{f}\phi$ is invertible, and ϕ is a split injection. The case of a split surjection is similar.

Lemma 6.3. *Let X and Y be projective functors in $Mack_R(G,1)$, and ϕ be a morphism from X to Y. The following conditions are equivalent :*

(1) *The morphism ϕ is a split injection (resp. a split surjection).*
(2) *For any p subgroup P of G, the morphism $\bar{\phi}(P)$ of $\overline{X}(P)$ in $\overline{Y}(P)$ is injective (resp. surjective).*

Since one can lift projective objects and morphisms between them from k to R, I can assume that R is a field k of characteristic p. In this case, the module $\overline{X}(P)$ identifies with $X(1)[P]$, and the morphism $\bar{\phi}(P)$ to $\phi(1)[P]$. I will prove simultaneously Lemma 6.3 and the following proposition :

Proposition 6.4. *Let M and N be p-permutation kG-modules, and f be a morphism from M to N. The following conditions are equivalent :*

(1) *The morphism f is a split injection (resp. a split surjection).*
(2) *For any p subgroup P of G, the morphism $f[P]$ of $M[P]$ in $N[P]$ is injective (resp. surjective).*

Indeed let X, Y and ϕ be as in Lemma 6.3. It is clear that if ϕ is a split surjection, then for any P, the morphism $\bar{\phi}(P)$ is (split) surjective. Conversely, if $\bar{\phi}(P)$ is surjective for all P, then $\phi(1) = \bar{\phi}(1)$ is surjective. Let Q be a p-subgroup such that $\phi(R)$ is surjective for all proper subgroup R of Q. Let moreover $v \in Y(P)$.

Since $\bar{\phi}(P)$ is surjective, there exists elements $v_R \in Y(R)$, for $R \subset Q$, and an element $w \in X(Q)$ such that

$$v = \phi(Q)(w) + \sum_{R \subset Q} t_R^Q v_R$$

Then each v_R can be written $v_R = \phi(R)(w_R)$ since $\phi(R)$ is surjective, and then

$$v = \phi(Q)(W) + \sum_{R \subset Q} t_R^Q \phi(R)(w_R) = \phi(Q)(w) + \sum_{R \subset Q} \phi(Q)(t_R^Q w_R)$$

which proves that $\phi(Q)$ is surjective for all Q. Then ϕ is surjective, hence ϕ is a split surjection since Y is projective. This shows the equivalence of conditions of Lemma 6.3 in the surjective case.

Then let M, N and f be as in Proposition 6.4. If f is a split surjection, it is clear that $f[P]$ is (split) surjective for all P. Conversely, if $f[P]$ is surjective for all P, let L_M (resp. L_N) the projective Mackey functor in $Mack_k(G,1)$ such that $L_M(1) = M$ (resp. $L_N(1) = N$). The morphism f can be extended to a morphism ϕ of L_M in L_N, such that $\phi(1) = f$. The hypothesis implies that $\bar{\phi}(P)$, which identifies with $f[P]$, is surjective for all P. Then ϕ is a split surjection, and $f = \phi(1)$ is also split surjective. Which proves the equivalence of conditions of Proposition 6.4 in the surjective case.

But by duality, this proves the equivalence of the conditions of Proposition 6.4 in the injective case[5].

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[5]This proof is not correct for infinitely generated modules. This was fixed in *Le complexe de chaînes d'un G-complexe simplicial acyclique* J. Alg. 220 (1999) 415-436

Let then again X, Y, and ϕ be as in Lemma 6.3. It is clear that if ϕ is a split injection, then $\bar{\phi}(P)$ is (split) injective for all P. Conversely, if $\bar{\phi}(P)$ is injective for all P, let $f = \phi(1)$. Then f is a morphism from $X(1)$ to $Y(1)$ such that $f[P]$ is injective for all P, hence f is a split injection, and Lemma 6.2 implies that ϕ is a split injection, which completes the proof of the Lemma 6.3 and Proposition 6.4.

By the same argument as above about lifting of projective modules, it suffices to prove Theorem 6.1 in the case where R is a filed k of characteristic p.

If M is a kG-module, and P a p-subgroup of G, I will denote by Br_P the projection morphism from M^P to $M[P]$. If X is an $r\mu_R(G)$-module, as the image of r_1^P is contained in $X(1)^P$, I can abuse notation, and denote by $Br_P r_1^P$ the composite map from $X(P)$ to $X(1)[P]$. With this notation :

Proposition 6.5. *Let X be an $r\mu_k(G)$-module. The following conditions are equivalent :*

(1) *The module X has a finite projective resolution.*
(2) *The module $X(1)$ is a p-permutation kG-module, and for any p-subgroup P of G, the map $Br_P r_1^P$ is a isomorphism from $X(P)$ to $X(1)[P]$.*

I will show that Assertion 1 of the proposition implies Assertion 2 by induction on the length of a finite projective resolution of X.

I already observed (in Section 2) that Assertion 2 is true if X is projective : indeed, if P is a p-subgroup of G, if E is a projective $kN_G(P)/P$-module projective, and if $X = L_{P,E}$, then

$$X(Q) = \oplus_{x\in T_G(Q,P)/N_G(P)} x \otimes E$$

On the other hand, the module $X(1)$ identifies with $\mathrm{Ind}_{N_G(P)}^G E$. Hence an element v of $X(1)^Q$ can be written $\sum_{x\in G/P} x\otimes v_x$, the sequence v_x being such that $v_{\sigma_q(x)} = h_{q,x}v_x$, denoting by $x \mapsto \sigma_q(x)$ the permutation of G/P induced by q, and $h_{q,x}$ the element of $N_G(P)$ defined by $qx = \sigma_q(x)h_{q,x}$. In particular, the element v_x is invariant by $Q \cap {}^xP$, and v can also be written

$$v = \sum_{x\in Q\backslash G/N_G(P)} Tr_{Q\cap {}^xP}^Q (x \otimes v_x)$$

It is then clear that $r_1^Q X(Q)$ is a supplement in $X(1)^Q$ of the kernel of Br_Q.

So let X be an $r\mu_k(G)$-module having a finite projective resolution. There exists a projective $r\mu_k(G)$-module L and an $r\mu_k(G)$-module Y having a strictly shorter finite projective resolution than the one of X, and an exact sequence

$$0 \to Y \xrightarrow{i} L \to X \to 0$$

The induction hypothesis implies that $Y(1)$ is a p-permutation module, and that $Br_P r_1^P$ is an isomorphism from $Y(P)$ to $Y(1)[P]$. Then the commutative diagram

$$\begin{array}{ccccc} 0 & \to & Y(P) & \stackrel{i(P)}{\to} & L(P) \\ & & \downarrow Br_P r_1^P & & \downarrow Br_P r_1^P \\ & & Y(1)[P] & \stackrel{i(1)[P]}{\to} & L(1)[P] \end{array}$$

where the vertical arrows are isomorphisms, shows that the map $i(1)$ is an injection from $Y(1)$ in $L(1)$ such that $i(1)[P]$ is injective for all P. Thus $i(1)$ is a split injection by Proposition 6.4, and the sequence

$$0 \to Y(1) \overset{i(1)}{\to} L(1) \to X(1) \to 0$$

is split exact. In particular, the module $X(1)$ is a direct summand of $L(1)$. Hence it is a p-permutation module. Then the completed diagram

$$\begin{array}{ccccccccc} 0 & \to & Y(P) & \overset{i(P)}{\to} & L(P) & \to & X(P) & \to & 0 \\ & & \downarrow Br_P r_1^P & & \downarrow Br_P r_1^P & & \downarrow Br_P r_1^P & & \\ 0 & \to & Y(1)[P] & \overset{i(1)[P]}{\to} & L(1)[P] & \to & X(1)[P] & \to & 0 \end{array}$$

shows that the vertical arrow on the right is an isomorphism, which proves Assertion 2 of the proposition.

Now I will show that Assertion 2 of the proposition implies Assertion 1. First a notation : if X is an $r\mu_k(G)$-module, I denote by $Supp(X)$ the set of subgroups P such that $X(P) \neq 0$, and $\overline{Supp}(X)$ its "downwards closure" for inclusion, i.e. the set of subgroups which are contained in a element of $Supp(X)$.

Let X be an $r\mu_k(G)$-module with the properties of Assertion 2 of the proposition. I will proceed by induction on the cardinality of $\overline{Supp}(X)$.

There is nothing to prove if this cardinality is zero, since then X is zero, hence projective. I postpone the proof of the following lemma :

Lemma 6.6. *Let X be an $r\mu_k(G)$-module, and for any p-subgroup P of G, let E_P be a projective cover of the $kN_G(P)/P$-module*

$$X(P)/(\sum_{Q\supset P} r_P^Q X(Q))$$

Then $\oplus_{P\in \underline{s}_p(G)/G} L_{P,E_P}$ is a projective cover of X.

The properties of Assertion 2 show indeed that if P is a maximal element of $Supp(X)$ (or equivalently of $\overline{Supp}(X)$), then the module $X(P)$ is a p-permutation $kN_G(P)/P$-module. Moreover, if $P \subset Q \subseteq N_G(P)$, then $X(P)[Q/P]$, isomorphic to $X(1)[P][Q/P]$, hence to $X(1)[Q]$, hence to $X(Q)$, is zero. Then $X(P)$ is a projective $kN_G(P)/P$-module. Since moreover $r_P^Q X(Q)$ is zero if Q strictly contains P, I see that $X(P)$ is equal to E_P for any maximal element P of $Supp(X)$.

Let then $L = \oplus_P L_{P,E_P}$ be a projective cover of X, and Y the kernel of a surjection s from L to X. Since $L_{P,E_P}(P) = E_P = X(P)$, I see that $Y(P)$ is zero if P is maximal in $Supp(X)$. Moreover $L(Q)$ is zero if Q is not in $\overline{Supp(X)}$: indeed, the support of L_{P,E_P} is the set of subgroups of G which are contained in P up to conjugation, if E_P, hence $X(P)$ are non zero.

Hence the cardinality of $\overline{Supp(Y)}$ is strictly smaller than the cardinality of $\overline{Supp(X)}$. On the other hand, the commutative diagram

$$\begin{array}{ccccc} L(P) & \overset{s(P)}{\to} & X(P) & \to & 0 \\ \downarrow Br_P r_1^P & & \downarrow Br_P r_1^P & & \\ L(1)[P] & \overset{s(1)[P]}{\to} & X(1)[P] & & \end{array}$$

shows as above that s is a split surjection. Similarly, the completed diagram

$$\begin{array}{ccccccccc} 0 & \to & Y(P) & \to & L(P) & \overset{s(P)}{\to} & X(P) & \to & 0 \\ & & \downarrow Br_P r_1^P & & \downarrow Br_P r_1^P & & \downarrow Br_P r_1^P & & \\ 0 & \to & Y(1)[P] & \to & L(1)[P] & \overset{s(1)[P]}{\to} & X(1)[P] & \to & 0 \end{array}$$

shows then that Y has the properties of Assertion 2. The induction hypothesis implies that Y has a finite projective resolution. It follows that X has a finite projective resolution, and this completes the proof of Proposition 6.5.

Remark 6.7. The previous proof also shows that if p^n is the p-part of the order of G, then any $r\mu_k(G)$-module having a finite projective resolution has a resolution of length smaller than n. The length of minimal finite projective resolutions is hence bounded[6].

I must now prove to Lemma 6.6. Recall that the simple $r\mu_k(G)$-modules, that I have denoted by $N_{P,V}$, are indexed by pairs $\{P, V\}$, where P is a p-subgroup of G and V a simple $kN_G(P)/P$-module. They are defined by $N_{P,V}(Q) = 0$ if Q is not conjugate to P, and $N_{P,V}(P) = V$. It is then easy to see that for all $r\mu_k(G)$-module X

$$\mathrm{Hom}(X, N_{P,V}) = \mathrm{Hom}_{kN_G(P)/P}(X(P)/\sum_{Q\supset P} r_P^Q X(Q), V)$$

which shows that, denoting by $J(X)$ the radical of X,

$$J(X)(P) = J(X(P)) + \sum_{Q\supset P} r_P^Q X(Q)$$

Lemma 6.6 follows easily.

I can now prove Theorem 6.1 : let X be an $r\mu_k(G)$-module such that $\mathcal{I}(X)$ is projective. Then, since $X(1) = \mathcal{I}(X)(1)$, the module $X(1)$ is a p-permutation module. On the other hand, the module $\overline{\mathcal{I}(X)}(P)$ identifies with $X(1)[P]$ if $\mathcal{I}(X)$ is projective, by Lemma 5.10. But for any X, it also identifies with $X(P)$. Moreover the diagram

$$\begin{array}{ccc} X(P) & \to & \mathcal{I}(X)(P) \\ \downarrow r_1^P & & \downarrow r_1^P \\ X(1) & \to & \mathcal{I}(X)(1) \end{array}$$

is commutative, hence X fulfills the conditions of Assertion 2 of Proposition 6.5. Thus X has a finite projective resolution, which proves Assertion 2 of the theorem.

To show that Assertion 2 of the theorem implies Assertion 1, I proceed by induction on the length of a finite projective resolution of X. If X is projective, then $\mathcal{I}(X)$ is projective, because the functor $\mathcal{I}$ is left adjoint to an exact functor.

If X has a finite projective resolution, then there exists a projective $r\mu_k(G)$-module L, an $r\mu_k(G)$-module Y having a finite projective resolution, strictly shorter than the one of X, and an exact sequence

$$0 \to Y \xrightarrow{i} L \to X \to 0$$

Then $\mathcal{I}(Y)$ and $\mathcal{I}(L)$ are projective Mackey functors. Moreover, since $\overline{\mathcal{I}(i)}(P) = i(P)$ is injective for all P, the morphism $\mathcal{I}(i)$ is a split injection by Lemma 6.3. Since the functor $\mathcal{I}$ is right exact, it follows that the sequence

$$0 \to \mathcal{I}(Y) \xrightarrow{\mathcal{I}(i)} \mathcal{I}(L) \to \mathcal{I}(X) \to 0$$

is split exact, hence the functor $\mathcal{I}(X)$ is projective, which completes the proof of Theorem 6.1.

[6] In other words, the *finitistic dimension* of $r\mu_k(G)$ is at most equal to n

6.2. Examples. Let M be a p-permutation module. There exists a unique projective Mackey functor L_M in $Mack_k(G,1)$, such that $L_M(1)=M$. Proposition 6.5 and Lemma 5.10 give a necessary and sufficient condition for the existence of an $r\mu_k(G)$-module X such that L_M is isomorphic to $\mathcal{I}(X)$: indeed in this case, the module $X(P)$ must be isomorphic to $\overline{L_M}(P)$, hence to $M[P]$. Hence there must exist maps r_Q^P, defined for $Q\subseteq P$, from $M[P]$ to $M[Q]$, such that

- $r_S^Q r_Q^P = r_S^P$ if $S\subseteq Q\subseteq P$.
- $r_P^P = Id$ for all $P\in \underline{s}_p(G)$.
- $xr_Q^P x^{-1} = r_{{}^xQ}^{{}^xP}$ for any $x\in G$ and any $Q\subseteq P$.
- r_1^P is injective and its image is a supplement of $\mathrm{Ker}Br_P$ in $M^P = M[1]^P$

Conversely, if such maps exist, then they define an $r\mu_k(G)$-module X such that $\mathcal{I}(X)$ is projective, and moreover $\mathcal{I}(X)(1)=X(1)=M$, hence $\mathcal{I}(X)$ is isomorphic to L_M.

6.2.1. *Permutation modules.* A simple example of this situation is the case where M is a *permutation* module : indeed, if B is a G-stable basis of M, then the inclusion of B^P in B^Q yields the required map r_Q^P. For example, if $M=k$, the functor L_M is the functor $b_p(G)$ of $Mack_k(G,1)$ associated Burnside to the Burnside functor (i.e. the subfunctor of the functor of Burnside such that $b_p(H)$ is generated by the elements H/P, where P is a p-subgroup of H). The associated $r\mu_k(G)$-module X is such that $X(P)=k$ for all P, the maps r_Q^P being identity maps, as well as the conjugations by the elements of G. In other words, the module X is isomorphic to $\mathcal{R}(FP_k)$. Thus

PROPOSITION 6.8. *The module $\mathcal{R}(FP_k)$ has a finite projective resolution.*

I will now give other examples of this situation.

6.2.2. *Some indecomposable p-permutation modules.* Recall (cf.[**2**]) that the indecomposable p-permutation modules can be indexed by the pairs (P,E), where P is a p-subgroup of G and E is an indecomposable projective $kN_G(P)/P$-module : the module $M(P,E)$ corresponding to the pair (P,E) is the unique indecomposable p-permutation module with vertex P such that $M(P,E)[P]=E$. The multiplicity of $M(P,E)$ as a direct summand of a p-permutation module N is equal to the multiplicity of E as a direct summand of $N[P]$. This multiplicity is given by the following :

LEMMA 6.9. *Let N be a kG-module, and E be a projective kG-module. The multiplicity of E as a direct summand of N is equal to*

$$\dim_k Tr_1^G \mathrm{Hom}_k(N,E/J(E))/\dim_k \mathrm{End}_{kG}(E/J(E))$$

Indeed $\dim_k Tr_1^G\mathrm{Hom}_k(N,E/J(E))$ is the dimension of the space of kG-homomorphisms f from N to $E/J(E)$ which factor through a projective module, i.e. which can be written $f=\pi g$, where π is the projection from E to $E/J(E)$ (since E is a projective cover of $E/J(E)$). Since π is essential, the morphism g is then surjective if f is non zero, hence surjective. Then E is a direct summand of N. Conversely, if N can be written $N=E^n\oplus M$, where the module M has no direct summand isomorphic to E, then $\dim_k Tr_1^G\mathrm{Hom}_k(M,E/J(E))$ is equal to zero, and $\dim_k Tr_1^G \mathrm{Hom}_k(N, E/J(E))$ is equal to $n\dim_k Tr_1^G \mathrm{Hom}_k(E, E/J(E))$, hence to $n\dim_k \mathrm{End}_{kG}(E/J(E))$.

Proposition 6.10. *Let P (resp. Q) be a p-subgroup of G, and E (resp. F) be an indecomposable projective $kN_G(P)/P$-module (resp. $kN_G(Q)/Q$-module) If $M(P,E)$ is direct summand of $\mathrm{Ind}_{N_G(Q)}^G F$, then there exists an element $x \in G$ such that $Q^x \cap O_p(N_G(P)) = P$, and in particular*

$$Q \cap O_p(G) \subseteq {}^x P \subseteq Q$$

Corollary 6.11. *If $Q \subseteq O_p(G)$, then the module $\mathrm{Ind}_{N_G(Q)}^G F$ is indecomposable (equal to $M(Q,F)$), and the functor $L_{M(Q,F)}$ is isomorphic to $\mathcal{I}(L_{Q,F})$.*

Indeed (cf. [**1**] Lemme 3), if $N = \mathrm{Ind}_{N_G(Q)}^G F$, then

$$N[P] = \sum_{x \in N_G(P)\backslash T_G(P,Q)/N_G(Q)} \mathrm{Ind}_{N_G(P,{}^xQ)/P}^{N_G(P)/P}\, {}^xF$$

moreover assuming for simplicity $x = 1$, hence $P \subset Q$

$$\dim_k Tr_1^{N_G(P)/P} \mathrm{Hom}_k(\mathrm{Ind}_{N_G(P,Q)/P}^{N_G(P)/P} F, E/J(E)) = \dots$$
$$\dots = \dim_k Tr_1^{N_G(P,Q)/P} \mathrm{Hom}_k(F, E/J(E))$$

But $N_Q(P)/P$ acts trivially on F, hence

$$\dim_k Tr_1^{N_G(P,Q)/P} \mathrm{Hom}_k(F, E/J(E)) = \dots$$
$$\dots = \dim_k Tr_{N_Q(P)/P}^{N_G(P,Q)/P} \mathrm{Hom}_{N_Q(P)/P}(F, Tr_1^{N_Q(P)/P}(E/J(E))$$

But $O_p(N_G(P)/P)$ acts trivially on the simple $kN_G(P)/P$ module $E/J(E)$. This expression is hence equal to zero if $N_Q(P) \cap O_p(N_G(P))$ is different from P, hence if $Q \cap O_p(N_G(P)) \neq P$.

Since $N_G(P)$ normalizes $PO_p(G)$, I have $N_{PO_p(G)}(P) \subseteq O_p(N_G(P))$, hence I must have $Q \cap N_{PO_p(G)}(P) = P$, i.e. $Q \cap PO_p(G) = P$, or $Q \cap O_p(G) \subseteq P$, which proves the proposition.

The first part of the corollary follows, since the only indecomposable summand with vertex Q of N is $M(Q,F)$, with multiplicity 1. On the other hand the functor $\mathcal{I}(L_{Q,F})$ is projective, and its value at the trivial subgroup is $\mathrm{Ind}_{N_G(Q)}^G F = M(Q,F)$.

Proposition 6.12. *The following conditions are equivalent :*

1. *For any p-subgroup of G and any indecomposable projective $kN_G(P)/P$-module, the module $\mathrm{Ind}_{N_G(P)}^G E$ is indecomposable.*
2. *For any indecomposable p-permutation module M, there exists a p-subgroup P and an indecomposable projective $kN_G(P)/P$-module E such that M is isomorphic to $\mathrm{Ind}_{N_G(P)}^G E$.*
3. *The group G has a normal Sylow p-subgroup.*
4. *Any $r\mu_k(G)$-module having a finite projective resolution is projective.*

If Assertion 1) holds, as $M(P,E)$ is a direct summand of $\mathrm{Ind}_{N_G(P)}^G E$, which is indecomposable, these modules are isomorphic, and Assertion 2) holds.

It is clear that Assertion 2 implies Assertion 3 : indeed, if the module k is isomorphic to $\mathrm{Ind}_{N_G(P)}^G E$, then $[G : N_G(P)] \dim_k E = 1$ by consideration of dimensions, hence P is a normal subgroup of G. Since p does not divide the dimension of the projective $kN_G(P)/P$-module E, the group $N_G(P)/P$ is a p'-group, and P is a Sylow p-subgroup of G.

Similarly, if G has a normal Sylow p-subgroup, equal to $O_p(G)$, then all the p-subgroups of G are contained in $O_p(G)$, and Assertion 1) holds by the previous proposition.

Assertion 4) implies Assertion 3), since if $\mathcal{R}(FP_k)$ has a finite projective resolution, it is a projective $r\mu_k(G)$-module, indecomposable since $\mathcal{R}(FP_k)(1) = k$. Hence there exists P and E such that $\mathcal{R}(FP_k)$ is isomorphic to $L_{P,E}$. Then k is isomorphic to $\mathrm{Ind}_{N_G(P)}^G E$, and G has a normal Sylow p-subgroup by the above argument.

To show that Assertion 1) implies Assertion 4), I will use the following lemma :

LEMMA 6.13. *Let P be a p-subgroup of G and E be a projective $kN_G(P)/P$-module such that the module $\mathrm{Ind}_{N_G(P)}^G E$ is indecomposable. Let X be an $r\mu_k(G)$-module having a finite projective resolution. If $M(P,E) = \mathrm{Ind}_{N_G(P)}^G E$ is a direct summand of $X(1)$, then there exists a submodule of X isomorphic to $L_{P,E}$.*

Let indeed α be a split injection from $\mathrm{Ind}_{N_G(P)}^G E$ into $X(1)$: the map α is determined by a $N_G(P)$-homomorphism from E to $X(1)$, i.e. by a $N_G(P)/P$-homomorphism β from E to $X(1)^P$. Composing this map with the projection Br_P onto $X(1)[P]$, and next with the isomorphism σ, inverse of $Br_P r_1^P$, I get the map $\phi = \sigma Br_P \beta$ from E to $X(P)$, which gives by adjunction a morphism Φ from $L_{P,E}$ to X. The morphism $\Phi(1)$ is defined by

$$\Phi(1)(x \otimes e) = x r_1^P \sigma Br_P \beta(e)$$

whereas α is defined by

$$\alpha(x \otimes e) = x\beta(e)$$

Let then γ be a morphism from $X(1)$ to $\mathrm{Ind}_{N_G(P)}^G E$ such that $\gamma\alpha = Id$. In particular, I have $\gamma\beta(e) = 1 \otimes e$. The morphism $\gamma\Phi(1)$ is an endomorphism of the indecomposable module $\mathrm{Ind}_{N_G(P)}^G E$. Hence it is hence invertible or nilpotent. But

$$\gamma\Phi(1)(1 \otimes e) = \gamma r_1^P \sigma Br_P \beta(e)$$

and

$$Br_P r_1^P \sigma Br_P \beta(e) = Br_P \beta(e)$$

by definition of σ. Thus $r_1^P \sigma Br_P \beta(e) - \beta(e) \in \mathrm{Ker} Br_P$, hence

$$\gamma\Phi(1)(1 \otimes e) - 1 \otimes e \in \mathrm{Ker} Br_P$$

i.e.

$$\gamma\Phi(1)[P] = Id_E$$

proving that $\gamma\Phi(1)$ is not nilpotent, hence that it is invertible. Thus $\Phi(1)$ is a split injection, which proves that $\Phi(1)[Q]$ is injective for all Q, hence that Φ is injective, which proves the lemma.

Then if the conditions of Assertion 1) of the proposition hold, and if X is a minimal counter example to Assertion 4), let $M(P,E) = \mathrm{Ind}_{N_G(P)}^G E$ be an indecomposable direct summand of $X(1)$, and L be a submodule of X isomorphic to $L_{P,E}$. The quotient Y of X by L has a finite projective resolution : indeed, a module Y has a finite projective resolution if and only if there exists an integer n such that $Ext^m(Y,Z) = 0$ for all Z and any $m \geq n$. Then if

$$0 \to A \to B \to C \to 0$$

is an exact sequence, the associated long exact sequence of *Ext* groups shows that if two of the modules A, B, C have a finite projective resolution, then so does the third.

Then the minimality of X implies that Y is projective, hence direct summand of X, which is hence a direct sum of two projective modules, hence projective. This contradicts the hypothesis on X, and completes the proof of the proposition.

6.2.3. *Quasi trivial intersection.* I will say that a p-subgroup P of G is a *quasi trivial intersection subgroup* if for any $x \in G$, either $P = P^x$, or $P \cap P^x \subseteq O_p(G)$. In the case where $O_p(G) = (1)$, a quasi trivial intersection subgroup is a trivial intersection subgroup.

Proposition 6.14. *Let P be a quasi trivial intersection p-subgroup of G, and M be an indecomposable p-permutation kG-module with vertex P. Then there exists an $r\mu_k(G)$-module X such that $\mathcal{I}(X) = L_M$.*

Indeed, if $M(Q,F)$ is a direct summand of $\mathrm{Ind}_{N_G(P)}^G E$, then there exists $x \in G$ such that $P^x \cap O_p(N_G(Q)) = Q$. Then if $Q \not\subseteq O_p(G)$, the group Q is contained in a unique conjugate P^x of P, hence $N_G(Q)$ normalizes P^x, which proves that $N_{P^x}(Q) \subseteq O_p(N_G(Q))$. Then it follows that $P^x = Q$. Hence the module $\mathrm{Ind}_{N_G(P)}^G E$ is the direct sum of $M(P,E)$ and indecomposable modules with vertex contained in $O_p(G)$.

Let then M be an indecomposable p-permutation kG-module with vertex P. I denote by E the projective $kN_G(P)/P$-module $M[P]$. If P is contained in $O_p(G)$, then $\mathrm{Ind}_{N_G(P)}^G E$ is indecomposable, equal to $M(P,E)$, and the $r\mu_k(G)$-module $X = L_{P,E}$ is such that $\mathcal{I}(X)$ is projective, and $X(1) = \mathrm{Ind}_{N_G(P)}^G E = M(P,E)$. Thus X is a solution to the question for the module $M = M(P,E)$.

If P is not contained in $O_p(G)$, let X be a minimal quotient of $L_{P,E}$ such that X has a finite projective resolution, and that $X(P) = E$. Such a quotient exists, since $L_{P,E}(P) = E$.

Since X is a quotient of $L_{P,E}$, and since X has a finite projective resolution, I know that $X(1)$ is a direct summand of $\mathrm{Ind}_{N_G(P)}^G E = L_{P,E}(1)$.

Then if $M(Q,F)$ is an indecomposable direct summand of $X(1)$, either Q is conjugate to P, or Q is contained in $O_p(G)$. In this case, the module $\mathrm{Ind}_{N_G(Q)}^G F$ is indecomposable, equal to $M(Q,F)$, and the split injection from $M(Q,F)$ to $X(1)$ yields an injection from $L_{Q,F}$ into X, hence an exact sequence

$$0 \to L_{Q,F} \to X \to Y \to 0$$

Then Y is a quotient of $L_{P,E}$, since X is, and moreover $Y(P)$ is equal to E, since $L_{Q,F}(P) = 0$. Since Y has a finite projective resolution, this contradicts the hypothesis on X, which proves that Q is conjugate to P, hence that $F = E$, and that $X(1)$ is indecomposable with vertex P, such that $X(1)[P] = E$. Hence X is a solution to the question, which proves the proposition.

Proposition 6.15. *If the Sylow p-subgroups of G are trivial intersection p-subgroups, then :*

(1) *For any p-subgroup P of G and any projective $kN_G(P)/P$-module E, there exists a projective $kN_G(P \cap O_p(G))/(P \cap O_p(G))$-module F such that*

$$\mathrm{Ind}_{N_G(P)}^G E = M(P,E) \oplus \mathrm{Ind}_{N_G(P\cap O_p(G))}^G F$$

(2) *For any p-permutation kG-module M, there exists an $r\mu_k(G)$-module X such that $\mathcal{I}(X) = L_M$.*

Indeed, if the Sylow p-subgroups of $G/O_p(G)$ are trivial intersection p-subgroups, then they are quasi trivial intersection p-subgroups. If P is a p-subgroup of G, not contained in $O_p(G)$, if S is a Sylow p-subgroup of G containing P, then S is the only Sylow subgroup of G containing P.

Then if $M(Q,F)$ is an indecomposable direct summand of $\mathrm{Ind}_{N_G(P)}^G E$, there exists $x \in G$ such that $P^x \cap O_p(N_G(Q)) = Q$, and $P^x \cap O_p(G) \subseteq Q$. If Q is not contained in $O_p(G)$, then S^x is the only Sylow p-subgroup of G containing Q. In particular $N_G(Q) \subseteq N_G(S^x)$, and $N_{S^x}(Q) \subseteq O_p(N_G(Q))$. Thus $N_{P^x}(Q) \subseteq O_p(N_G(Q))$, which proves that $Q = P^x$. And if Q is contained in $O_p(G)$, then $Q = P^x \cap O_p(G)$, which proves Assertion 1).

Let then M be a p-permutation indecomposable module with vertex P, and $E = M[P]$. Let as above X be a minimal quotient of $L_{P,E}$ having a finite projective resolution, and such that $X(P) = E$.

In particular, the module $X(1)$ is a direct summand of $\mathrm{Ind}_{N_G(P)}^G E$. Thus if $P \subseteq O_p(G)$, then $X(1)$ is equal to $\mathrm{Ind}_{N_G(P)}^G E = M(P,E)$, and X fulfills the conditions of Assertion 2).

If P is not contained in $O_p(G)$, let $M(Q,F)$ be an indecomposable direct summand of $X(1)$. Then $M(Q,F)$ is a direct summand of $\mathrm{Ind}_{N_G(P)}^G E$. Thus if Q is not conjugate to P, then $Q \subseteq O_p(G)$, and $M(Q,F) = \mathrm{Ind}_{N_G(Q)}^G F$ is a direct summand of $X(1)$. The module $L_{Q,F}$ is then a submodule of X, and the quotient Y is a quotient of $L_{P,E}$ strictly smaller than X, having a finite projective resolution, and such that $Y(P) = X(P) = E$, since $L_{Q,F}(P) = 0$. This contradicts the definition of X, and proves that $Q = P$, hence that $F = E$, and that $X(1)$ is indecomposable. Since $X(1)[P] = X(P) = E$, this prove that $X(1)$ is isomorphic to M, and Assertion 2) follows.

6.2.4. *The case of cyclic groups.* Let M is a p-permutation indecomposable module with vertex P, then M is a direct summand of $\mathrm{Ind}_P^G k$. This is equivalent to saying that there exists a P-invariant linear form ϕ on M and a vector f of M^P, such that

$$Id_M = Tr_P^G(\phi \otimes f)$$

denoting by $\phi \otimes f$ the endomorphism of M defined by

$$(\phi \otimes f)(v) = \phi(v)f$$

Let then Q be a p-subgroup of G. I define an endomorphism α_Q of M by

$$\alpha_Q(v) = \sum_{x \in T_G(Q,P)/P} \phi(x^{-1}v)xf$$

so that α_1 is the identity map. Then :

Lemma 6.16. *With this notation :*

(1) *If $Q \in \underline{s}_p(G)$ and $x \in G$, then $x\alpha_Q x^{-1} = \alpha_{{}^xQ}$.*
(2) *The image of α_Q is contained in M^Q, and its kernel contains $[Q,M]$.*
(3) *If $A \supseteq B \subseteq C$, then $\alpha_A\alpha_B\alpha_C = \alpha_A\alpha_C$*
(4) *In particular, for all p-subgroup Q of G, I have*

$$\alpha_Q^3 = \alpha_Q^2$$

and α_Q^2 is a projector whose image is isomorphic to $M[Q]$.

Assertion 1) follows from the fact that $T_G({}^xQ, P) = T_G(Q, P)x$. For Assertion 2), I observe that if $q \in Q$, then

$$q\alpha_Q(v) = q \sum_{x \in T_G(Q,P)/P} \phi(x^{-1}v)xf = \sum_{x \in T_G(Q,P)/P} \phi(x^{-1}v)xq^x f = \alpha_Q(v)$$

since f is P-invariant. The second part of Assertion 2) follows then from Assertion 1), since $\alpha_Q(qv) = q\alpha_Q(v) = \alpha_Q(v)$.
Under the assumptions of Assertion 3), let $v \in M^B$. Since α_1 is the identity map, I can write

$$v = \sum_{x \in B\backslash G/P} \sum_{y \in B/B\cap {}^xP} \phi(x^{-1}y^{-1}v)yxv$$

i.e., since v is B-invariant

$$v = \sum_{x \in B\backslash G/P} \phi(x^{-1}v)Tr_{B\cap {}^xP}^B(xv)$$

But $\alpha_A(Tr_{B\cap {}^xP}^B(xv)) = 0$ if $B \not\subseteq {}^xP$, and hence

$$\alpha_A(v) = \alpha_A\alpha_B(v) \text{ si } \ v \in M^B$$

and Assertion 3) follows, as the image of α_C is contained in M^C, hence in M^B.

The first part of Assertion 4) follows, taking $A = B = C = Q$. The second one follows from the fact that for $v \in M^Q$, the vector $v - \alpha_Q(v)$ is in the kernel of Br_Q, which is contained in the kernel of α_Q.

In the case where P is cyclic, this gives :

Proposition 6.17. *Let P a cyclic p-subgroup of G, and M be an indecomposable p-permutation kG-module with vertex P. Then there exists an $r\mu_k(G)$-module X such that $\mathcal{I}(X) = L_M$.*

If the Sylow p-subgroups of G are cyclic, then for any indecomposable p-permutation module M there exists an $r\mu_k(G)$-module X such that $\mathcal{I}(X) = L_M$.

With the previous notation, I set, for any p-subgroup Q of G

$$\beta_Q = \alpha_{\Phi^n(Q)}\alpha_{\Phi^{n-1}(Q)} \ldots \alpha_{\Phi(Q)}\alpha_Q^2$$

where $\Phi(Q)$ is the Frattini subgroup of Q, the integer n being chosen such that $\Phi^{n+1}(Q) = (1)$ (the definition of β_Q does not depend on the choice of such an integer n, since $\alpha_1 = Id$). It is clear that if $x \in G$, then

$$x\beta_Q x^{-1} = \beta_{{}^xQ}$$

Moreover, if S is a subgroup of Q, there exists k such that $S = \Phi^k(Q)$, and

$$\alpha_S\beta_Q = \alpha_{\Phi^k(Q)}\alpha_{\Phi^n(Q)}\alpha_{\Phi^{n-1}(Q)} \ldots \alpha_{\Phi(Q)}\alpha_Q^2$$

hence

$$\alpha_S\beta_Q = \alpha_{\Phi^k(Q)}\alpha_{\Phi^{k+1}(Q)} \ldots \alpha_Q^2$$

which gives

$$\beta_S\beta_Q = \beta_Q$$

It follows that β_Q is a projector. Moreover, it is clear that $\beta_Q\alpha_Q = \beta_Q$, and that $\alpha_Q\beta_Q = \alpha_Q^2$ (it is the case $S = Q$, hence $k = 0$, of the above equality). In particular, the projectors β_Q and α_Q^2 have the same kernel, hence images isomorphic to $M[Q]$.

Moreover if S is a subgroup of Q, then $\beta_S\beta_Q=\beta_Q$, and the image of β_Q is contained in the image of β_S.

Denoting by r_S^Q this inclusion, I get the required maps from $M[Q]$ to $M[S]$ I define that way an $r\mu_k(G)$-module X such that $\mathcal{I}(X)$ is projective and $\mathcal{I}(X)(1)=M$, hence $\mathcal{I}(X)$ is isomorphic to L_M, which proves the proposition. The corollary follows trivially.

6.2.5. *A counter example.* Let G be the group symmetric S_5, and k be the field with two elements[7]. I will build a p-permutation kG-module M for which there is no $r\mu_k(G)$-module X such that $\mathcal{I}(X)$ is isomorphic to L_M.

Let M be a 6-dimensional vector space over k. The standard generators of S_5 act on M by the following correspondence ρ :

$$\rho((12))=\begin{pmatrix}1&0&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&0&0&0&1\end{pmatrix}\quad \rho((23))=\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{pmatrix}$$

$$\rho((34))=\begin{pmatrix}1&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\end{pmatrix}\quad \rho((45))=\begin{pmatrix}1&0&0&1&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}$$

Then $\rho((12))$, $\rho((23))$, and $\rho((34))$ act on the basis of M via the following permutations

$$\rho((12))\mapsto(23)(45)\qquad \rho((23))\mapsto(12)(56)\qquad \rho((34))\mapsto(24)(35)$$

It is then clear that the restriction of ρ to the subgroup S_4 generated by (12), (23), and (34) is a permutation representation of S_4.

An elementary computation shows that $\rho((45))$ commutes with $\rho((12)$ and $\rho((23))$, and that the product $\rho((34))\rho((45))$ has order 3. It follows that ρ is a representation of S_5, and that M is a kS_5 module.

Since the restriction of M to S_4 is a permutation module, and since S_4 contains a Sylow 2-subgroup of S_5, the module M is a 2-permutation kS_5-module.

Let S the Sylow 2-subgroup of S_5 generated by (12), (34) and (13)(24). Then S stabilizes the canonical basis $B=\{e_1,\ldots,e_6\}$ of M. Since

$$(13)(24)=(23)(34)(12)(23)$$

the element (13)(24) of S acts on B by the permutation

$$(12)(56)(24)(35)(23)(45)(12)(56)=(16)(34)$$

The orbit of S on B are hence $\{e_1,e_6\}$ and $\{e_2,e_3,e_4,e_5\}$. The stabilizer of e_1 in S is the group P generated by (12) and (34), and the stabilizer of e_2 in S is the group T generated by (13)(24). It follows that

$$\mathrm{Res}_S^{S_5}M=\mathrm{Ind}_P^S k\oplus\mathrm{Ind}_T^S k$$

[7] so $p=2$ here

The indecomposable direct summands of M are hence of vertex contained in P or in T up to conjugation. The module M has no projective direct summand, since its dimension is smaller than the 2-part of the order of S_5. On the other hand, the module $M[T]$ has a basis in one to one correspondence with $B^T = \{e_2, e_5\}$. Since $N_{S_5}(T)/T$ has order 4, it follows that $M[T]$ has no $kN_{S_5}(T)/T$-projective direct summand. Hence the module M only has indecomposable direct summands with non trivial vertex contained in P.

The module $M[P]$ has a basis in one to one correspondence with B^P, consisting of the vectors e_1 and e_6. The normalizer of P in S_5 is equal to S, and the group S switches e_1 and e_6. It follows that the module $M[P]$ is $kN_{S_5}(P)/P$-indecomposable and projective, hence that M has a direct summand isomorphic to $M(P, E_k)$, where E_k denotes a projective cover of the trivial $kN_{S_5}(P)/P$-trivial.

Let then Q be the subgroup of P generated by (12). The module $M[Q]$ has a basis in one to one correspondence with B^Q, i.e. $\{e_1, e_6\}$. The normalizer of Q, isomorphic to $\mathbf{Z}/2\mathbf{Z} \times S_3$, is generated by Q and the elements (34) and (45), which act trivially on the quotient of M by the subspace generated by $\{e_2, e_3, e_4, e_5\}$. Thus $N_{S_5}(Q)/Q$ acts trivially on $M[Q]$, hence this $kN_{S_5}(Q)/Q$-module has no projective direct summand. It follows that M has no direct summand with vertex Q.

The only proper non trivial subgroup of P which is not conjugate to Q is the group U generated by $(12)(34)$, which acts on B by the permutation $(25)(34)$. Thus $M[U]$ has dimension 2. Since $N_{S_5}(U)/U$ has order 4, the module $M[U]$ has no projective direct summand as $kN_{S_5}(U)/U$-module, and the module M has no direct summand with vertex U.

The module M is hence indecomposable of vertex P. Since $M[P]$ is the projective cover of the trivial module, the module M is the Scott module of S_5 for the subgroup P (cf. [**2**]).

Then if it is possible of find suitable restriction maps, in particular, the map r_1^P must be injective, as well as that the map r_Q^P, since $r_1^P = r_1^Q r_Q^P$. Since $M[P]$ and $M[Q]$ have the same dimension, the map r_Q^P must be surjective. Then r_1^P and r_1^Q have the same image W, which must be a subspace of M^P invariant by the group H generated by $N_{S_5}(P)$ and $N_{S_5}(Q)$.

The group H is a transitive subgroup of S_5 which contains a transposition. Thus $H = S_5$. On the other hand, a vector v of M^P is of the form

$$v = \begin{pmatrix} a \\ b \\ b \\ b \\ b \\ c \end{pmatrix}$$

The image of v by $\rho((23))$ is the vector

$$\rho((23))(v) = \begin{pmatrix} b \\ a \\ b \\ b \\ c \\ b \end{pmatrix}$$

Thus if $v \in W$, then $\rho((23))(v) \in W$ and then $a = b = c$. Hence the only subspace of M^P which is invariant by S_5 has dimension 1, generated by the vector

$$v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

It is the module M^{S_5}. The space W cannot have dimension 2, hence it is impossible to find an $r\mu_k(G)$-module X such that $\mathcal{I}(X) = L_M$.

7. Complexes of projective Mackey functors

I suppose here that the ring R is a complete local ring, whose residue field k has characteristic p.

7.1. Split complexes. Let A be a ring and L^* be a complex of A-modules, whose differential d has degree 1. In [**7**], Webb gives the following definition of a *split* complex : the complex L^* is split if there exists maps α_n from L^n to L^{n-1} such that for all n,

$$d^n\alpha_{n+1}d^n = d^n$$

Lemma 7.1. *The following conditions are equivalent :*

(1) *The complex L^* is split.*
(2) *The complex L^* is homotopic to a complex with zero differentials.*

Indeed, if M^* is a complex with zero differentials, and if f is a homotopy equivalence from L^* to M^*, with inverse g, then there exists maps α_n from L^n to L^{n-1} such that

$$Id - g_nf_n = d^{n-1}\alpha_n + \alpha_{n+1}d^n$$

Then $d^n\alpha_{n+1}d^n = d^n - d^ng_nf_n$, and d^ng_n is zero since g is a morphism of complexes and since the differential of M^* is zero.

Conversely, if L^* is split, let $M^n = H^n(L^*)$ its n-th homology group. I consider M^* as a complex with zero differentials. Let Z^n be the kernel of d^n, let i_n be the injection from Z^n to L^n, and p_n the projection from Z^n to M^n. The image of $Id - \alpha_{n+1}d^n$ is contained in Z^n by hypothesis, and I set

$$f_n = p_n(Id - \alpha_{n+1}d^n)$$

Conversely, the map $Id - d^{n-1}\alpha_n$ sends Z^n inside itself, and its restriction to Z^n factors through M^n, i.e.

$$Id - d^{n-1}\alpha_n = h_np_n$$

and I denote by g_n the composition of h_n and i_n. It is then clear that if $u = p_n(v) \in M^n$

$$f_ng_n(u) = p_n(Id - \alpha_{n+1}d^n)i_nh_np_n(v) = p_n(Id - \alpha_{n+1}d^n)(Id - d^{n-1}\alpha_n)(v)$$

i.e., since $d^n(v) = 0$

$$f_ng_n(u) = p_n(v - d^{n-1}\alpha_n(v)) = p_n(v) = u$$

Conversely, if $w \in L^n$, then

$$g_nf_n(w) = i_nh_np_n(Id - \alpha_{n+1}d^n)(w) = i_n(Id - d^{n-1}\alpha_n)(Id - \alpha_{n+1}d^n)(w)$$

or

$$g_n f_n = Id - \alpha_{n+1} d^n - d^{n-1}(\alpha_n - \alpha_n \alpha_{n+1} d^n)$$

and setting $\beta_n = \alpha_n - \alpha_n \alpha_{n+1} d^n$, I have indeed

$$g_n f_n = Id - \beta_{n+1} d^n - d^{n-1} \beta_n$$

which shows that L^* is homotopic to M^*.

It follows from this lemma that L^* is homotopic to the zero complex if and only if L^* is acyclic and split in the sense of the above definition.

Corollary 7.2. *Let L^* be a complex of A-modules. If for any integer n, there exists a complex M^* (possibly depending on n) homotopic to L^* and such that $M^n = 0$, then L^* is homotopic to the zero complex.*

Indeed in those conditions, the complex L^* is certainly acyclic, since $H^n(L^*) = H^n(M^*) = 0$. Moreover, if f is a homotopy equivalence from L^* to M^*, with inverse g, then there exists maps α_m from L^m to L^{m-1} and β_m from M^m to M^{m-1} such that for all m

$$Id - g_m f_m = \alpha_{m+1} d^m + d^{m-1} \alpha_m$$

In particular, I see that $d^n - d^n g_n f_n = d^n \alpha_{n+1} d^n$, and since $M^n = 0$, I have $d^n g_n = 0$. Then the complex L^* is split, hence it is homotopic to the zero complex.

7.2. Split complexes of projective Mackey functors. Lemma 6.3 and Proposition 6.4 have the following generalization :

Proposition 7.3. *Let L^* be a complex of projective functors in $Mack_R(G,1)$, such that there exists n with $L^n = 0$. The following conditions :*

1. *The complex L^* is acyclic and split.*
2. *For any p-subgroup P of G, the complex $\overline{L}^*(P)$ is acyclic.*

It is clear that Assertion 1) implies Assertion 2). Conversely, I can e.g. suppose that L^{-1} is zero. Lemma 6.3 shows then that d^0 is a split injection, which amounts to saying that the complex L^* is homotopic to the complex

$$\dots L^{-2} \xrightarrow{d^{-2}} 0 \to 0 \to 0 \to L^1/L^0 \xrightarrow{d'^1} L^2 \to \dots$$

Lemma 6.3 also shows that d^{-3} is a split surjection. By induction, it follows that the complex L^* is homotopic to a complex with an arbitrary number of consecutive zero modules around L_0, and the result follows by Corollary 7.2.

Similarly :

Proposition 7.4. *Let M^* a complex of p-permutation RG-modules, such that there exists n with $M^n = 0$. The following conditions are equivalent :*

1. *The complex M^* is acyclic and split.*
2. *For any p-subgroup of G, the complex $M^*[P]$ is acyclic.*

If K^* and L^* are complexes of modules over an arbitrary ring, to any morphism f from K^* to L^* is associated a third complex, the cone of f, denoted by $C^*(K, L, f)$ with the following properties :

- The complex $C^*(K, L, f)$ is acyclic if and only if $H^n(f)$ is a isomorphism for all n (i.e. if f is a *quasi isomorphism*)
- The complex $C^*(K, L, f)$ is acyclic and split if and only if f is a homotopy equivalence.

- The module $C^n(K,L,f)$ is the direct sum of K^n and L^{n-1}.

The third property shows that if K^* and L^* are complexes of Mackey functors in $Mack_R(G,1)$, then so is $C^*(K,L,f)$. Similarly, if K^* and L^* are complexes of p-permutations modules, then so is $C^*(K,L,f)$. The above propositions and remarks yield then the following two propositions :

Proposition 7.5. *Let K^* and L^* be complexes of projective Mackey functors in $Mack_R(G,1)$, such that there exists n with $K^n = L^n = 0$ and $K^{n-1} = L^{n-1} = 0$. Let f be a morphism from K^* to L^*. The following conditions are equivalent :*

(1) *The morphism f is a homotopy equivalence.*
(2) *For any p-subgroup P of G, the morphism $\overline{f}(P)$ is a quasi isomorphism.*

Proposition 7.6. *Let M^* and N^* be complexes of p-permutation kG-modules, such that there exists m with $M^n = N^n = 0$ and $M^{n-1} = N^{n-1} = 0$. Let f be a morphism from M^* to N^*. The following conditions are equivalent :*

(1) *The morphism f is a homotopy equivalence.*
(2) *For any p-subgroup P of G, the morphism $f[P]$ is a quasi isomorphism.*

7.3. Complexes of p-permutations modules. I will try to see how to extend the previous results when the hypothesis “for all P” is replaced by “for all non trivial P” : the consequence of this change will be the replacement of zero modules by projective modules, of zero Mackey functors by projective Mackey functors with trivial vertex, of split acyclic complexes by complexes homotopic to complexes of projective modules.

I suppose that R is a field k. I call *p-permutations complex* a complex of p-permutations kG-modules.

I will try to find here under which conditions such a complex is homotopic to a complex of projective kG-modules. First, I can suppose that G is a p-group

Lemma 7.7. *Let L^* be a complex of finitely generated kG-modules, and S be a Sylow p-subgroup of G. The following conditions are equivalent :*

(1) *The complex L^* is homotopic to a complex of projective kG-modules.*
(2) *The restriction of L^* to S is homotopic to a complex of projective kS-modules.*

It is clear that Assertion 1) implies Assertion 2). Conversely, let E^* be a complex of projective kS-modules, and a be a homotopy equivalence from E^* to $\mathrm{Res}_S^G L^*$, with inverse b. Then the complex $\mathrm{Ind}_S^G E^*$ is a complex of projective kG-modules, and by adjunction, the morphism a and b yield morphisms A and B between $\mathrm{Ind}_S^G E^*$ and L^*. It is easy to see that the morphism AB is homotopic to $[G:S]Id$. The lemma follows then from the following lemma :

Lemma 7.8. *Let M^* and L^* be complexes of finitely generated kG-modules. Let A be a morphism from M^* to L^* and B be a morphism from L^* to M^* such that AB is homotopic to the identity of L^*. Then L^* is homotopic to a direct summand of M^*.*

Indeed, let $L_1^* = \cap_n (AB)^n(L^*)$, and $L_2^* = \sum_n Ker(AB)^n$. Then L_1^* and L_2^* are subcomplexes of L^*, invariant by AB, and L^* identifies with the direct sum of the complexes L_1^* and L_2^*. The restriction of AB to L_1^* and L_2^* is homotopic to the identity. Since in each degree n, the restriction of AB to L_2^n is nilpotent, it follows that for any n, there exists a complex K^* homotopic to L_2^*, such that $K^n = 0$. Then the complex L_2^* is homotopic to the zero complex, and the complex L^* is homotopic to the complex L_1^*.

On the other hand, the complex L_1^* is isomorphic to the complex $B(L_1^*)$, which is homotopic to $\cap_n (BA)^n(M^*)$, which is direct summand of M^*. This proves the lemma.

The previous lemmas allow to show the

Proposition 7.9. *Let L^* be a complex of p-permutation kG-modules, such that there exists an integer n for which L^n is projective. The following conditions are equivalent :*

1. *The complex L^* is homotopic to a complex of projective kG-modules.*
2. *For any non trivial p-subgroup P of G, the complex $L^*[P]$ is acyclic.*

It is clear that Assertion 1) implies Assertion 2). To prove the converse, I need a notation :

If X and Y are kG-modules, I will denote by $J_{kG}(X, Y)$ the set of kG-homomorphisms f from X to Y such that for any kG-homomorphism g from Y to X, the morphism gf is nilpotent.

For example, $J_{kG}(X, X)$ is the Jacobson radical of $\mathrm{End}_{kG}(X)$. It is easy to see on the other hand that this definition is additive with respect to X and Y : the canonical isomorphism between $\mathrm{Hom}_{kG}(X, Y \oplus Z)$ and $\mathrm{Hom}_{kG}(X, Y) \oplus \mathrm{Hom}_{kG}(X, Z)$ induces indeed an isomorphism between $J_{kG}(X, Y \oplus Z)$ and $J_{kG}(X, Y) \oplus J_{kG}(X, Z)$. Similarly, if X^* and Y^* are the respective duals of X and Y, then $\phi \in J_{kG}(X, Y)$ if and only if $\phi^* \in J_{kG}(Y^*, X^*)$: indeed, the morphism $\psi^*\phi^*$ is nilpotent if and only if the morphism $\phi^*\psi^* = (\psi\phi)^*$ is.

By the previous lemma, I can suppose that G is a p-group. Let then n be an arbitrary integer, and ϕ be a morphism from L^{n+1} to L^n, such that ϕd^n is not nilpotent. The diagram

$$\begin{array}{ccccccc}
\ldots L^{n-1} & \rightarrow & L^n & \stackrel{d^n}{\rightarrow} & L^{n+1} & \rightarrow & L^{n+2}\ldots \\
0 \downarrow & & \phi d^n \downarrow & \stackrel{\phi}{\swarrow} & \downarrow d^n\phi & & \downarrow 0 \\
\ldots L^{n-1} & \rightarrow & L^n & \stackrel{d^n}{\rightarrow} & L^{n+1} & \rightarrow & L^{n+2}\ldots
\end{array}$$

defines then a endomorphism γ of L^*, homotopic to 0. I can then replace L^* by its direct summand $\cap_m (Id+\gamma)^m(L^*)$, which is homotopic to it. The only modules modified by this operation are L^n and L^{n+1}, which are replaced by some direct summands L'^n and L'^{n+1}. The differentials d^{n-1}, d^n and d^{n+1} become respectively d'^{n-1}, d'^n and d'^{n+1}. If $d^{n-1} \in J_{kG}(L^{n-1}, L^n)$, then $d'^{n-1} \in J_{kG}(L'^{n-1}, L'^n)$. Similarly, if $d^{n+1} \in J_{kG}(L^{n+1}, L^{n+2})$, then $d'^{n+1} \in J_{kG}(L'^{n+1}, L'^{n+2})$.

Hence I can suppose that $d_n \in J_{kG}(L^n, L^{n+1})$ for all n.

Let then l be an integer such that L^l is a projective kG-module. If one of the modules L^n, for $n > l$, is not projective, let n be the smallest integer greater than l such that L^n is not projective.

Let moreover Q be a maximal subgroup of G which is a vertex of some indecomposable direct summand of L^n. The only indecomposable p-permutation kG-module with vertex Q is $\mathrm{Ind}_Q^G k$, because G is a p-group. The module M is then the sum of its part A with vertex Q, isomorphic to a sum of copies of $\mathrm{Ind}_Q^G k$, and a module X such that $X[Q] = 0$. On the other hand, the module L^{n+1} is the sum of its part B of vertex Q, of a module Z such that $Z[Q]=0$, and of a module Y having only indecomposable summands with vertex strictly bigger than Q.

Then d^n can be represented by a matrix

$$\begin{pmatrix} \phi_{B,A} & \phi_{B,X} \\ \phi_{Y,A} & \phi_{Y,X} \\ \phi_{Z,A} & \phi_{Z,X} \end{pmatrix}$$

The morphism $d^n[Q]$ can then be represented by the matrix

$$\begin{pmatrix} \phi_{B,A}[Q] \\ \phi_{Y,A}[Q] \end{pmatrix}$$

Then if $Q \neq (1)$, as the complex $L^*[Q]$ is acyclic, and as $L^{n-1}[Q]=0$, the map $d^n[Q]$ must be injective.

The morphism $\phi_{B,A}$ is a matrix with coefficients in $\mathrm{End}_{kG}(\mathrm{Ind}_Q^G k)$, whose every coefficient is actually in $J(\mathrm{End}_{kG}(\mathrm{Ind}_Q^G k))$, hence maps the socle of $\mathrm{Ind}_Q^G k$ (isomorphic to k) to zero, since $\phi \in J_{kG}(M,N)$. Thus $\phi_{B,A}$ maps the socle of A to zero.

But this socle maps onto the socle of $A[Q]$. Thus $\phi_{Y,A}[Q]$ restricted to this socle must be injective, hence $\phi_{Y,A}[Q]$ must be injective. Then $A[Q]$, which is projective as an $N_G(Q)/Q$ module, maps injectively into $Y[Q]$, hence $A[Q]$ is a direct summand of $Y[Q]$. But since Y only has direct summands with vertex strictly bigger than Q, this is impossible, and this contradiction proves that $Q=(1)$, hence that L^n is projective.

Thus L^n is projective for all $n \geq l$. I can then apply the same argument to the dual complex $\mathrm{Hom}_k(L^*,k)$ to show that L_n is projective for all $n \leq l$, hence that L^* is a complex of projective modules. The proposition follows.

It yields the following corollary, which is a slightly more precise version of a precise a result of Webb (cf.[**8**]) :

Corollary 7.10. *Let Δ be a simplicial complex on which acts the group G. The following conditions are equivalent :*

(1) *The chain complex $C(\Delta)$ of Δ over k is homotopic to a complex of projective kG-modules.*
(2) *For any subgroup P of order p of G, the set Δ^P is acyclic modulo p.*

Indeed, a classical argument shows that if Assertion 2) holds, then Δ^P is acyclic modulo p for all non trivial p-subgroup P of G. On the other hand, the complex $C(\Delta)[P]$ is the chain complex of Δ^P.

7.4. Complexes of projective Mackey functors. I suppose here again that R is a field k of characteristic p. Propositions 7.4 and 7.6 have the following consequence :

Proposition 7.11. *Let X^* be a complex of projective functors in $Mack_k(G,1)$, such that there exists an integer n for which $X^n = X^{n+1} = 0$. The following conditions are equivalent :*

(1) *There exists a complex L^* of projective kG-modules such that X^* is homotopic to the complex FQ_{L^*}.*
(2) *For any non trivial p-subgroup P of G, the complex $\overline{X}^*(P)$ is acyclic.*
(3) *The complex $X^*(1)$ is homotopic to a complex of projective modules.*
(4) *The complex X^* is homotopic to the complex $FQ_{X^*(1)}$.*

Indeed, if M is a kG-module and Y is a Mackey functor of the form FQ_M, it is easy to see that $\overline{Y}(P)$ is zero when P is non trivial, because the traces t_R^P are surjective for $R \neq P$. It is then clear that Assertion 1) implies the Assertion 2). Similarly, Assertion 4) implies Assertion 2).

Similarly, Assertion 2) implies that the complex $X^*(1)$ fulfills the hypothesis of Assertion 2) of Proposition 7.9. Then there exists a complex L^* of projective kG-modules such that $X^*(1)$ is homotopic to L^*. Thus Assertion 2) implies Assertion 3).

It is clear that Assertion 3) implies Assertion 2), since $\overline{X}^*(P)$ identifies with $X^*(1)[P]$.

Let then L^* be a complex of projective modules homotopic to the complex $X^*(1)$. Up to replacing L^* by a direct summand, I can suppose that $L^n = L^{n+1} = 0$. Let then a be a homotopy equivalence from L^* to $X^*(1)$. By adjunction, the morphism a yields a morphism A from the complex FQ_{L^*} to the complex X^* : the morphism $A^n(H)$ from $(L^n)_H$ to $X^n(H)$ is given by

$$A^n(v) = t_1^H a^n(v)$$

The complex $X^*(1)[P]$ is homotopic to the complex $\overline{X}^*(P)$, and to the complex $L^*[P]$, which is zero if $P \neq (1)$. The complex $\overline{X}^*(P)$ is hence acyclic and split in this case. The complex $\overline{FQ_L^*}(P)$ is zero if $P \neq (1)$. Then the morphism $\overline{A}(P)$ is trivially a quasi isomorphism if $P \neq (1)$. Since the morphism $\overline{A}(1)$ is equal to $A(1)$, hence to a, it is a homotopy equivalence, hence a quasi isomorphism.

Then the complexes FQ_{L^*} and X^* fulfill the hypotheses of Assertion 2 of Proposition 7.5, and A is a homotopy equivalence, which proves that Assertion 3) implies Assertion 1). Hence Assertions 1), 2) and 3) are equivalent.

Then if Assertion 1) is true, i.e. if X^* is homotopic to the complex FQ_{L^*}, the complex $X^*(1)$ is homotopic to the complex $FQ_{L^*}(1)$, i.e. to the complex L^*, hence the complex $FQ_{X^*(1)}$ is homotopic to the complex FQ_{L^*}, hence also to the complex X^*. Thus Assertion 1) implies Assertion 4), and this completes the proof of the proposition.

I will use this result to give a slightly more precise version of a theorem of Webb (cf.[**8**] Theorem 1) in a particular case. For this, I need a notation : if the group G acts on the set X, then X can be decomposed in a disjoint union of G-orbits, which are G-sets of the form G/H. Then if M is a Mackey functor, Webb denotes by M_X the Mackey functor defined by

$$M_{X \coprod Y} = M_X + M_Y \qquad\qquad M_{G/H} = \mathrm{Ind}_H^G \mathrm{Res}_H^G M$$

On the other hand, if Δ is a simplicial complex, then Δ_i denotes the set of simplices of Δ of dimension i (or of cardinal $i+1$). Webb's Theorem is then the following :

Theorem (Webb) : Let Δ be a simplicial complex on which acts the group G. Let $Y \subset X$ be sets of subgroups of G, closed under inclusion and G-conjugation. Suppose that :

(1) For any simplex s of Δ, the vertices of s are in distinct G-orbits.
(2) The functor M is projective relative to X.
(3) For any $H \in X - Y$, the complex Δ^H is contractible.
(4) For any $H \in Y$, the module $M(H)$ is zero.

Then there exists a split exact sequence of Mackey functors

$$0 \to M \to M_{\Delta_0} \to \ldots \to M_{\Delta_i} \to \ldots \to 0$$

I will consider here the case where X is the set of p-subgroups of G, and Y consists only of the trivial subgroup of G. Then a Mackey functor projective relative to X is a functor in $Mack_k(G,1)$. Denoting by $C_i(\Delta)$ the kG-module with k-basis Δ_i,

Proposition 7.12. *Let M be a Mackey functor in $Mack_k(G,1)$, and Δ be a simplicial complex on which G acts. If :*

(1) *For any simplex s of Δ, the vertices of s are in distinct G-orbits,*
(2) *For any p-subgroup P of order p of G, the complex Δ^P is acyclic modulo p,*

then the complex

$$0 \to M \to M_{\Delta_0} \to \ldots \to M_{\Delta_i} \to \ldots \to 0$$

is homotopic to the complex

$$0 \to FQ_{M(1)} \to FQ_{M(1)\otimes C_0(\Delta)} \to \ldots \to FQ_{M(1)\otimes C_i(\Delta)} \to \ldots \to 0$$

To prove this proposition, I will apply Proposition 7.11 to the complex b^*

$$0 \to (b_p) \to (b_p)_{\Delta_0} \to \ldots \to (b_p)_{\Delta_i} \to \ldots \to 0$$

where I denote by b_p the functor $z(f_1^G)b_G$. It is a projective functor in $Mack_k(G,1)$, and since the above complex is finite, it vanishes in two consecutive degrees. The value at (1) of this complex is the chain complex of Δ, which is homotopic to a complex of projective modules by Corollary 7.10. Proposition 7.11 now shows that the complex b^* is homotopic to the complex $FQ_{b^*(1)} = FQ_{C^*(\Delta)}$.

But for any Mackey functor M in $Mack_k(G,1)$, the functor $H(b_p, M)$ is isomorphic to the functor M. Then the functor $H((b_p)_{\Delta_i}, M)$, isomorphic to $H(b_p, M_{\Delta_i})$, is isomorphic to M_{Δ_i}. Then if N denotes the dual functor of M, the complex

$$0 \leftarrow H((b_p), N) \leftarrow H((b_p)_{\Delta_0}, N) \leftarrow \ldots \leftarrow H((b_p)_{\Delta_i}, N) \leftarrow \ldots \leftarrow 0$$

is isomorphic to the complex

$$0 \leftarrow N \leftarrow N_{\Delta_0} \leftarrow \ldots \leftarrow N_{\Delta_i} \leftarrow \ldots \leftarrow 0$$

and this complex is homotopic to the complex $H(FQ_{C_*(\Delta)}, N)$. It is clear on the other hand that for any kG-module V, the functor $H(FQ_V, N)$ is isomorphic to the functor $FP_{\mathrm{Hom}_k(V,N(1))}$. Since the dual of this functor is the functor $FQ_{\mathrm{Hom}_k(N(1),V)}$, and since $\mathrm{Hom}(N(1), C_i(\Delta))$ is isomorphic to $C_i(\Delta) \otimes M(1)$, the dual of the above complex, which is the complex

$$0 \to M \to M_{\Delta_0} \to \ldots \to M_{\Delta_i} \to \ldots \to 0$$

is indeed homotopic to the complex

$$0 \to FQ_{M(1)} \to FQ_{M(1)\otimes C_0(\Delta)} \to \ldots \to FQ_{M(1)\otimes C_i(\Delta)} \to \ldots \to 0$$

which proves the proposition.

Remark 7.13. Hypothesis 1) plays no role in the proof, but it is necessary to build the complex of the proposition (cf.[**8**]).

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