# A remark on the Dade group and the Burnside group

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**Abstract:** The object of this note is to show that the formula for tensor induction of relative syzygies in the Dade group, stated in [3], can be viewed as a special case of a functorial homomorphism from the dual $B^*$ of the Burnside group to the subgroup $D^\Omega$ of the Dade group generated by relative syzygies. It follows that there exists a short exact sequence of functors

$$0 \to R_{\mathbb{Q}}^* \to B^* \to D^\Omega/D^\Omega_{tors} \to 0 \quad ,$$

where $R_{\mathbb{Q}}$ is the functor of rational representations. This may be viewed as an improvement (from $\mathbb{Q}$ to $\mathbb{Z}$) of Theorem D of [5].

## 1. Introduction

**1.1.** Let $p$ be a prime number, and $k$ be a field of characteristic $p$. In [6],[7], E.C. Dade defined a group structure on the set $D(P) = D_k(P)$ of equivalence classes of endo-permutation $kP$-modules, which is now called the Dade group of $P$. This group can also be viewed as the set of equivalence classes of permutation $P$-algebras over $k$, under a suitable relation.

Most examples of endo-permutation modules are provided by relative syzygies, defined as follows : let $X$ be a finite $P$-set, then the relative syzygy $\Omega_X$ of the trivial module with respect to $X$ is defined as the kernel of the augmentation map $kX \to k$ sending each element of the set $X$ to 1.

In [3], it was shown that the subgroup $D^\Omega(P)$ generated by these relative syzygies is invariant under the natural functorial operations of restriction, inflation, deflation, and tensor induction on the Dade group. These operations can be defined using the corresponding operations of restriction, inflation, Brauer quotient, and tensor induction of permutation algebras. In particular, a rather complicated formula for tensor induction of relative syzygies was stated :

**1.2. Theorem :** [**[3] Theorem 5.1.2**] *Let $Q \subseteq P$ be $p$-groups. Let $X$ be a non-empty finite $Q$-set. Then in the Dade group $D(P)$*

$$\mathrm{Ten}_Q^P \Omega_X = \sum_{\substack{U,V\in[s_P]\\ U\leq_P V}} \mu_P(U,V)\ |\{a \in V\backslash P/Q \mid X^{V^a\cap Q} \neq \emptyset\}|\ \Omega_{P/U} \quad ,$$

*where $[s_P]$ is a set of representatives of the poset $s_P$ of conjugacy classes of subgroups of $P$, where $\mu_P$ is the Möbius function of the poset $s_P$, and $X^{V^a\cap Q}$ is the set of fixed points of $X$ under $V^a \cap Q$.*

**1.3.** The object of this note is to give an interpretation of the previous formula in perhaps more conceptual terms, using the following definition :

Date : April 17, 2003

**1.4. Definition :** *Let $B(P)$ denote the Burnside group of $P$, and let*

$$B^*(P) = \mathrm{Hom}_{\mathbb{Z}}\big(B(P), \mathbb{Z}\big)$$

*denote the dual group.*

*If $X$ is a finite $P$-set, let $\omega_X$ be the element of $B^*(P)$ defined on the canonical basis of $B(P)$ by*

$$\omega_X(P/Q) = \begin{cases} 1 & \text{if } X^Q \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad ,$$

*where $Q$ is a subgroup of $P$.*

**1.5.** Now let $\mathcal{C}_p$ denote the following category :

- The objects of $\mathcal{C}_p$ are the finite $p$-groups.
- If $P$ and $Q$ are finite $p$-groups, then $\mathrm{Hom}_{\mathcal{C}_p}(P, Q)$ is the Grothendieck group of finite $(Q, P)$-bisets, or equivalently the Burnside group $B(Q \times P^{op})$.
- The composition in $\mathcal{C}_p$ is bilinear, and if $U : P \to Q$ is a finite $(Q, P)$-biset and $V : Q \to R$ is a finite $(R, Q)$-biset, then the composition $V \circ U$ is equal to $V \times_Q U$.

The category $\mathcal{C}_p$ is preadditive, in the sense of Mac Lane ([8]), and the correspondence sending a $p$-group $P$ to its Burnside group $B(P)$ is a functor from $\mathcal{C}_p$ to the category $\mathcal{A}b$ of abelian groups : if $Q$ is another finite $p$-group and $U$ is a finite $(Q, P)$-biset, then the map $B(U) : B(P) \to B(Q)$ is the linear map sending the class of the $P$-set $X$ to the class of the $Q$-set $U \times_P X$.

Similarly, the correspondence $P \mapsto B^*(P)$ is an additive functor from $\mathcal{C}_p$ to $\mathcal{A}b$ : with the same notation, the map $B^*(U) : B^*(P) \to B^*(Q)$ is defined by

$$B^*(U) = {}^tB(U^{op}) \quad ,$$

where ${}^tB(U^{op})$ denote the transposed map of $B(U^{op})$, and $U^{op}$ is the $(P, Q)$-biset with underlying set equal to $U$, and $(P, Q)$-action defined by

$$\forall (g, h) \in P \times Q,\ \forall u \in U,\ g.u.h \text{ (in } U^{op}) = h^{-1}ug^{-1} \text{ (in } U) \quad .$$

**1.6.** In [5] Section 3, we attached to any endomorphism $a$ of the field $k$, and to any finite $p$-group $P$, a map $\gamma_a : D(P) \to D(P)$, and we observed in Example (3.3) that the element $\Omega_{P/1}$ of $D(P)$ is invariant under $\gamma_a$. The same argument shows more generally that if $X$ is a finite $P$-set, then $\gamma_a(\Omega_X) = \Omega_X$. It now follows from Proposition (3.10) of [5], and from Sections 4 and 5 of [3], that the correspondence sending $P$ to $D^\Omega(P)$ is also an additive functor from $\mathcal{C}_p$ to $\mathcal{A}b$.

The main results of this note can now be stated :

**1.7. Theorem :** *There is a unique natural transformation $\Theta : B^* \to D^\Omega$ of additive functors from $\mathcal{C}_p$ to $\mathcal{A}b$, with the property that*

$$\Theta_P(\omega_X) = \Omega_X$$

*for any finite $p$-group $P$ and any finite $P$-set $X$.*

Let $R_\mathbb{Q}$ denote the functor from $\mathcal{C}_p$ to $\mathcal{A}b$ sending a $p$-group $P$ to the group $R_\mathbb{Q}(P)$ of its rational represention, and let $R_\mathbb{Q}^*$ denote the dual functor. It follows from a theorem of Ritter and Segal ([4]) that the natural transformation $B \to R_\mathbb{Q}$ sending the finite $P$-set $X$ to the permutation module $\mathbb{Q}X$ is surjective. This gives by duality an injective natural transformation $i : R_\mathbb{Q}^* \to B^*$.

**1.8. Theorem :** *The image of the natural transformation $\Theta \circ i$ is equal to the torsion part $D^\Omega_{tors}$ of $D^\Omega$. In other words, there is an exact sequence of functors from $\mathcal{C}_p$ to $\mathcal{A}b$*

$$0 \to R^*_{\mathbb{Q}} \to B^* \to D^\Omega / D^\Omega_{tors} \to 0 \quad .$$

**1.9. Remark:** Theorem 1.8 gives in some sense an explanation for the exact sequence of functors of Theorem D of [5] (see also Proposition 7.6.2 of [3]) : this sequence can be obtained by applying the functor $\mathrm{Hom}_{\mathcal{A}b}(-, \mathbb{Q})$ to the previous one, giving finally

$$0 \to \mathbb{Q}D \to \mathbb{Q}B \to \mathbb{Q}R_{\mathbb{Q}} \to 0 \quad ,$$

since moreover $\mathbb{Q}D \cong \mathbb{Q}D^*$ (see Remark 10.3 of [5]), and since $\mathbb{Q}D \cong \mathbb{Q}D^\Omega$ (by Proposition 7.4.9 of [3]).

## 2. Proof of Theorem 1.7

**2.1.** Theorem 1.2 follows easily from Theorem 1.7, by expressing that the map $\Theta$ commutes with (tensor) induction. Unfortunately, I couldn't find any direct proof of Theorem 1.7, and the only proof I know uses Theorem 1.2.

**2.2. Lemma :** *The set of elements $\omega_{P/Q}$, for $Q \in [s_P]$, is a $\mathbb{Z}$-basis of $B^*(P)$.*
**Proof:** Let $\delta_{P/Q}$ be the element of $B^*(P)$ defined by

$$\delta_{P/Q}(P/R) = \begin{cases} 1 & \text{if } Q \text{ and } R \text{ are conjugate in } P \\ 0 & \text{otherwise} \end{cases} \quad .$$

Then the set of elements $\delta_{P/Q}$, for $Q \in [s_P]$, is the dual basis of the canonical basis of $B(P)$. Now if $R$ is a subgroup of $P$, one has that

$$\omega_{P/R} = \sum_{\substack{Q \in [s_P] \\ Q \leq_P R}} \delta_{P/Q} \quad ,$$

since for any subgroup $S$ of $P$, the set $(P/R)^S$ is non empty if and only if $S \leq_P R$. Hence the set of elements $\omega_{P/R}$, for $R \in [s_P]$, is obtained from the basis $(\delta_{P/R})_{R \in [s_P]}$ of $B^*(P)$ by a matrix which is triangular with 1 on the diagonal, for a suitable ordering of the set $[s_P]$. The lemma follows. □

**2.3. Remark:** By definition of the Möbius function of $s_P$, it follows that

$$\delta_{P/R} = \sum_{\substack{Q \in [s_P] \\ Q \leq_P R}} \mu_P(Q, R) \omega_{P/Q} \quad ,$$

thus for any $\varphi \in B^*(P)$

$$\varphi = \sum_{\substack{Q, R \in [s_P] \\ Q \leq_P R}} \varphi(P/R) \mu_P(Q, R) \omega_{P/Q} \quad .$$

**2.4.** The uniqueness assumption in Theorem 1.7 is now obvious : indeed by Lemma 2.2, the map $\Theta_P : B^*(P) \to D^\Omega(P)$ is uniquely defined by

$$\Theta_P(\omega_{P/Q}) = \Omega_{P/Q} \quad .$$

It remains to check that this definition implies

$$\Theta_P(\omega_X) = \Omega_X \tag{2.5}$$

for any finite $P$-set $X$, and that the maps $\Theta_P$ define a natural transformation of functors from $B^*$ to $D^\Omega$.

**2.6.** In order to check relation 2.5, first note that by Remark 2.3

$$\omega_X = \sum_{\substack{U,V\in[s_P]\\ U\leq_P V\\ X^V\neq\emptyset}} \mu_P(U,V)\omega_{P/U}$$

in $B^*(P)$. Hence, it remains to check that a similar relation holds in $D^\Omega(P)$, namely

$$\Omega_X = \sum_{\substack{U,V\in[s_P]\\ U\leq_P V\\ X^V\neq\emptyset}} \mu_P(U,V)\Omega_{P/U} \quad .$$

But this is precisely the content of Lemma 5.2.3 of [3], i.e. the case $Q = P$ of Theorem 1.2.

**2.7.** Since every morphism in $\mathcal{C}_p$ is a linear combination of transitive bisets, and since any transitive biset is the composition of a restriction, followed by a deflation, followed by an isomorphism, followed by an inflation, and followed by an induction (see Lemme 3 of [1] or Lemma 7.4 of [5]), it is enough to check that

$$F\big(\Theta_P(\omega_X)\big) = \Theta_Q\big(F(\omega_X)\big) \quad ,$$

whenever $P$ and $Q$ are finite $p$-groups, when $X$ is any finite $P$-set, and $F : P \to Q$ is one of restriction, deflation, isomorphism, inflation, or induction. Hence there are essentially three cases :

- There is a group homomorphism $f : Q \to P$, and $F$ is restriction along $f$. This case involves restriction, inflation and isomorphism. It corresponds to the morphism from $P$ to $Q$ in $\mathcal{C}_p$ defined by the $(Q,P)$-biset $P$, acted on the right by multiplication by $P$, and on the left by first taking image by $f$ and multiplying on the left in $P$. If $R$ is any subgroup of $Q$, one has that

  $$(\mathrm{Res}_f\omega_X)(Q/R) = \omega_X(P^{op} \times_Q Q/R) \quad .$$

  Now the map from $P^{op} \times_Q Q/R$ to $P/f(R)$ sending $(g, qR)$ to $gf(q)f(R)$, for $g \in P$ and $q \in Q$, is an isomorphism of $P$-sets. Thus

  $$(\mathrm{Res}_f\omega_X)(Q/R) = \omega_X\big(P/f(R)\big) = \begin{cases} 1 & \text{if } X^{f(R)} \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad .$$

  Thus $\mathrm{Res}_f\omega_X = \omega_{\mathrm{Res}_f X}$ in this case, as was to be shown, since $\mathrm{Res}_f\Omega_X = \Omega_{\mathrm{Res}_f X}$ by Lemma 4.1.1 of [3].

- The group $Q$ is equal to $P/N$, for some normal subgroup $N$ of $P$, and $F$ is deflation from $P$ to $Q$. This case corresponds to the morphism from $P$ to $Q$ in $\mathcal{C}_p$ defined by the $(Q,P)$-biset $Q$, acted on the left by multiplication by $Q$, and on the right by first taking image in $Q$ and multiplying on the right in $Q$. If $R$ is any subgroup of $P$ containing $N$, and if $\overline{R}=R/N$, one has that

$$(\mathrm{Def}_Q^P\omega_X)(Q/\overline{R})=\omega_X(Q\times_Q Q/\overline{R})=\omega_X(P/R)=\left\{\begin{array}{ll}1 & \text{if } X^R\neq\emptyset\\ 0 & \text{otherwise}\end{array}\right. .$$

Thus $(\mathrm{Def}_Q^P\omega_X)=\omega_{X^N}$ in this case, as was to be shown, since $\mathrm{Def}_Q^P\Omega_X=\Omega_{X^N}$ by Lemma 4.2.1 of [3].

- The group $P$ is a subgroup of $Q$, and $F$ is induction from $P$ to $Q$. This case corresponds to the morphism from $P$ to $Q$ in $\mathcal{C}_p$ defined by the $(Q,P)$-biset $Q$, acted on the left by multiplication by $Q$, and on the right by multiplication by $P$. If $R$ is any subgroup of $Q$, one has that

$$\begin{aligned}(\mathrm{Ind}_P^Q\omega_X)(Q/R) &= \omega_X(Q\times_Q Q/R)=\omega_X(\mathrm{Res}_P^Q Q/R)\\ &= |\{x\in P\backslash Q/R \mid X^{P\cap {}^xR}\neq\emptyset\}| \ .\end{aligned}$$

By Theorem 1.2, the equality to check in this case is

$$\mathrm{Ind}_P^Q\omega_X=\sum_{\substack{U,V\in[s_Q]\\ U\leq_Q V}}\mu_Q(U,V)\,|\{a\in V\backslash Q/P \mid X^{V^a\cap P}\neq\emptyset\}|\,\omega_{Q/U} \quad ,$$

Let $\omega$ denote the right hand side of this relation. One has that

$$\begin{aligned}\omega(Q/R) &= \sum_{\substack{U,V\in[s_Q]\\ R\leq_Q U\leq_Q V}}\mu_Q(U,V)\,|\{a\in V\backslash Q/P \mid X^{V^a\cap P}\neq\emptyset\}|\\ &= \sum_{V\in[s_Q]}\delta_{R,V}^Q\,|\{a\in V\backslash Q/P \mid X^{V^a\cap P}\neq\emptyset\}| \quad ,\end{aligned}$$

where $\delta_{R,V}^Q=1$ if $R$ and $V$ are conjugate in $Q$, and $\delta_{R,V}^Q=0$ otherwise. Thus

$$\omega(Q/R)=|\{a\in R\backslash Q/P \mid X^{R^a\cap P}\neq\emptyset\}| \quad ,$$

and $\omega=\mathrm{Ind}_P^Q\omega_X$, as was to be shown.

This completes the proof of Theorem 1.7 □

**2.8. Remark:** The last part of this proof shows that conversely, Theorem 1.7 implies Theorem 1.2.

## 3. Proof of Theorem 1.8

**3.1.** If $P$ is a finite $p$-group, and let $\varphi \in B^*(P)$. I must show that $\Theta_P(\varphi)$ is a torsion element of $D(P)$ if and only if $\varphi$ belongs to $i_P\big(R_{\mathbb{Q}}^*(P)\big)$.

By Remark 2.3

$$\varphi = \sum_{\substack{Q,R\in[s_P]\\ Q\leq_P R}} \varphi(P/R)\mu_P(Q,R)\omega_{P/Q} \quad ,$$

thus

$$\Theta_P(\varphi) = \sum_{\substack{Q,R\in[s_P]\\ Q\leq_P R}} \varphi(P/R)\mu_P(Q,R)\Omega_{P/Q}$$

in $D(P)$. The element $\Theta_P(\varphi)$ is a torsion element of $D(P)$ if and only if the element $2|P|\Theta_P(\varphi)$ is, i.e. if there exists an integer $n > 0$ such that

$$2n \sum_{\substack{Q,R\in[s_P]\\ Q\leq_P R}} \varphi(P/R)\mu_P(Q,R)|P|\Omega_{P/Q} = 0 \quad .$$

By Proposition 6.5.1 of [3], for any finite $P$-set $X$, one has that

$$|P|\Omega_X = -\sum_{\substack{U\subseteq V\subseteq P\\ X^U\neq\emptyset}} |U|\mu(U,V)\mathrm{Ten}_V^P\Delta(V) \quad ,$$

where $\mu$ is the Möbius function of the poset of all subgroups of $P$, ordered by inclusion, and $\Delta(P)$ is defined in Notation 6.2.1 of [3] by

$$\Delta(P) = \Omega_{M(P)} \quad ,$$

where $M(P)$ is the disjoint union of sets $P/Q$, for maximal proper subgroups $Q$ of $P$. It follows that

$$2n \sum_{\substack{Q,R\in[s_P]\\ Q\leq_P R}} \varphi(P/R)\mu_P(Q,R) \sum_{\substack{U\subseteq V\subseteq P\\ U\leq_P Q}} |U|\mu(U,V)\mathrm{Ten}_V^P\Delta(V) = 0 \quad .$$

Summing first on $Q$ gives, by the defining property of the Möbius function $\mu_P$

$$2n \sum_{U\subseteq V\subseteq P} \varphi(P/U)|U|\mu(U,V)\mathrm{Ten}_V^P\Delta(V) = 0 \quad .$$

In this expression, for $V \subseteq P$, the coefficient of $\mathrm{Ten}_V^P\Delta(V)$ is equal to

$$2n \sum_{U\subseteq V} \varphi(P/U)|U|\mu(U,V) \quad ,$$

hence it is constant on the conjugacy class of $V$ in $P$. It follows that the whole previous summation can be written as

$$\sum_{V\in[s_P]} 2n|P:N_P(V)|\big(\sum_{U\subseteq V}\varphi(P/U)|U|\mu(U,V)\big)\mathrm{Ten}_V^P\Delta(V) = 0 \quad .$$

By Corollary 6.5.2 and Proposition 6.4.1 of [3], this is equivalent to requiring that for any non-cyclic subgroup $V$ of $P$

$$2n|P:N_P(V)|\sum_{U\subseteq V}\varphi(P/U)|U|\mu(U,V)=0 \quad ,$$

or equivalently, since $n>0$

$$\sum_{U\subseteq V}\varphi(P/U)|U|\mu(U,V)=0 \quad .$$

Now Theorem 1.8 follows from the following :

**3.2. Lemma :** *Let $P$ be a finite $p$-group, and let $\varphi\in B^*(P)$. Then $\varphi\in i_P\big(R_{\mathbb{Q}}^*(P)\big)$ if and only if, for any non-cyclic subgroup $Q$ of $P$*

$$\sum_{U\subseteq Q}\varphi(P/U)|U|\mu(U,Q)=0 \quad .$$

**Proof:** The group $R_{\mathbb{Q}}^*(P)$ is a free abelian group, with canonical basis $(V^*)_{V\in \mathrm{Irr}_{\mathbb{Q}}(P)}$ indexed by the set $\mathrm{Irr}_{\mathbb{Q}}(P)$ of rational irreducible representations of $P$. The value of the linear form $V^*$ on a (finitely generated) $\mathbb{Q}P$-module $W$ is equal to the multiplicity $m(V,W)$ of $V$ as a summand of $W$. Since $B^*(P)$ is also a free abelian group, there is a commutative diagram

$$\textbf{(3.3)}\qquad \begin{array}{ccc} R_{\mathbb{Q}}^*(P) & \xrightarrow{\;i_P\;} & B^*(P) \\ {\scriptstyle r_P}\downarrow & & \downarrow{\scriptstyle b_P} \\ \mathrm{Hom}_{\mathbb{Z}}\big(R_{\mathbb{Q}}(P),\mathbb{Q}\big) & \xrightarrow[I_P]{} & \mathrm{Hom}_{\mathbb{Z}}\big(B(P),\mathbb{Q}\big) \end{array} \quad ,$$

where $r_P$ and $b_P$ are the canonical maps, which may be viewed as inclusions, and $I_P=\mathrm{Hom}_{\mathbb{Z}}(i_P,\mathbb{Q})$. This square is obviously cartesian : if $s:B(P)\to R_{\mathbb{Q}}(P)$ is the canonical map sending a $P$-set to the corresponding permutation $\mathbb{Q}P$-module, and if $i:\mathbb{Z}\to\mathbb{Q}$ is the canonical injection, then for $\alpha\in\mathrm{Hom}_{\mathbb{Z}}\big(R_{\mathbb{Q}}(P),\mathbb{Q}\big)$ and $\beta\in B^*(P)$, the equality $\alpha\circ s=i\circ\beta$ implies $\alpha(\mathrm{Im}(s))\subseteq\mathrm{Im}(i)=\mathbb{Z}$, hence $\mathrm{Im}(\alpha)\subseteq\mathbb{Z}$ since $s$ is surjective.

Hence to complete the proof of Lemma 3.2, it suffices to prove the following claim :

**3.4. Claim :** *An element $\varphi\in\mathrm{Hom}_{\mathbb{Z}}\big(B(P),\mathbb{Q}\big)$, i.e. a linear form on $B(P)$ with values in $\mathbb{Q}$, lies in the image of the map $I_P$ if and only if for any non-cyclic subgroup $Q$ of $P$*

$$\textbf{(3.5)}\qquad \sum_{U\subseteq Q}\varphi(P/U)|U|\mu(U,Q)=0 \quad .$$

Suppose first that $\varphi\in\mathrm{Im}(I_P)$. Then $\varphi$ is a linear combination with rational coefficients of the elements $I_P(V^*)$, for $V\in\mathrm{Irr}_{\mathbb{Q}}(P)$. It suffices to prove that Equation 3.5 holds for these elements.

If $E$ and $F$ are $\mathbb{Q}P$-modules, define

$$\langle E,F\rangle_P = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}P}(E,F) \quad ,$$

and extend this scalar product to a bilinear form on $R_{\mathbb{Q}}(P)$, with values in $\mathbb{Z}$. With this notation, one has that

$$\langle V,V\rangle_P V^*(W) = \langle V,W\rangle_P \quad ,$$

for any finite dimensional $\mathbb{Q}P$-module $W$. Thus if $\varphi = I_P(V^*)$

$$\begin{aligned}\langle V,V\rangle_P \sum_{U\subseteq Q} \varphi(P/U)|U|\mu(U,Q) &= \sum_{U\subseteq Q} |U|\mu(U,Q)\langle V,\mathbb{Q}P/U\rangle_P \\ &= \sum_{U\subseteq Q} |U|\mu(U,Q)\langle \mathrm{Res}_U^P V,\mathbb{Q}\rangle_U \\ &= \sum_{U\subseteq Q} \mu(U,Q) \sum_{x\in U} \chi_V(x) \quad ,\end{aligned}$$

where $\chi_V$ is the character of $V$. This can also be written as

$$\langle V,V\rangle_P \sum_{U\subseteq Q} \varphi(P/U)|U|\mu(U,Q) = \sum_{x\in Q} \chi_V(x)\Big( \sum_{<x>\subseteq U\subseteq Q} \mu(U,Q)\Big) \quad ,$$

and this is zero if $Q$ is not cyclic. Hence Equation 3.5 holds.

Now $\mathrm{Hom}_{\mathbb{Z}}\big(R_{\mathbb{Q}},\mathbb{Q}\big)$ and $\mathrm{Hom}_{\mathbb{Z}}\big(B,\mathbb{Q}\big)$ both are Mackey functors for $P$ over $\mathbb{Q}$, and the maps $I_Q$, for $Q\subseteq P$, form a morphism of Mackey functors $I : \mathrm{Hom}_{\mathbb{Z}}\big(R_{\mathbb{Q}},\mathbb{Q}\big) \to \mathrm{Hom}_{\mathbb{Z}}\big(B,\mathbb{Q}\big)$. In particular, there is a natural action of the Burnside algebra $\mathbb{Q}\otimes_{\mathbb{Z}} B(P)$ on $\mathrm{Hom}_{\mathbb{Z}}\big(R_{\mathbb{Q}}(P),\mathbb{Q}\big)$ and $\mathrm{Hom}_{\mathbb{Z}}\big(B(P),\mathbb{Q}\big)$. For example, the action of $\mathbb{Q}\otimes_{\mathbb{Z}} B(P)$ on $\mathrm{Hom}_{\mathbb{Z}}\big(B(P),\mathbb{Q}\big)$ is given by

$$(Y\varphi)(X) = \varphi(X\times Y) \quad ,$$

for $X$ and $Y$ in $B(P)$, and $\varphi\in \mathrm{Hom}_{\mathbb{Z}}\big(B(P),\mathbb{Q}\big)$.

This algebra $\mathbb{Q}\otimes_{\mathbb{Z}} B(P)$ is a split semi-simple commutative algebra, with primitive idempotents

$$e_Q^P = \frac{1}{|N_P(Q)|}\sum_{R\subseteq Q} |R|\mu(R,Q)P/R \quad ,$$

for $Q\in [s_P]$ (see for example [2] Theorem 3.3.2).

The proof of Claim 3.4 can be completed by induction on the order of $P$ : first observe that there is nothing to prove if $P$ is cyclic, since in that case $i_P$ and $I_P$ are isomorphisms, and since the condition of the claim is void if $P$ is cyclic. This starts induction.

Now suppose that $P$ is non-cyclic, and that Claim 3.4 holds for any proper subgroup of $P$. Let $\varphi\in \mathrm{Hom}_{\mathbb{Z}}\big(B(P),\mathbb{Q}\big)$, and suppose that relation 3.5 holds for any non-cyclic subgroup $Q$ of $P$. If $R\subseteq Q$ are subgroups of $P$, then

$$\mathrm{Res}_Q^P\varphi(Q/R) = \varphi(P/R) \quad .$$

Hence the induction hypothesis implies that $\mathrm{Res}_Q^P\varphi$ is in the image of the map $I_Q$, for any proper subgroup $Q$ of $P$. Moreover

$$(e_P^P\varphi)(X) = \varphi(e_P^P X) = |X^P|\varphi(e_P^P) \quad ,$$

by the defining property of the idempotent $e_P^P$. This shows that

$$e_P^P\varphi = \varphi(e_P^P)\delta_{P/P} \quad .$$

Moreover

$$\varphi(e_P^P) = \frac{1}{|P|}\sum_{R\subseteq P} |R|\mu(R,Q)\varphi(P/R) = 0$$

by assumption, since $P$ is not cyclic. Hence $e_P^P\varphi = 0$, and $\varphi = \sum_{Q\in[s_P]-\{P\}} e_Q^P\varphi$. But $e_Q^P\varphi$ is a linear combination with rational coefficients of elements of $\mathrm{Hom}_{\mathbb{Z}}\big(B(P),\mathbb{Q}\big)$ of the form $\mathrm{Ind}_R^P\mathrm{Res}_R^P\varphi$, for $R\subseteq Q$. Hence $e_Q^P\varphi\in\mathrm{Im}(I_P)$, for any proper subgroup $Q$ of $P$. Hence $\varphi\in\mathrm{Im}(I_P)$, as was to be shown. □

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