Relative B-groups

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Abstract : This paper extends the notion of B-group to a relative context. For a finite group K and a field $\mathbb{F}$ of characteristic 0, the lattice of ideals of the Green biset functor $\mathbb{F}B_K$ obtained by shifting the Burnside functor $\mathbb{F}B$ by K is described in terms of B_K*-groups*. It is shown that any finite group (L, φ) over K admits a *largest quotient* B_K*-group* $\beta_K(L, \varphi)$. The simple subquotients of $\mathbb{F}B_K$ are parametrized by B_K-groups, and their evaluations can be precisely determined. Finally, when p is a prime, the restriction $\mathbb{F}B_K^{(p)}$ of $\mathbb{F}B_K$ to finite p-groups is considered, and the structure of the lattice of ideals of the Green functor $\mathbb{F}B_K^{(p)}$ is described in full detail. In particular, it is shown that this lattice is always finite.

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1. Introduction

In the study of the lattice of biset-subfunctors of the Burnside functor $\mathbb{F}B$ over a field $\mathbb{F}$ of characteristic 0 (cf. Section 7.2 of [1], or Chapter 5 of [2]), a special class of finite groups, called B*-groups*, plays an important role: indeed, the simple subquotients of the biset functor $\mathbb{F}B$ are exactly the functors $S_{H,\mathbb{F}}$, where H is such a B-group. It was shown moreover that each finite group G has a largest quotient B-group $\beta(G)$.

Let K be a fixed finite group. This paper proposes a generalization of the above methods and notions, in order to study the lattice of ideals of the *shifted Burnside functor* $\mathbb{F}B_K$. We start by introducing a category $\mathsf{grp}_{\Downarrow K}$ of groups over K, similar to the comma category of finite groups over K, in which morphisms are obtained by allowing diagrams to commute up to inner automorphisms of K.

To each such group (L, φ), where $\varphi : L \to K$, is attached a specific ideal $\mathsf{e}_{L,\varphi}$ of $\mathbb{F}B_K$, and it is shown that every ideal of $\mathbb{F}B_K$ is equal to the sum of the ideals $\mathsf{e}_{L,\varphi}$ it contains. A special class of groups over K is introduced, called B_K-groups, and it is shown that for each group (L, φ) over K, there exists a largest B_K-group $\beta_K(L, \varphi)$ quotient of (L, φ). Moreover $\mathsf{e}_{L,\varphi} = \mathsf{e}_{\beta_K(L,\varphi)}$. It follows that the lattice of ideals of $\mathbb{F}B_K$ can be described in terms of *closed families* of B_K-groups.

Moreover, each ideal $\mathsf{e}_{L,\varphi}$ associated to a B_K-group (L, φ) has a unique maximal proper subideal $\mathsf{e}^0_{L,\varphi}$. The quotient $S_{L,\varphi} = \mathsf{e}_{L,\varphi}/\mathsf{e}^0_{L,\varphi}$ is a simple

$\mathbb{F}B_K$-module. The evaluations of this simple module can be precisely described, as well as its minimal groups, and this yields a new example of a simple module over a Green biset functor with several isomorphism classes of minimal groups.

Finally, when p is a prime number, we consider the restriction $\mathbb{F}B_K^{(p)}$ of $\mathbb{F}B_K$ to finite p-groups, and we describe completely the lattice of ideals of this Green biset functor. We show in particular that this lattice is always finite. As a byproduct, we get some examples of Green p-biset functors without non zero proper ideals.

2. Review of shifted Green biset functors

We quickly recall some definitions and basic notions on biset functors for finite groups, and refer to [2] for details. Let $\mathbb{F}$ be a field of characteristic 0. The biset category $\mathbb{F}\mathcal{C}$ of finite groups has all finite groups as objects. If G and H are finite groups, then $\mathrm{Hom}_{\mathbb{F}\mathcal{C}}(G,H) = \mathbb{F}\otimes_{\mathbb{Z}} B(H,G)$, where $B(H,G)$ is the Grothendieck group of finite (H,G)-bisets. Composition in $\mathbb{F}\mathcal{C}$ is induced by the product $(V,U)\mapsto V\times_H U = (V\times U)/H$, where V is a (K,H)-biset and U a (H,G)-biset, and H acts on $(V\times U)$ by $(v,u)\cdot h = (vh, h^{-1}u)$. A biset functor over $\mathbb{F}$ is an $\mathbb{F}$-linear functor from $\mathbb{F}\mathcal{C}$ to the category of $\mathbb{F}$-vector spaces.

Any biset is a disjoint union of transitive ones, and any transitive (H,G)-biset is of the form $(H\times G)/L$, where L is a subgroup of $(H\times G)$. Denoting by $p_1 : H\times G\to H$ and $p_2 : H\times G\to G$ the first and second projections, we set $k_1(L) = p_1(L\cap \mathrm{Ker}\, p_2)$ and $k_2(L) = p_2(L\cap\mathrm{Ker}\, p_1)$. The biset $(H\times G)/L$ factors as the composition

$$(H\times G)/L \cong \mathrm{Ind}_{p_1(L)}^H \circ \mathrm{Inf}_{p_1(L)/k_1(L)}^{p_1(L)} \circ \mathrm{Iso}(\alpha)\circ \mathrm{Def}_{p_2(L)/k_2(L)}^{p_2(L)}\circ \mathrm{Res}_{p_2(L)}^G$$

of elementary bisets called *induction*, *inflation*, *isomorphism*, *deflation*, and *restriction*, where $\alpha : p_2(L)/k_2(L)\to p_1(L)/k_1(L)$ is the canonical isomorphism sending $bk_2(L)$ to $ak_1(L)$ for $(a,b)\in L$. These elementary morphisms generate all morphisms in the category $\mathbb{F}\mathcal{C}$.

A Green biset functor A over $\mathbb{F}$ (cf. Section 8.5 of [2]) is a biset functor with additional bilinear products $A(G)\times A(H)\to A(G\times H)$, denoted by $(\alpha,\beta)\mapsto \alpha\times\beta$, which are associative and bifunctorial. There is also an identity element $\varepsilon_A\in A(\mathbf{1})$.

A left A-module M is then defined similarly as a biset functor with products $A(G)\times M(H)\to M(G\times H)$ which are associative, bifunctorial, and unital. Left A-modules form an abelian category denoted by A-Mod. A left ideal of A is an A-submodule of the left A-module A.

When A is a Green functor, each evaluation $A(G)$ is an $\mathbb{F}$-algebra for the product

$$\alpha, \beta \in A(G) \mapsto \alpha \cdot \beta = A\big(\mathrm{Iso}(\delta) \circ \mathrm{Res}_{\Delta}^{G\times G}\big)(\alpha \times \beta) \ \ ,$$

where Δ is the diagonal subgroup of $G \times G$, and $\delta : \Delta \to G$ the canonical isomorphism. The identity element of this algebra is $A(\mathrm{Inf}_{\mathbf{1}}^{G})(\varepsilon_A)$. If M is an A-module, each evaluation $M(G)$ is endowed with an $A(G)$-module structure defined similarly. By Proposition 2.16 of [7], a biset subfunctor I of A is an ideal if and only if $I(G)$ is an ideal of the algebra $A(G)$, for any finite group G.

A Green biset functor A is called *commutative* (cf. [3] for details) if the algebra $A(G)$ is commutative, for any G.

A fundamental example of Green biset functor is the Burnside functor sending a finite group G to $\mathbb{F}B(G) = \mathbb{F}B(G, 1)$, where $B(G)$ is the Burnside group of G. The products maps $\mathbb{F}B(G) \times \mathbb{F}B(H) \to \mathbb{F}B(G \times H)$ are induced by the cartesian product sending a G-set X and an H-set Y to the $(G\times H)$-set $X \times Y$. An $\mathbb{F}B$-module is precisely a biset functor over $\mathbb{F}$.

Let K be a finite group. A Green biset functor A over $\mathbb{F}$ can be *shifted* by K. This gives a new Green biset functor A_K defined for a finite group G by

$$A_K(G) = A(G \times K) \ \ .$$

For finite groups G and H and a finite (H, G)-biset U, the map

$$A_K(U) : A_K(G) \to A_K(H)$$

is the map $A(U \times K)$, where $U \times K$ is viewed as a $(H \times K, G \times K)$-biset in the obvious way, letting K act on both sides on $U \times K$ by multiplication on the second component. For an arbitrary element $\alpha \in \mathbb{F}B(H, G)$, that is an $\mathbb{F}$-linear combination of (H, G)-bisets, the map $A_K(\alpha) : A_K(G) \to A_K(H)$ is defined by $\mathbb{F}$-linearity.

This endows A_K with a biset functor structure. Moreover, for finite groups G and H, the product

$$\times_{A_K} : A_K(G) \times A_K(H) \to A_K(G \times H)$$

is defined as follows: if $\alpha \in A_K(G) = A(G\times K)$ and $\beta \in A_K(H) = A(H\times K)$, then $\alpha \times \beta \in A(G \times K \times H \times K)$. We set

$$\alpha \times_{A_K} \beta = A\big(\mathrm{Iso}(\delta) \circ \mathrm{Res}_{\Delta}^{G\times K\times H\times K}\big)(\alpha \times \beta) \ \ ,$$

where $\Delta = \{(g, k, h, k) \mid g \in G, h \in H, k \in K\}$, and δ is the isomorphism $\Delta \to G \times H \times K$ sending (g, k, h, k) to (g, h, k). The identity element ε_{A_K} is $A(\mathrm{Inf}_{\mathbf{1}}^{K})(\varepsilon_K)$.

For a finite group G, the algebra structure on $A_K(G)$ is simply the algebra structure on $A(G\times K)$ defined for the Green functor A.

All these notion can be extended to functors from an admissible subcategory $\mathcal{D}$ of the biset category (cf. Chapter 4 of [2]), which is moreover closed under taking direct products of finite groups. We have then the notions of $\mathcal{D}$-biset functors and $\mathcal{D}$-Green biset functors, as well as modules over them.

In this paper, we will consider the shifted Burnside functor $\mathbb{F}B_K$, and its restriction $\mathbb{F}B_K^{(p)}$ to finite p-groups, for a prime p. A fundamental classical result is that for any finite group G, the algebra $\mathbb{F}B_K(G)$ is a split semisimple commutative algebra, with primitive idempotents $e_L^{G\times K}$ indexed by subgroups L of $G\times K$, up to conjugation. The explicit formula for $e_L^{G\times K}$, due to Gluck ([4]) and Yoshida ([8]) is

$$e_L^{G\times K}=\frac{1}{|N_{G\times K}(L)|}\sum_{X\leq L}|X|\,\mu(X,L)\,[(G\times K)/X]\ \ ,$$

where X runs through all subgroups of L, where μ is the Möbius function of the poset of subgroups of $G\times K$, and $[(G\times K)/X]$ is the isomorphism class of the transitive $(G\times K)$-set $(G\times K)/X$.

2.1. Notation: *When N is a normal subgroup of a finite group L, let*

$$m_{L,N}=\frac{1}{|L|}\sum_{\substack{X\leq L\\ XN=L}}|X|\mu(X,L)\ \ .$$

2.2. Lemma: *Let G be a finite group, and L be a subgroup of $G\times K$. If N is a normal subgroup of G, then*

$$\mathbb{F}B_K\big(\mathrm{Def}_{G/N}^G\big)(e_L^{G\times K})=\lambda\, m_{L,L\cap(N\times\mathbf{1})}\,e_{\overline{L}}^{(G/N)\times K}\quad ,$$

where $\overline{L}$ is the image of L by the projection $G\times K\to(G/N)\times K$, and $\lambda=\frac{|N_{(G/N)\times K}(\overline{L}):\overline{L}|}{|N_{G\times K}(L):L|}$.

Proof: Indeed

$$\mathbb{F}B_K\big(\mathrm{Def}_{G/N}^G\big)(e_L^{G\times K})=\mathbb{F}B\big(\mathrm{Def}_{(G\times K)/(N\times\mathbf{1})}^{G\times K}\big)(e_L^{G\times K})\ \ .$$

The result now follows from Assertion 4 of Theorem 5.2.4 of [2]. □

3. Ideals generated by idempotents

We now introduce a category $\mathsf{grp}_{\Downarrow K}$, similar to the *comma category* over K: its objects are the same, but morphisms are slightly different.

3.1. Definition:

- *For a finite group K, let $\mathsf{grp}_{\Downarrow K}$ denote the following category:*
  - *The objects are* finite groups over K, *i.e. pairs (L, φ), where L is a finite group and $\varphi : L \to K$ is a group homomorphism.*
  - *A morphism $f : (L, \varphi) \to (L', \varphi')$ of groups over K in the category $\mathsf{grp}_{\Downarrow K}$ is a group homomorphism $f : L \to L'$ such that there exists some inner automorphism i of K with $i \circ \varphi = \varphi' \circ f$.*
  - *The composition of morphisms in $\mathsf{grp}_{\Downarrow K}$ is the composition of group homomorphisms, and the identity morphism of (L, φ) is the identity automorphism of L.*
- *If (L, φ) and (L', ϕ') are groups over K, we say that (L', φ') is a* quotient *of (L, φ), and we note $(L, \varphi) \twoheadrightarrow (L', \varphi')$, if there exists a morphism $f \in \mathrm{Hom}_{\mathsf{grp}_{\Downarrow K}}\big((L, \varphi), (L', \varphi')\big)$ with $f : L \to L'$ surjective. In this case, we will say that f is a* surjective morphism *from (L, φ) to (L', φ').*

3.2. Remarks:

1. Using the well known fact that the epimorphisms in the category of (finite) groups are the surjective group homomorphisms (cf. [5] I.5 Exercise 5), one can show that a morphism $f \in \mathrm{Hom}_{\mathsf{grp}_{\Downarrow K}}\big((L, \varphi), (L', \varphi')\big)$ is an epimorphism in $\mathsf{grp}_{\Downarrow K}$ if and only if $f : L \to L'$ is surjective, that is, if f is a surjective morphism. We will not use this fact here, except as a motivation to the use of the word "quotient" in Definition 3.1.
2. A morphism $f : (L, \varphi) \to (L', \varphi')$ in $\mathsf{grp}_{\Downarrow K}$ is an isomorphism if and only if $f : L \to L'$ is an isomorphism of groups.
3. If (L', φ') is a quotient of (L, φ), and if (L, φ) is a quotient of (L', φ'), then (L, φ) and (L', φ') are isomorphic in $\mathsf{grp}_{\Downarrow K}$. Indeed any surjective morphism from (L, φ) to (L', φ') is an isomorphism, for L and L' have the same order.
4. Clearly, the relation "being quotient of" on the class of groups over K is transitive. In particular, any group over K isomorphic in $\mathsf{grp}_{\Downarrow K}$ to a quotient of (L, φ) is itself a quotient of (L, φ), and also a quotient of any group over K isomorphic to (L, φ) in $\mathsf{grp}_{\Downarrow K}$.

3.3. Notation: *When (L,φ) is a group over K, we denote by L_φ the subgroup of $L\times K$ defined by*

$$L_\varphi = \{\big(l,\varphi(l)\big) \mid l\in L\}\ \ .$$

3.4. Theorem: *Let I be an ideal of the Green biset functor $\mathbb{F}B_K$. If G is a finite group and L is a subgroup of $G\times K$, the following conditions are equivalent:*

1. *The idempotent $e_L^{G\times K}$ belongs to $I(G)$.*
2. *The idempotent $e_{L_{p_2}}^{L\times K}$ belongs to $I(L)$, where $p_2: L\to K$ is the restriction to L of the second projection homomorphism $G\times K\to K$.*

Proof: $\boxed{1\Rightarrow 2}$ Let $\widehat{L} = L_{p_1}\subseteq L\times G$, where $p_1: L\to G$ is the restriction to L of the first projection homomorphism $G\times K\to G$. Thus

$$p_1(\widehat{L}) = L,\ k_1(\widehat{L}) = \mathbf{1}\times k_2(L),\ p_2(\widehat{L}) = p_1(L),\ k_2(\widehat{L}) = \mathbf{1}\ \ .$$

It follows that the (L,G)-biset $U=(L\times G)/\widehat{L}$ factors as

$$U\cong \mathrm{Inf}_{L/N}^L\circ \mathrm{Iso}(\theta^{-1})\circ \mathrm{Res}^G_{p_1(L)}\ \ ,$$

where $N=\mathbf{1}\times k_2(L)\trianglelefteq L$ and $\theta: L/N\to p_1(L)$ is the canonical isomorphism induced by the first projection $p_1: L\to G$.

Now if $e_L^{G\times K}$ belongs to $I(G)$, its restriction $\mathbb{F}B_K(\mathrm{Res}^G_{p_1(L)})(e_L^{G\times K})$ belongs to $I(G)$. But

$$\begin{aligned}\mathbb{F}B_K(\mathrm{Res}^G_{p_1(L)})(e_L^{G\times K}) &= \mathbb{F}B\big(\mathrm{Res}^{G\times K}_{p_1(L)\times K}\big)(e_L^{G\times K})\\ &= \sum_{L'} e_{L'}^{p_1(L)\times K},\end{aligned}$$

where L' runs through a set of representatives of $\big(p_1(L)\times K\big)$-conjugacy classes of subgroups of $p_1(L)\times K$ which are conjugate to L in $G\times K$ (cf. [2], Theorem 5.2.4, Assertion 1). In particular, the group L is one of them, and

$$e_L^{p_1(L)\times K}\cdot \mathbb{F}B_K(\mathrm{Res}^G_{p_1(L)})(e_L^{G\times K}) = e_L^{p_1(L)\times K}\in I\big(p_1(L)\big)\ \ .$$

It follows that $\mathbb{F}B_K\big(\mathrm{Iso}(\theta^{-1})\big)(e_L^{p_1(L)\times K})\in I(L/N)$.

But $\mathbb{F}B_K\big(\mathrm{Iso}(\theta^{-1})\big) = \mathbb{F}B\big(\mathrm{Iso}(\theta_K^{-1})\big)$, where $\theta_K = \theta \times \mathrm{Id}_K$ is the isomorphism from $(L/N) \times K$ to $p_1(L) \times K$ deduced from θ. It follows that $e_{\overline{L}}^{(L/N)\times K} \in I(L/N)$, where $\overline{L} = \theta_K^{-1}(L) = \{\big(lN, p_2(l)\big) \mid l \in L\}$. Now

$$\begin{aligned}\mathbb{F}B_K(\mathrm{Inf}_{L/N}^L)(e_{\overline{L}}^{(L/N)\times K}) &= \mathbb{F}B\big(\mathrm{Inf}_{(L/N)\times K}^{L\times K}\big)(e_{\overline{L}}^{(L/N)\times K})\\ &= \sum_X e_X^{L\times K} \in I(L) \ ,\end{aligned}$$

where X runs through a set of representatives of $(L \times K)$-conjugacy classes of subgroups of $L \times K$ which map to a conjugate of $\overline{L}$ through the surjection $L \times K \to (L/N) \times K$ (cf. [2], Theorem 5.2.4, Assertion 3).

The group L_{p_2} is one of these subgroups, hence

$$e_{L_{p_2}}^{L\times K} \cdot \mathbb{F}B_K(\mathrm{Inf}_{L/N}^L)(e_{\overline{L}}^{(L/N)\times K}) = e_{L_{p_2}}^{L\times K} \in I(L) \ ,$$

as was to be shown.

$\boxed{2 \Rightarrow 1}$ We now consider the opposite (G, L)-biset $U^{op} \cong (G \times L)/\widetilde{L}$, where $\widetilde{L} = \{\big(p_1(l), l\big) \mid l \in L\}$, which factors as

$$U^{op} \cong \mathrm{Ind}_{p_1(L)}^G \circ \mathrm{Iso}(\theta) \circ \mathrm{Def}_{L/N}^L \ .$$

If $e_{L_{p_2}}^{L\times K} \in I(L)$, then $u = \mathbb{F}B_K(U^{op})(e_{L_{p_2}}^{L\times K})$ belongs to $I(G)$. By Lemma 2.2

$$\mathbb{F}B_K\big(\mathrm{Def}_{L/N}^L\big)(e_{L_{p_2}}^{L\times K}) = \lambda\, m_{L_{p_2}, L_{p_2}\cap(N\times\mathbf{1})}\, e_{\overline{L_{p_2}}}^{(L/N)\times K} \ ,$$

where $\overline{L_{p_2}}$ is the image of L_{p_2} by the projection $L \times K \to (L/N) \times K$, and λ is some non zero rational number. Now the intersection

$$L_{p_2} \cap (N \times \mathbf{1}) = \{\big((a,b),b\big) \mid (a,b) \in L\} \cap \Big(\big(\mathbf{1} \times k_2(L)\big) \times \mathbf{1}\Big)$$

is trivial. It follows that $m_{L_{p_2}, L_{p_2}\cap(N\times\mathbf{1})} = 1$, and

$$\begin{aligned}u &= \lambda\, \mathbb{F}B_K\big(\mathrm{Ind}_{p_1(L)}^G \circ \mathrm{Iso}(\theta)\big)(e_{\overline{L_{p_2}}}^{(G/N)\times K)})\\ &= \lambda\, \mathbb{F}B\big(\mathrm{Ind}_{p_1(L)\times K}^{G\times K} \circ \mathrm{Iso}(\theta_K)\big)(e_{\overline{L_{p_2}}}^{(G/N)\times K)}) \ .\end{aligned}$$

Now for $(a, b) \in L$, the image by $\theta_K = \theta \times \mathrm{Id}_K$ of $\big((a,b),b\big)(N \times \mathbf{1}) \in \overline{L_{p_2}}$ is the element $\big(p_1(a,b),b\big) = (a,b)$ of $p_1(L) \times K$. Hence $\theta_K(\overline{L_{p_2}})$ identifies with L, viewed as a subgroup of $p_1(L) \times K$, and

$$u = \lambda\, \mathbb{F}B\big(\mathrm{Ind}_{p_1(L)\times K}^{G\times K}\big)(e_L^{p_1(L)\times K}) = \lambda\lambda' e_L^{G\times K} \ ,$$

for some non zero rational number λ' (cf. [2], Theorem 5.2.4, Assertion 2). Since $u \in I(G)$ and $\lambda\lambda' \neq 0$, it follows that $e_L^{G\times K} \in I(G)$, as was to be shown. □

3.5. Corollary: *Let G be a finite group, and L be a subgroup of $G \times K$. Then the ideal of $\mathbb{F}B_K$ generated by $e_L^{G\times K}$ is equal to the ideal of $\mathbb{F}B_K$ generated by $e_{L_{p_2}}^{L\times K}$*

Proof: Indeed, denoting by I the ideal generated by $e_L^{G\times K}$, and by J the ideal generated by $e_{L_{p_2}}^{L\times K}$, we have

$$\begin{aligned} &e_L^{K\times G} \in I(G) \Rightarrow e_{L_{p_2}}^{L\times K} \in I(L) \Rightarrow J \subseteq I \ \ , \\ &e_{L_{p_2}}^{L\times K} \in J(L) \Rightarrow e_L^{G\times K} \in J(G) \Rightarrow I \subseteq J \ \ , \end{aligned}$$

so $I = J$. □

3.6. Notation: *Let (L, φ) be a group over K. We denote by $\mathsf{e}_{L,\varphi}$ the ideal of $\mathbb{F}B_K$ generated by $e_{L_\varphi}^{L\times K} \in \mathbb{F}B_K(L)$.*

3.7. Lemma: *Let (L, φ) and (M, ψ) be groups over K.*

1. *If $(M, \psi) \twoheadrightarrow (L, \varphi)$, then $\mathsf{e}_{M,\psi} \subseteq \mathsf{e}_{L,\varphi}$.*
2. *In particular, if (M, ψ) is isomorphic to (L, φ), then $\mathsf{e}_{M,\psi} = \mathsf{e}_{L,\varphi}$.*

Proof: 1. Let $s : M \to L$ be a surjective group homomorphism, and i be an inner automorphism of K such that $i \circ \psi = \varphi \circ s$. Let U denote the set L, viewed as an (M, L)-biset for the action given by $m \cdot u \cdot l = s(m)ul$, for $m \in M$ and $u, l \in L$. There is an isomorphism of (M, L)-bisets

$$U \cong \mathrm{Inf}_{M/\mathrm{Ker}\, s}^{M} \circ \mathrm{Iso}(\alpha^{-1}) \ \ ,$$

where $\alpha : \overline{M} = M/\mathrm{Ker}\, s \to L$ is the group isomorphism induced by s.

Let $u = \mathbb{F}B_K(U)(e_{L_\varphi}^{L\times K}) \in \mathsf{e}_{L,\varphi}(M)$. Then

$$u = \mathbb{F}B\big(\mathrm{Inf}_{\overline{M}\times K}^{M\times K} \circ \mathrm{Iso}(\alpha_K^{-1})\big)(e_{L_\varphi}^{L\times K}) \ \ ,$$

where $\alpha_K = \alpha \times \mathrm{Id}_K : \overline{M} \times K \to L \times K$. Then

$$\alpha_K^{-1}(L_\varphi) = \{\big(\alpha^{-1}(l), \varphi(l)\big) \mid l \in L\} = \{\big(m\mathrm{Ker}\, s, \varphi \circ s(m)\big) \mid m \in M\} \ \ .$$

It follows that $\mathbb{F}B\big(\mathrm{Iso}(\alpha_K^{-1})\big)(e_{L_\varphi}^{L\times K}) = e_{\overline{M}_\theta}^{\overline{M}\times K}$, where $\theta : \overline{M} \to K$ is defined by $\theta(m\mathrm{Ker}\, s) = \varphi \circ s(m)$. In particular $e_{\overline{M}_\theta}^{\overline{M}\times K} \in \mathsf{e}_{L,\varphi}(\overline{M})$. Now

$$u = \mathbb{F}B\big(\mathrm{Inf}_{\overline{M}\times K}^{M\times K}\big)(e_{\overline{M}_\theta}^{\overline{M}\times K}) = \sum_X e_X^{M\times K} \quad ,$$

where X runs through a set of representatives of conjugacy classes of subgroups of $M \times K$ such that the projection of X in $\overline{M} \times K$ is conjugate to $\overline{M}_\theta$. The subgroup $M_{\varphi\circ s}$ is one of these subgroups, so $e_{M_{\varphi\circ s}}^{M\times K} \cdot u$ is a non zero scalar multiple of $e_{M_{\varphi\circ s}}^{M\times K}$ lying in $\mathsf{e}_{L,\varphi}(M)$. Hence $e_{M_{\varphi\circ s}}^{M\times K} \in \mathsf{e}_{L,\varphi}(M)$. Now $\varphi \circ s = i \circ \psi$, where i is an inner automorphism of K. This implies readily that the subgroups $M_{i\circ\psi}$ and M_ψ of $M \times K$ are conjugate. It follows that

$$e_{M_\psi}^{M\times K} = e_{M_{i\circ\psi}}^{M\times K} = e_{M_{\varphi\circ s}}^{M\times K} \in \mathsf{e}_{L,\varphi}(M) \quad ,$$

that is $\mathsf{e}_{M,\psi} \subseteq \mathsf{e}_{L,\varphi}$, proving Assertion 1.

Now if $f : (L,\varphi) \to (M,\psi)$ is an isomorphism in $\mathsf{grp}_{\Downarrow K}$, the group homomorphism $f : M \to L$ is an isomorphism. Then (M,ψ) and (L,φ) are quotient of one another, so $\mathsf{e}_{M,\psi} = \mathsf{e}_{L,\varphi}$, proving Assertion 2. □

3.8. Notation: *We fix a set $\mathcal{S}_K$ of representatives of isomorphism classes of objects in the category $\mathsf{grp}_{\Downarrow K}$.*

3.9. Proposition: *Let I be an ideal of $\mathbb{F}B_K$. Then I is equal to the sum of the ideals $\mathsf{e}_{L,\varphi}$ it contains. More precisely, if*

$$\mathcal{A}_I = \{(L,\varphi) \in \mathcal{S}_K \mid \mathsf{e}_{L,\varphi} \subseteq I\} \quad ,$$

we have $I = \sum_{(L,\varphi)\in\mathcal{A}_I} \mathsf{e}_{L,\varphi}$. It follows that the ideals of $\mathbb{F}B_K$ form a set.

Proof: Let $J = \sum_{\substack{(L,\varphi)\in\mathcal{S}_K \\ \mathsf{e}_{L,\varphi}\subseteq I}} \mathsf{e}_{L,\varphi}$. Then obviously $J \subseteq I$. Moreover, if (M,ψ) is a group over K such that $\mathsf{e}_{M,\psi} \subseteq I$, then $\mathsf{e}_{M,\psi} \subseteq J$: indeed, there is some $(L,\varphi) \in \mathcal{S}_K$ isomorphic to (M,ψ), and $\mathsf{e}_{M,\psi} = \mathsf{e}_{L,\varphi}$ by Lemma 3.7. Conversely, let G be a finite group, and $u \in I(G)$. Then u is a linear combination

$$u = \sum_L \lambda_L e_L^{G\times K}$$

with coefficients λ_L in $\mathbb{F}$, of idempotents $e_L^{G\times K}$, where L runs through a set S of representatives of conjugacy classes of subgroups of $G \times K$. Then for any

$L \in S$, we have $e_L^{G\times K} \cdot u = \lambda_L e_L^{G\times K} \in I(G)$, hence $e_L^{G\times K} \in I(G)$ if $\lambda_L \neq 0$. So in this case, the ideal of $\mathbb{F}B_K$ generated by $e_L^{G\times K}$ is contained in I. This ideal is equal to e_{L,p_2}, by Corollary 3.5, thus $\mathsf{e}_{L,p_2} \subseteq J$ by the above observation. Hence $e_L^{G\times K} \in \mathsf{e}_{L,p_2}(G) \subseteq J(G)$. It follows that

$$u = \sum_{\substack{L\in S\\ \lambda_L\neq 0}} \lambda_L e_L^{G\times K}$$

also belongs to $J(G)$. Hence $I(G) \subseteq J(G)$, so $I(G) = J(G)$ since $J \subseteq I$. As G was arbitrary, it follows that $I = J$.

Now an ideal I of $\mathbb{F}B_K$ is determined by the subset $\mathcal{A}_I$ of $\mathcal{S}_K$, so the class of ideals of $\mathbb{F}B_K$ is in one to one correspondence with a set of subsets of $\mathcal{S}_K$. Hence this class is a set. □

3.10. Lemma: *Let $\mathcal{A}$ be a set of ideals of $\mathbb{F}B_K$, and (M,ψ) be a group over K. The following are equivalent:*

1. $\mathsf{e}_{M,\psi} \subseteq \sum_{I\in\mathcal{A}} I$.
2. *There exists $I \in \mathcal{A}$ such that $\mathsf{e}_{M,\psi} \subseteq I$.*

Proof: Clearly 2 implies 1. Now 1 is equivalent to saying that

$$e_{M_\psi}^{M\times K} \in \sum_{I\in\mathcal{A}} I(M) \ .$$

If this holds, there exists $I \in \mathcal{A}$ and $u \in I(M)$ such that $e_{M_\psi}^{M\times K} \cdot u \neq 0$. Now $e_{M_\psi}^{M\times K} \cdot u \in I(M)$, and moreover there is a scalar $\lambda \in \mathbb{F}$ such that $e_{M_\psi}^{M\times K} \cdot u = \lambda e_{M_\psi}^{M\times K} \neq 0$. Hence $\lambda \neq 0$, and $e_{M_\psi}^{M\times K} \in I(M)$. In other words $\mathsf{e}_{M,\psi} \subseteq I$, so 1 implies 2. □

4. B_K-groups

In view of Proposition 3.9, every ideal of $\mathbb{F}B_K$ is a sum of ideals $\mathsf{e}_{L,\varphi}$, where (L,φ) runs in some subset of $\mathcal{S}_K$. In view of Lemma 3.10, to describe the inclusions between such sum of ideals $\mathsf{e}_{L,\varphi}$, it suffices to describe elementary inclusions of the form $\mathsf{e}_{M,\psi} \subseteq \mathsf{e}_{L,\varphi}$, where (L,φ) and (M,ψ) are groups over K. Lemma 3.7 shows that it is the case if $(M,\psi) \twoheadrightarrow (L,\varphi)$. Moreover:

4.1. Theorem: *Let $s : (M,\psi) \to (L,\varphi)$ be a surjective morphism in $\mathsf{grp}_{\Downarrow K}$. If $m_{M,\mathrm{Ker}\, s} \neq 0$, then $\mathsf{e}_{M,\psi} = \mathsf{e}_{L,\varphi}$.*

Proof: We already know from Lemma 3.7 that $\mathsf{e}_{M,\psi} \subseteq \mathsf{e}_{L,\varphi}$, so it suffices to prove the reverse inclusion. We first observe that since there exists an inner automorphism i of K such that $i \circ \psi = \varphi \circ s$, we have $\mathrm{Ker}\, s \leq \mathrm{Ker}\,(i \circ \psi) = \mathrm{Ker}\, \psi$. So there is a group homomorphism $\overline{\psi} : \overline{M} = M/\mathrm{Ker}\, s \to K$ such that $\psi = \overline{\psi} \circ \pi$, where $\pi : M \to \overline{M}$ is the projection map.

Now let V be the set L, viewed as an (L, M)-biset for the action defined by $l \cdot v \cdot m = lvs(m)$, for $l, v \in V$ and $m \in M$ (in other words $V = U^{op}$, where U is the (M, L)-biset introduced in the proof of Lemma 3.7). Then there is an isomorphism of (L, M)-bisets

$$V \cong \mathrm{Iso}(\alpha) \circ \mathrm{Def}^M_{M/\mathrm{Ker}\, s} \ \ ,$$

where $\alpha : \overline{M} \to L$ is the group isomorphism induced by s, i.e. such that $s = \alpha \circ \pi$.

Let $v = \mathbb{F}B_K(V)(e^{M\times K}_{M_\psi}) \in \mathsf{e}_{M,\psi}(L)$. By Lemma 2.2

$$\mathbb{F}B_K\big(\mathrm{Def}^M_{M/\mathrm{Ker}\, s}\big)(e^{M\times K}_{M_\psi}) = \lambda\, m_{M_\psi, M_\psi \cap (\mathrm{Ker}\, s \times \mathbf{1})}\, e^{\overline{M}\times K}_{\overline{M_\psi}} \ \ ,$$

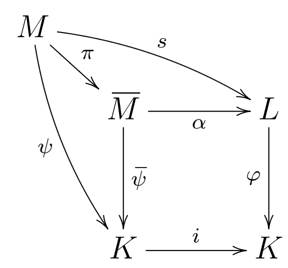
where $\overline{M_\psi}$ is the image of M_ψ by the projection $M \times K \to \overline{M} \times K$, and λ is some non zero rational number. Then

$$v = \lambda\, \mathbb{F}B_K\big(\mathrm{Iso}(\alpha)\big)(e^{\overline{M}\times K}_{\overline{M_\psi}}) = \lambda\, \mathbb{F}B\big(\mathrm{Iso}(\alpha_K)\big)(e^{\overline{M}\times K}_{\overline{M_\psi}}) \ \ ,$$

where $\alpha_K = \alpha \times \mathrm{Id}_K : \overline{M} \times K \to L \times K$. The image of $\overline{M}_{\overline{\psi}}$ under α_K is the subgroup

$$\alpha_K(\overline{M}_{\overline{\psi}}) = \big\{\big(\alpha(\overline{m}), \overline{\psi}(\overline{m})\big) \mid \overline{m} \in \overline{M}\big\} = \big\{\big(l, \overline{\psi} \circ \alpha^{-1}(l)\big) \mid l \in L\big\} \ \ .$$

Moreover, we have a diagram



where the two triangles and the outer "square" commute. It follows that

$$\varphi \circ \alpha \circ \pi = \varphi \circ s = i \circ \psi = i \circ \overline{\psi} \circ \pi \ \ ,$$

hence $\varphi \circ \alpha = i \circ \overline{\psi}$ since π is surjective. Hence $\overline{\psi} \circ \alpha^{-1} = i^{-1} \circ \varphi$, and $\alpha_K(\overline{M}_{\overline{\psi}}) = L_{i^{-1}\circ\varphi}$.

It follows that $v = \lambda\, e_{L_{i^{-1}\circ\varphi}}^{L\times K}$, and moreover $e_{L_{i^{-1}\circ\varphi}}^{L\times K} = e_{L_\varphi}^{L\times K}$ since $L_{i^{-1}\circ\varphi}$ and L_φ are conjugate in $L\times K$. Finally $v = \lambda\, e_{L_\varphi}^{L\times K}$, so $e_{L_\varphi}^{L\times K} \in \mathsf{e}_{M,\psi}(L)$, since $v \in \mathsf{e}_{M,\psi}(L)$ and $\lambda \neq 0$. In other words $\mathsf{e}_{L,\varphi} \subseteq \mathsf{e}_{M,\psi}$, and finally $\mathsf{e}_{L,\varphi} = \mathsf{e}_{M,\psi}$, as was to be shown. □

4.2. Notation: *When (M,ψ) is a group over K, and Q is a normal subgroup of M with $Q \leq \operatorname{Ker}\psi$, let $\psi/Q : M/Q \to K$ be the group homomorphism defined by $\psi = (\psi/Q)\circ\pi$, where π is the projection $M \to M/Q$.*

Thus for any group (M,ψ) over K, if Q is a normal subgroup of M contained in $\operatorname{Ker}\psi$, we get a surjective morphism $\pi : (M,\psi) \to (M/Q, \psi/Q)$ in $\mathsf{grp}_{\Downarrow K}$, with $\operatorname{Ker}\pi = Q$. If moreover $m_{M,Q} \neq 0$, we have $\mathsf{e}_{M,\psi} = \mathsf{e}_{M/Q,\psi/Q}$. This motivates the following:

4.3. Definition: *Let (L,φ) be a group over K. We say that (L,φ) is a B_K-group, or a B-group relative to K, if $m_{L,N} = 0$ for every non-trivial normal subgroup N of L contained in $\operatorname{Ker}\varphi$.*

4.4. Examples:

1. If $\varphi : L \to K$ is injective, then (L,φ) is a B_K-group.
2. On the other hand, if $K = \mathbf{1}$, then a group over K is a pair (L,φ), where L is a finite group and $\varphi : L \to \mathbf{1}$ is the unique morphism. Moreover the category $\mathsf{grp}_{\Downarrow\mathbf{1}}$ clearly identifies with the usual category of finite groups. With this identification, a $B_{\mathbf{1}}$-group is just a B-group (cf. Section 7.2 of [1], or Chapter 5 of [2]).

4.5. Lemma: *Let (L,φ) be a B_K-group. If (M,ψ) is a group over K, and (M,ψ) is isomorphic to (L,φ) in $\mathsf{grp}_{\Downarrow K}$, then (M,ψ) is a B_K-group.*

Proof: Since (M,ψ) is isomorphic to (L,φ) in $\mathsf{grp}_{\Downarrow K}$, there exists a group isomorphism $f : L \to M$ and an inner automorphism i of K such that $\psi \circ f = i \circ \varphi$. If P is a normal subgroup of M contained in $\operatorname{Ker}\psi$, then $f^{-1}(P)$ is a normal subgroup of L contained in $\operatorname{Ker}\varphi$, because

$$i \circ \varphi\big(f^{-1}(P)\big) = \psi \circ f\big(f^{-1}(P)\big) = \psi(P) = \mathbf{1}$$

and i is an automorphism. Moreover $m_{L,f^{-1}(P)} = m_{M,P}$. If P is non trivial, then $f^{-1}(P)$ is non trivial, so $m_{L,f^{-1}(P)} = m_{M,P} = 0$, as was to be shown. □

4.6. Theorem: *Let (L, φ) be a group over K.*

1. *If Q is a normal subgroup of L, contained in $\mathrm{Ker}\,\varphi$, and maximal such that $m_{L,Q} \neq 0$, then $(L/Q, \varphi/Q)$ is a B_K-group, quotient of (L, φ).*
2. *If (P, ψ) is a B_K-group which is quotient of (L, φ), and if N is a normal subgroup of L contained in $\mathrm{Ker}\,\varphi$ and such that $m_{L,N} \neq 0$, then (P, ψ) is a quotient of $(L/N, \varphi/N)$.*
3. *In particular, if P and Q are normal subgroups of L, contained in $\mathrm{Ker}\,\varphi$, and maximal such that $m_{L,P} \neq 0 \neq m_{L,Q}$, then $(L/P, \varphi/P)$ and $(L/Q, \varphi/Q)$ are isomorphic in $\mathsf{grp}_{\Downarrow K}$.*

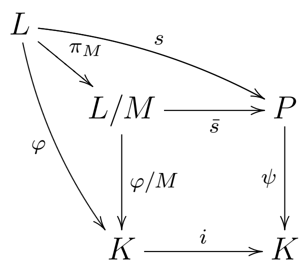
Proof: 1. Let P/Q be a normal subgroup of L/Q contained in $\mathrm{Ker}\,(\varphi/Q) = \mathrm{Ker}\,\varphi/Q$. Then P is a normal subgroup of L, and $Q \leq P \leq \mathrm{Ker}\,\varphi$. If $P/Q \neq \mathbf{1}$, i.e. if $Q < P$, then by maximality of Q and Proposition 5.3.1 of [2]

$$m_{L,P} = 0 = m_{L,Q} m_{L/Q,P/Q} \quad .$$

Since $m_{L,Q} \neq 0$, it follows that $m_{L/Q,P/Q} = 0$, so $(L/Q, \varphi/Q)$ is a B_K-group, quotient of (L, φ).

2. Since (P, ψ) is a quotient of (L, φ), there exists a surjective group homomorphism $s : L \to P$ and an inner automorphism i of K such that $\psi \circ s = i \circ \varphi$. It follows that $M = \mathrm{Ker}\,s$ is a normal subgroup of L contained in $\mathrm{Ker}\,(i \circ \varphi) = \mathrm{Ker}\,\varphi$.

We have a diagram



where the two triangles and the outer “square” commute, and $\bar{s}$ is an isomorphism, the map $\pi_M : L \to L/M$ being the projection. As in the proof of Theorem 4.1, we have

$$\psi \circ \bar{s} \circ \pi_M = \psi \circ s = i \circ \varphi = i \circ (\varphi/M) \circ \pi_M \quad ,$$

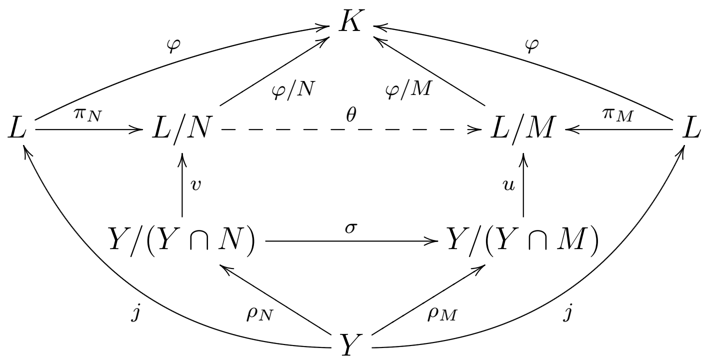
so $\psi \circ \bar{s} = i \circ (\varphi/M)$ since π_M is surjective. It follows that $\bar{s}$ is an isomorphism from $(L/M, \varphi/M)$ to (P, ψ) in $\mathsf{grp}_{\Downarrow K}$, so $(L/M, \varphi/M)$ is a B_K-group by Lemma 4.5.

Now by Proposition 5.3.3 of [2]

$$m_{L,N} = \frac{1}{|L|} \sum_{YN=YM=L} |Y|\mu(Y,L) m_{L/M,(Y\cap N)M/M} \quad .$$

In particular, if $m_{L,N} \neq 0$, there exists $Y \leq L$ such that $YN = YM = L$ and $m_{L/M,(Y\cap N)M/M} \neq 0$. But since $N \subseteq \operatorname{Ker}\varphi$, the group $(Y \cap N)M/M$ is a normal subgroup of L/M contained in $\operatorname{Ker}(\varphi/M) = \operatorname{Ker}\varphi/M$. Then since $m_{L/M,(Y\cap N)M/M} \neq 0$ and since $(L/M, \varphi/M)$ is a B_K-group, we have $(Y \cap N)M/M = \mathbf{1}$, i.e. $Y \cap N \subseteq Y \cap M$.

Consider now the following diagram:



where

- $j : Y \to L$ is the inclusion map,
- $\rho_N : Y \to Y/(Y \cap N)$ and $\rho_M : Y \to Y/(Y \cap M)$ are the projection maps,
- $u : Y/(Y \cap M) \to L/M$ and $v : Y/(Y \cap N) \to L/N$ are the canonical isomorphisms $Y/(Y \cap M) \cong YM/M = L/M$ and $Y/(Y \cap N) \cong YN/N = L/N$, respectively,
- $\sigma : Y/(Y\cap N) \to Y/(Y\cap M)$ is the projection map (as $Y\cap N \subseteq Y\cap M$),
- $\theta : L/N \to L/M$ is defined as $\theta = u \circ \sigma \circ v^{-1}$. In particular θ is surjective.

We have $\pi_N \circ j = v \circ \rho_N$, since for any $y \in Y$

$$\pi_N \circ j(y) = \pi_N(y) = yN = v\big(y(Y \cap N)\big) = v \circ \rho_N(y) \quad .$$

Similarly $\pi_M \circ j = u \circ \rho_M$. We also have $\sigma \circ \rho_N = \rho_M$. Then

$$\theta \circ \pi_N \circ j = \theta \circ v \circ \rho_N = u \circ \sigma \circ \rho_N = u \circ \rho_M = \pi_M \circ j \quad .$$

Hence

$$(\varphi/M) \circ \theta \circ \pi_N \circ j = (\varphi/M) \circ \pi_M \circ j = \varphi \circ j = (\varphi/N) \circ \pi_N \circ j \ .$$

Since $\pi_N \circ j = v \circ \rho_N : Y \to L/N$ is surjective, it follows that

$$(\varphi/M) \circ \theta = (\varphi/N) \ .$$

Hence θ is a surjective morphism from $(L/N, \varphi/N)$ to $(L/M, \varphi/M)$ in $\mathsf{grp}_{\Downarrow K}$. As the latter is isomorphic to (P, ψ) in $\mathsf{grp}_{\Downarrow K}$, it follows that (P, ψ) is a quotient of $(L/N, \varphi/N)$, as was to be shown.

3. If P and Q are normal subgroups of L, contained in $\operatorname{Ker}\varphi$, and maximal such that $m_{L,P} \neq 0 \neq m_{L,Q}$, then $(L/P, \varphi/P)$ and $(L/Q, \varphi/Q)$ are both B_K-groups by Assertion 1, and they are quotient of one another by Assertion 2. Hence they are isomorphic in $\mathsf{grp}_{\Downarrow K}$. □

4.7. Notation: *Let (L, φ) be a group over K. If Q is a normal subgroup of L, contained in* $\operatorname{Ker}\varphi$*, and maximal such that $m_{L,Q} \neq 0$, we denote by $\beta_K(L, \varphi)$ the quotient $(L/Q, \varphi/Q)$ of (L, φ).*

4.8. Remark: As observed in Example 4.4, when K is trivial, a B_K-group is simply a B-group. Moreover, for any finite group L, if $u : L \to \mathbf{1}$ is the unique group homomorphism, then $\beta_{\mathbf{1}}(L, u) = \beta(L)$.

The following corollary shows that $\beta_K(L, \varphi)$ is the largest B_K-group quotient of (L, φ):

4.9. Corollary: *Let (L, φ) be a group over K.*

1. *$\beta_K(L, \varphi)$ is well defined up to isomorphism in $\mathsf{grp}_{\Downarrow K}$.*
2. *$\beta_K(L, \varphi)$ is a B_K-group, quotient of (L, φ).*
3. *If (P, ψ) is a B_K-group, quotient of (L, φ), then (P, ψ) is a quotient of $\beta_K(L, \varphi)$.*
4. *$\mathsf{e}_{L,\varphi} = \mathsf{e}_{\beta_K(L,\varphi)}$.*

Proof: 1. This follows from Assertion 3 of Theorem 4.6.

2. This follows from Assertion 1 of Theorem 4.6.

3. This follows from Assertion 2 of Theorem 4.6.

4. This follows from Theorem 4.1, by definition of $\beta_K(L, \varphi)$ □

4.10. Corollary: *Let $s : (M,\psi) \twoheadrightarrow (L,\varphi)$ be a surjective morphism in $\mathsf{grp}_{\Downarrow K}$. Then $\beta_K(M,\psi) \cong \beta_K(L,\varphi)$ if and only if $m_{M,\mathrm{Ker}\, s} \neq 0$.*

Proof: Indeed $\beta_K(L,\varphi)$ is a quotient of (M,ψ), as it is a quotient of (L,φ) and s is surjective. Hence $\beta_K(L,\varphi)$ is a quotient of $\beta_K(M,\psi)$. Set $N = \mathrm{Ker}\, s$, so that $(L,\varphi) \cong (M/N, \psi/N)$.

If $m_{M,N} \neq 0$, then since $\beta_K(M,\psi)$ is a B_K-group quotient of (M,ψ), Assertion 2 of Theorem 4.6 implies that $\beta_K(M,\psi)$ is a quotient of $(M/N,\psi/N) \cong (L,\varphi)$, hence of $\beta_K(L,\varphi)$. It follows that $\beta_K(M,\psi) \cong \beta_K(L,\varphi)$, as they are quotient of one another.

Conversely, suppose that $\beta_K(M,\psi) \cong \beta_K(L,\varphi)$, and let P/N be a normal subgroup of M/N contained in $\mathrm{Ker}\,(\psi/N) = \mathrm{Ker}\,\psi/N$ and maximal such that $m_{M/N,P/N} \neq 0$. Then the quotient $\big((M/N)/(P/N), (\psi/N)/(P/N)\big) \cong (M/P, \psi/P)$ is isomorphic to $\beta_K(M/N, P/N) \cong \beta_K(L,\varphi)$, hence to $\beta_K(M,\psi)$. Now if Q is a normal subgroup of M contained in $\mathrm{Ker}\,\psi$ and maximal such that $m_{M,Q} \neq 0$, then the quotient $(M/Q, \psi/Q)$ is isomorphic to $\beta_K(M,\psi) \cong (M/P,\psi/P)$. In particular $M/Q \cong M/P$, and then $m_{M,P} = m_{M,Q}$ by Proposition 5.3.4 of [2], so $m_{M,P} \neq 0$. But $m_{M,P} = m_{M,N} m_{M/N,P/N}$, so $m_{M,N} \neq 0$, as was to be shown. □

5. The ideals of $\mathbb{F}B_K$

5.1. Notation and Definition:

1. *We let $\mathcal{B}_K$-gr denote the subset of $\mathcal{S}_K$ consisting of B_K-groups.*
2. *A subset $\mathcal{P}$ of $\mathcal{B}_K$-gr is said to be* closed *if*

$$\forall (L,\varphi) \in \mathcal{P},\ \forall (M,\psi) \in \mathcal{B}_K\text{-gr},\ (M,\psi) \twoheadrightarrow (L,\varphi) \implies (M,\psi) \in \mathcal{P} \ .$$

5.2. Proposition: *Let I be an ideal of $\mathbb{F}B_K$, and*

$$\mathcal{P}_I = \{(L,\varphi) \in \mathcal{B}_K\text{-gr} \mid \mathsf{e}_{L,\varphi} \subseteq I\} \ .$$

Then $\mathcal{P}_I$ is a closed subset of $\mathcal{B}_K$-gr, and $I = \sum\limits_{(L,\varphi)\in\mathcal{P}_I} \mathsf{e}_{L,\varphi}$

Proof: The subset $\mathcal{P}_I$ of $\mathcal{B}_K$-gr is closed by Lemma 3.7. The second assertion follows from Proposition 3.9 and Assertion 4 of Corollary 4.9. □

5.3. Theorem: *Let (L,φ) be a B_K-group. Then for any finite group G*

$$\mathsf{e}_{L,\varphi}(G)=\sum_X \mathbb{F}e_X^{G\times K}\quad ,$$

where X runs through all subgroups of $G\times K$ such that $(X,p_2)\twoheadrightarrow(L,\varphi)$.

Proof: If $X\leq G\times K$ and $(X,p_2)\twoheadrightarrow(L,\varphi)$, then $\mathsf{e}_{X,p_2}\subseteq\mathsf{e}_{L,\varphi}$ by Lemma 3.7. Equivalently $e_{X_{p_2}}^{X\times K}\in\mathsf{e}_{L,\varphi}(X)$, which is equivalent to $e_X^{G\times K}\in\mathsf{e}_{L,\varphi}(G)$, by Theorem 3.4. This proves that for each finite group G, the sum $E(G)=\sum\limits_X \mathbb{F}e_X^{G\times K}$, where $X\leq G\times K$ and $(X,p_2)\twoheadrightarrow(L,\varphi)$, is a subset of $\mathsf{e}_{L,\varphi}(G)$.

Moreover the map $\big(l,\varphi(l)\big)\in L_\varphi\mapsto l\in L$ is clearly an isomorphism $(L_\varphi,p_2)\to(L,\varphi)$ in $\mathsf{grp}_{\Downarrow K}$. In particular $(L_\varphi,p_2)\twoheadrightarrow(L,\varphi)$, and then by definition $e_{L_\varphi}^{L\times K}\in E(L)$. If we can prove that $G\mapsto E(G)$ defines an ideal E of $\mathbb{F}B_K$, then we are done, because $E\subseteq\mathsf{e}_{L,\varphi}$ since $E(G)\subseteq\mathsf{e}_{L,\varphi}(G)$ for any G, and $\mathsf{e}_{L,\varphi}\subseteq E$ because the generator $e_{L_\varphi}^{L\times K}$ of $\mathsf{e}_{L,\varphi}$ belongs to $E(L)$.

Since $E(G)$ is obviously an ideal of the algebra $\mathbb{F}B_K(G)$, for any G, all we have to do is to show that E is a biset subfunctor of $\mathbb{F}B_K$, in other words that it is preserved by the elementary biset operations of induction, restriction, inflation, deflation, and transport by group isomorphism. For this, in what follows, we refer to Theorem 5.2.4 of [2].

Let $X\leq G\times K$ be such that $(X,p_2)\twoheadrightarrow(L,\varphi)$, and suppose first that G is a subgroup of a group H. Then

$$\mathbb{F}B_K\big(\mathrm{Ind}_G^H\big)(e_X^{G\times K})=\mathbb{F}B\big(\mathrm{Ind}_{G\times K}^{H\times K}\big)(e_X^{G\times K})=\lambda\, e_{X'}^{H\times K}$$

for some scalar λ, where X' is the group X, viewed as a subgroup of $H\times K$. Clearly $(X',p_2)=(X,p_2)$, so $(X',p_2)\twoheadrightarrow(L,\varphi)$ and $e_{X'}^{H\times K}\in E(H)$. Hence E is preserved by induction.

Assume now that H is a subgroup of G. Then

$$\mathbb{F}B_K\big(\mathrm{Res}_H^G\big)(e_X^{G\times K})=\mathbb{F}B\big(\mathrm{Res}_{H\times K}^{G\times K}\big)(e_X^{G\times K})=\sum_Y e_Y^{H\times K}\quad ,$$

where Y runs through a set of representatives of $(H\times K)$-conjugacy classes of subgroups of $H\times K$ which are conjugate to X in $G\times K$. If Y is such a subgroup, there exists $(g,k)\in G\times K$ such that $Y=X^{(g,k)}$. Then we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & X\\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2}\\ K & \xrightarrow{\beta} & K \end{array}$$

where α is (left-)conjugation by (g,k) and β is (left-)conjugation by k. Since β is an inner automorphism of K, and since α is a group isomorphism, it follows that $\alpha : (Y,p_2) \to (X,p_2)$ is an isomorphism in $\mathsf{grp}_{\Downarrow K}$. Hence $(Y,p_2) \twoheadrightarrow (L,\varphi)$, and $e_Y^{H\times K} \in E(H)$. It follows that E is preserved by restriction.

Assume next that G is a quotient of a group H by a normal subgroup N. Then

$$\mathbb{F}B_K\big(\mathrm{Inf}_G^H\big)(e_X^{G\times K}) = \mathbb{F}B\big(\mathrm{Inf}_{G\times K}^{H\times K}\big)(e_X^{G\times K}) = \sum_Y e_Y^{H\times K} \;\;,$$

where Y runs through a set of $(H\times K)$ conjugacy classes of subgroup of $H\times K$ which map to a conjugate of X under the projection $\pi\times\mathrm{Id}_K : H\times K \to G\times K$, where $\pi : H \to G$ is the projection. Replacing Y by a conjugate, which does not change $e_Y^{H\times K}$, we can assume that Y is mapped to X by $\pi \times \mathrm{Id}_K$. This gives a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\pi\times\mathrm{Id}_K} & X \\ & {\scriptstyle p_2}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & K & \end{array}$$

showing that $(Y,p_2) \twoheadrightarrow (X,p_2)$. Hence $(Y,p_2) \twoheadrightarrow (L,\varphi)$, so $e_Y^{H\times K} \in E(H)$, and E is preserved by inflation.

As for deflation, we assume now that $H = G/N$, where $N \trianglelefteq G$. Let $\pi : G \to H$ be the projection map. Then by Lemma 2.2

$$\mathbb{F}B_K\big(\mathrm{Def}_H^G\big)(e_X^{G\times K}) = \lambda\, m_{X,X\cap(N\times\mathbf{1})}\, e_{\overline{X}}^{H\times K} \;\;,$$

where $\overline{X}$ is the image of X under the projection $\pi \times \mathrm{Id}_K : G\times K \to H\times K$, and λ is some non zero scalar. As above, we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & \overline{X} \\ & {\scriptstyle p_2}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & K & \end{array}$$

where s is the restriction of $\pi \times \mathrm{Id}_K$ to X. Then $s : (X,p_2) \to (\overline{X},p_2)$ is a surjective morphism in $\mathsf{grp}_{\Downarrow K}$. Setting $P = \mathrm{Ker}\, s = X \cap (N \times \mathbf{1})$, we get an isomorphism $(\overline{X},p_2) \cong (X/P, p_2/P)$ in $\mathsf{grp}_{\Downarrow K}$. Moreover (L,φ) is a B_K-group quotient of (X,p_2) by assumption. Then there are two cases: either $m_{X,P} = 0$, and then $\mathbb{F}B_K\big(\mathrm{Def}_H^G\big)(e_X^{G\times K}) = 0 \in E(H)$. Or $m_{X,P} \neq 0$, and then (L,φ) is a quotient of $(X/P,p_2/P) \cong (\overline{X},p_2)$, by Assertion 2 of

Theorem 4.6. It follows that $e_X^{H\times K} \in E(H)$, so $\mathbb{F}B_K\big(\mathrm{Def}_H^G\big)(e_X^{G\times K}) \in E(H)$ as well. This shows that E is preserved by deflation.

Finally, it is clear that E is preserved by group isomorphisms. This completes the proof of Theorem 5.3. □

5.4. Remark: Theorem 5.3 implies that the set of idempotents $e_X^{G\times K}$, where X runs through a set of *representatives of conjugacy classes* of subgroups of $G\times K$ such that $(X, p_2) \twoheadrightarrow (L, \varphi)$, is an $\mathbb{F}$-*basis* of $\mathsf{e}_{L,\varphi}(G)$.

5.5. Corollary: *Let (L, φ) be a B_K-group, and (M, ψ) be a group over K. Then $\mathsf{e}_{M,\psi} \subseteq \mathsf{e}_{L,\varphi}$ if and only if $(M, \psi) \twoheadrightarrow (L, \varphi)$.*

Proof: Indeed $\mathsf{e}_{M,\psi} \subseteq \mathsf{e}_{L,\varphi}$ if and only if $e_{M_\psi}^{M\times K} \in \mathsf{e}_{L,\varphi}(M)$, i.e. if and only if $(M_\psi, p_2) \twoheadrightarrow (L, \varphi)$. But we have already noticed at the beginning of the proof of Theorem 5.3 that the map $\big(m, \psi(m)\big) \in M_\psi \mapsto m \in M$ is an isomorphism from (M_ψ, p_2) to (M, ψ) in $\mathsf{grp}_{\Downarrow K}$. □

5.6. Remark: It was shown in Section 5.2.2 of [3] that the category $\mathbb{F}B_K$-Mod splits as a product

$$\mathbb{F}B_K\text{-Mod} \cong \prod_H = e_H^K \mathbb{F}B_K\text{-Mod} \;\; ,$$

of categories of modules over smaller Green biset functors $e_H^K\mathbb{F}B_K$, where H runs through a set of representatives of conjugacy classes of subgroups of K. The functor $e_H^K\mathbb{F}B_K$ is the direct summand of $\mathbb{F}B_K$ obtained from the idempotent e_H^K of $\mathbb{F}B_K(\mathbf{1}) \cong \mathbb{F}B(K)$. Its value at a group G is the set of $\mathbb{F}$-linear combinations of idempotents $e_L^{G\times K}$ associated to subgroups L for which $p_2(L)$ is conjugate to H in K. This condition is equivalent to the existence of a surjective morphism $(L, p_2) \twoheadrightarrow (H, j_H)$, where $j_H : H \hookrightarrow K$ is the inclusion morphism. Since (H, j_H) is a B_K-group by Example 4.4, it follows that $e_H^K\mathbb{F}B_K = \mathsf{e}_{H,j_H}$.

5.7. Theorem: *Let $\mathcal{I}_{\mathbb{F}B_K}$ be the lattice of ideals of $\mathbb{F}B_K$, ordered by inclusion of ideals, and $\mathcal{C}l_{\mathcal{B}_K\text{-gr}}$ be the lattice of closed subsets of $\mathcal{B}_K$-gr, ordered by inclusion of subsets. Then the map*

$$I \in \mathcal{I}_{\mathbb{F}B_K} \mapsto \mathcal{P}_I = \{(L, \varphi) \in \mathcal{B}_K\text{-gr} \mid \mathsf{e}_{L,\varphi} \subseteq I\}$$

is an isomorphism of lattices from $\mathcal{I}_{\mathbb{F}B_K}$ to $\mathcal{C}l_{\mathcal{B}_K\text{-gr}}$. The inverse isomorphism is the map

$$\mathcal{P} \in \mathcal{C}l_{\mathcal{B}_K\text{-gr}} \mapsto I_\mathcal{P} = \sum_{(L,\varphi)\in\mathcal{P}} \mathsf{e}_{L,\varphi} \;\; .$$

In particular $\mathcal{I}_{\mathbb{F}B_K}$ is completely distributive.

Proof: By Proposition 5.2, if I is an ideal of $\mathbb{F}B_K$, then $\mathcal{P}_I$ is a closed subset of $\mathcal{B}_K$-gr, so the map $\alpha : I \mapsto \mathcal{P}_I$ from $\mathcal{I}_{\mathbb{F}B_K}$ to $\mathcal{C}l_{\mathcal{B}_K\text{-gr}}$ is well defined. It is moreover clearly order preserving. The map $\beta : \mathcal{P} \mapsto \mathcal{P}_I$ from $\mathcal{C}l_{\mathcal{B}_K\text{-gr}}$ is also well defined and order preserving. By Proposition 5.2 again, the composition $\beta \circ \alpha$ is the identity map of $\mathcal{I}_{\mathbb{F}B_K}$. Conversely, if $\mathcal{P} \in \mathcal{C}l_{\mathcal{B}_K\text{-gr}}$, then

$$\alpha \circ \beta = \big\{(M, \psi) \in \mathcal{B}_K\text{-gr} \mid \mathsf{e}_{M,\psi} \subseteq \sum_{(L,\varphi)\in\mathcal{P}} \mathsf{e}_{L,\varphi}\big\} \enspace .$$

Then clearly $\mathcal{P} \subseteq \alpha \circ \beta(\mathcal{P})$. Conversely, if $\mathsf{e}_{M,\psi} \subseteq \sum\limits_{(L,\varphi)\in\mathcal{P}} \mathsf{e}_{L,\varphi}$, then by Lemma 3.10 there exists $(L, \varphi) \in \mathcal{P}$ such that $\mathsf{e}_{M,\psi} \subseteq \mathsf{e}_{L,\varphi}$. Then (L, φ) is a B_K-group, and by Corollary 5.5, this implies $(M, \psi) \twoheadrightarrow (L, \varphi)$. Hence $(M, \psi) \in \mathcal{P}$, since $\mathcal{P}$ is closed. Thus $\alpha \circ \beta(\mathcal{P}) \subseteq \mathcal{P}$, proving that $\alpha \circ \beta$ is the identity map of $\mathcal{C}l_{\mathcal{B}_K\text{-gr}}$. The last assertion follows from the fact that $\mathcal{C}l_{\mathcal{B}_K\text{-gr}}$ is clearly completely distributive, since its join and meet operation are union and intersection of closed subsets, respectively, and since arbitrary unions (resp. intersections) distribute over arbitrary intersections (resp. unions). □

6. Some simple $\mathbb{F}B_K$-modules

6.1. Theorem:

1. *Let (L, φ) be a B_K-group. Then $\mathsf{e}_{L,\varphi}$ admits a unique maximal proper subideal $\mathsf{e}^0_{L,\varphi}$, defined by*

$$\mathsf{e}^0_{L,\varphi} = \sum_{\substack{(M,\psi)\in\mathcal{B}_K\text{-gr} \\ (M,\psi)\twoheadrightarrow(L,\varphi) \\ (M,\psi)\not\cong(L,\varphi)}} \mathsf{e}_{M,\psi} \enspace .$$

2. *The quotient $S_{L,\varphi} = \mathsf{e}_{L,\varphi}/\mathsf{e}^0_{L,\varphi}$ is a simple $\mathbb{F}B_K$-module.*
3. *For any finite group G, let A_G be a set of representatives of conjugacy classes of subgroups X of $G \times K$ such that $\beta_K(X, p_2) \cong (L, \varphi)$. Then the set $\{e^{G\times K}_X \mid X \in A_G\}$ maps to an $\mathbb{F}$-basis of $S_{L,\varphi}(G)$ under the projection map $\mathsf{e}_{L,\varphi}(G) \to S_{L,\varphi}(G)$.*
4. *If $I' \subset I$ are ideals of $\mathbb{F}B_K$ such that I/I' is a simple $\mathbb{F}B_K$-module, then there exists a B_K-group (L, φ) such that $I/I' \cong S_{L,\varphi}$.*

Proof: 1. Without loss of generality, we can assume that $(L,\varphi) \in \mathcal{B}_K$-gr. Using Theorem 5.7, saying that $\mathsf{e}_{L,\varphi}$ admits a unique maximal proper subideal is equivalent to saying that the closed subset $\mathcal{P}_{\mathsf{e}_{L,\varphi}}$ contains a unique maximal proper closed subset. But

$$\mathcal{P}_{\mathsf{e}_{L,\varphi}} = \{(M,\psi) \in \mathcal{B}_K\text{-gr} \mid (M,\psi) \twoheadrightarrow (L,\varphi)\} \ ,$$

so $\mathcal{P}^0 = \mathcal{P}_{\mathsf{e}_{L,\varphi}} - \{(L,\varphi)\}$ is the unique maximal proper closed subset of $\mathcal{P}_{\mathsf{e}_{L,\varphi}}$. It follows that $I_{\mathcal{P}^0} = \mathsf{e}^0_{L,\varphi}$ is the unique maximal proper subideal of $\mathsf{e}_{L,\varphi}$.

2. This is clear, from 1.

3. We know from Remark 5.4 that $\mathsf{e}_{L,\varphi}(G)$ has a basis consisting of the idempotents $e_X^{G\times K}$, for X in a set of representatives of conjugacy classes of subgroups of $G\times K$ such that $(X,p_2) \twoheadrightarrow (L,\varphi)$, or equivalently, by Corollary 4.9, such that $\beta_K(X,p_2) \twoheadrightarrow (L,\varphi)$. Now saying that $e_X^{G\times K} \in \mathsf{e}^0_{L,\varphi}(G)$ amounts to saying that $e_{X_{p_2}}^{X\times K} \in \mathsf{e}^0_{L,\varphi}(X)$, by Theorem 3.4, i.e. that $\mathsf{e}_{X,p_2} \subseteq \mathsf{e}_{M,\psi}$ for some $(M,\psi) \in \mathcal{B}_K$-gr such that $(M,\psi) \twoheadrightarrow (L,\varphi)$, but $(M,\psi) \ncong (L,\varphi)$. This in turn is equivalent to saying that $\beta_K(X,p_2) \twoheadrightarrow (L,\varphi)$, but $\beta_K(X,p_2) \ncong (L,\varphi)$. Hence $S_{L,\varphi}(G)$ has a basis consisting of the idempotents $e_X^{G\times K}$, for X in a set of representatives of conjugacy classes of subgroups of $G \times K$ such that $\beta_K(X,p_2) \cong (L,\varphi)$. Assertion 2 follows.

4. Let $I' \subset I$ be ideals of $\mathbb{F}B_K$ such that $S = I/I'$ is a simple $\mathbb{F}B_K$-module, or equivalently, such that I' is a maximal subideal of I. Then there exists $(L,\varphi) \in \mathcal{B}_K$-gr such that $\mathsf{e}_{L,\varphi} \subseteq I$ but $\mathsf{e}_{L,\varphi} \nsubseteq I'$. Hence $\mathsf{e}_{L,\varphi} + I' = I$, and $S = I/I' \cong \mathsf{e}_{L,\varphi}/(\mathsf{e}_{L,\varphi} \cap I')$. Then $\mathsf{e}_{L,\varphi} \cap I'$ is a proper subideal of $\mathsf{e}_{L,\varphi}$, so $\mathsf{e}_{L,\varphi} \cap I' \subseteq \mathsf{e}^0_{L,\varphi}$, and then S maps surjectively onto $\mathsf{e}_{L,\varphi}/\mathsf{e}^0_{L,\varphi} = S_{L,\varphi}$. Since S and $S_{L,\varphi}$ both are simple $\mathbb{F}B_K$-modules, the surjection $S \to S_{L,\varphi}$ is an isomorphism. □

6.2. Remark: By Corollary 4.10, the condition $\beta_K(X,p_2) \cong (L,\varphi)$ in Assertion 3 is equivalent to the existence of a surjective morphism s from (X,p_2) to (L,φ) such that $m_{X,\mathrm{Ker}\,s} \neq 0$. By Theorem 5.4.11 of [2], or by Corollary 4.10 applied to the case $K = \mathbf{1}$, this is equivalent to the condition $\beta(X) \cong \beta(L)$.

6.3. Corollary: *Let (L,φ) and (M,ψ) be B_K-groups. Then the simple $\mathbb{F}B_K$-modules $S_{L,\varphi}$ and $S_{M,\psi}$ are isomorphic if and only if (L,φ) and (M,ψ) are isomorphic in $\mathsf{grp}_{\Downarrow K}$.*

Proof: Clearly if $(L,\varphi) \cong (M,\psi)$ in $\mathsf{grp}_{\Downarrow K}$, then $S_{L,\varphi} \cong S_{M,\psi}$. Conversely, if $\theta : S_{L,\varphi} \to S_{M,\psi}$ is an isomorphism of $\mathbb{F}B_K$-modules, then for any finite group G, we get an isomorphism $\theta_G : S_{L,\varphi}(G) \to S_{M,\psi}(G)$ of $\mathbb{F}B_K(G)$-modules. Choose G such that $S_{L,\varphi}(G) \neq 0$ (e.g. $G = L$), and a subgroup X of $G \times K$ such that $\beta_K(X, p_2) \cong (L, \varphi)$. Then the image u of $a = e_X^{G\times K} \in \mathbb{F}B_K(G)$ in $S_{L,\varphi}(G)$ is non zero, and moreover $a \cdot u = u$. It follows that $\theta_G(a \cdot u) = a \cdot \theta_G(u) = \theta_G(u)$ is also non zero in $S_{M,\psi}(G)$. So there is a subgroup $Y \leq G\times K$ with $\beta_K(Y, p_2) \cong (M, \psi)$, such that the image v of $e_Y^{G\times K}$ in $S_{M,\psi}(G)$ satisfies $a \cdot v \neq 0$. This forces X and Y to be conjugate in $G \times K$, so $(L, \varphi) \cong \beta_K(X, p_2) \cong \beta_K(Y, p_2) \cong (M, \psi)$ in $\mathsf{grp}_{\Downarrow K}$, as was to be shown. □

Recall that a *minimal group* for a (non zero) biset functor F is a finite group G of minimal order such that $F(G) \neq \{0\}$.

6.4. Lemma: *Let (L, φ) be a group over K.*

1. *If $N \trianglelefteq L$, and $N \cap \mathrm{Ker}\,\varphi = \mathbf{1}$, then $\mathsf{e}_{L,\varphi}(L/N) \neq \{0\}$.*
2. *If moreover (L, φ) is a B_K-group, then $S_{L,\varphi}(L/N) \neq \{0\}$.*

Proof: Indeed the map

$$\theta : l \in L \mapsto \big(lN, \varphi(l)\big) \in (L/N) \times K$$

is injective. Let $\overline{L} \leq (L/N) \times K$ denote the image of θ. Then we have a commutative diagram

$$\begin{array}{ccccc} & & \overline{L} & \xrightarrow{\ t\ } & L \\ & \swarrow{\scriptstyle p_1} & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \varphi} \\ L/N & & K & \xrightarrow{\ i\ } & K\,, \end{array}$$

where $t : \overline{L} \to L$ is the inverse of the isomorphism $L \to \overline{L}$ induced by θ, and i is the identity map of K. Hence $(\overline{L}, p_2) \cong (L, \varphi)$ in $\mathsf{grp}_{\Downarrow K}$, and $e_{L,\varphi} = e_{\overline{L},p_2}$. In particular $e_{\overline{L}_{p_2}}^{(L/N)\times K} \in e_{L,\varphi}(L/N)$ by Theorem 3.4, hence $\mathsf{e}_{L,\varphi}(L/N) \neq \{0\}$. This proves 1.

If moreover (L, φ) is a B_K-group, then $\beta_K(\overline{L}, p_2) \cong (L, \varphi)$. It follows from Theorem 6.1 that $e_{\overline{L}}^{(L/N)\times K} \in \mathsf{e}_{L,\varphi}(L/N)$ maps to an element of a basis of $S_{L,\varphi}(L/N)$, so $S_{L,\varphi}(L/N) \neq \{0\}$, proving 2. □

6.5. Theorem: *Let (L, φ) be a B_K-group, and G be a finite group. The following are equivalent:*

1. *The group G is a minimal group for $S_{L,\varphi}$.*
2. *The group G is isomorphic to L/N, where N is a normal subgroup of L*

of maximal order such that $N \cap \operatorname{Ker}\varphi = \mathbf{1}$.

Moreover in this case, the images in $S_{L,\varphi}(G)$ of the idempotents $e_X^{G\times K}$, where X runs through a set of representatives of conjugacy classes of subgroups of $G \times K$ such that $(X, p_2) \cong (L, \varphi)$, form an $\mathbb{F}$-basis of $S_{L,\varphi}(G)$.

Proof: By Theorem 6.1, saying that $S_{L,\varphi}(G) \neq \{0\}$ for a finite group G amounts to saying that there exists a subgroup X of $G \times K$ such that $\beta_K(X, p_2) \cong (L, \varphi)$ in $\mathsf{grp}_{\Downarrow K}$. Equivalently, there is a commutative diagram

$$(\mathbf{6.6})\qquad \begin{array}{ccccc} & & X & \xrightarrow{\;s\;} & L \\ & \swarrow^{p_1} & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \varphi} \\ G & & K & \xrightarrow{\;i\;} & K\,, \end{array}$$

where

- s is surjective and $m_{X,\operatorname{Ker} s} \neq 0$,
- i is an inner automorphism of K,
- the map $(p_1, p_2) : X \to (G \times K)$ is injective.

Now we proceed with the proof of Theorem 6.5.

$\boxed{1 \Rightarrow 2}$ If G is minimal for $S_{L,\varphi}$, then $S_{L,\varphi}(G) \neq \{0\}$, so we have a diagram (6.6). Let $H = p_1(G)$. Replacing G by H in this diagram gives a diagram for the group H with the same properties, so $S_{L,\varphi}(H) \neq 0$. Hence $H = G$ by minimality of G. In other words p_1 is surjective, so $G \cong X/\operatorname{Ker} p_1$. Let $N = s(\operatorname{Ker} p_1)$. If $u \in N \cap \operatorname{Ker}\varphi$, then $u = s(x)$ for some $x \in X$, and then $\varphi \circ s(x) = i \circ p_2(x) = 1$, so $p_2(x) = 1$. Thus $x = 1$ since $\operatorname{Ker} p_1 \cap \operatorname{Ker} p_2 = \mathbf{1}$. Moreover N is normal in L, since s is surjective. Lemma 6.4 shows that $S_{L,\varphi}(L/N) \neq \{0\}$, and by minimality of G, the surjection $\bar{s} : G \cong X/\operatorname{Ker} p_1 \twoheadrightarrow L/N$ induced by s must be an isomorphism. Lemma 6.4 also implies that N is a normal subgroup of maximal order of L such that $N \cap \operatorname{Ker}\varphi$. Hence 2 holds.

Observe that it also follows that $\operatorname{Ker} s \leq \operatorname{Ker} p_1$, so $\operatorname{Ker} s = \mathbf{1}$ since $\operatorname{Ker} s \leq \operatorname{Ker} p_2$ as $\varphi \circ s = i \circ p_2$, and $\operatorname{Ker} p_1 \cap \operatorname{Ker} p_2 = \mathbf{1}$. So s is an isomorphism $X \to L$. This proves the last assertion of the theorem.

$\boxed{2 \Rightarrow 1}$ Suppose that 2 holds. Then $S_{L,\varphi}(G) \neq 0$, by the above claim. By the first part of the proof, if H is a minimal group for $S_{L,\varphi}$, then $H \cong L/M$, where M is a normal subgroup of maximal order such that $M \cap \operatorname{Ker}\varphi = \mathbf{1}$. Then $|M| = |N|$, so $|G| = |H|$, and $S_{L,\varphi}(G') = \{0\}$ for any group G' of order smaller than $|G| = |H|$. Hence G is minimal for $S_{L,\varphi}$, and 1 holds. □

6.7. Corollary: *Let (L,φ) be a group over K. The following conditions are equivalent:*

1. *$\varphi : L \to K$ is injective.*
2. *(L,φ) is a B_K-group and $S_{L,\varphi}(\mathbf{1}) \neq \{0\}$.*

Proof: $\boxed{1 \Rightarrow 2}$ If φ is injective, then (L,φ) is a B_K-group (cf. Example 4.4). Moreover $L \cap \mathrm{Ker}\,\varphi = \mathbf{1}$, so $S_{L,\varphi}(L/L) = S_{L,\varphi}(\mathbf{1}) \neq \{0\}$.

$\boxed{2 \Rightarrow 1}$ If (L,φ) is a B_K-group and $S_{L,\varphi}(\mathbf{1}) \neq \{0\}$, then $\mathbf{1}$ is a minimal group for $S_{L,\varphi}$. So there is a normal subgroup N of L of maximal order such that $N \cap \mathrm{Ker}\,\varphi = \mathbf{1}$, such that moreover $L/N \cong \mathbf{1}$. Hence $N = L$, and $\mathrm{Ker}\,\varphi = N \cap \mathrm{Ker}\,\varphi = \mathbf{1}$. □

6.8. Example: Let $L = C_2 \times (C_3 \rtimes C_4)$ be a direct product of a group of order 2, generated by the element a, and a semidirect product of a group of order 3, generated by b, and a cyclic group of order 4, generated by c (so $cbc^{-1} = b^{-1}$). Let P be the subgroup of L generated by a and b. Then P is cyclic of order 6, and the factor group $K = L/P$ is cyclic of order 4, generated by the class cP. Let $\varphi : L \to K$ be the projection map. One can check that (L,φ) is a B_K-group, i.e. that $m_{L,Q} = 0$ when Q is any of the non trivial subgroups of P (these subgroups are all normal in L, as P is cyclic).

Then the subgroups $M = \langle ac^2\rangle$ and $N = \langle c^2\rangle$ both are normal (central, in fact) subgroups of L of maximal order (equal to 2) intersecting trivially $P = \mathrm{Ker}\,\varphi$. So the groups $G = L/M$ and $H = L/N$ are both minimal groups[1] for the simple $\mathbb{F}B_K$-module $S_{L,\varphi}$, but they are not isomorphic, as $G \cong C_3 \rtimes C_4$ but $H \cong C_2 \times S_3$, where S_3 is the symmetric group of degree 3. This gives yet another counterexample to a conjecture I made in 2010, saying that the minimal groups for a Green biset functor should form a single isomorphism class of groups. The first counterexample to this conjecture was found by Nadia Romero in 2013 (cf. [6]). Another counterexample was found recently by Ibrahima Tounkara (cf. [7]).

7. Restriction to p-groups

In this section, we fix a prime number p, and restrict the functor $\mathbb{F}B_K$ to finite p-groups. We obtain a Green p-biset functor $\mathbb{F}B_K^{(p)}$. We do not assume that K is itself a p-group.

[1]One can show moreover that $S_{L,\varphi}(G)$ and $S_{L,\varphi}(H)$ are both one dimensional.

In order to study the ideals of $\mathbb{F}B_K^{(p)}$, it is natural to try to determine those groups (L,φ) over K for which the restriction of $\mathsf{e}_{L,\varphi}$ to p-groups does not vanish. This motivates the following definition:

7.1. Definition: *Let K be a finite group. Then a group (L,φ) over K is called* p-persistent *if there is a finite p-group P such that $\mathsf{e}_{L,\varphi}(P) \neq \{0\}$.*

We denote by $\mathsf{grp}_{\Downarrow K}^{(p)}$ the full subcategory of $\mathsf{grp}_{\Downarrow K}$ consisting of p-persistent groups over K.

7.2. Remarks:

1. If X is a subgroup of $P \times K$, where P is a p-group, then (X, p_2) is p-persistent: indeed $e_X^{P\times K} \in \mathsf{e}_{X,p_2}(P)$ by Corollary 3.5.
2. Any quotient of a p-persistent group over K is p-persistent: indeed is $s : (M,\psi) \twoheadrightarrow (L,\varphi)$ is a surjective morphism in $\mathsf{grp}_{\Downarrow K}$, then $\mathsf{e}_{M,\psi} \subseteq \mathsf{e}_{L,\varphi}$ by Lemma 3.7. It follows that $\mathsf{e}_{L,\varphi}(P) \neq \{0\}$ if P is a p-group such that $\mathsf{e}_{M,\psi}(P) \neq \{0\}$. In particular, if (L,φ) is p-persistent, then $\beta_K(L,\varphi)$ is a p-persistent B_K-group.

7.3. Notation: *When L is a finite group, we denote by $O^p(L)$ the subgroup of L generated by p'-elements, and by $L^{[p]}$ the quotient $L/O^p(L)$.*

Recall that $O^p(L)$ is the smallest normal subgroup N of L such that L/N is a p-group. Also recall that if $s : M \to L$ is a surjective group homomorphism, then $s\big(O^p(M)\big) = O^p(L)$. Indeed $N = s\big(O^p(M)\big) \trianglelefteq L$, and s induces a surjection $M^{[p]} \to L/N$. So L/N is a p-group, thus $N \geq O^p(L)$. But N is generated by p'-elements, as $O^p(M)$ is, so $N \leq O^p(L)$.

7.4. Proposition: *Let (L,φ) be a group over K. The following are equivalent:*

1. *(L,φ) is p-persistent.*
2. *$\mathsf{e}_{L,\varphi}\big(L^{[p]}\big) \neq \{0\}$.*
3. *$m_{L,O^p(L)\cap \mathrm{Ker}\,\varphi} \neq 0$.*

Proof: Indeed if 3 holds, then setting $N = O^p(L) \cap \mathrm{Ker}\,\varphi$, we have $\mathsf{e}_{L,\varphi} = \mathsf{e}_{L/N,\varphi/N}$ by Theorem 4.1. Moreover $O^p(L/N) = O^p(L)/N$, and $\mathrm{Ker}\,(\varphi/N) = \mathrm{Ker}\,\varphi/N$. Thus $O^p(L/N) \cap \mathrm{Ker}(\varphi/N) = \mathbf{1}$, so $\mathsf{e}_{L/N,\varphi/N}\big((L/N)/O^p(L/N)\big)$ is non zero by Lemma 6.4. But

$$\mathsf{e}_{L/N,\varphi/N}\big((L/N)/O^p(L/N)\big) \cong \mathsf{e}_{L/N,\varphi/N}\big(L/O^p(L)\big) = \mathsf{e}_{L,\varphi}(L^{[p]}) \ ,$$

so 2 holds. Clearly 2 implies 1, as $L^{[p]}$ is a p-group. Now if 1 holds, let P be a p-group such that $\mathsf{e}_{L,\varphi}(P) \neq \{0\}$. Let N be a normal subgroup of L contained in $\mathrm{Ker}\,\varphi$, and maximal such that $m_{L,N} \neq 0$. Then setting $\bar{L} = L/N$ and $\bar{\varphi} = \varphi/N$, we have $\beta_K(L,\varphi) \cong (\bar{L},\bar{\varphi})$, and $\mathsf{e}_{L,\varphi} = \mathsf{e}_{\bar{L},\bar{\varphi}}$ by Theorem 4.1. Moreover as $(\bar{L},\bar{\varphi})$ is a B_K-group, by Theorem 5.3, there exists a subgroup X of $P \times K$, and a commutative diagram

$$\begin{array}{ccccc} & & X & \overset{s}{\twoheadrightarrow} & \bar{L} \\ & {}^{p_1}\swarrow & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \bar{\varphi}} \\ P & & K & \overset{i}{\longrightarrow} & K\,, \end{array}$$

where s is surjective and i is an inner automorphism of K. Then $N = s(\mathrm{Ker}\,p_1)$ is a normal subgroup of $\bar{L}$, as s is surjective. Moreover if $l \in N \cap \mathrm{Ker}\,\bar{\varphi}$, then $l = s(x)$ for some $x \in \mathrm{Ker}\,p_1$, so $p_1(x) = 1$ and $i \circ p_2(x) = \varphi \circ s(x) = 1$, so $p_2(x) = 1$. Hence $x = 1$, and $l = 1$, so $N \cap \mathrm{Ker}\,\bar{\varphi} = 1$. Now s induces a surjection $X/\mathrm{Ker}\,p_1 \cong p_1(X) \twoheadrightarrow L/N$, so L/N is a p-group, thus $N \geq O^p(L)$. It follows that $O^p(\bar{L}) \cap \mathrm{Ker}\,\bar{\varphi} = \mathbf{1}$. Now if $\pi : L \to \bar{L} = L/N$ is the projection map, we have $\bar{\varphi} \circ \pi = \varphi$, so

$$\pi\big(O^p(L) \cap \mathrm{Ker}\,\varphi\big) \leq O^p(\bar{L}) \cap \mathrm{Ker}\,\bar{\varphi} = \mathbf{1} \;\;,$$

that is $O^p(L) \cap \mathrm{Ker}\,\varphi \leq N = \mathrm{Ker}\,\pi$. Then if $M = O^p(L) \cap \mathrm{Ker}\,\varphi$, we have $m_{L,N} = m_{L,M} m_{L/M,N/M} \neq 0$, hence $m_{L,M} \neq 0$, so 3 holds. □

7.5. Corollary: *Let (L,φ) be a p-persistent B_K-group. Then*

$$O^p(L) \cap \mathrm{Ker}\,\varphi = \mathbf{1} \;\;.$$

Proof: Indeed $m_{L,O^p(L)\cap \mathrm{Ker}\,\varphi} \neq 0$, and (L,φ) is a B_K-group. □

7.6. Notation: *When (L,φ) is a p-persistent group over K, we denote by $L_\varphi^{(p)}$ the subgroup of $L^{[p]} \times K$ defined by*

$$L_\varphi^{(p)} = \big\{\big(lO^p(L), \varphi(l)\big) \mid l \in L\big\} \;\;.$$

The following theorem is analogous to Theorem 3.4:

7.7. Theorem: *Let I be an ideal of the Green biset functor $\mathbb{F}B_K^{(p)}$. If G is a finite p-group and L is a subgroup of $G \times K$, the following conditions are equivalent:*

1. *The idempotent $e_L^{G\times K}$ belongs to $I(G)$.*
2. *The idempotent $e_{L_{p_2}^{(p)}}^{L^{[p]}\times K}$ belongs to $I(L^{[p]})$.*

Proof: The proof is similar to the proof of Theorem 3.4, so we only sketch it. If $L \leq G \times K$, denote by $\widehat{L}$ the image of L in the group $L^{[p]} \times G$ by the map $l \mapsto \big(lO^p(L), p_1(l)\big)$. Recall that $\mathrm{Ker}\, p_1 \geq O^p(L)$, since G is a p-group. Furthermore $p_1(\widehat{L}) = L^{[p]}$, $k_1(\widehat{L}) = \mathrm{Ker}\, p_1/O^p(L)$, $p_2(\widehat{L}) = p_1(L)$, and $k_2(\widehat{L}) = p_1\big(O^p(L)\big) = \mathbf{1}$. The $(L^{[p]}, G)$-biset $U = (L^{[p]} \times G)/\widehat{L}$ factors as

$$U \cong \mathrm{Inf}_{L^{[p]}/k_1(\widehat{L})}^{L^{[p]}} \circ \mathrm{Iso}(\theta^{-1}) \circ \mathrm{Res}_{p_1(L)}^{G} \;\;,$$

where $\theta : L^{[p]}/k_1(\widehat{L}) \to p_1(G)$ is the isomorphism induced by the map $lO^p(L) \mapsto p_1(l)$ from $L^{[p]}$ to G.

If $e_L^{G\times K}$ belongs to $I(G)$, then $\mathbb{F}B_K^{(p)}(U)(e_L^{G\times K})$ belongs to $I(L^{[p]})$. As in the proof of Theorem 3.4, one can check that the product $e_{L_{p_2}^{(p)}}^{L^{[p]}\times K} \cdot \mathbb{F}B_K^{(p)}(e_L^{G\times K})$ is non zero. As it is a scalar multiple of $e_{L_{p_2}^{(p)}}^{L^{[p]}\times K}$, we get that $e_{L_{p_2}^{(p)}}^{L^{[p]}\times K} \in I(L^{[p]})$, thus 1 implies 2.

Conversely, assume that $e_{L_{p_2}^{(p)}}^{L^{[p]}\times K} \in I(L^{[p]})$. Then, as in the proof of Theorem 3.4 again, the opposite biset U^{op} factors as

$$U^{op} \cong \mathrm{Ind}_{p_1(L)}^{G} \circ \mathrm{Iso}(\theta) \circ \mathrm{Def}_{L^{[p]}/k_1(\widehat{L})}^{L^{[p]}} \;\;,$$

and the element $\mathbb{F}B_K^{(p)}(U^{op})\big(e_{L_{p_2}^{(p)}}^{L^{[p]}\times K}\big)$ belongs to $I(G)$. One can can check moreover that there is a non zero scalar λ such that

$$\mathbb{F}B_K^{(p)}(U^{op})\big(e_{L_{p_2}^{(p)}}^{L^{[p]}\times K}\big) = \lambda\, m_{L_{p_2}^{(p)}, L_{p_2}^{(p)}\cap(N\times\mathbf{1})}\, e_L^{G\times K} \;\;,$$

where $N = k_1(\widehat{L}) = \mathrm{Ker}\, p_1/O^p(L) \leq L^{[p]}$.

But if $\big(lO^p(L), p_2(l)\big) \in L_{p_2}^{(p)} \cap (N \times \mathbf{1})$, then $l \in \mathrm{Ker}\, p_2 \cap \mathrm{Ker}\, p_1 = \mathbf{1}$. It follows that $m_{L_{p_2}^{(p)}, L_{p_2}^{(p)}\cap(N\times\mathbf{1})} = m_{L_{p_2}^{(p)}, \mathbf{1}} = 1$, and $e_L^{G\times K} \in I(G)$, as $\lambda \neq 0$. Hence 2 implies 1. □

7.8. Corollary: *Let G be a finite p-group, and L be a subgroup of $G\times K$. Then the ideal of $\mathbb{F}B_K^{(p)}$ generated by $e_L^{G\times K}$ is equal to the ideal of $\mathbb{F}B_K^{(p)}$ generated by $e_{L_{p_2}^{(p)}}^{L^{[p]}\times K}$*

Proof: The proof is the same as the proof of Corollary 3.5. □

7.9. Notation: *Let (L,φ) be a p-persistent group over K. We denote by $\mathsf{e}_{L,\varphi}^{(p)}$ the ideal of $\mathbb{F}B_K^{(p)}$ generated by $e_{L_\varphi^{(p)}}^{L^{[p]}\times K}\in\mathbb{F}B_K^{(p)}(L^{[p]})$.*

7.10. Theorem: *Let $s:(M,\psi)\twoheadrightarrow(L,\varphi)$ be a surjective morphism in $\mathsf{grp}_{\Downarrow K}$, and assume that (M,ψ) is p-persistent. Then:*

1. *(L,φ) is p-persistent, and $\mathsf{e}_{M,\psi}^{(p)}\subseteq\mathsf{e}_{L,\varphi}^{(p)}$.*
2. *If $m_{M,\mathrm{Ker}\,s}\neq 0$, then $\mathsf{e}_{M,\psi}^{(p)}=\mathsf{e}_{L,\varphi}^{(p)}$.*

Proof: 1. We already observed in Remarks 7.2 that any quotient of a p-persistent group over K is itself p-persistent, hence (L,φ) is p-persistent. Let i be an inner automorphism of K such that $i\circ\psi=\varphi\circ s$. The surjection $s:M\to L$ induces a surjection $s^{[p]}:M^{[p]}\to L^{[p]}$, hence a surjection

$$s^{[p]}\times\mathrm{Id}_K:M^{[p]}\times K\to L^{[p]}\times K\ \ .$$

Let $u=\big(mO^p(M),\psi(m)\big)$ be the image of $m\in M$ in $M_\psi^{(p)}$. Then

$$(s^{[p]}\times\mathrm{Id}_K)(u)=\big(s(m)O^p(L),\psi(m)\big)=\Big(s(m)O^p(L),i^{-1}\circ\varphi\big(s(m)\big)\Big)\ \ ,$$

which shows that $s^{[p]}\times\mathrm{Id}_K$ maps $M_\psi^{(p)}$ to a conjugate of $L_\varphi^{(p)}$ in $L^{[p]}\times K$. Then the idempotent $e_{M_\psi^{(p)}}^{M^{[p]}\times K}$ appears in the decomposition of

$$\mathbb{F}B_K^{(p)}\big(\mathrm{Inf}_{M^{[p]}/\mathrm{Ker}\,s^{[p]}}^{M^{[p]}}\circ\mathrm{Iso}(\alpha^{-1})\big)(e_{L_\varphi^{(p)}}^{L^{[p]}\times K})\ \ ,$$

where $\alpha:M^{[p]}/\mathrm{Ker}\,s^{[p]}\to L^{[p]}$ is the canonical isomorphism. It follows that $e_{M_\psi^{(p)}}^{M^{[p]}\times K}\in\mathsf{e}_{L,\varphi}^{(p)}(M^{[p]})$, hence $\mathsf{e}_{M,\psi}^{(p)}\subseteq\mathsf{e}_{L,\varphi}^{(p)}$.

2. Consider now $v=\mathbb{F}B_K^{(p)}\big(\mathrm{Iso}(\alpha)\circ\mathrm{Def}_{M^{[p]}/\mathrm{Ker}\,s^{[p]}}^{M^{[p]}}\big)(e_{M_\psi^{(p)}}^{M^{[p]}\times K})\in\mathsf{e}_{M,\psi}^{(p)}(L^{[p]})$. By Lemma 2.2, there is a non zero scalar λ such that

$$v=\lambda\,m_{M_\psi^{(p)},M_\psi^{(p)}\cap(\mathrm{Ker}\,s^{[p]}\times\mathbf{1})}e_{L_\varphi^{(p)}}^{L^{[p]}\times K}\ \ . \qquad \textbf{(7.11)}$$

Now the projection $m \in M \mapsto \big(mO^p(M), \psi(m)\big) \in M_\psi^{(p)}$ induces an isomorphism $M_\psi^{(p)} \cong M/\big(O^p(M) \cap \mathrm{Ker}\,\psi\big)$. As $\mathrm{Ker}\,s^{[p]} = \mathrm{Ker}\,s\,O^p(L)/O^p(L)$, the subgroup $M_\psi^{(p)} \cap (\mathrm{Ker}\,s^{[p]} \times \mathbf{1})$ maps to $(\mathrm{Ker}\,s\,O^p(M) \cap \mathrm{Ker}\,\psi)/\big(O^p(M) \cap \mathrm{Ker}\,\psi\big)$ under this isomorphism.

Moreover $\mathrm{Ker}\,s\,O^p(M) \cap \mathrm{Ker}\,\psi = \mathrm{Ker}\,s\big(O^p(M) \cap \mathrm{Ker}\,\psi\big)$ as $\mathrm{Ker}\,s \leq \mathrm{Ker}\,\psi$. It follows that

$$m_{M_\psi^{(p)}, M_\psi^{(p)} \cap (\mathrm{Ker}\,s^{[p]} \times \mathbf{1})} = m_{M/(O^p(M) \cap \mathrm{Ker}\,\psi), \mathrm{Ker}\,s(O^p(M) \cap \mathrm{Ker}\,\psi)/(O^p(M) \cap \mathrm{Ker}\,\psi)} \ .$$

Multiplying by $m_{M, O^p(M) \cap \mathrm{Ker}\,\psi}$, which is non zero by Proposition 7.4, since (M, ψ) is p-persistent, this gives

$$\begin{aligned} m_{M, O^p(M) \cap \mathrm{Ker}\,\psi}\, m_{M_\psi^{(p)}, M_\psi^{(p)} \cap (\mathrm{Ker}\,s^{[p]} \times \mathbf{1})} &= m_{M, \mathrm{Ker}\,s(O^p(M) \cap \mathrm{Ker}\,\psi)} \\ &= m_{M, \mathrm{Ker}\,s}\, m_{M/\mathrm{Ker}\,s, \mathrm{Ker}\,s(O^p(M) \cap \mathrm{Ker}\,\psi)/\mathrm{Ker}\,s} \\ &= m_{M, \mathrm{Ker}\,s}\, m_{L, O^p(L) \cap \mathrm{Ker}\varphi} \ \ , \end{aligned}$$

as the canonical isomorphism $M/\mathrm{Ker}\,s \to L$ maps $\mathrm{Ker}\,s(O^p(M) \cap \mathrm{Ker}\,\psi)/\mathrm{Ker}\,s$ to $O^p(L) \cap \mathrm{Ker}\varphi$. Since $m_{L, O^p(L) \cap \mathrm{Ker}\varphi} \neq 0$ as (L, φ) is p-persistent, and since $m_{M, \mathrm{Ker}\,s} \neq 0$ by assumption, it follows that $m_{M_\psi^{(p)}, M_\psi^{(p)} \cap (\mathrm{Ker}\,s^{[p]} \times \mathbf{1})} \neq 0$, hence $e_{L_\varphi^{(p)}}^{L^{[p]} \times K}$ is a non zero scalar multiple of v, by 7.11. It follows that $e_{L_\varphi^{(p)}}^{L^{[p]} \times K}$ belongs to $\mathsf{e}_{M,\psi}^{(p)}(L^{[p]})$, so $\mathsf{e}_{L,\varphi}^{(p)} \subseteq \mathsf{e}_{M,\psi}^{(p)}$, and $\mathsf{e}_{L,\varphi}^{(p)} = \mathsf{e}_{M,\psi}^{(p)}$, as was to be shown. □

7.12. Corollary: *Let (L, φ) be a p-persistent group over K. Then the restriction of $\mathsf{e}_{L,\varphi}$ to finite p-groups is equal to $\mathsf{e}_{L,\varphi}^{(p)}$.*

Proof: Since $\mathsf{e}_{L,\varphi} = \mathsf{e}_{\beta_K(L,\varphi)}$ by Corollary 4.9, and since $\mathsf{e}_{L,\varphi}^{(p)} = \mathsf{e}_{\beta_K(L,\varphi)}^{(p)}$ by Theorem 7.10, we may assume that (L, φ) is a B_K-group. By Corollary 7.5, we have $O^p(L) \cap \mathrm{Ker}\,\varphi = \mathbf{1}$. Thus the projection $L \to L_\varphi^{(p)}$ is an isomorphism, and it induces an isomorphism $(L, \varphi) \cong (L_\varphi^{(p)}, p_2)$. Hence $e_{L_\varphi^{(p)}}^{L^{(p)} \times K} \in \mathsf{e}_{L,\varphi}(L^{[p]})$, and $\mathsf{e}_{L,\varphi}^{(p)}$ is contained in the restriction of $\mathsf{e}_{L,\varphi}$ to p-groups.

Conversely, if G is a p-group and $e_X^{G \times K} \in \mathsf{e}_{L,\varphi}(G)$, then $(X, p_2) \twoheadrightarrow (L, \varphi)$ by Theorem 5.3. Then $\mathsf{e}_{X,p_2}^{(p)} \subseteq \mathsf{e}_{L,\varphi}^{(p)}$, hence $e_X^{G \times K} \in \mathsf{e}_{L,\varphi}^{(p)}$ by Corollary 7.8. Hence the restriction of $\mathsf{e}_{L,\varphi}$ is contained in $\mathsf{e}_{L,\varphi}^{(p)}$, which completes the proof. □

7.13. Corollary: *Let (L, φ) be a p-persistent B_K-group, and (M, ψ) be a p-persistent group over K. Then $\mathsf{e}_{M,\psi}^{(p)} \subseteq \mathsf{e}_{L,\varphi}^{(p)}$ if and only if $(M, \psi) \twoheadrightarrow (L, \varphi)$.*

Proof: Indeed if $(M,\psi) \twoheadrightarrow (L,\varphi)$, then $\mathsf{e}^{(p)}_{M,\psi} \subseteq \mathsf{e}^{(p)}_{L,\varphi}$ by Theorem 7.10. Conversely, if $\mathsf{e}^{(p)}_{M,\psi} \subseteq \mathsf{e}^{(p)}_{L,\varphi}$, showing that $(M,\psi) \twoheadrightarrow (L,\varphi)$ amounts to showing that $\beta_K(M,\psi) \twoheadrightarrow (L,\varphi)$, because (L,φ) is a B_K-group. Now $\mathsf{e}_{M,\psi} = \mathsf{e}_{\beta_K(M,\psi)}$, hence $\mathsf{e}^{(p)}_{M,\psi} = \mathsf{e}^{(p)}_{\beta_K(M,\psi)}$ by Corollary 7.12, and we can assume that (M,ψ) is also a B_K-group.

If $\mathsf{e}^{(p)}_{M,\psi} \subseteq \mathsf{e}^{(p)}_{L,\varphi}$, then $e^{M^{[p]}\times K}_{M^{(p)}_\psi} \in \mathsf{e}^{(p)}_{L,\varphi}(M^{[p]})$, and $\mathsf{e}^{(p)}_{L,\varphi}(M^{[p]}) = \mathsf{e}_{L,\varphi}(M^{[p]})$ by Corollary 7.12. Hence $(M^{(p)}_\psi, p_2) \twoheadrightarrow (L,\varphi)$ by Theorem 5.3. But the projection $M \to M^{(p)}_\psi$ is a group isomorphism, since (M,ψ) is a B_K-group. It is in fact an isomorphism from (M,ψ) to $(M^{(p)}_\psi, p_2)$ in $\mathsf{grp}_{\Downarrow K}$. It follows that $(M,\psi) \twoheadrightarrow (L,\varphi)$. □

The following is analogous to Lemma 3.10, and the proof is the same:

7.14. Lemma: *Let $\mathcal{A}$ be a set of ideals of $\mathbb{F}B^{(p)}_K$, and (M,ψ) be a p-persistent group over K. The following are equivalent:*

1. $\mathsf{e}^{(p)}_{M,\psi} \subseteq \sum_{I\in\mathcal{A}} I$.
2. *There exists $I \in \mathcal{A}$ such that $\mathsf{e}^{(p)}_{M,\psi} \subseteq I$.*

Proof: Clearly 2 implies 1. Now 1 is equivalent to saying that

$$e^{M^{[p]}\times K}_{M^{(p)}_\psi} \in \sum_{I\in\mathcal{A}} I(M^{[p]}) \quad .$$

If this holds, there exists $I \in \mathcal{A}$ and $u \in I(M^{[p]})$ such that $e^{M^{[p]}\times K}_{M^{(p)}_\psi} \cdot u \neq 0$. Now $e^{M^{[p]}\times K}_{M^{(p)}_\psi} \cdot u \in I(M^{[p]})$, and moreover there is a scalar $\lambda \in \mathbb{F}$ such that $e^{M^{[p]}\times K}_{M^{(p)}_\psi} \cdot u = \lambda e^{M^{[p]}\times K}_{M^{(p)}_\psi} \neq 0$. Hence $\lambda \neq 0$, and $e^{M^{[p]}\times K}_{M^{(p)}_\psi} \in I(M^{[p]})$. In other words $\mathsf{e}^{(p)}_{M,\psi} \subseteq I$, so 1 implies 2. □

7.15. Notation: *Let $\mathcal{B}^{(p)}_K$-gr denote the subset of $\mathcal{B}_K$-gr consisting of p-persistent B_K-groups.*

As before, a subset $\mathcal{P}$ of $\mathcal{B}^{(p)}_K$-gr is called *closed* if

$$\forall (L,\varphi) \in \mathcal{P},\ \forall (M,\psi) \in \mathcal{B}^{(p)}_K\text{-gr},\ (M,\psi) \twoheadrightarrow (L,\varphi) \implies (M,\psi) \in \mathcal{P} \quad .$$

7.16. Theorem: *Let $\mathcal{I}_{\mathbb{F}B_K^{(p)}}$ be the lattice of ideals of $\mathbb{F}B_K^{(p)}$, ordered by inclusion of ideals, and $\mathcal{C}l_{\mathcal{B}_K^{(p)}\text{-gr}}$ be the lattice of closed subsets of $\mathcal{B}_K^{(p)}$-gr, ordered by inclusion of subsets. Then the map*

$$I \in \mathcal{I}_{\mathbb{F}B_K^{(p)}} \mapsto \mathcal{P}_I = \{(L,\varphi) \in \mathcal{B}_K^{(p)}\text{-gr} \mid \mathsf{e}_{L,\varphi}^{(p)} \subseteq I\}$$

is an isomorphism of lattices from $\mathcal{I}_{\mathbb{F}B_K^{(p)}}$ to $\mathcal{C}l_{\mathcal{B}_K^{(p)}\text{-gr}}$. The inverse isomorphism is the map

$$\mathcal{P} \in \mathcal{C}l_{\mathcal{B}_K^{(p)}\text{-gr}} \mapsto I_{\mathcal{P}} = \sum_{(L,\varphi)\in\mathcal{P}} \mathsf{e}_{L,\varphi}^{(p)} \enspace .$$

In particular $\mathcal{I}_{\mathbb{F}B_K^{(p)}}$ is completely distributive.

Proof: First the map $I \in \mathcal{I}_{\mathbb{F}B_K^{(p)}} \mapsto \mathcal{P}_I \in \mathcal{C}l_{\mathcal{B}_K^{(p)}\text{-gr}}$ is well defined: indeed $\mathcal{P}_I \in \mathcal{C}l_{\mathcal{B}_K^{(p)}\text{-gr}}$ by Theorem 7.10. This map is obviously order preserving. Similarly, the map $\mathcal{P} \in \mathcal{C}l_{\mathcal{B}_K^{(p)}\text{-gr}} \mapsto I_{\mathcal{P}} = \sum_{(L,\varphi)\in\mathcal{P}} \mathsf{e}_{L,\varphi}^{(p)}$ is also well defined and order preserving.

Hence all we need to show is that if I is an ideal of $\mathbb{F}B_K^{(p)}$, then

$$I = \sum_{(L,\varphi)\in\mathcal{P}_I} \mathsf{e}_{L,\varphi}^{(p)} \enspace , \tag{7.17}$$

and that if $\mathcal{P}$ is a closed subset of $\mathcal{B}_K^{(p)}$-gr, and $(M,\psi) \in \mathcal{B}_K^{(p)}$-gr, then

$$\mathsf{e}_{M,\psi}^{(p)} \subseteq \sum_{(L,\varphi)\in\mathcal{P}} \mathsf{e}_{L,\varphi}^{(p)} \;\Leftrightarrow\; (M,\psi) \in \mathcal{P} \enspace . \tag{7.18}$$

For 7.17, let $J = \sum_{(L,\varphi)\in\mathcal{P}_I} \mathsf{e}_{L,\varphi}^{(p)}$. Then $J \subseteq I$ by definition of $\mathcal{P}_I$. Conversely, let G be a finite p-group, and $u = \sum_{X\in E} \lambda_X e_X^{G\times K}$ be an element of $I(G)$, where $\lambda_X \in \mathbb{F}$, and E is a set of representatives of conjugacy classes of subgroups of $G \times K$. Then $e_X^{G\times K} \cdot u = \lambda_X e_X^{G\times K} \in I(G)$, for any $X \in E$. So if $\lambda_X \neq 0$, then $e_X^{G\times K} \in I(G)$. Equivalently, by Theorem 7.7, $e_{X_{p_2}^{(p)}}^{X^{[p]}\times K} \in I(X^{[p]})$, that is $\mathsf{e}_{X,p_2}^{(p)} \subseteq I$. Let (L,φ) be the element of $\mathcal{B}_K^{(p)}$-gr isomorphic to $\beta_K(X,p_2)$. Then $\mathsf{e}_{X,p_2}^{(p)} = \mathsf{e}_{L,\varphi}^{(p)}$ by Theorem 7.10, and $(L,\varphi) \in \mathcal{P}_I$.

Moreover $e_{X_{p_2}^{(p)}}^{X^{[p]}\times K} \in \mathsf{e}_{L,\varphi}^{(p)}(X^{[p]})$, or equivalently $e_X^{G\times K} \in \mathsf{e}_{L,\varphi}^{(p)}(G) \subseteq J(G)$. As this holds for any $X \in E$ such that $\lambda_X \neq 0$, we have also $u \in J(G)$, so

$J(G) = I(G)$, as u was arbitrary in $I(G)$, and $J = I$, as G was an arbitrary finite p-group. This completes the proof of 7.17.

As for 7.18, clearly if $(M,\psi) \in \mathcal{P}$, then $\mathsf{e}_{M,\psi}^{(p)} \subseteq \sum\limits_{(L,\varphi)\in\mathcal{P}} \mathsf{e}_{L,\varphi}^{(p)}$. Conversely if $\mathsf{e}_{M,\psi}^{(p)} \subseteq \sum\limits_{(L,\varphi)\in\mathcal{P}} \mathsf{e}_{L,\varphi}^{(p)}$, then by Lemma 7.14, there exists $(L,\varphi) \in \mathcal{P}$ such that $\mathsf{e}_{M,\psi}^{(p)} \subseteq \mathsf{e}_{L,\varphi}^{(p)}$. Hence $(M,\psi) \twoheadrightarrow (L,\varphi)$, by Corollary 7.13. Since $(L,\varphi) \in \mathcal{P}$ and $\mathcal{P}$ is closed, we get that $(M,\psi) \in \mathcal{P}$, as was to be shown. □

7.19. Theorem: *Let (L,φ) be a p-persistent B_K-group. Let $[s_K]$ be a set of representatives of conjugacy classes of subgroups of K. Let H be the unique element of $[s_K]$ conjugate to $\varphi(L)$, and $j_H : H \hookrightarrow K$ be the inclusion map. Then one and one only of the following holds:*

1. $\mathrm{Ker}\,\varphi = \mathbf{1}$, *and* $(L,\varphi) \cong (H, j_H)$ *in* $\mathsf{grp}_{\Downarrow K}$.
2. $\mathrm{Ker}\varphi \cong C_p$, *the group $H^{[p]}$ is cyclic and non trivial, and* $(L,\varphi) \cong (C_p \times H, j_H \circ \pi_H)$ *in* $\mathsf{grp}_{\Downarrow K}$, *where $\pi_H : C_p \times H \to K$ is the projection onto H.*
3. $\mathrm{Ker}\varphi \cong C_p \times C_p$, *the group $H^{[p]}$ is trivial - in other words H is a p-perfect subgroup of K - and* $(L,\varphi) \cong (C_p \times C_p \times H, j_H \circ \pi_H)$ *in* $\mathsf{grp}_{\Downarrow K}$, *where $\pi_H : C_p \times C_p \times H \to K$ is the projection onto H.*

Proof: Since $O^p(L) \cap \mathrm{Ker}\,\varphi = \mathbf{1}$ by Corollary 7.5, the group $\mathrm{Ker}\,\varphi$ embeds into $L^{[p]}$, so it is a p-group. Let F denote the Frattini subgroup of $\mathrm{Ker}\,\varphi$. Then F is a normal subgroup of L. Moreover if X is a subgroup of L such that $XF = L$, then $F \leq \mathrm{Ker}\,\varphi \leq XF$, so $\mathrm{Ker}\,\varphi = (\mathrm{Ker}\,\varphi \cap X)F$, hence $\mathrm{Ker}\,\varphi \cap X = \mathrm{Ker}\,\varphi$, and then $XF = X = L$ since $F \leq \mathrm{Ker}\,\varphi \leq X$. It follows that $m_{L,F} = 1$, thus $F = \mathbf{1}$ as (L,φ) is a B_K-group. This shows that $\mathrm{Ker}\,\varphi$ is elementary abelian.

Let now $N = \bigcap\limits_{P\in\mathcal{M}} P$, where $\mathcal{M}$ is the set of normal subgroups of L which are contained in $\mathrm{Ker}\,\varphi$, and maximal for these conditions (in other words the factor group $\mathrm{Ker}\,\varphi/P$ is a simple $\mathbb{F}_pL$-module). If X is a subgroup of L such that $XN = L$, then $N \leq \mathrm{Ker}\,\varphi \leq XN$, so $\mathrm{Ker}\,\varphi = (\mathrm{Ker}\,\varphi \cap X)N$. But $\mathrm{Ker}\,\varphi \cap X$ is normalized by X and $\mathrm{Ker}\,\varphi$, so it is normal in L. If $\mathrm{Ker}\,\varphi \cap X < \mathrm{Ker}\,\varphi$, then there is $P \in \mathcal{M}$ such that $\mathrm{Ker}\,\varphi \cap X \leq P$. Then $N \leq P$ also, and $\mathrm{Ker}\,\varphi = (\mathrm{Ker}\,\varphi \cap X)N \leq P$, contradicting $P < \mathrm{Ker}\,\varphi$. It follows that $\mathrm{Ker}\,\varphi \leq X$, and $XN = L$ implies $X = L$. Thus $m_{L,N} = 1$ and $N = \mathbf{1}$.

But then the product of the projection maps $\mathrm{Ker}\,\varphi \to \prod\limits_{P\in\mathcal{M}} \mathrm{Ker}\,\varphi/P$ is injective, and the latter is a semisimple $\mathbb{F}_pL$-module. Hence $\operatorname{Ker}\varphi$ is also a semisimple $\mathbb{F}_pL$-module. Now since $O^p(L)$ and $\operatorname{Ker}\varphi$ are normal subgroups of L with trivial intersection, they centralize each other. In other words $\operatorname{Ker}\varphi$ is a module for the factor group $L^{[p]} = L/O^p(L)$. Then $\operatorname{Ker}\varphi$ is a semisimple $\mathbb{F}_pL^{[p]}$-module. As $L^{[p]}$ is a p-group, the action of $L^{[p]}$ on $\operatorname{Ker}\varphi$ has to be trivial. Hence $\operatorname{Ker}\varphi$ is central in L.

Let Z be any subgroup of order p of $\operatorname{Ker}\varphi$. Then $0 = m_{L,Z} = 1 - \frac{k_L(Z)}{p}$, by Proposition 5.6.4 of [2], where $k_L(Z)$ denotes the number of complements of Z in L. It follows that $k_L(Z) = p$, so in particular there is a subgroup H of L such that $L = Z \times H$. Then the complements of Z in L are the groups of the form $\{\big(f(h), h\big) \mid h \in H\}$, where $f : H \to Z$ is any group homomorphism. It follows that there are exactly p homomorphisms from H to $Z \cong C_p$. Equivalently, there are exactly p homomorphisms from the p-group $H^{[p]}$ to C_p, so $H^{[p]}$ is cyclic and non trivial. Since $\operatorname{Ker}\varphi$ embeds in $L^{[p]} \cong Z \times H^{[p]}$, the rank of $\operatorname{Ker}\varphi$ is at most 2.

We now observe that if $(M, \psi) \twoheadrightarrow (L, \varphi)$ is a surjective morphism of groups over K - in particular if it is an isomorphism -, then $\psi(M)$ and $\varphi(L)$ are conjugate in K. Then there are three disjoint cases:

1. $\operatorname{Ker}\varphi = \mathbf{1}$. In this case, denoting by π_H the inclusion map $H \hookrightarrow K$ and by $\varphi^0 : L \to H$ the isomorphism induced by φ, we have $i \circ \varphi = \pi_H \circ \varphi^0$ for some inner automorphism i of K which conjugates $\varphi(L)$ to H. So φ^0 is an isomorphism from (L, φ) to (H, π_H) in $\mathsf{grp}_{\Downarrow K}$, and we are in Case 1 of Theorem 7.19.

2. $\operatorname{Ker}\varphi = Z \cong C_p$. Then we have seen that $L = Z \times H_1$, where H_1 is a subgroup of L such that $H_1^{[p]}$ is cyclic and non trivial. In this case φ induces an isomorphism $\varphi^0 : H_1 \to H = \varphi(L)$, and $\mathrm{Id}_Z \times \varphi^0$ is an isomorphism from (L, φ) to $(Z \times H, j_H \circ \pi_H)$, where $\pi_H : Z \times H \to K$ is the projection onto H. Hence we are in Case 2 of Theorem 7.19.

3. $\operatorname{Ker}\varphi \cong C_p \times C_p$. Then let Z be a subgroup of order p of $\operatorname{Ker}\varphi$. Then we have seen that $L = Z \times H_1$, where H_1 is a subgroup of L such that $H_1^{[p]}$ is cyclic and non trivial. In this case $Z_1 = \operatorname{Ker}\varphi \cap H_1$ has order p, and $m_{L,Z_1} = 0$ since (L, φ) is a B_K-group. It follows that Z_1 must have also exactly p complements in L. In particular, there is a subgroup J of L such that $L = Z_1 \times J$. But then $Z_1 \leq H_1 \leq Z_1 J$ implies that $H_1 = Z_1 \times H_2$, where $H_2 = H_1 \cap J$. Hence $L = Z \times Z_1 \times H_2$, and moreover $H_2^{[p]} = \mathbf{1}$ since $H_1^{[p]} \cong Z_1 \times H_2^{[p]}$ is cyclic. Then φ induces an isomorphism $\varphi^0 : H_2 \to H = \varphi(L)$, and $\mathrm{Id}_{Z\times Z_1} \times \varphi^0$ is an isomorphism from (L, φ) to $(Z \times Z_1 \times H, j_H \circ \pi_H)$, where $\pi_H : Z \times Z_1 \times H \to K$ is the projection onto H. Hence we are in Case 3 of Theorem 7.19.

This completes the proof of Theorem 7.19. □

7.20. Corollary: *Let* $\underline{1} = \{0, 1\}$ *and* $\underline{2} = \{0, 1, 2\}$ *be totally ordered lattices of cardinality 2 and 3, respectively. Let* c_K *(resp.* nc_K*) be the number of conjugacy classes of subgroups* H *of* K *such that* $H^{[p]}$ *is cyclic (resp. non cyclic). Then the lattice* $\mathcal{I}_{\mathbb{F}B_K^{(p)}}$ *of ideals of* $\mathbb{F}B_K^{(p)}$ *is isomorphic to the direct product of* c_K *copies of* $\underline{2}$ *and* nc_K *copies of* $\underline{1}$*. In particular it is a finite distributive lattice.*

Proof: By Theorem 7.16, the lattice $\mathcal{I}_{\mathbb{F}B_K^{(p)}}$ is distributive, isomorphic to the lattice $\mathcal{C}l_{\mathcal{B}_K^{(p)}\text{-gr}}$ of closed subsets of $\mathcal{B}_K^{(p)}$-gr. Moreover, the join-irreducible elements of $\mathcal{I}_{\mathbb{F}B_K^{(p)}}$ are the ideals $\mathsf{e}_{L,\varphi}$, for $(L, \varphi) \in \mathcal{I}_{\mathbb{F}B_K^{(p)}}$. By Theorem 7.19, the set $\mathcal{B}_K^{(p)}$-gr is finite, and contains three types of elements:

1. the elements (H, j_H) of the first type, for $H \in [s_K]$.
2. the elements $(C_p \times H, j_H \circ \pi_H)$ of the second type, for $H \in [s_K]$ such that $H^{[p]}$ is cyclic and non-trivial.
3. the elements $(C_p \times C_p \times H, j_H \circ \pi_H)$ of the third type, for $H \in [s_K]$ such that $H^{[p]}$ is trivial.

The only possible surjective morphisms between elements of $\mathcal{B}_K^{(p)}$-gr are of the following form:

- $(C_p \times H, j_H \circ \pi_H) \twoheadrightarrow (H, j_H)$, where $H \in [s_K]$ is such that $H^{[p]}$ is cyclic and non trivial.
- $(C_p \times C_p \times H, j_H \circ \pi_H) \twoheadrightarrow (H, j_H)$, where $H \in [s_K]$ is such that $H^{[p]} = \mathbf{1}$.

It follows that the poset $\mathcal{B}_K^{(p)}$-gr has as many connected components as conjugacy classes of subgroups of K. The connected components corresponding to subgroups H for which $H^{[p]}$ is cyclic - trivial or not - are isomorphic to a totally ordered poset of size 2, and the other ones are posets with one element. Hence $\mathcal{B}_K^{(p)}$-gr is a disjoint union of c_K-components which are totally ordered of size 2, and nc_K isolated points. The lattice of closed subsets of a totally ordered poset of size n is a totally ordered lattice of size $n + 1$, and the lattice of closed subsets of a disjoint union of posets is the direct product of the lattices of closed subsets of the pieces. This completes the proof. □

7.21. Remark: As in Remark 5.6, it follows from Section 5.2.2 of [3] that the category $\mathbb{F}B_K^{(p)}$-Mod splits as a product

$$\mathbb{F}B_K^{(p)}\text{-Mod} \cong \prod_H e_H^K \mathbb{F}B_K^{(p)}\text{-Mod} \;\; ,$$

of categories of modules over smaller Green biset functors $e_H^K \mathbb{F}B_K^{(p)}$, where $H \in [s_K]$. The above connected components correspond to this decomposition. In particular, when H is a subgroup of K such that $H^{[p]}$ is non cyclic, then the (commutative) Green functor $e_H^K \mathbb{F}B_K^{(p)}$ *has no non zero proper ideals*. It might therefore be called a *Green field*.

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