The functor of rational representations for p-groups

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Abstract : Let k be a field of characteristic $p > 0$. Let $\mathcal{C}_{p,k}$ be the category whose objects are the finite p-groups, morphisms are the k-linear combinations of bisets, and composition of morphisms is obtained by k-linear extension from the usual product of bisets. Let $\mathcal{F}_{p,k}$ denote the category of k-linear functors from $\mathcal{C}_{p,k}$ to the category of k-vector spaces.

This paper investigates the structure of the functor $kR_{\mathbb{Q}}$ mapping a p-group P to $kR_{\mathbb{Q}}(P) = k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(P)$, as an object of $\mathcal{F}_{p,k}$, where $R_{\mathbb{Q}}(P)$ is the Grothendieck group of the category of finite dimensional $\mathbb{Q}P$-modules. The main result is an explicit description of the lattice of all subfunctors of $kR_{\mathbb{Q}}$. In particular for p odd, it is shown that $kR_{\mathbb{Q}}$ is a uniserial object in $\mathcal{F}_{p,k}$. For $p = 2$, the lattice of subfunctors of $kR_{\mathbb{Q}}$ can be described as the lattice of closed subsets of a graph whose vertices are the 2-groups of normal 2-rank 1.

In both cases a composition series of $kR_{\mathbb{Q}}$ is obtained, which leads to a formula giving the dimension of the evaluations $S_{Q,k}(P)$ of the simple objects $S_{Q,k}$ of $\mathcal{F}_{p,k}$ associated to p-groups Q of normal p-rank 1, different from C_p. This formula can be phrased in terms of rational representations of P, but also in terms of the geometry of the lattice of subgroups of P, using the notions of basic subgroups and origins. For example, if $p = 2$, the dimension of $S_{\mathbf{1},k}(P)$ is equal to the number of absolutely irreducible $\mathbb{Q}P$-modules.

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1 Introduction

The formalism of bisets for finite groups, exposed in [1], provides a single framework involving the usual operations of induction, restriction, inflation, and deflation naturally associated to the usual representation groups of finite groups. With this formalism, these representation groups appear as functors from a suitable category, whose objects are (some specific classes of) finite groups, and morphisms are (some classes of) bisets, or linear combination of bisets, with values in the category of abelian groups, or more generally of modules over some fixed commutative ring.

The context of this article is the following special case, also exposed in [4] (Section 7) : let p be a prime number, and k be a field. Let $\mathcal{C} = \mathcal{C}_{p,k}$ denote the following category :

- The objects of $\mathcal{C}$ are the finite p-groups.
- If P and Q are finite p-groups, then
$$\mathrm{Hom}_{\mathcal{C}}(P, Q) = k \otimes_{\mathbb{Z}} B(Q \times P^{op}) \quad ,$$
where $B(Q \times P^{op})$ is the Burnside group of $Q \times P^{op}$, viewed as the Grothendieck group of the category of finite Q-sets-P.
- The composition of morphisms in $\mathcal{C}$ is obtained by k-bilinearity from the usual product of bisets : if P, Q and R are finite p-groups, if X is a Q-set-P, and Y is an R-set-Q, then the composition of $X \in \mathrm{Hom}_{\mathcal{C}}(P, Q)$ and $Y \in \mathrm{Hom}_{\mathcal{C}}(Q, R)$ is equal to $Y \circ X = Y \times_Q X$.

Let $\mathcal{F} = \mathcal{F}_{p,k}$ denote the category of k-linear functors from $\mathcal{C}_{p,k}$ to the category of k-vector spaces. Then $\mathcal{F}$ is an abelian category. The simple objects of $\mathcal{F}$ are parametrized by pairs (Q, V) consisting of a finite p-group Q and a

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simple $k\mathrm{Out}(Q)$-module V ([1] Proposition 2 page 678). The simple functor corresponding to the pair (Q,V) is denoted by $S_{Q,V}$.

Most of the natural representation groups, such as the groups $R_{\mathbb{Q}}(P)$ of rational representations of the p-group P, or the Burnside group $B(P)$, provide examples of objects of $\mathcal{F}$, after tensoring with k : in particular, the correspondence $P \mapsto kR_{\mathbb{Q}}(P) = k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(P)$ is a k-linear functor on $\mathcal{C}$, with values in k-vector spaces. If Q and P are finite p-groups, and if U is a finite Q-set-P, viewed as an element of $\mathrm{Hom}_{\mathcal{C}}(P,Q)$, then

$$kR_{\mathbb{Q}}(U) : kR_{\mathbb{Q}}(P) \to kR_{\mathbb{Q}}(Q)$$

is the map obtained by k-linearity from the map sending the class of the $\mathbb{Q}P$-module V to the class of the $\mathbb{Q}Q$-module $\mathbb{Q}U \otimes_{\mathbb{Q}P} V$.

Remark 1.1 The reason for restricting the attention to p-groups is twofold : firstly, for p-groups, the Ritter-Segal theorem holds : any rational representation of a p-group is a (virtual) permutation representation, thus combinatorial or geometric properties of p-groups give information on their rational representations. Secondly, some important functors appearing naturally in modular representations of finite groups are defined only for p-groups : the main examples here are provided by groups of endo-permutation modules (see [4] for details), i.e. subgroups of the Dade group, such as the group of relative syzygies ([2]), which are functorial in the above sense.

Example 1.2 Recall in particular, if Q is a subgroup of P, then the operation associated to the set $U = P$, viewed as a Q-set-P, is ordinary restriction Res_Q^P of modules. If U is viewed as a P-set-Q, the corresponding operation is induction Ind_Q^P of modules. Similarly, if $Q = P/N$ is a factor group of P, then the set $U = Q$, viewed as a P-set-Q, corresponds to inflation Inf_Q^P of modules. And when U is viewed as a Q-set-P, the associated operation is called deflation, and denoted by Def_Q^P. The effect on a module is the construction of co-invariants by N. Finally, if $\varphi : P \to Q$ is a group isomorphism, then the obvious associated operation of change of group is denoted by Iso_P^Q, and corresponds to the set $U = Q$, viewed as a Q-set-P.

The structure of $kR_{\mathbb{Q}}$ as an object of $\mathcal{F}_{p,k}$ depends on the characteristic q of k. The case $q \neq p$ is solved by Theorem 8.2 and Corollaries 8.3 and 8.4 of [4] : in this case, if q does not divide $p-1$, the functor $kR_{\mathbb{Q}}$ is simple, isomorphic to $S_{\mathbf{1},k}$. And if q divides $p-1$, then $kR_{\mathbb{Q}}$ has a unique non zero proper subfunctor F, isomorphic to $S_{C_p,k}$, and the quotient $kR_{\mathbb{Q}}/F$ is isomorphic to $S_{\mathbf{1},k}$.

This article deals with the remaining case $q = p$. The situation in this case is quite different. The possible methods to attack this question are also completely different : the main tool in [4] was the fact that the Burnside algebra $kB(P) = k \otimes_{\mathbb{Z}} B(P)$ of a p-group P is semi-simple if $q \neq p$, with explicit formulae for primitive idempotents. In the case $q = p$ on the contrary, the algebra $kB(P)$ is local, and it has no non-trivial idempotents.

The main results stated here (Theorem 6.1, Theorem 6.2, Corollary 6.5) describe the lattice of subfunctors of $kR_{\mathbb{Q}}$, in terms of groups of normal p-rank 1. Any such group P has a unique irreducible faithful rational module Φ_P, whose image in $kR_{\mathbb{Q}}(P)$ generates a subfunctor H_P of $kR_{\mathbb{Q}}$. Then any subfunctor of $kR_{\mathbb{Q}}$ is the sum of those functors H_P it contains. When p is odd, the situation is simple, since $kR_{\mathbb{Q}}$ is uniserial. When $p = 2$, the lattice of subfunctors of $kR_{\mathbb{Q}}$ can be described as the lattice of closed subsets of an explicit graph, whose vertices are the isomorphism classes of groups of normal p-rank 1.

The main consequence of this description is the following :

Theorem 1.3 *Let k be a field of characteristic $p > 0$.*

1. *If $p \neq 2$, then the functor $kR_{\mathbb{Q}}$ is a uniserial object of $\mathcal{F}_{p,k}$. If $F_0 = kR_{\mathbb{Q}} \supset F_1 \supset F_2 \supset \ldots \supset F_n \supset \ldots$ is the set of its non-zero subfunctors, then*
$$F_0/F_1 \cong S_{\mathbf{1},k} \qquad F_i/F_{i+1} \cong S_{C_{p^{i+1}},k} \text{ for } i \geq 1 \quad ,$$
where $C_{p^{i+1}}$ denotes a cyclic group of order p^{i+1}.

2. *If $p = 2$, then the functor $kR_{\mathbb{Q}}$ has a filtration $F_0 = kR_{\mathbb{Q}} \supset F_1 \supset F_2 \supset \ldots \supset F_n \supset \ldots$ by subfunctors F_n such that $\bigcap_{n\geq 0} F_n = 0$ and*
$$\begin{aligned} F_0/F_1 &\cong S_{\mathbf{1},k} \\ F_1/F_2 &\cong S_{C_4,k} \oplus S_{D_{16},k} \oplus S_{SD_{16},k} \\ F_i/F_{i+1} &\cong S_{C_{2^{i+1}},k} \oplus S_{Q_{2^{i+1}},k} \oplus S_{D_{2^{i+3}},k} \oplus S_{SD_{2^{i+3}},k} \ \text{ if } i \geq 2, \end{aligned}$$
where C_{2^i}, Q_{2^i}, D_{2^i} and SD_{2^i} denote respectively the cyclic, generalized quaternion, dihedral, and semi-dihedral group of order 2^i.

This theorem can be considered as a functorial refinement of methods initiating in P. Roquette's work ([7]). Assertion 1 has been conjectured by I. Bourizk ([5] Conjecture 1).

For $p = 2$, the description of the lattice of subfunctors of $kR_{\mathbb{Q}}$ shows in particular that the subfunctor H_{Q_8} of $kR_{\mathbb{Q}}$ attached to the quaternion group Q_8 is a uniserial object of $\mathcal{F}$. This was conjecture 2 in [5].

The other main result of this article (Theorem 5.12) is a computation of the k-dimension of the evaluation of the simple functors $S_{P,k}$, when P is a p-group of normal p-rank 1, different from C_p (the notion of *type* of an irreducible rational module for a p-group is introduced in Definition 3.5) :

Theorem 1.4 *Let Q be a finite p-group.*

1. *The dimension of $S_{\mathbf{1},k}(Q)$ is equal to*

$$\begin{aligned}\dim_k S_{\mathbf{1},k}(Q) &= |\{V \in \mathrm{Irr}_{\mathbb{Q}}(Q) \mid V \textit{ has type } \mathbf{1} \textit{ or } C_p\}|\\ &= |\{V \in \mathrm{Irr}_{\mathbb{Q}}(Q) \mid \langle V,V\rangle_Q \leq p-1\}|\\ &= |\{V \in \mathrm{Irr}_{\mathbb{Q}}(Q) \mid \langle V,V\rangle_Q \not\equiv 0\ (p)\}|\end{aligned}$$

2. *If P has normal p-rank 1, and if $|P| \geq p^2$, then*

$$\dim_k S_{P,k}(Q) = |\{V \in \mathrm{Irr}_{\mathbb{Q}}(Q) \mid V \textit{ has type } P\}| \quad .$$

This gives a partial answer to a question already raised in [1] (Section 8, page 713).

Remark 1.5 The particular status of C_p in this theorem will be explained in Remark 5.7.

The paper is organized as follows : in section 2, I will recall some results on the rational irreducible representations of p-groups. Section 3 deals with the irreducible representations of p-groups of normal p-rank 1.

This leads to the definition of the subfunctors H_P of $kR_{\mathbb{Q}}$, in section 4. In section 5, it is shown that the functor H_P has a unique simple quotient, isomorphic to $S_{P,k}$ if $|P| \neq p$. An expression of the k-dimension of $S_{P,k}(Q)$ is given, for a p-group Q and a p-group P of normal p-rank 1 (different from C_p), in terms of rational representations of Q.

Section 6 describes the possible mutual inclusions of subfunctors H_P, and this gives the structure of the lattice of subfunctors of $kR_{\mathbb{Q}}$. The last section exposes an interpretation of the dimension of the evaluations of the simple functors obtained in section 4, in terms of the bilinear forms introduced in [1] Proposition 16 page 717. The result is an expression of these dimensions purely in terms of the structure of the lattice of subgroups of the given p-group, without any reference to rational representations.

2 Rational representations of p-groups

Recall the following theorem from [3] :

Theorem 2.1 *([3] Theorem 1) Let p be a prime number, and P be a finite p-group. If V is a non-trivial simple $\mathbb{Q}P$-module, then there exist subgroups $R \supset Q$ of P, with $|R : Q| = p$, and an isomorphism of $\mathbb{Q}P$-modules*

$$V \cong \mathrm{Ind}_R^P \mathrm{Inf}_{R/Q}^R \Omega_{R/Q}$$

where $\Omega_{R/Q}$ is the augmentation ideal of the group algebra $\mathbb{Q}(R/Q)$.

In other words there is an exact sequence of $\mathbb{Q}P$-modules

$$0 \to V \to \mathbb{Q}(P/Q) \to \mathbb{Q}(P/R) \to 0$$

where the map $\mathbb{Q}(P/Q) \to \mathbb{Q}(P/R)$ is the natural projection. In particular in $R_{\mathbb{Q}}(P)$

$$V = \mathbb{Q}(P/Q) - \mathbb{Q}(P/R)$$

Conversely, the following proposition is a test for irreducibility of the module $\mathrm{Ind}_R^P \mathrm{Inf}_{R/Q}^R \Omega_{R/Q}$:

Proposition 2.2 *([3] Proposition 4) Let p be a prime number, and P be a p-group. Let $R \supset Q$ be subgroups of P, with $|R : Q| = p$. Then the following conditions are equivalent:*

1. *The module $\mathrm{Ind}_R^P \mathrm{Inf}_{R/Q}^R \Omega_{R/Q}$ is an irreducible $\mathbb{Q}P$-module.*
2. *If S is any subgroup of P such that $R \cap S \subseteq Q$, then $|S| < |R|$.*
3. *The group $N_P(Q)/Q$ is cyclic or generalized quaternion, the group R/Q is its unique subgroup of order p, and if S is any subgroup of P such that $|S| > |Q|$, then $S \cap N_P(Q) \not\subseteq Q$.*

This proposition shows in particular that R is determined by Q.

Definition 2.3 *Let P be a finite p-group. By definition, a basic subgroup of P is a subgroup Q of P such that the following two conditions hold :*

1. *The group $N_P(Q)/Q$ is cyclic or generalized quaternion.*
2. *If S is any subgroup of P such that $|S| > |Q|$, then $S \cap N_P(Q) \not\subseteq Q$.*

The group P itself is a basic subgroup of P.

Notation 2.4 *Let P be a finite p-group.*

- *Let $Irr_{\mathbb{Q}}(P)$ denote the set of isomorphism classes of irreducible $\mathbb{Q}P$-modules, and let $\mathcal{B}_P$ denote the set of basic subgroups of P.*

- *If Q is a proper basic subgroup of P, then let V_Q denote the simple $\mathbb{Q}P$-module $\mathrm{Ind}_{\tilde{Q}}^P \mathrm{Inf}_{\tilde{Q}/Q}^{\tilde{Q}} \Omega_{\tilde{Q}/Q}$, where $\tilde{Q}/Q$ is the unique subgroup of order p of $N_P(Q)/Q$. Also let V_P denote the trivial $\mathbb{Q}P$-module. The simple module V_Q is called the module associated to the basic subgroup Q.*

- *If V and W are finite dimensional $\mathbb{Q}P$-modules, let*

  $$\langle V, W\rangle_P = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}P}(V, W) \quad .$$

  This extends uniquely to a bilinear form on $R_{\mathbb{Q}}(P)$, with values in $\mathbb{Z}$, still denoted by $\langle\ ,\ \rangle_P$.

- *If V is a simple $\mathbb{Q}P$-module, denote by $m(V, W)$ the multiplicity of V in W, i.e. the number of times V appears in a decomposition of W as a direct sum of simple $\mathbb{Q}P$-modules.*

- *If S and Q are subgroups of P, set*

  $$I_P(S, Q) = \{x \in P \mid S^x \cap N_P(Q) \subseteq Q\}$$

  and

  $$\mathcal{I}_P(S, Q) = S\backslash I_P(S, Q)/N_P(Q) \quad .$$

The following proposition lists some properties of basic subgroups and associated representations :

Proposition 2.5 *Let P be a finite p-group.*

1. *If Q is a proper basic subgroup of P, and S is any subgroup of P, then*

   $$I_P(S, Q) = \{x \in P \mid S^x \cap \tilde{Q} \subseteq Q\} \quad ,$$

   and

   $$\begin{aligned}\langle \mathbb{Q}(P/S), V_Q\rangle_P &= (p-1)|\{x \in S\backslash P/\tilde{Q} \mid S^x \cap \tilde{Q} \subseteq Q\}| \\ &= (p-1)\frac{|N_P(Q)/Q|}{p}|\mathcal{I}_P(S, Q)|\end{aligned}$$

2. *If Q is a basic subgroup of P, and if S is any subgroup of P, then*

   $$m\big(V_Q, \mathbb{Q}(P/S)\big) = \frac{|\mathcal{I}_P(S, Q)|}{|\mathcal{I}_P(Q, Q)|}$$

3. *If Q is a basic subgroup of P, then the cardinality of $\mathcal{I}_P(Q,Q)$ is a power of p. If moreover $Q \neq P$, then*

$$\langle V_Q, V_Q\rangle_P = (p-1)\frac{|N_P(Q)/Q|}{p}|\mathcal{I}_P(Q,Q)| \quad .$$

4. *If Q and Q' are basic subgroups of P, then the corresponding simple $\mathbb{Q}P$-modules V_Q and $V_{Q'}$ are isomorphic if and only if $|Q| = |Q'|$ and $\mathcal{I}_P(Q,Q') \neq \emptyset$.*

Proof : Observe first that if Q is a proper basic subgroup of P, if $x \in P$ and $S^x \cap N_P(Q) \subseteq Q$, then $S^x \cap \tilde{Q} \subseteq Q$, since $\tilde{Q} \subseteq N_P(Q)$. Conversely, if $S^x \cap N_P(Q) \not\subseteq Q$, then $\tilde{Q} \subseteq (S^x \cap N_P(Q))Q$, since $\tilde{Q}/Q$ is the only subgroup of order p of $N_P(Q)/Q$. Thus

$$\tilde{Q} = \Big((S^x \cap N_P(Q)) \cap \tilde{Q}\Big)Q = (S^x \cap \tilde{Q}).Q$$

and $S^x \cap \tilde{Q} \not\subseteq Q$. This shows that

$$S^x \cap \tilde{Q} \subseteq Q \quad \text{if and only if} \quad S^x \cap N_P(Q) \subseteq Q \quad .$$

Now recall that if R and S are subgroups of P, then

$$\langle \mathbb{Q}(P/R), \mathbb{Q}(P/S)\rangle_P = |R\backslash P/Q| \quad .$$

If Q is a proper basic subgroup of P, since $V_Q = \mathbb{Q}(P/Q) - \mathbb{Q}(P/\tilde{Q})$ in $R_\mathbb{Q}(P)$, it follows that

$$\langle \mathbb{Q}(P/S), V_Q\rangle_P = |S\backslash P/Q| - |S\backslash P/\tilde{Q}| \quad .$$

Let $x \in P$. There are two cases : either $S^x \cap \tilde{Q} \not\subseteq Q$, and then $\tilde{Q} = (S^x \cap \tilde{Q})Q$ since $|\tilde{Q} : Q| = p$. Hence $\tilde{Q} \subseteq S^x.Q$ in this case, and $Sx\tilde{Q} = SxQ$. Or $S^x \cap \tilde{Q} \subseteq Q$, and in this case

$$|Sx\tilde{Q}| = \frac{|S||\tilde{Q}|}{|S^x \cap \tilde{Q}|} = p\frac{|S||Q|}{|S^x \cap Q|} = p|SxQ| \quad .$$

This shows that the map $\varphi : S\backslash P/Q \to S\backslash P/\tilde{Q}$ sending SxQ to $Sx\tilde{Q}$ is such that $|\varphi^{-1}(u)| = 1$ or p, for all $u \in S\backslash P/\tilde{Q}$. Hence

$$\begin{aligned} |S\backslash P/Q| - |S\backslash P/\tilde{Q}| &= \sum_{u \in S\backslash P/\tilde{Q}} \Big(|\varphi^{-1}(u)| - 1\Big) \\ &= (p-1)|\{u \in S\backslash P/\tilde{Q} \mid |\varphi^{-1}(u)| = p\}| \end{aligned}$$

and finally

$$\langle \mathbb{Q}(P/S), V_Q\rangle_P = (p-1)|\{x \in S\backslash P/\tilde{Q} \mid S^x \cap \tilde{Q} \subseteq Q\}| \qquad \textbf{(2.6)}$$

proving the second formula in Assertion 1.

Now the group $N_P(Q)$ normalizes $\tilde{Q}$, and acts on the right on the set

$$U = \{x \in S\backslash P/\tilde{Q} \mid S^x \cap \tilde{Q} \subseteq Q\}$$

by right multiplication. The stabilizer in $N_P(Q)$ of $Sx\tilde{Q}$ is the set of elements $n \in N_P(Q)$ such that $Sxn\tilde{Q} = Sx\tilde{Q}$, or equivalently

$$n \in S^x.\tilde{Q} \cap N_P(Q) = (S^x \cap N_P(Q)).\tilde{Q} = \tilde{Q} \quad .$$

In other words, the group $N_P(Q)/\tilde{Q}$ acts freely on U, and the set of its orbits is precisely the set of elements $x \in S\backslash P/N_P(Q)$ such that $S^x \cap N_P(Q) \subseteq Q$. Hence

$$|U| = |N_P(Q)/\tilde{Q}||\mathcal{I}_P(S,Q)|$$

proving the third formula of Assertion 1.

It follows in particular that $\langle \mathbb{Q}(P/S), V_Q\rangle_P = 0$ if $|S| > |Q|$, by definition of a basic subgroup. Thus

$$\begin{aligned}\langle V_Q, V_Q\rangle_P &= \langle \mathbb{Q}(P/Q), V_Q\rangle_P - \langle \mathbb{Q}(P/\tilde{Q}), V_Q\rangle_P = \langle \mathbb{Q}(P/Q), V_Q\rangle_P \\ &= (p-1)\frac{|N_P(Q)/Q|}{p}|\mathcal{I}_P(Q,Q)|\end{aligned}$$

proving the second part of Assertion 3.

Now for any $\mathbb{Q}P$-module W

$$\langle W, V_Q\rangle_P = m(V_Q, W)\langle V_Q, V_Q\rangle_P$$

since the irreducible modules are mutually orthogonal for $\langle\ ,\ \rangle_P$. Taking $W = \mathbb{Q}(P/S)$ gives Assertion 2, in the case $Q \neq P$. The case $Q = P$ is obvious, since the trivial module V_P has multiplicity 1 in $\mathbb{Q}(P/S)$.

Taking now $S = \mathbf{1}$ in Assertion 2 gives

$$m(V_Q, \mathbb{Q}(P/\mathbf{1})) = \frac{|P : N_P(Q)|}{|\mathcal{I}_P(Q,Q)|}$$

showing that the denominator is a power of p, and completing the proof of Assertion 3 in the case $Q \neq P$. The case $Q = P$ is trivial.

Finally, suppose that Q and Q' are basic subgroups of P such that the $\mathbb{Q}P$-modules V_Q and $V_{Q'}$ are isomorphic. Then $Q = P$ if and only if $Q' = P$, since

V_Q is not the trivial module for $Q \neq P$. And if Q and Q' are proper basic subgroups of P, then $|Q| = |Q'|$, since $\dim_{\mathbb{Q}} V_Q = |P : Q|(p-1)/p = \dim_{\mathbb{Q}} V_{Q'} = |P : Q'|(p-1)/p$. Hence $|\tilde{Q}| > |Q'|$, and $\langle V_Q, V_{Q'}\rangle_P = \langle \mathbb{Q}(P/Q), V_{Q'}\rangle_P$. This is non zero if $V_Q \cong V_{Q'}$. By Assertion 1, the set $|\mathcal{I}_P(Q, Q')|$ is not empty.

Conversely, if $|Q| = |Q'|$, then either $Q = Q' = P$, and $V_Q = V_{Q'} = V_P$, or Q and Q' are proper basic subgroups of P. Since $|\tilde{Q}| > |Q'|$, it follows as above that $\langle V_Q, V_{Q'}\rangle_P = \langle \mathbb{Q}(P/Q), V_{Q'}\rangle_P$, and this is non zero by Assertion 1, if moreover $|\mathcal{I}_P(Q, Q')| \neq \emptyset$. Hence $V_Q \cong V_{Q'}$ since V_Q and $V_{Q'}$ are both simple. This completes the proof of Assertion 4. □

Notation 2.7 *If P is a finite p-group, define the following relation on $\mathcal{B}_P$:*

$$Q \doteqdot_P Q' \overset{\text{def.}}{\Longleftrightarrow} |Q| = |Q'| \text{ and } I_P(Q, Q') \neq \emptyset \quad .$$

Theorem 2.8 *Let P be a finite p-group.*

1. *The relation $\doteqdot_P$ is an equivalence relation on $\mathcal{B}_P$.*

2. *The correspondence $Q \in \mathcal{B}_P \mapsto V_Q \in Irr_{\mathbb{Q}}(P)$ induces a one to one correspondence between the set of equivalence classes of basic subgroups of P for the relation $\doteqdot_P$ and the set of isomorphism classes of rational irreducible representations of P.*

Proof : Assertion 1 follows from Assertion 4 of Proposition 2.5. The same argument shows that the correspondence $Q \in \mathcal{B}_P \mapsto V_Q \in \mathrm{Irr}_{\mathbb{Q}}(P)$ induces an injective correspondence between the set of equivalence classes of basic subgroups of P for the relation $\doteqdot$ and the set $\mathrm{Irr}_{\mathbb{Q}}(P)$. This induced correspondence is moreover surjective, since any irreducible $\mathbb{Q}P$-module is isomorphic to V_Q, for some basic subgroup Q of P, by Theorem 2.1 and Notation 2.4. □

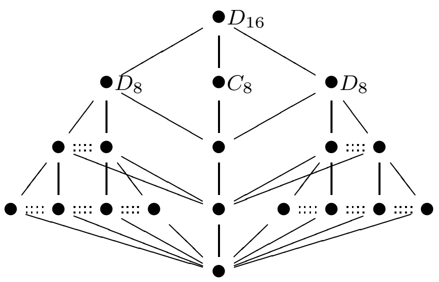
Remark 2.9 It it easy to check directly (i.e. without using the modules V_Q) that the relation $\doteqdot_P$ is reflexive and symmetric, but it's far from obvious to prove transitivity without this.

3 Groups of normal p-rank 1

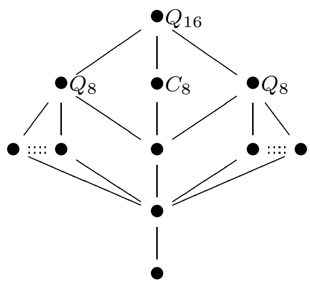
Definition 3.1 *A p-group P is said to have normal p-rank 1 if it does not have any normal subgroup isomorphic to $(C_p)^2$. Denote by $\mathcal{N}$ the class of finite p-groups of normal p-rank 1, and by $[\mathcal{N}]$ a set of isomorphism classes of elements of $\mathcal{N}$. More generally, if $\mathcal{A}$ is a subclass of $\mathcal{N}$, set $[\mathcal{A}] = \mathcal{A} \cap [\mathcal{N}]$.*

If $P \in \mathcal{N}$, then by Theorem 4.10 of Chapter 5 of [6], the group P is cyclic if p is odd, and if $p = 2$, the group P is cyclic, or generalized quaternion of order at least 8, dihedral of order at least 16, or semi-dihedral of order at least 16.

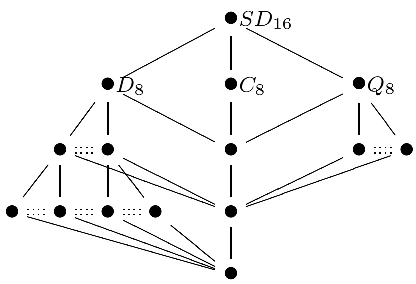
The following schematic diagram represents the lattice of subgroups of the dihedral group D_{16}, the quaternion group Q_{16}, and the semi-dihedral group SD_{16} (an horizontal dotted link between two vertices means that the corresponding subgroups are conjugate) :



D_{16}



Q_{16}



SD_{16}

This diagram gives a good idea of the general case, quoted without proof in the following lemma :

Lemma 3.2 *Let P a non-cyclic group in $\mathcal{N}$. Then $p = 2$, and P has exactly 3 maximal subgroups A, B, and C. The group C is cyclic, and moreover :*

1. *If P is dihedral, then A and B are dihedral.*

2. *If $P \cong Q_8$, then A and B are cyclic.*

3. *If P is generalized quaternion of order at least 16, then A and B are generalized quaternion.*

4. *If P is semi-dihedral, then A is dihedral and B is generalized quaternion.*

There are 0,1, or 2 conjugacy classes of non-central involutions in P, according to P being generalized quaternion, semi-dihedral, or dihedral.

Remark 3.3 It follows in particular that if R is a subgroup of $P \in \mathcal{N}$, then $R \in \mathcal{N}$, except if $p = 2$ and $R \cong (C_2)^2$ or $R \cong D_8$.

The following theorem gives another way, different from Theorem 2.8, to build irreducible representations of a finite p-group. It is more or less well known, and already implicit in Roquette's methods ([7]) :

Theorem 3.4 *Let p be a prime number, and P be a p-group. If V is an irreducible $\mathbb{Q}P$-module, then there exist subgroups S and T of P, with $S \trianglelefteq T$, and a faithful irreducible $\mathbb{Q}(T/S)$-module W such that :*

1. *The module V is isomorphic to $\mathrm{Ind}_T^P \mathrm{Inf}_{T/S}^T W$.*

2. *This isomorphism induces an isomorphism of $\mathbb{Q}$-algebras*

$$\mathrm{End}_{\mathbb{Q}P}(V) \cong \mathrm{End}_{\mathbb{Q}(T/S)}(W)$$

3. *The group T/S has normal p-rank 1.*

Proof : For convenience, let me sketch a proof, which is very similar to the proof of Theorem 2.1 (see [3]), and goes by induction on the order of P. One can suppose V faithful, since otherwise V is inflated from a factor group of P, for which the theorem holds by induction. Moreover inflation preserves the required properties.

Now if P has an elementary abelian normal subgroup $E \cong (C_p)^2$, let L be a direct summand of $\mathrm{Res}_E^P V$. Let I denote the inertial subgroup of L in P, and let $\tilde{L}$ denote the isotypic component of L in $\mathrm{Res}_E^P V$. Then $V \cong \mathrm{Ind}_I^P \tilde{L}$ by Clifford theory. Moreover I contains the centralizer $C_P(E)$ of E, which has index at most p in P, since $P/C_P(E)$ is isomorphic to a subgroup of the automorphism group of E. Thus $|P : I| \leq p$, and in particular $I \trianglelefteq P$.

If $I = E$, then $\tilde{L} \cong V$ is faithful, thus L is faithful. But E has no faithful irreducible rational representations. Hence I has index p in P. Moreover $\tilde{L}$ is an irreducible representation of I, and if $x \in P$ is such that ${}^x\tilde{L} = \tilde{L}$, then ${}^xL \cong L$, thus $x \in I$. It follows that $\mathrm{End}_{\mathbb{Q}P}(V) \cong \mathrm{End}_{\mathbb{Q}I}(\tilde{L})$. Now the induction hypothesis applies to I and $\tilde{L}$, completing the proof. □

Definition 3.5 *In the situation of Theorem 3.4, the pair (T, S) will be called a genetic section of P for V, and if $R \cong T/S$ is a p-group of normal p-rank 1, then V will be said to have type R.*

The set of all sections (T, S) appearing as genetic sections of P for some $V \in Irr_{\mathbb{Q}}(P)$, will be denoted by $\mathcal{G}_P$.

Remark 3.6 A natural question is then to ask if the type of an irreducible $\mathbb{Q}P$-module V is well defined : the answer is affirmative, and a proof will be given in Corollary 5.9. Another more direct proof will be sketched in Remark 3.9, using the following result on the irreducible faithful rational representations of p-groups of normal p-rank 1, which also shows that once the genetic section (T, S) of P for V is given, then the module W appearing in Theorem 3.4 is uniquely determined up to isomorphism :

Proposition 3.7 *Let p be a prime number. If P is a finite p-group of normal p-rank 1, then P admits a unique faithful irreducible rational representation, up to isomorphism.*

Proof : By Theorem 2.8, this amounts to show that there is a single equivalence class of basic subgroups Q of P (for the relation $\div_P$) for which V_Q is faithful. If $P = \mathbf{1}$, this is trivial. If not, let Q be a basic subgroup of P. If V_Q is faithful, then $Q \neq P$. Moreover since $V_Q = \mathrm{Ind}_{\tilde{Q}}^P \mathrm{Inf}_{\tilde{Q}/Q}^{\tilde{Q}} \Omega_{\tilde{Q}/Q}$ is faithful, the intersection of conjugates of Q in P is trivial. Equivalently Q intersects the center of P trivially.

If P is cyclic or generalized quaternion, this forces $Q = \mathbf{1}$. Conversely, in that case, it follows from the definition that the trivial subgroup is basic. The corresponding simple $\mathbb{Q}P$-module $V_{\mathbf{1}}$ is faithful, and it is the only faithful simple $\mathbb{Q}P$-module in this case, up to isomorphism.

If P is dihedral or semi-dihedral, and if Q intersects the center of P trivially, then Q is trivial, or non-central of order 2. In that case, the trivial subgroup is not basic (since $N_P(\mathbf{1})/\mathbf{1} = P$ is neither cyclic nor generalized quaternion !). Hence Q is non central, of order 2. Conversely if Q is non-central and of order 2, then $N_P(Q)$ has order 4, equal to $Q.Z$, where Z is the center of P. Thus $N_P(Q)/Q$ is cyclic, and Q is basic, since any subgroup of P of order at least 4 contains Z.

Now if P is semi-dihedral, then there is a single conjugacy class of non-central subgroups of order 2 in P. Such a subgroup Q is basic, and V_Q is the only faithful simple $\mathbb{Q}P$-module, up to isomorphism.

And if P is dihedral, there are two conjugacy classes of non-central subgroups of order 2 in P. Let Q and Q' be such subgroups in different classes. Then $N_P(Q)$ has order 4, and is equal to $Q.Z$, where Z is the center of P. Since $Q' \cap N_P(Q) = \mathbf{1} \subseteq Q$, it follows that $Q \doteqdot_P Q'$. Hence $V_Q \cong V_{Q'}$ is the only faithful simple $\mathbb{Q}P$-module in that case, up to isomorphism.

Notice that this argument holds also for the dihedral group D_8 of order 8, showing that it has a unique faithful rational simple module.

Notation 3.8 *If R is a finite p-group of normal p-rank 1, or if $R \cong D_8$, let Φ_R denote its unique faithful irreducible rational representation.*

If P is a finite p-group, and if $(T,S) \in \mathcal{G}_P$, let $V(T,S) = \mathrm{Ind}_T^P \mathrm{Inf}_{T/S}^T \Phi_{T/S}$ denote the corresponding simple $\mathbb{Q}P$-module.

Remark 3.9 One can then prove directly that if P is a p-group, the type of a simple $\mathbb{Q}P$-module V is well defined, up to isomorphism : indeed, if V has type R, then $\mathrm{End}_{\mathbb{Q}R}\Phi_R \cong \mathrm{End}_{\mathbb{Q}P}V$. Let (T,S) and (T',S') be genetic sections of P for V, and set $R = T/S$ and $R' = T'/S'$. Then $\mathrm{End}_{\mathbb{Q}R}\Phi_R \cong \mathrm{End}_{\mathbb{Q}R'}\Phi_{R'}$. But the structure of the algebra $\mathrm{End}_{\mathbb{Q}R}\Phi_R$ (actually a skew-field) can be explicitly obtained from the above description of Φ_R, and then it is easy to see that for R and R' in $\mathcal{N}$, the algebra isomorphism $\mathrm{End}_{\mathbb{Q}R}\Phi_R \cong \mathrm{End}_{\mathbb{Q}R'}\Phi_{R'}$ is equivalent to the group isomorphism $R \cong R'$.

There is now an obvious equivalence relation on the set $\mathcal{G}_P$ of genetic sections of a finite p-group P : say that the elements (T,S) and (T',S') of $\mathcal{G}_P$ are equivalent if and only if the corresponding simple $\mathbb{Q}P$-modules $V(T,S)$ and $V(T',S')$ are isomorphic. It turns out that this equivalence relation can be translated in a purely "geometric" form : first recall the following definition from [1] :

Definition 3.10 *([1] page 685) Let P be a p-group. Two pairs (T,S) and (V,U) of subgroups of P, with $S \trianglelefteq T$ and $U \trianglelefteq V$ are said to be linked (notation $(T,S) \mathrel{\textbf{---}} (V,U)$) if*

$$S(T\cap V) = T \qquad U(T\cap V) = V \qquad S\cap V = T\cap U$$

or equivalently if $T.U = S.V$ (as subsets of P) and $T \cap U = S \cap V$.

Two such pairs (V,U) and (T,S) are said to be linked modulo P (notation $(V,U) \mathrel{\textbf{---}}_P (T,S)$ if there exists $x \in P$ such that $(V,U) \mathrel{\textbf{---}} ({}^xT, {}^xS)$.

Theorem 3.11 *Let P be a finite p-group.*

1. *The relation $\longrightarrow_P$ is an equivalence relation on $\mathcal{G}_P$.*

2. *The correspondence $(T,S) \in \mathcal{G}_P \mapsto V(T,S) \in Irr_{\mathbb{Q}}(P)$ induces a one to one correspondence between the set of equivalence classes of genetic sections of P for the relation $\longrightarrow_P$ and the set of isomorphism classes of irreducible $\mathbb{Q}P$-modules.*

Proof : This is another form of Theorem 7.11, proved in Section 7. □

The following shows in particular that if $R \in \mathcal{N}$ (or if $R \cong D_8$), then the proper deflations of Φ_P are zero :

Lemma 3.12 *Let P be a finite group. If N is a non-trivial normal subgroup of P, and Φ is a faithful simple $\mathbb{Q}P$-module, then*

$$\mathrm{Def}^P_{P/N}\Phi = 0 \quad .$$

Proof : If V is any $\mathbb{Q}(P/N)$-module, then $\mathrm{Inf}^P_{P/N}V$ is a sum of non-faithful irreducible representations of P. Thus $\langle \Phi, \mathrm{Inf}^P_{P/N}V\rangle_P = 0$, since Φ is faithful. By adjunction, it follows that $\langle \mathrm{Def}^P_{P/N}\Phi, V\rangle_{P/N} = 0$ for any V, hence $\mathrm{Def}^P_{P/N}\Phi = 0$. □

Notation 3.13 *Denote by $\mathcal{N}_1$ the subclass of $\mathcal{N}$ consisting of groups of order at least p^2.*

Lemma 3.14 *Let $P \in \mathcal{N}_1$, and R be a maximal subgroup of P.*

1. *If P is cyclic or generalized quaternion, then $\mathrm{Res}^P_R\Phi_P = p\Phi_R$.*

2. *If P is dihedral or semi-dihedral, then*

$$\mathrm{Res}^P_R\Phi_P = \begin{cases} \Phi_R & \text{if } R \text{ is cyclic or generalized quaternion} \\ 2\Phi_R & \text{if } R \text{ is dihedral} \end{cases} \quad .$$

Proof : Let Z denote the only central subgroup of P of order p. There are several cases :

- If P is cyclic or generalized quaternion, then so is R. Moreover $\Phi_P = \mathbb{Q}(P/\mathbf{1}) - \mathbb{Q}(P/Z)$. Then clearly

$$\mathrm{Res}^P_R\Phi_P = |P:R|\big(\mathbb{Q}(R/\mathbf{1}) - \mathbb{Q}(R/Z)\big) = p\Phi_R$$

  in this case.

- If $p = 2$ and P is dihedral or semi-dihedral, then

$$\Phi_P = \mathbb{Q}(P/S) - \mathbb{Q}(P/SZ) \quad ,$$

where S is non-central of order 2. If R contains S, then R is dihedral, and

$$\mathrm{Res}^P_R \Phi_P = \sum_{x \in P/R} \big(\mathbb{Q}(R/{}^xS) - \mathbb{Q}(R/{}^xSZ)\big) = 2\Phi_R$$

since ${}^xS$ is a non-central subgroup of R of order 2, and Z is the unique central subgroup of order 2 of R. The conclusion is the same if P is dihedral, and R is dihedral, but does not contain S. In that case indeed, one has that $\Phi_P = \mathbb{Q}(P/S') - \mathbb{Q}(P/S'Z)$ for a non-central subgroup S' of P contained in R.

- The only remaining cases are when P is dihedral or semi-dihedral, and R is cyclic or generalized quaternion. In that case

$$\mathrm{Res}^P_R \Phi_P = \mathbb{Q}(R/\mathbf{1}) - \mathbb{Q}(R/Z) = \Phi_R \quad ,$$

and this completes the proof of the lemma. □

4 The subfunctors of $kR_{\mathbb{Q}}$

Notation 4.1 *If P is a finite p-group, and if V is any finitely generated $\mathbb{Q}P$-module, let $\overline{V}$ denote the image of V in $kR_{\mathbb{Q}}(P)$.*

The set $\{\overline{V} \mid V \in \mathrm{Irr}_{\mathbb{Q}}(P)\}$ is a k-basis of $kR_{\mathbb{Q}}(P)$. If $u \in kR_{\mathbb{Q}}(P)$ and $V \in \mathrm{Irr}_{\mathbb{Q}}(P)$, let $\gamma(V, u)$ denote the element of k defined by

$$u = \sum_{V \in \mathrm{Irr}_{\mathbb{Q}}(P)} \gamma(V, u)\overline{V} \quad .$$

If P has normal p-rank 1, let H_P denote the subfunctor of $kR_{\mathbb{Q}}$ generated by $\overline{\Phi}_P \in kR_{\mathbb{Q}}(P)$.

This means that H_P is the intersection of all subfunctors F of $kR_{\mathbb{Q}}$ such that $F(P) \ni \overline{\Phi}_P$. Then clearly, for any any finite p-group Q

$$H_P(Q) = \{kR_{\mathbb{Q}}(\varphi)(\overline{\Phi}_P) \mid \varphi \in \mathrm{Hom}_{\mathcal{C}}(P, Q)\} \quad .$$

Equivalently $H_P(Q)$ is the k-subspace of $kR_{\mathbb{Q}}(Q)$ generated by the images of the elements $\mathbb{Q}U \otimes_{\mathbb{Q}P} \Phi_P$, where U is a finite Q-set-P.

Lemma 4.2 *Let P be a finite p-group. Let $V \in \mathrm{Irr}_{\mathbb{Q}}(P)$, and let (T, S) denote a genetic section of P for V.*

1. *Let W be any $\mathbb{Q}P$-module. Then $\gamma(V, \overline{W})$ is equal to the image $\overline{m(V, W)}$ of the integer $m(V, W)$ in k.*

2. *Let $u \in kR_{\mathbb{Q}}(P)$. Then $\gamma(V, u) = \gamma\Big(\Phi_{T/S}, \mathrm{Def}^T_{T/S}\mathrm{Res}^P_T(u)\Big)$.*

Proof : Assertion 1 is obvious. The equality in Assertion 2 is k-linear in u, hence it suffices to check the case $u = \overline{W}$, for some $\mathbb{Q}P$-module W. But again by adjunction

$$\begin{aligned}\langle V, V\rangle_P \, m(V, W) &= \langle V, W\rangle_P = \langle \mathrm{Ind}_T^P \mathrm{Inf}_{T/S}^T \Phi(T/S), W\rangle_P \\ &= \langle \Phi_{T/S}, \mathrm{Def}^T_{T/S}\mathrm{Res}^P_T W\rangle_{T/S} \\ &= \langle \Phi_{T/S}, \Phi_{T/S}\rangle_{T/S} \, m(\Phi_{T/S}, \mathrm{Def}^T_{T/S}\mathrm{Res}^P_T W) \quad .\end{aligned}$$

The equality $m(V, W) = m(\Phi_{T/S}, \mathrm{Def}^T_{T/S}\mathrm{Res}^P_T W)$ follows, since moreover $\langle V, V\rangle_P = \langle \Phi_{T/S}, \Phi_{T/S}\rangle_{T/S}$. Now the result follows from Assertion 1. □

Lemma 4.3 *Let F be a subfunctor of $kR_{\mathbb{Q}}$, and P be a finite p-group. Let $u \in F(P)$, and $V \in \mathrm{Irr}_{\mathbb{Q}}(P)$ such that $\gamma(V, u) \neq 0$. If V has type R, then $\overline{\Phi}_R \in F(R)$.*

Proof : Let (T, S) be a genetic section of P for V. Thus $T/S \cong R$. Set $n_V = \gamma(V, u)$. One has that $n_V = \gamma(\Phi_{T/S}, v)$ by Lemma 4.2, where $v = \mathrm{Def}^T_{T/S}\mathrm{Res}^P_T(u)$. In particular $v \in F(T/S)$ since F is a subfunctor of $kR_{\mathbb{Q}}$. By isomorphism, it follows that there is an element $w \in F(R)$ such that $n_V = \gamma(\Phi_R, w) \neq 0$.

Now proceed by induction on $|R|$: if $|R| = 1$, since $kR_{\mathbb{Q}}(\mathbf{1})$ is one dimensional, it follows that $\overline{\Phi}_{\mathbf{1}} \in F(\mathbf{1})$.

Otherwise, the element w is equal to

$$w = n_V \overline{\Phi}_R + \sum_{\substack{W \in \mathrm{Irr}_{\mathbb{Q}}(R) \\ W \neq \Phi_R}} m_W \overline{W} \quad ,$$

where $m_W = \gamma(W, w)$. But all the simple $\mathbb{Q}R$-modules W different from Φ_R are of type S, with $|S| < |R|$, by Proposition 3.7. By induction hypothesis, one has that $\overline{\Phi}_S \in F(S)$ whenever $m_W \neq 0$. Thus $\overline{W} \in F(R)$, since W is obtained from Φ_R by inflation followed by induction, and since F is a subfunctor of $kR_{\mathbb{Q}}$.

By difference, it follows that $n_V \overline{\Phi}_R \in F(R)$, hence $\overline{\Phi}_R \in F(R)$, as was to be shown. □

Theorem 4.4 *If F is a subfunctor of $kR_{\mathbb{Q}}$, then*

$$F = \sum_{P \in [\mathcal{N}_F]} H_P \quad ,$$

where $\mathcal{N}_F$ is the subclass of $\mathcal{N}$ consisting of p-groups R for which $\overline{\Phi}_R \in F(R)$.

Proof : Observe first that the inclusion

$$\sum_{P \in [\mathcal{N}_F]} H_P \subseteq F$$

is obvious : if $\overline{\Phi}_R \in F(R)$, then F contains the subfunctor H_R generated by $\overline{\Phi}_R$.

Conversely, let Q be a finite p-group, and $u = \sum_{V \in \mathrm{Irr}_{\mathbb{Q}}(Q)} \gamma(V,u)\overline{V}$ be an element of $F(Q)$. Then by Lemma 4.3, if $V \in \mathrm{Irr}_{\mathbb{Q}}(Q)$ has type R and if $\gamma(V,u) \neq 0$, one has that $\overline{\Phi}_R \in F(R)$, i.e. $R \in \mathcal{N}_F$. Now $\overline{V} \in H_R(Q)$ since V is obtained from Φ_R by inflation followed by induction. It follows that $u \in \sum_{R \in \mathcal{N}_F} H_R(Q)$, as was to be shown. □

Proposition 4.5 *Let $\mathcal{F}$ be a subclass of $\mathcal{N}$. If P is any element of $\mathcal{N}$ such that $H_P \subseteq \sum\limits_{R \in [\mathcal{F}]} H_R$, then there exists $R \in \mathcal{F}$ such that $H_P \subseteq H_R$.*

Proof : Indeed, if $H_P \subseteq \sum_{R \in [\mathcal{F}]} H_R$, then there is a finite subset $\mathcal{S}$ of $[\mathcal{F}]$, and for $R \in \mathcal{S}$, there is an element $\varphi_R \in \mathrm{Hom}_{\mathcal{C}}(R,P)$, such that

$$\overline{\Phi}_P = \sum_{R \in \mathcal{S}} kR_{\mathbb{Q}}(\varphi_R)(\overline{\Phi}_R) \quad .$$

It follows in particular that there is some element $R \in \mathcal{S}$ with $\gamma(\Phi_P, v) \neq 0$ for some element $v = kR_{\mathbb{Q}}(\varphi_R)(\overline{\Phi}_R)$ of $H_R(P)$. Lemma 4.3 shows then that $\overline{\Phi}_P \in H_R(P)$, i.e. that $H_P \subseteq H_R$. □

Notation 4.6 *Denote by $P \preceq Q$ the relation on $\mathcal{N}$ defined by*

$$P \preceq Q \Leftrightarrow H_P \subseteq H_Q \quad .$$

A subclass $\mathcal{A}$ of $\mathcal{N}$ is said to be closed if

$$\forall P \in \mathcal{N}, \forall Q \in \mathcal{A}, P \preceq Q \Rightarrow P \in \mathcal{A} \quad .$$

A subset $\mathcal{S}$ of $[\mathcal{N}]$ is said to be closed if $\mathcal{S} = [\mathcal{A}]$ for some closed subclass $\mathcal{A}$ of $\mathcal{N}$.

Theorem 4.7 *The correspondences sending a subfunctor F of $kR_{\mathbb{Q}}$ to the subset $[\mathcal{N}_F]$ of $[\mathcal{N}]$, and the subset $\mathcal{S}$ of $[\mathcal{N}]$ to the subfunctor $\sum\limits_{R\in\mathcal{S}} H_R$, are mutual inverse isomorphisms between the lattice of subfunctors of $kR_{\mathbb{Q}}$ (ordered by inclusion of subfunctors), and the lattice of closed subsets of $[\mathcal{N}]$ (ordered by inclusion of subsets).*

Proof : First, if F is a subfunctor of $kR_{\mathbb{Q}}$, then $\mathcal{N}_F$ is a closed subclass of $\mathcal{N}$, by definition of $\mathcal{N}_F$.

Denote by A the first correspondence, and by B the second one. Then clearly A and B are maps of posets. Theorem 4.4 shows that $B \circ A$ is equal to the identity, and Proposition 4.5 shows that $A \circ B$ is equal to the identity : indeed if $\mathcal{S} \subseteq [\mathcal{N}]$, and if $P \in \mathcal{N}$ is such that $H_P \subseteq \sum_{R\in\mathcal{S}} H_R$, then there exists $R \in \mathcal{S}$ with $H_P \subseteq H_R$. Hence $P \in \mathcal{S}$ since $\mathcal{S}$ is a closed subset of $[\mathcal{N}]$. Thus $\mathcal{N}_{B(\mathcal{S})} = \mathcal{S}$. □

5 The functors H_P

Proposition 5.1 *Let $P \in \mathcal{N}$.*

1. *The functor H_P has a unique maximal (proper) subfunctor J_P, defined by*
$$J_P = \sum_R H_R \quad ,$$
where the sum runs through the elements R of $[\mathcal{N}]$ for which H_R is a proper subfunctor of H_P.

2. *The quotient functor H_P/J_P is isomorphic to the simple functor $S_{Q,k}$, where Q is an element of $\mathcal{N}$ of minimal order such that $H_Q = H_P$.*

3. *If Q is any element of $\mathcal{N}$ of minimal order such that $H_Q = H_P$, then Q is isomorphic to a section of P.*

Proof : Let F be a proper subfunctor of H_P. Then F is equal to the sum of the functors H_Q it contains. Those are all proper subfunctors of H_P. Thus $F \subseteq J_P$. And J_P is indeed a proper subfunctor of H_P, by Proposition 4.5. This proves Assertion 1.

Now the quotient $S = H_P/J_P$ is a simple functor, isomorphic to a simple functor $S_{Q,V}$, for some p-group Q and some $k\mathrm{Out}(Q)$-module V : the group Q is of minimal order subject to $S(Q) \neq 0$, and then $V = S(Q)$, as $k\mathrm{Out}(Q)$-module.

Let $u \in H_P(Q) - J_P(Q)$, and write $u = \sum_{V \in \mathrm{Irr}_{\mathbb{Q}}(Q)} \gamma(V,u)\overline{V}$. Then if $\gamma(V,u) \neq 0$, and V has type R, Lemma 4.3 shows that $\overline{\Phi}_R \in H_P(R)$. If $|R| < |Q|$ then $H_P(R) = J_P(R)$ by minimality of Q. In this case $\overline{V} \in J_P(Q)$. Since $u \neq 0$, it means that there must exist a $V \in \mathrm{Irr}_{\mathbb{Q}}(Q)$ of type Q, and such that $\gamma(V,u) \neq 0$. In other words $Q \in \mathcal{N}$, and $\gamma(\Phi_Q, u) \neq 0$. Moreover $H_P(Q) = J_P(Q) + k\overline{\Phi}_Q$. Now $\overline{\Phi}_Q \in H_P(Q)$, hence $H_Q \subseteq H_P$. And since $\overline{\Phi}_Q \notin J_P(Q)$, it follows that H_Q is not contained in J_P. Hence $H_Q = H_P$.

Now if $Q' \in \mathcal{N}$ is such that $|Q'| < |Q|$ and $H_{Q'} = H_P$, then also $J_{Q'} = J_P$, hence $S \cong H_{Q'}/J_{Q'}$. But $S(Q') = 0$ by minimality of Q. In particular $\overline{\Phi}_{Q'} \in J_{Q'}(Q')$, hence $H_{Q'} \subseteq J_{Q'}$, which is not possible since $J_{Q'}$ is a proper subfunctor of $H_{Q'}$. Thus Q has minimal order such that $H_Q = H_P$.

Finally $S(Q) = H_P(Q)/J_P(Q)$ is one dimensional over k, generated by the image of $\overline{\Phi}_Q$. This element is clearly invariant by $\mathrm{Out}(Q)$. Hence $S(Q)$ is the trivial module, and $S \cong S_{Q,k}$.

Conversely, if R is any element of $\mathcal{N}$ such that $H_R = H_P$, then also $J_R = J_P$, and $H_R/J_R \cong S_{Q,k}$. In particular $S_{Q,k}(R) \neq 0$, since otherwise $H_R(R) = J_R(R)$, and $\overline{\Phi}_R \in J_R(R)$, hence $H_R = J_R$, a contradiction. By Lemme 7 page 678 of [1], this implies that Q is isomorphic to a section of R (hence also of P). In particular, any two elements R of minimal order such that $H_R = H_P$ are isomorphic. □

Lemma 5.2 *Let P and Q be finite p-groups, and let U be a finite Q-set-P. Let V be a simple $\mathbb{Q}P$-module such that $\langle V,V\rangle_P \geq p$, and let W be a simple $\mathbb{Q}Q$-module.*

If $\langle W,W\rangle_Q < \langle V,V\rangle_P$, then $m(W, \mathbb{Q}U \otimes_{\mathbb{Q}P} V)$ is a multiple of p.

Proof : Indeed by adjunction

$$\begin{aligned} \langle W, \mathbb{Q}U \otimes_{\mathbb{Q}P} V\rangle_Q &= \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}Q}(W, \mathbb{Q}U \otimes_{\mathbb{Q}P} V) \\ &= \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}P}(\mathbb{Q}U^{op} \otimes_{\mathbb{Q}Q} W, V) \\ &= \langle \mathbb{Q}U^{op} \otimes_{\mathbb{Q}Q} W, V\rangle_P \end{aligned}$$

where U^{op} denotes the opposite biset, i.e. the set U viewed as a P-set-Q by $x.u.y(\text{in } U^{op}) = y^{-1}ux^{-1}(\text{in } U)$, for $x \in P$, $y \in Q$, $u \in U$.

It follows that

$$\langle W,W\rangle_Q \, m(W, \mathbb{Q}U \otimes_{\mathbb{Q}P} V) = \langle V,V\rangle_P \, m(V, \mathbb{Q}U^{op} \otimes_{\mathbb{Q}Q} W) \quad .$$

But since $\langle V,V\rangle_P \geq p$, it follows from Assertion 3 of Proposition 2.5 that there is an integer $a > 0$ such that

$$\langle V,V\rangle_P = (p-1)p^a \quad .$$

Now either W is trivial and then

$$m(W, \mathbb{Q}U \otimes_{\mathbb{Q}P} V) = (p-1)p^a \, m(V, \mathbb{Q}U^{op} \otimes_{\mathbb{Q}Q} W) \equiv 0 \; (p)$$

or W is non-trivial, and there exists an integer $b \geq 0$ such that

$$\langle W, W \rangle_Q = (p-1)p^b \quad .$$

Moreover $b < a$ since $\langle W, W \rangle_Q < \langle V, V \rangle_P$. Then

$$m(W, \mathbb{Q}U \otimes_{\mathbb{Q}P} V) = p^{a-b} m(V, \mathbb{Q}U^{op} \otimes_{\mathbb{Q}Q} W) \equiv 0 \quad (p)$$

as was to be shown. □

Notation 5.3 *Let $m \in \mathbb{N}$. If P is a finite p-group, denote by $N_m(P)$ the k-subspace of $kR_{\mathbb{Q}}(P)$ generated by the elements $\overline{V}$, for $V \in Irr_{\mathbb{Q}}(P)$, such that $\langle V, V \rangle_P \equiv 0 \; (p^m)$.*

Lemma 5.4 *With this notation N_m is a subfunctor of $kR_{\mathbb{Q}}$.*

Proof : If $m = 0$ there is nothing to prove, since $N_0 = kR_{\mathbb{Q}}$. One can suppose $m \geq 1$, and the result follows then from Lemma 5.2 : if P and Q are finite p-groups, if $\varphi \in \mathrm{Hom}_{\mathcal{C}}(P, Q)$, if $V \in \mathrm{Irr}_{\mathbb{Q}}(P)$ is such that $\langle V, V \rangle_P \equiv 0 \; (p^m)$, and if $W \in \mathrm{Irr}_{\mathbb{Q}}(Q)$ are such that $\gamma\Big(W, kR_{\mathbb{Q}}(\varphi)(\overline{V})\Big) \neq 0$, then there is a finite Q-set-P, say U, such that the multiplicity of W in $\mathbb{Q}U \otimes_{\mathbb{Q}P} V$ is not divisible by p. Since $\langle V, V \rangle_P \geq p$, it follows from Lemma 5.2 that $\langle W, W \rangle_Q \geq \langle V, V \rangle_P$. But there are integers a and b such that $\langle V, V \rangle_P = (p-1)p^a$, and $\langle W, W \rangle_Q = (p-1)p^b$. Thus $b \geq a$, and $a \geq m$ if $\langle V, V \rangle_P \equiv 0 \; (p^m)$. Hence $\langle W, W \rangle_Q \equiv 0 \; (p^m)$, as was to be shown. □

Proposition 5.5 *Let P be a p-group of normal p-rank 1, of order p^d.*

1. *If P is cyclic or generalized quaternion, then*

$$\langle \Phi_P, \Phi_P \rangle_P = \begin{cases} 1 & \text{if } d = 0 \\ (p-1)p^{d-1} & \text{if } d \geq 1 \end{cases} \quad .$$

2. *If P is dihedral or semi-dihedral, then*

$$\langle \Phi_P, \Phi_P \rangle_P = 2^{d-3} \quad .$$

Proof : Suppose first that P is cyclic or generalized quaternion. If $P = \mathbf{1}$ the result is trivial. Otherwise the proof of Proposition 3.7 shows that $\Phi_P = V_{\mathbf{1}}$. Now by Assertion 3 of Proposition 2.5

$$\langle V_{\mathbf{1}}, V_{\mathbf{1}} \rangle_P = (p-1)\frac{|N_P(\mathbf{1}) : \mathbf{1}|}{p} |\mathcal{I}_P(\mathbf{1}, \mathbf{1})| = (p-1)\frac{|P|}{p} \quad .$$

Assertion 1 follows.

Similarly, if P is dihedral or semi-dihedral, and if Q is a non-central subgroup of P of order 2, then $\Phi_P = V_Q$, and

$$\langle V_Q, V_Q \rangle_P = \frac{|N_P(Q) : Q|}{2} |\mathcal{I}_P(Q, Q)| \quad .$$

Moreover $\frac{|N_P(Q):Q|}{2} = 1$ in this case, since $N_P(Q)$ has order 4. And if $x \in P$ is such that $Q^x \cap N_P(Q) \subseteq Q$, then either $x \in N_P(Q)$, or $x \notin N_P(N_P(Q))$, and then $Q^x \cap N_P(Q) = \mathbf{1}$, or equivalently $|QxN_P(Q)| = 8$. Since $N_P(N_P(Q))$ has order 8, there are $(|P| - 8)/8$ such double cosets $QxN_P(Q)$. This shows that $|\mathcal{I}_P(Q, Q)| = |P|/8 = 2^{d-3}$, and Assertion 2 follows. Note that the argument still holds if P is a dihedral group of order 8 (which is not of normal 2 rank 1) : if Q is a non-central subgroup of P of order 2, then Q is basic, and $\langle V_Q, V_Q \rangle_P = 2^{3-3} = 1$. □

Theorem 5.6 *Let P and Q be finite p-groups of normal p-rank 1. Then $H_P = H_Q$ if and only if one of the following holds :*

1. *One has that $|P| \leq p$ and $|Q| \leq p$. In this case $H_P = H_Q = kR_{\mathbb{Q}}$.*
2. *One has $|P| \geq p^2$, and $Q \cong P$.*

Proof : If $H_P = kR_{\mathbb{Q}}$, then in particular H_P is not contained in N_1, which is a proper subfunctor of $kR_{\mathbb{Q}}$ (since $\langle \Phi_{\mathbf{1}}, \Phi_{\mathbf{1}} \rangle_{\mathbf{1}} = 1 \not\equiv 0 \ (p)$). It follows that $\langle \Phi_P, \Phi_P \rangle_P$ is not divisible by p. By Proposition 5.5, this can only happen if $|P| \leq p$.

Conversely, if $|P| \leq p$, then $H_P = kR_{\mathbb{Q}}$: first suppose that $P = \mathbf{1}$. Let Q be a finite p-group. If R is a subgroup of Q, then

$$\mathbb{Q}(Q/R) = \mathrm{Ind}_R^Q \mathrm{Inf}_{R/R}^R \mathbb{Q} \quad .$$

In other words, if U is the set Q/R, viewed as a Q-set-$\mathbf{1}$, then $\mathbb{Q}(Q/R) = R_{\mathbb{Q}}(U)(\mathbb{Q})$. Since $\mathbb{Q} = \Phi_{\mathbf{1}}$, and since any element of $kR_{\mathbb{Q}}(Q)$ is a k-linear combination of elements $\overline{\mathbb{Q}(Q/R)}$, it follows that $H_{\mathbf{1}} = kR_{\mathbb{Q}}$.

Now if P is cyclic of order p, then

$$\Phi_P = \mathbb{Q}(P/\mathbf{1}) - \mathbb{Q}(P/P) \quad .$$

Restriction to the trivial group gives

$$\mathrm{Res}_{\mathbf{1}}^P \Phi_P = p\mathbb{Q} - \mathbb{Q} = (p-1)\mathbb{Q} = (p-1)\Phi_{\mathbf{1}} \quad .$$

Now $p - 1 = -1$ in k, proving that $\overline{\Phi}_{\mathbf{1}} \in H_P(\mathbf{1})$. Thus $H_{\mathbf{1}} \subseteq H_P$, and $H_P = kR_{\mathbb{Q}}$.

Now suppose P has order at least p^2. Then $\langle \Phi_P, \Phi_P \rangle \equiv 0 \ (p)$, thus $H_P \subseteq N_1$. Hence if $H_Q = H_P$, then $H_Q \subseteq N_1$, hence also $|Q| \geq p^2$. There are positive integers m and l such that $\langle \Phi_P, \Phi_P \rangle_P = (p-1)p^m$, and $\langle \Phi_Q, \Phi_Q \rangle_Q = (p-1)p^l$. Thus $H_P \subseteq N_m$, and then $H_Q \subseteq N_m$. Hence $\langle \Phi_Q, \Phi_Q \rangle_Q \equiv 0 \ (p^m)$, and $l \geq m$. By symmetry $l = m$.

To prove that $Q \cong P$, it suffices to prove that the hypothesis $H_P = H_Q$ and $|P| \geq p^2$ implies $|P| = |Q|$. Indeed, in that case, the groups P and Q will both be of minimal order such that $H_P = H_Q$. Hence each will be isomorphic to a section of the other, by Assertion 3 of Proposition 5.1.

Suppose then that $|Q| < |P|$, and that $\langle \Phi_Q, \Phi_Q \rangle_Q = \langle \Phi_P, \Phi_P \rangle_P$. By Proposition 5.5, this can only happen if $p = 2$, if P is dihedral or semi-dihedral of order 2^d, and Q is cyclic or generalized quaternion of order 2^{d-2}. The following lemma shows that $\overline{\Phi}_Q \notin H_P(Q)$ in that case, completing the proof. □

Remark 5.7 This proof gives the reason why the cyclic group of order p plays a special rôle in this work : the group C_p is the only non trivial element P of $\mathcal{N}$ for which the dimension of Φ_P is not divisible by p, or equivalently, for which $\mathrm{Res}_{\mathbf{1}}^P \overline{\Phi}_P \neq 0$.

Lemma 5.8 *Let P be a dihedral or semi-dihedral group of order 2^d, and Q be a cyclic or generalized quaternion group of order 2^{d-2}. Then if U is a finite Q-set-P, the multiplicity of Φ_Q in $\mathbb{Q}U \otimes_{\mathbb{Q}P} \Phi_P$ is even.*

Proof : One can suppose that U is a transitive biset, and in this case (Lemme 3 of [1], or Lemma 7.4 of [4]), there are subgroups X of P and Y of Q, there is a normal subgroup R of X, and a normal subgroup R' of Y, with $X/R \cong Y/R'$, such that

$$\mathbb{Q}U \otimes_{\mathbb{Q}P} \Phi_P = \mathrm{Ind}_Y^Q \mathrm{Inf}_{Y/R'}^Y \mathrm{Iso}_{X/R}^{Y/R'} \mathrm{Def}_{X/R}^X \mathrm{Res}_X^P(\Phi_P) \quad .$$

Since $|Q| < |P|$, then $|X/R| < |P|$, which forces $|X| < |P|$ or $|R| \neq 1$. Suppose first that $X = P$. Then R is a non-trivial normal subgroup of P, and $\mathrm{Def}^P_{P/R}\Phi_P = 0$ by Lemma 3.12 in this case.

Now if $X \neq P$, the restriction Res^P_X factors through the restriction to some maximal subgroup M of P. By Lemma 3.14, one has that $\mathrm{Res}^P_M\Phi_P = 2\Phi_M$ if M is dihedral, and $\mathrm{Res}^P_M\Phi_P = \Phi_M$ if M is cyclic or generalized quaternion. In the first case, all simple modules have even multiplicity in $\mathrm{Res}^P_M\Phi_P$. Thus $m(\Phi_Q, \mathbb{Q}U \otimes_{\mathbb{Q}P} \Phi_P)$ is even in that case.

In the second case, one has that $\langle\Phi_M, \Phi_M\rangle_M = 2^{d-2}$, since M is cyclic or quaternion of order 2^{d-1}. Now there is a Q-set-M, say V, such that

$$\mathbb{Q}U \otimes_{\mathbb{Q}P} \Phi_P = \mathbb{Q}V \otimes_{\mathbb{Q}M} \Phi_M \quad ,$$

If $m(\Phi_Q, \mathbb{Q}U \otimes_{\mathbb{Q}P} \Phi_P) = m(\Phi_Q, \mathbb{Q}V \otimes_{\mathbb{Q}M} \Phi_M)$ is odd, it follows by Lemma 5.2 that

$$\langle\Phi_Q, \Phi_Q\rangle \geq \langle\Phi_M, \Phi_M\rangle_M = 2^{d-2} \quad .$$

This is a contradiction since $\langle\Phi_Q, \Phi_Q\rangle = 2^{d-3}$ when Q is cyclic or generalized quaternion of order 2^{d-2}. This completes the proof of the lemma. □

Corollary 5.9 *Let P be a finite p-group, and let V be a simple $\mathbb{Q}P$-module. If (T, S) and (T', S') are genetic sections of P for V, then $T/S \cong T'/S'$.*

Proof : Set $R = T/S$ and $R' = T'/S'$. Since the trivial module is the only one of trivial type, one can suppose that V is non-trivial, and that R and R' are non-trivial. By assumption $V \cong \mathrm{Ind}^P_T\mathrm{Inf}^T_R\Phi_R$ and $V \cong \mathrm{Ind}^P_{T'}\mathrm{Inf}^{T'}_{R'}\Phi_{R'}$. The first isomorphism implies $\overline{V} = \mathrm{Ind}^P_T\mathrm{Inf}^T_R\overline{\Phi}_R$, and then the second one gives $\overline{\Phi}_R \in H_{R'}(R)$. Hence $H_R \subseteq H_{R'}$, thus $H_R = H_{R'}$ by symmetry. It follows that $R \cong R'$, by Theorem 5.6, since R and R' are non-trivial. □

The following corollary is a summary of the previous results :

Corollary 5.10 *Let $P \in \mathcal{N}$.*

1. *If $|P| \leq p$, then $H_P = kR_{\mathbb{Q}}$ has a unique maximal subfunctor $J_{\mathbf{1}}$, and the quotient $kR_{\mathbb{Q}}/J_{\mathbf{1}}$ is isomorphic to $S_{\mathbf{1},k}$.*

2. *If $|P| \geq p^2$, then H_P has a unique maximal subfunctor J_P, and the quotient H_P/J_P is isomorphic to $S_{P,k}$.*

Corollary 5.11 *Let F be a subfunctor of $kR_{\mathbb{Q}}$, and let F' be a maximal proper subfunctor of F. Then there exists $P \in \mathcal{N}$, with $P \not\cong C_p$, such that $F/F' \cong S_{P,k}$.*

Proof : Indeed since F' is a proper subfunctor of F, by Theorem 4.4, there must be $P \in \mathcal{N}$ such that $H_P \subseteq F$ but $H_P \not\subseteq F'$, and one can moreover suppose $P \not\cong C_p$. Then $H_P + F' = F$, and $F/F' \cong H_P/(H_P \cap F')$ is a simple quotient of H_P. Hence it is isomorphic to $S_{P,k}$. □

Theorem 5.12 *Let Q be a finite p-group.*

1. *The dimension of $S_{\mathbf{1},k}(Q)$ is equal to*

$$\begin{aligned}\dim_k S_{\mathbf{1},k}(Q) &= |\{V \in \mathrm{Irr}_{\mathbb{Q}}(Q) \mid V \textit{ has type } \mathbf{1} \textit{ or } C_p\}| \\ &= |\{V \in \mathrm{Irr}_{\mathbb{Q}}(Q) \mid \langle V, V\rangle_Q \leq p-1\}| \\ &= |\{V \in \mathrm{Irr}_{\mathbb{Q}}(Q) \mid \langle V, V\rangle_Q \not\equiv 0\ (p)\}|\end{aligned}$$

2. *If $P \in \mathcal{N}$ and $|P| \geq p^2$, then*

$$\dim_k S_{P,k}(Q) = |\{V \in \mathrm{Irr}_{\mathbb{Q}}(Q) \mid V \textit{ has type } P\}| \quad .$$

Proof : If $P \in \mathcal{N}$ and $|P| \neq p$, then $H_P/J_P \cong S_{P,k}$ by Corollary 5.10. Let $u \in H_P(Q)$. Then

$$u = \sum_{V \in \mathrm{Irr}_{\mathbb{Q}}(Q)} \gamma(V, u)\overline{V} \quad .$$

If $V \in \mathrm{Irr}_{\mathbb{Q}}(Q)$ has type R, and if $\gamma(V, u) \neq 0$, then by Lemma 4.3 one has that $\overline{\Phi}_R \in H_P(R)$, thus $H_R \subseteq H_P$. If the inclusion is proper, then $H_R \subseteq J_P$, and $\overline{\Phi}_R \in J_P(R)$. It follows that $\overline{V} \in J_P(Q)$.

Hence $H_P(Q)$ is generated by the elements $\overline{V}$, where V is of type $R \preceq P$, and moreover $\overline{V} \in J_P(Q)$ if H_R is properly contained in H_P. The quotient $H_P(Q)/J_P(Q)$ is then generated by the images of the elements $\overline{V}$, for $V \in \mathrm{Irr}_{\mathbb{Q}}(Q)$ of type R such that $H_R = H_P$. By Corollary 5.10, if $P = \mathbf{1}$, this implies $|R| \leq p$, and if $|P| \geq p^2$, this implies $R \cong P$.

Now suppose that $u \in H_P(Q)$ is a linear combination of elements $\overline{V}$, for $V \in \mathrm{Irr}_{\mathbb{Q}}(P)$, of type R such that $H_R = H_P$, and that $u \in J_P(Q)$. Then if $\gamma(V, u) \neq 0$ for some $V \in \mathrm{Irr}_{\mathbb{Q}}(Q)$, Lemma 4.3 shows that $\overline{\Phi}_R \in J_P(R)$, and then $H_R = H_P \subseteq J_P$, a contradiction. It follows that $u = 0$, and the images of the elements $\overline{V}$, for $V \in \mathrm{Irr}_{\mathbb{Q}}(P)$, of type R such that $H_R = H_P$, form a k-basis of $H_P(Q)/J_P(Q)$. The theorem follows then from Proposition 2.5 and Proposition 5.5. □

Remark 5.13 If $p = 2$, Theorem 5.12 implies that the k-dimension of $S_{\mathbf{1},k}(Q)$ is equal to the number of absolutely irreducible rational representations of Q.

Remark 5.14 I. Bourizk ([5] Proposition 4 and Proposition 6) has given upper and lower bounds for $\dim_k S_{P,k}(Q)$, when P and Q are finite p-groups with $|P| \neq p$. One can check that these bounds are compatible with the values given here, in the case $P \in \mathcal{N}$: for example, if P has order at least p^2, the lower bound in Proposition 4 of [5] is equal to the number l of normal subgroups R of Q such that $Q/R \cong P$. If $P \in \mathcal{N}$, then each of these quotients has a unique faithful rational irreducible representation, and inflating to P gives at least l non-isomorphic irreducible rational representations of Q of type P. The other verifications are similar.

6 The lattice of subfunctors of $kR_{\mathbb{Q}}$

The following is the first part of Theorem 1.3 :

Theorem 6.1 *If k is a field of odd characteristic p, then the functor $kR_{\mathbb{Q}}$ is a uniserial object of $\mathcal{F}_{p,k}$. If $F_0 = kR_{\mathbb{Q}} \supset F_1 \supset F_2 \supset \ldots \supset F_n \supset \ldots$ is the set of its non-zero subfunctors, then*

$$F_0/F_1 \cong S_{\mathbf{1},k} \qquad F_i/F_{i+1} \cong S_{C_{p^{i+1}},k} \text{ for } i \geq 1 \quad ,$$

where $C_{p^{i+1}}$ denotes a cyclic group of order p^{i+1}.

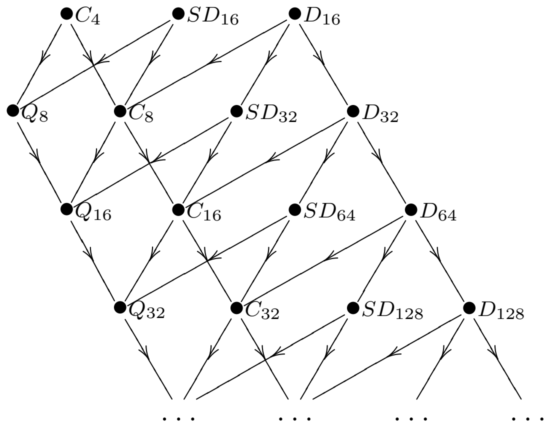
Proof : Indeed, if $p \neq 2$, then $\mathcal{N}$ just consists of cyclic groups. If P is cyclic of order $p^k \geq p^2$, then $\Phi_P = \mathbb{Q}(P/\mathbf{1}) - \mathbb{Q}(P/Z)$, where Z is the only subgroup of order p of P. If Q is the only subgroup of P of index p, then $\Phi_P = \mathrm{Ind}_Q^P \Phi_Q$. This shows that $\overline{\Phi}_P \in H_Q(P)$, i.e. that $H_P \subseteq H_Q$. By Corollary 5.10, the inclusion is proper.

Now the set of subfunctors $H_{C_{p^m}}$, for $m \geq 2$, is linearly ordered by inclusion. Moreover recall that $H_{\mathbf{1}} = H_{C_p} = kR_{\mathbb{Q}}$. Proposition 5.1 shows that $J_{\mathbf{1}} = H_{C_{p^2}}$, and that $J_P = H_{C_{p^{m+1}}}$ if P is cyclic of order $p^m \geq p^2$. Since J_P is the unique maximal subfunctor of H_P, it follows that $kR_{\mathbb{Q}}$ is uniserial, with filtration

$$kR_{\mathbb{Q}} \supset H_{C_{p^2}} \supset H_{C_{p^3}} \supset \ldots \supset H_{C_{p^m}} \supset \ldots \quad .$$

Moreover the quotient $kR_{\mathbb{Q}}/H_{C_{p^2}}$ is isomorphic to $S_{\mathbf{1},k}$, and the quotient $H_{C_{p^m}}/H_{C_{p^{m+1}}}$ is isomorphic to $S_{C_{p^m}}$, for $m \geq 2$. □

Theorem 6.2 *Suppose $p = 2$, and consider the following graph structure on $[\mathcal{N}_1]$:*



Then for P and Q in $[\mathcal{N}_1]$, one has that $P \preceq Q$ if and only if there is an (oriented) path from Q to P in $[\mathcal{N}_1]$.

Proof : First check that if $R \rightarrowtail P$ is an arrow in $[\mathcal{N}_1]$, then $P \preceq R$. There are several cases :

- $X_{2^n} \rightarrowtail X_{2^{n+1}}$, where X is one of Q, C, or D : in this case X_{2^n} is a subgroup of $X_{2^{n+1}}$, and $\Phi_{X_{2^{n+1}}} = \mathrm{Ind}_{X_{2^n}}^{X_{2^{n+1}}}\Phi_{X_{2^n}}$, thus $X_{2^{n+1}} \preceq X_{2^n}$.
- $C_{2^n} \rightarrowtail Q_{2^{n+1}}$: here again $\Phi_{Q_{2^{n+1}}} = \mathrm{Ind}_{C_{2^n}}^{Q_{2^{n+1}}}\Phi_{C_{2^n}}$, and it follows that $Q_{2^{n+1}} \preceq C_{2^n}$.
- $SD_{2^n} \rightarrowtail X_{2^{n-1}}$, where X is Q or C : here $X_{2^{n-1}}$ is a subgroup of SD_{2^n}, and $\mathrm{Res}_{X_{2^{n-1}}}^{SD_{2^n}}\Phi_{SD_{2^n}} = \Phi_{X_{2^{n-1}}}$, by Lemma 3.14. It follows that $X_{2^{n-1}} \preceq SD_{2^n}$.
- $D_{2^n} \rightarrowtail C_{2^{n-1}}$: here again $\mathrm{Res}_{C_{2^{n-1}}}^{D_{2^n}}\Phi_{D_{2^n}} = \Phi_{C_{2^{n-1}}}$, and it follows that $C_{2^{n-1}} \preceq D_{2^n}$.
- $D_{2^n} \rightarrowtail SD_{2^{n+1}}$: here again $\Phi_{SD_{2^{n+1}}} = \mathrm{Ind}_{SD_{2^{n+1}}}^{D_{2^n}}\Phi_{D_{2^n}}$, and it follows that $SD_{2^{n+1}} \preceq D_{2^n}$.

Conversely, suppose that P and R are in $[\mathcal{N}_1]$, and that $P \preceq R$. Then there exists $\varphi \in \mathrm{Hom}_{\mathcal{C}}(R, P)$ such that $\overline{\Phi}_P = kR_{\mathbb{Q}}(\varphi)(\overline{\Phi}_R)$. Hence there is a finite P-set-Q, say U, such that $m(\Phi_P, \mathbb{Q}U \otimes_{\mathbb{Q}R} \Phi_R)$ is odd. One can moreover suppose that U is a transitive biset. In other words, there are subgroups $Y \trianglelefteq X \subseteq P$ and $T \trianglelefteq Z \subseteq R$, with $X/Y \cong Z/T$, such that $\mathbb{Q}U \otimes_R \Phi_R$ is equal to

$$\Psi = \mathrm{Ind}_X^P \circ \mathrm{Inf}_{X/Y}^X \circ \mathrm{Iso}_{Z/T}^{X/Y} \circ \mathrm{Def}_{Z/T}^Z \circ \mathrm{Res}_Z^R \Phi_R \quad .$$

Since $m(\Phi_P, \Psi)$ is odd, it follows in particular that $\mathrm{Res}_Z^R \Phi_R \notin 2R_{\mathbb{Q}}(Z)$. But it follows easily from Lemma 3.14 and from Remark 3.3 that if $R \in \mathcal{N}_1$ and $|R : Z| \geq 4$, then $\mathrm{Res}_Z^R \in 2R_{\mathbb{Q}}(Z)$. And if $|R : Z| = 2$, the only case where $\mathrm{Res}_Z^R \notin 2R_{\mathbb{Q}}(Z)$ is when R is dihedral or semi-dihedral, and Z is cyclic or generalized quaternion.

In this case $\mathrm{Res}_Z^R \Phi_R = \Phi_Z$. Then $Z \preceq R$, and $P \preceq Z$, since $m(\Phi_P, \Psi)$ is odd and $\Psi = \mathrm{Ind}_X^P \mathrm{Inf}_{X/Y}^X \mathrm{Iso}_{Z/T}^{X/Y} \mathrm{Def}_{Z/T}^Z \Phi_Z$. Since moreover $R \twoheadrightarrow Z$ in $[\mathcal{N}_1]$, it suffices to consider the case $Z = R$.

Then $T = \mathbf{1}$ by Lemma 3.12, and one can suppose $R = X/Y$. Moreover

$$\langle \Phi_P, \Phi_P \rangle_P \, m(\Phi_P, \Psi) = \langle \Phi_R, \Phi_R \rangle_R \, m(\Phi_R, \mathrm{Def}_R^X \mathrm{Res}_X^P \Phi_P) \quad . \qquad \mathbf{(6.3)}$$

Now X is a subgroup of $P \in \mathcal{N}$, and X has a quotient R in $\mathcal{N}_1$. By Remark 3.3, it follows that $X \in \mathcal{N}_1$. Now by Lemma 3.14 and Lemma 3.12, the right hand side of equation 6.3 is zero if Def_R^X is a proper deflation, i.e. if $R \ncong X$.

Hence one can suppose that R is a subgroup of P, such that $m(\Phi_P, \mathrm{Ind}_R^P \Phi_R)$ is odd. There are several cases :

- If P is cyclic or generalized quaternion, then there is indeed a path from R to P in $[\mathcal{N}_1]$.
- If P is dihedral, then there is a path from R to P in $[\mathcal{N}_1]$ if R is dihedral. If R is cyclic, then R is contained in a cyclic subgroup C of index 2 in P. Since $\Phi_C = \mathrm{Ind}_R^C \Phi_R$, it follows that $m(\Phi_P, \mathrm{Ind}_C^P \Phi_C)$ is odd, and then $P \preceq C$. This cannot occur, since $C \preceq P$ in this case.
- Finally, if P is semi-dihedral, then there is a path from R to P in $[\mathcal{N}_1]$ if R is cyclic or generalized quaternion. And the case where R is dihedral cannot occur, since $R \preceq P$ in this case. □

Definition 6.4 *A subset $\mathcal{S}$ of $[\mathcal{N}_1]$ is called closed if for any arrow $R \twoheadrightarrow P$ in $[\mathcal{N}_1]$ with $R \in \mathcal{S}$, one has that $P \in \mathcal{S}$.*

Corollary 6.5 *The lattice of proper subfunctors of $kR_{\mathbb{Q}}$ is isomorphic to the lattice of closed subsets of the graph $[\mathcal{N}_1]$, ordered by inclusion of subsets.*

Proof : This follows from Theorem 4.7. □

Remark 6.6 This shows in particular that the functor $kR_{\mathbb{Q}}$ is not a noetherian object of $\mathcal{F}_{2,k}$, since the sequence $S_n = \sum_{l=4}^{l=n} H_{SD_{2^l}}$, for $n \geq 4$, is an infinitely increasing sequence of subobjects of $kR_{\mathbb{Q}}$.

Corollary 6.7 *The functor H_{Q_8} is a uniserial object of $\mathcal{F}_{2,k}$. The set of its non-zero subfunctors is*

$$H_{Q_8} \supset H_{Q_{16}} \supset \ldots \supset H_{Q_{2^n}} \supset \ldots \quad .$$

Moreover $H_{Q_{2^n}}/H_{Q_{2^{n+1}}} \cong S_{Q_{2^n},k}$, for $n \geq 3$.

Proof : Indeed, if $Q \in \mathcal{N}_1$ and $Q \preceq Q_8$, then Q is generalized quaternion. Thus $H_{Q_{2^{n+1}}} = J_{Q_{2^n}}$. □

Remark 6.8 This result was Conjecture 2 in [5], since (with notation of [5]) one has that $\varepsilon_{Q_8} = \overline{\Phi}_{Q_8}$.

The following theorem implies part 2 of Theorem 1.3 :

Theorem 6.9 *Suppose that $p = 2$. If $l \in \mathbb{N}$, then*

$$N_l = \begin{cases} kR_{\mathbb{Q}} & \text{if } l = 0 \\ H_{C_4} + H_{SD_{16}} + H_{D_{16}} & \text{if } l = 1 \\ H_{Q_{2^{l+1}}} + H_{C_{2^{l+1}}} + H_{SD_{2^{l+3}}} + H_{D_{2^{l+3}}} & \text{if } l \geq 2 \end{cases}$$

Moreover $\bigcap\limits_{l \in \mathbb{N}} N_l = 0$, and

$$N_l/N_{l+1} \cong \begin{cases} S_{\mathbf{1},k} & \text{if } l = 0 \\ S_{C_4,k} \oplus S_{SD_{16},k} \oplus S_{D_{16},k} & \text{if } l = 1 \\ S_{Q_{2^{l+1}}} \oplus S_{C_{2^{l+1}}} \oplus S_{SD_{2^{l+3}}} \oplus S_{D_{2^{l+3}}} & \text{if } l \geq 2 \end{cases} \quad .$$

Proof : Since any subfunctor of $kR_{\mathbb{Q}}$ is equal to the sum of the subfunctors H_P it contains, it follows from Theorem 6.2 that

$$J_{\mathbf{1}} = H_{C_4} + H_{SD_{16}} + H_{D_{16}} \quad .$$

Moreover if $P \in \mathcal{N}_1$, then $\langle \Phi_P, \Phi_P \rangle_P$ is even by Proposition 5.5. It follows that $\Phi_P \in N_1(P)$, hence that $H_P \subseteq N_1$. Thus $J_{\mathbf{1}} \subseteq N_1$, and $J_{\mathbf{1}} = N_1$ since N_1 is a proper subfunctor of $kR_{\mathbb{Q}}$. In particular, it follows that $N_0/N_1 \cong S_{\mathbf{1},k}$.

Now if $m \in \mathbb{N}$, with $m \geq 2$, and P is one of $Q_{2^{m+1}}$, $C_{2^{m+1}}$, $SD_{2^{m+3}}$ or $D_{2^{m+3}}$, then $\langle \Phi_P, \Phi_P \rangle_P = 2^m$, thus $H_P \subseteq N_l$, for $l \in \mathbb{N}$, if and only if $m \geq l$. Hence

$$\begin{aligned} N_l &= \sum_{m \geq l} H_{Q_{2^{m+1}}} + \sum_{m \geq l} H_{C_{2^{m+1}}} + \sum_{m \geq l} H_{SD_{2^{m+3}}} + \sum_{m \geq l} H_{D_{2^{m+3}}} \\ &= H_{Q_{2^{l+1}}} + H_{C_{2^{l+1}}} + H_{SD_{2^{l+3}}} + H_{D_{2^{l+3}}} \quad , \end{aligned}$$

since moreover for $m \geq l$

$$H_{Q_{2^{m+1}}} \subseteq H_{Q_{2^{l+1}}} \qquad H_{C_{2^{m+1}}} \subseteq H_{C_{2^{l+1}}} \qquad H_{SD_{2^{m+3}}} + H_{D_{2^{m+3}}} \subseteq H_{D_{2^{l+3}}} \quad .$$

Let $I = \bigcap_{l\in\mathbb{N}} N_l$. Then I cannot contain any functor H_P. Thus $I = 0$.

Now for $l \geq 2$ the functor H_P is a proper subfunctor of $H_{Q_{2^{l+1}}}$ if and only if $P \cong Q_{2^m}$, with $m \geq l+2$. Hence $J_{Q_{2^{l+1}}} = H_{Q_{2^{l+2}}}$, whence $J_{Q_{2^{l+1}}} \subseteq N_{l+1}$.

Similarly if $l \geq 1$, the functor H_P is a proper subfunctor of $H_{C_{2^{l+1}}}$ if and only if $P \cong Q_{2^m}$ or $P \cong C_{2^m}$, with $m \geq l+2$. Thus $J_{C_{2^{l+1}}} = H_{Q_{2^{l+2}}} + H_{C_{2^{l+2}}}$, whence $J_{C_{2^{l+1}}} \subseteq N_{l+1}$.

Again the functor H_P is a proper subfunctor of $H_{SD_{2^{l+3}}}$ if and only if $P \cong Q_{2^m}$ or $P \cong C_{2^m}$, with $m \geq l+2$. Thus $J_{SD_{2^{l+3}}} = H_{Q_{2^{l+2}}} + H_{C_{2^{l+2}}}$, and $J_{SD_{2^{l+3}}} \subseteq N_{l+1}$.

Finally H_P is a proper subfunctor of $H_{D_{2^{l+3}}}$ if and only if $P \cong Q_{2^m}$, with $m \geq l+3$, or $P \cong C_{2^m}$, with $m \geq l+2$, or $P \cong SD_{2^m}$, with $m \geq l+4$, or $P \cong D_{2^m}$, with $m \geq l+4$. Hence $J_{D_{2^{l+3}}} = H_{C_{2^{l+2}}} + H_{SD_{2^{l+4}}} + H_{D_{2^{l+4}}}$, and $J_{D_{2^{l+3}}} \subseteq N_{l+1}$.

Let σ_1 denote the canonical epimorphism

$$H_{C_4} \oplus H_{SD_{16}} \oplus H_{D_{16}} \to H_{C_4} + H_{SD_{16}} + H_{D_{16}} = N_1 \quad ,$$

and for $l \geq 2$, and let σ_l denote the canonical epimorphism

$$H_{Q_{2^{l+1}}} \oplus H_{C_{2^{l+1}}} \oplus H_{SD_{2^{l+3}}} \oplus H_{D_{2^{l+3}}} \to N_l \quad ,$$

where $N_l = H_{Q_{2^{l+1}}} + H_{C_{2^{l+1}}} + H_{SD_{2^{l+3}}} + H_{D_{2^{l+3}}}$. The above remarks show that

$$\sigma_1(J_{C_4} \oplus J_{SD_{16}} \oplus J_{D_{16}}) \subseteq N_2$$

and that

$$\sigma_l(J_{Q_{2^{l+1}}} \oplus J_{C_{2^{l+1}}} \oplus J_{SD_{2^{l+3}}} \oplus J_{D_{2^{l+3}}}) \subseteq N_{l+1}$$

for $l \geq 2$. Let $\pi_l : N_l \to N_l/N_{l+1}$ denote the canonical projection. It follows that $\pi_1 \circ \sigma_1$ factors through an epimorphism

$$\bar{\sigma}_1 : S_{C_4,k} \oplus S_{SD_{16},k} \oplus S_{D_{16},k} \to N_1/N_2$$

and that $\pi_l \circ \sigma_l$ factors through an epimorphism

$$\bar{\sigma}_l : S_{Q_{2^{l+1}}} \oplus S_{C_{2^{l+1}}} \oplus S_{SD_{2^{l+3}}} \oplus S_{D_{2^{l+3}}} \to N_l/N_{l+1}$$

for $l \geq 2$.

Let $K_l = \sigma_l^{-1}(N_{l+1})$ denote the kernel of $\pi_l \circ \sigma_l$. Then

$$\bigoplus_{P\in E_l} J_P \subseteq K_l \subseteq \bigoplus_{P\in E_l} H_P \quad ,$$

where

$$E_l = \left\{ \begin{array}{ll} \{C_4, SD_{16}, D_{16}\} & \text{if } l = 1 \\ \{Q_{2^{l+1}}, C_{2^{l+1}}, SD_{2^{l+3}}, D_{2^{l+3}}\} & \text{if } l \geq 2 \end{array} \right. .$$

Fix $P_0 \in E_l$. Then

$$J_{P_0} \subseteq H_{P_0} \cap (K_l + \underset{P \in E_l - \{P_0\}}{\oplus} H_P) \subseteq H_{P_0} \quad .$$

Since J_{P_0} is maximal in H_{P_0} it follows that $H_{P_0} \cap (K_l + \underset{P \in E_l - \{P_0\}}{\oplus} H_P)$ is equal to J_{P_0} or to H_{P_0}. In the latter case $H_{P_0} \subseteq K_l + \oplus_{P \in E_l - \{P_0\}} H_P$, and taking images by σ_l gives

$$H_{P_0} \subseteq \sigma_l(K_l + \underset{P \in E_l - \{P_0\}}{\oplus} H_P) = N_{l+1} + \sum_{P \in E_l - \{P_0\}} H_P \quad .$$

By Theorem 4.4 and Lemma 4.5, it follows that H_{P_0} is contained in N_{l+1}, or in H_P, for some $P \in E_l - \{P_0\}$. Both are impossible, thus

$$H_{P_0} \cap (K_l + \underset{P \in E_l - \{P_0\}}{\oplus} H_P) = J_{P_0} \quad .$$

This holds for any $P_0 \in E_l$, and it follows that

$$K_l = \underset{P \in E_l}{\oplus} J_P \quad .$$

Hence the epimorphisms $\bar{\sigma}_l$ are isomorphisms. □

7 Basic subgroups and genetic sections

Recall the following proposition from [1]

Proposition 7.1 *([1] Proposition 16 page 717) Let P and Q be finite p-groups. Let $B_Q(P)$ be the free $\mathbb{Z}$-module with basis the set of conjugacy classes of pairs (T, S) of subgroups of P with $S \trianglelefteq T$ and $T/S \cong Q$. If (T, S) and (V, U) are such pairs, set*

$$\langle (T, S) \mid (V, U) \rangle_{\mathbb{Z}} = |\{x \in T\backslash P/V \mid (T, S) — ({}^xV, {}^xU)\}|$$

and denote by $\langle (T, S) \mid (V, U) \rangle_k$ the image of $\langle (T, S) \mid (V, U) \rangle_{\mathbb{Z}}$ in k.

Then there is an isomorphism of k-vector spaces

$$S_{Q,k}(P) \cong kB_Q(P)/\mathrm{Rad}\langle \mid \rangle_k \quad ,$$

where $kB_Q(P) = k \otimes B_Q(P)$*, and* $\mathrm{Rad}\langle\ |\ \rangle_k$ *is the radical of the* k*-valued bilinear form* $\langle\ |\ \rangle_k$ *on* $kB_Q(P) = k \otimes B_Q(P)$*. In particular, the dimension of* $S_{Q,k}(P)$ *is equal to the rank of the form* $\langle\ |\ \rangle_k$ *on* $kB_Q(P)$*, and this bilinear form induces a non-degenerate bilinear form on* $S_{Q,k}(P)$*, still denoted by* $\langle\ |\ \rangle_k$*.*

This section describes the relations between basic subgroups and genetic sections, and gives an interpretation of the above proposition in these terms.

Lemma 7.2 *Let* $P \in \mathcal{N}$*. If* Q *is a basic subgroup of* P *such that* $V_Q \cong \Phi_P$*, then*

$$<I_P(Q,Q)> = P \quad , \qquad \bigcap_{x\in P} {}^xQ = \mathbf{1} \quad .$$

Proof : There are two cases :

- If P is cyclic or generalized quaternion, then $Q = \mathbf{1}$, and $I_P(Q,Q)$ is equal to P. Then both equalities are obvious.
- If P is dihedral or semi-dihedral, then Q is non-central of order 2. The second equality is clear. For the first one, observe that in this case

  $$I_P(Q,Q) = N_P(Q) \cup \Big(P - N_P\Big(N_P(Q)\Big)\Big)$$

  (as in the proof of Proposition 5.5), and that this set generates P. □

Definition 7.3 *Let* P *be any finite p-group. If* Q *is a basic subgroup of* P *set*

$$T_Q = <I_P(Q,Q)> \quad , \qquad S_Q = \bigcap_{x\in T_Q} {}^xQ \quad .$$

The basic subgroup Q *of* P *is called an origin (in* P*) if* T_Q/S_Q *has normal p-rank 1. In this case if* V *is a simple* $\mathbb{Q}P$*-module, and if* $V \cong V_Q$*, then* Q *is called an origin of* V *(or* V *is said to have origin* Q*).*

Proposition 7.4 *Let* P *be a finite p-group. Then :*

1. *If* V *is a simple* $\mathbb{Q}P$*-module, let* (T,S) *be a genetic section of* P *for* V*. If* Q/S *is a basic subgroup of* T/S *such that*

   $$Q/S \cap Z(T/S) = \mathbf{1} \quad ,$$

   then Q *is an origin of* V *in* P*, and* $(T_Q,S_Q) = (T,S)$*.*

2. *If Q is an origin in P, then (T_Q, S_Q) is a genetic section of V_Q. Moreover Q/S_Q is a basic subgroup of T_Q/S_Q, and*

$$Q/S_Q \cap Z(T/S_Q) = \mathbf{1} \quad .$$

Proof : Let V be a simple $\mathbb{Q}P$-module. Assertion 1 is trivial if $V \cong \mathbb{Q}$, since (P, P) is the only genetic section of P for V in this case. Hence one can suppose that V is non-trivial. If (T, S) is a genetic section of P for V, then

$$V \cong \mathrm{Ind}_T^P \mathrm{Inf}_{T/S}^T \Phi_{T/S}$$

Let $\overline{Q} = Q/S$ be a basic subgroup of $\overline{T} = T/S$ such that $\overline{Q} \cap Z(\overline{T}) = \mathbf{1}$. This means that the simple $\mathbb{Q}\overline{T}$-module corresponding to $\overline{Q}$ is faithful, hence it is isomorphic to $\Phi_{\overline{T}}$. In other words

$$\Phi_{\overline{T}} \cong \mathrm{Ind}_{\overline{R}}^{\overline{T}} \mathrm{Inf}_{\overline{R}/\overline{Q}}^{\overline{R}} \Omega_{\overline{R}/\overline{Q}} \quad ,$$

where $\overline{R}/\overline{Q}$ is the unique subgroup of order p of $N_{\overline{T}}(\overline{Q})/\overline{Q}$. Then

$$V \cong \mathrm{Ind}_R^P \mathrm{Inf}_{R/Q}^R \Omega_{R/Q} \quad .$$

Since V is irreducible, it follows that Q is a basic subgroup of P, that $R = \tilde{Q}$, and that $V \cong V_Q$. Now by Proposition 2.5

$$\langle V, V \rangle_P = (p-1)|\{x \in Q\backslash P/\tilde{Q} \mid Q^x \cap \tilde{Q} \subseteq Q\}|$$

But similarly, the group T has an irreducible rational module

$$V' = \mathrm{Inf}_{T/S}^T \Phi_{\overline{T}} \cong \mathrm{Ind}_{\tilde{Q}}^T \mathrm{Inf}_{\tilde{Q}/Q}^{\tilde{Q}} \Omega_{\tilde{Q}/Q} \quad ,$$

thus Q is a basic subgroup of T and

$$\langle V', V' \rangle_T = (p-1)|\{x \in Q\backslash T/\tilde{Q} \mid Q^x \cap \tilde{Q} \subseteq Q\}|$$

Since $\langle V, V \rangle_P = \langle \Phi_{\overline{T}}, \Phi_{\overline{T}} \rangle_{\overline{T}} = \langle V', V' \rangle_T$, it follows that

$$|\{x \in Q\backslash P/\tilde{Q} \mid Q^x \cap \tilde{Q} \subseteq Q\}| = |\{x \in Q\backslash T/\tilde{Q} \mid Q^x \cap \tilde{Q} \subseteq Q\}| \quad .$$

In other words $I_P(Q, Q) = I_T(Q, Q)$. Now clearly

$$I_{\overline{T}}(\overline{Q}, \overline{Q}) = \{xS \mid x \in I_P(Q, Q)\} \quad ,$$

and $\overline{T}$ is generated by $I_{\overline{T}}(\overline{Q}, \overline{Q})$ by Lemma 7.2. Hence T is generated by $I_P(Q, Q)$, i.e. $T = T_Q$, and moreover

$$S = \bigcap_{x \in T} {}^xQ \quad ,$$

i.e. $S = S_Q$. Thus Q is an origin of V in P, and $(T_Q, S_Q) = (T, S)$. This proves Assertion 1.

Conversely, let Q be an origin in P. If $Q = P$, then $V_Q = \mathbb{Q}$, and $(T_Q, S_Q) = (P, P)$. Assertion 2 is trivial in this case, and one can suppose that Q is a proper subgroup of P. In particular Q is a proper basic subgroup of P, and $T_Q \supseteq N_P(Q) \supseteq \tilde{Q}$. Thus

$$V_Q \cong \mathrm{Ind}_{T_Q}^{P} \mathrm{Inf}_{T_Q/S_Q}^{T_Q} W_Q \quad ,$$

where

$$W_Q = \mathrm{Ind}_{\tilde{Q}/S_Q}^{T_Q/S_Q} \mathrm{Inf}_{\tilde{Q}/Q}^{\tilde{Q}/S_Q} \Omega_{\tilde{Q}/Q} \quad .$$

Now the intersection of conjugates of Q/S_Q in T_Q/S_Q is equal to $S_Q/S_Q = \mathbf{1}$. This is equivalent to saying that $Q/S_Q \cap Z(T_Q/S_Q) = \mathbf{1}$. Now W_Q is a faithful $\mathbb{Q}(T_Q/S_Q)$-module, and W_Q is irreducible, since V_Q is an irreducible $\mathbb{Q}P$-module. It follows from Proposition 2.2 that $\overline{Q} = Q/S_Q$ is a basic subgroup of $\overline{T} = T_Q/S_Q$, and that $\widetilde{Q/S_Q} = \tilde{Q}/S_Q = \tilde{\overline{Q}}$. Now by Proposition 2.5

$$\langle W_Q, W_Q \rangle_{\overline{T}} = (p-1)|\{xS_Q \in \overline{Q}\backslash \overline{T}/\tilde{\overline{Q}} \mid \overline{Q}^x \cap \tilde{\overline{Q}} \subseteq \overline{Q}\}| \quad .$$

Now the map

$$\theta : \{x \in Q\backslash T_Q/\tilde{Q} \mid Q^x \cap \tilde{Q} \subseteq Q\} \to \{xS_Q \in \overline{Q}\backslash \overline{T}/\tilde{\overline{Q}} \mid \overline{Q}^x \cap \tilde{\overline{Q}} \subseteq \overline{Q}\}$$

defined by $\theta(x) = xS_Q$ is one to one. Hence

$$\langle W_Q, W_Q \rangle_{\overline{T}} = (p-1)|\{x \in Q\backslash T_Q/\tilde{Q} \mid Q^x \cap \tilde{Q} \subseteq Q\}| \quad .$$

But again by Proposition 2.5

$$\langle V_Q, V_Q \rangle_P = (p-1)|\{x \in Q\backslash P/\tilde{Q} \mid Q^x \cap \tilde{Q} \subseteq Q\}| \quad .$$

Now if $x \in P$ and $Q^x \cap \tilde{Q} \subseteq Q$, then $x \in I_P(Q, Q) \subseteq T_Q$. Thus

$$\langle V_Q, V_Q \rangle_P = (p-1)|\{x \in Q\backslash T_Q/\tilde{Q} \mid Q^x \cap \tilde{Q} \subseteq Q\}| = \langle W_Q, W_Q \rangle_{\overline{T}} \quad ,$$

showing that (T_Q, S_Q) is a genetic section of P for V_Q. □

Corollary 7.5 *1. If $V \in Irr_{\mathbb{Q}}(P)$, then there exists an origin for V in P. In other words, each equivalence class of $\mathcal{B}_P$ for $\doteqdot_P$ contains an origin.*

2. If Q and R are origins in P, and if $Q \doteqdot_P R$, then $T_Q/S_Q \cong T_R/S_R$.

Proof : Each assertion follows from the corresponding assertion of Proposition 7.4 : for Assertion 1, by Theorem 3.4, there exists a genetic section (T, S) of P for V, and moreover T/S always has a basic subgroup Q/S intersecting the center $Z(T/S)$ trivially, by Proposition 3.7. For Assertion 2, if Q and R are origins in P, and if $Q \div_P R$, then $V_Q \cong V_R$. Thus (T_Q, S_Q) and (T_R, S_R) are both genetic sections of P for V_Q, and Assertion 2 follows from Corollary 5.9. □

Corollary 7.6 *Let Q be an origin in P.*

1. *If T_Q/S_Q is cyclic or generalized quaternion, then $Q = S_Q$, and*
$$T_Q = N_P(Q) = I_P(Q, Q) \quad .$$
2. *If T_Q/S_Q is dihedral or semi-dihedral, then $N_P(Q)/Q$ has order 2. In other words $N_P(Q) = \tilde{Q}$.*

Proof : Suppose first that T_Q/S_Q is cyclic or generalized quaternion. Then $Q = S_Q$, since Q/S_Q intersects the center of T_Q/S_Q trivially. Now always $N_P(Q) \subseteq I_P(Q, Q) \subseteq T_Q$. Moreover T_Q normalizes $S_Q = Q$, thus $T_Q \subseteq N_P(Q)$.

And if T_Q/S_Q is dihedral or semi-dihedral, then $N_P(Q) = N_{T_Q}(Q)$, and $N_{T_Q}(Q)/Q \cong N_{T_Q/S_Q}(Q/S_Q) \cong C_2$. Hence $N_P(Q)/Q \cong C_2$, or equivalently $N_P(Q) = \tilde{Q}$. □

Remark 7.7 It may happen that a basic subgroup is not an origin : for an example, consider the central product $P = D_{16} * C_4$ of order 32, and the subgroup Q which is the image in P of a non-central subgroup of D_{16} of order 2. Then Q is basic in P, but $T_Q = P$ and $S_Q = \mathbf{1}$, thus $T_Q/S_Q \cong P$ is not in $\mathcal{N}$.

But if R is the subgroup of P generated by ab, where a is an element of order 4 in D_{16} and b is a generator of C_4, one can check that $R \div_P Q$, and that R is an origin in P. It should be noted in that case that the groups $N_P(Q)/Q$ and $N_P(R)/R$ are *not* isomorphic : both are cyclic, but the first one has order 4, and the second one has order 8.

This latter point is general : suppose that V is a non-trivial irreducible $\mathbb{Q}P$-module for some finite p-group P, and that the type of V is cyclic or generalized quaternion. Let Q be a basic subgroup for V of P, and R be an origin of V in P. Then $(T,S) = \big(N_P(R),R\big)$ is a genetic section of P for V, by Corollary 7.6, thus $V \cong V(T,S)$, and in particular

$$\langle V,V\rangle_P = \langle \Phi_{T/S},\Phi_{T/S}\rangle_{T/S} = (1-\frac{1}{p})|T:S| \quad ,$$

by Proposition 5.5. On the other hand, the group $U = N_P(Q)/Q$ is cyclic or generalized quaternion, since Q is basic, and it is easy to see that $V \cong \mathrm{Ind}_{N_P(Q)}^P \mathrm{Inf}_U^{N_P(Q)} \Phi_U$, thus

$$\langle V,V\rangle_P \geq \langle \Phi_U,\Phi_U\rangle_U = (1-\frac{1}{p})|U| \quad ,$$

with equality if and only if the section $\big(N_P(Q),Q\big)$ is a genetic section of P for V. Thus $|N_P(Q):Q| = |U| \leq |T:S| = |N_P(R):R|$, with equality if and only if Q is also an origin of V in P.

Lemma 7.8 *Let $P,Q\in\mathcal{N}$, and let R be a finite p-group. Let U be a finite R-set-P and V be a finite R-set-Q such that $\langle \mathbb{Q}U\otimes_{\mathbb{Q}P}\Phi_P, \mathbb{Q}V\otimes_{\mathbb{Q}Q}\Phi_Q\rangle_R \neq 0$. Then*

$$\langle \mathbb{Q}U\otimes_{\mathbb{Q}P}\Phi_P, \mathbb{Q}V\otimes_{\mathbb{Q}Q}\Phi_Q\rangle_R^2 \geq \langle \Phi_P,\Phi_P\rangle_P\langle \Phi_Q,\Phi_Q\rangle_Q \quad .$$

If equality holds, then $\mathbb{Q}U\otimes_{\mathbb{Q}P}\Phi_P$ and $\mathbb{Q}V\otimes_{\mathbb{Q}Q}\Phi_Q$ have a unique irreducible common direct summand W. Moreover if W is of type S, then $H_S = H_P = H_Q$, thus either P, Q and S are of order at most p, or $P\cong Q\cong S$.

Proof : Let $W \in \mathrm{Irr}_{\mathbb{Q}}(R)$, and set

$$m_W = m(W,\mathbb{Q}U\otimes_{\mathbb{Q}P}\Phi_P) \qquad m'_W = m(\Phi_P,\mathbb{Q}U^{op}\otimes_{\mathbb{Q}R}W)$$

$$n_W = m(W,\mathbb{Q}V\otimes_{\mathbb{Q}Q}\Phi_Q) \qquad n'_W = m(\Phi_P,\mathbb{Q}V^{op}\otimes_{\mathbb{Q}R}W) \quad .$$

Then m_W, m'_W, n_W and n'_W are non-negative integers, and

$$\mathbb{Q}U\otimes_{\mathbb{Q}P}\Phi_P \;\cong\; \bigoplus_{W\in\mathrm{Irr}_{\mathbb{Q}}(R)} m_W W$$

$$\mathbb{Q}V\otimes_{\mathbb{Q}Q}\Phi_Q \;\cong\; \bigoplus_{W\in\mathrm{Irr}_{\mathbb{Q}}(R)} n_W W$$

thus

$$\langle \mathbb{Q}U\otimes_{\mathbb{Q}P}\Phi_P, \mathbb{Q}V\otimes_{\mathbb{Q}Q}\Phi_Q\rangle_R = \sum_{W\in\mathrm{Irr}_{\mathbb{Q}}(R)} m_W n_W\langle W,W\rangle_R \quad . \qquad (\mathbf{7.9})$$

Moreover

$$\begin{aligned} m_W\langle W,W\rangle_R &= \langle W,\mathbb{Q}U\otimes_{\mathbb{Q}P}\Phi_P\rangle_R \\ &= \langle \mathbb{Q}U^{op}\otimes_{\mathbb{Q}R}W,\Phi_P\rangle_P \\ &= m'_W\langle\Phi_P,\Phi_P\rangle_P \quad , \end{aligned}$$

and similarly $n_W\langle W,W\rangle_R = n'_W\langle\Phi_Q,\Phi_Q\rangle_Q$. Hence

$$\langle\mathbb{Q}U\otimes_{\mathbb{Q}P}\Phi_P,\mathbb{Q}V\otimes_{\mathbb{Q}Q}\Phi_Q\rangle_R = \sum_{W\in\mathrm{Irr}_{\mathbb{Q}}(R)} m'_W n'_W \frac{\langle\Phi_P,\Phi_P\rangle_P\langle\Phi_Q,\Phi_Q\rangle_Q}{\langle W,W\rangle_R} \quad . \tag{7.10}$$

Set $\sigma = \langle\mathbb{Q}U\otimes_{\mathbb{Q}P}\Phi_P,\mathbb{Q}V\otimes_{\mathbb{Q}Q}\Phi_Q\rangle_R$. Multiplying equations (7.9) and (7.10) shows that σ^2 is equal to

$$\Big(\sum_{W\in\mathrm{Irr}_{\mathbb{Q}}(R)} m_W n_W\langle W,W\rangle_R\Big)\Big(\sum_{W\in\mathrm{Irr}_{\mathbb{Q}}(R)} m'_W n'_W \frac{\langle\Phi_P,\Phi_P\rangle_P\langle\Phi_Q,\Phi_Q\rangle_Q}{\langle W,W\rangle_R}\Big) \quad ,$$

hence

$$\sigma^2 \geq \sum_{W\in\mathrm{Irr}_{\mathbb{Q}}(R)} m_W n_W m'_W n'_W\langle\Phi_P,\Phi_P\rangle_P\langle\Phi_Q,\Phi_Q\rangle_Q \quad .$$

Now if $\langle\mathbb{Q}U\otimes_{\mathbb{Q}P}\Phi_P,\mathbb{Q}V\otimes_{\mathbb{Q}Q}\Phi_Q\rangle_R \neq 0$, then there exists $W\in\mathrm{Irr}_{\mathbb{Q}}(R)$ such that the integers m_W and n_W are both positive, which happens if and only if m'_W and n'_W are both positive too. It follows that

$$\sigma^2 \geq \langle\Phi_P,\Phi_P\rangle_P\langle\Phi_Q,\Phi_Q\rangle_Q \quad .$$

Moreover equality can hold only if there is a unique W with this property, and in that case moreover $m_W = m'_W = n_W = n'_W = 1$. Thus the coefficient $\gamma\Big(W,kR_{\mathbb{Q}}(U)(\overline{\Phi}_P)\Big)$ of $\overline{W}$ in the expression of $kR_{\mathbb{Q}}(U)(\overline{\Phi}_P)$ as a linear combination of images of irreducibles (see Notation 4.1) is equal to $\overline{m_W}=1$, and if W has type S, then $\overline{\Phi}_S\in H_P(S)$ by Lemma 4.3. Hence $H_S\subseteq H_P$. Similarly $H_S\subseteq H_Q$.

But also $\gamma\Big(\Phi_P,kR_{\mathbb{Q}}(U^{op})(\overline{W})\Big) = \overline{m'_W}=1$, thus $\overline{\Phi}_P\in H_S(P)$, or equivalently $H_P\subseteq H_S$. Similarly $H_Q\subseteq H_S$, and finally $H_P=H_Q=H_S$. The last assertion of the lemma follows from Theorem 5.6. □

Theorem 7.11 *Let P be a finite p-group, and let V and V' be irreducible $\mathbb{Q}P$-modules. If (T,S) is a genetic section of P for V, and (T',S') is a genetic section of P for V', then*

$$\langle (T,S)\mid(T',S')\rangle_{\mathbb{Z}} = \begin{cases} 1 & \text{if } V\cong V' \\ 0 & \text{otherwise}\end{cases} \quad .$$

Proof : Let R (resp. R') denote the factor group T/S (resp. T'/S'). The hypothesis of the theorem means that

$$V \cong \mathrm{Ind}_T^P \mathrm{Inf}_R^T \Phi_R \qquad \langle V, V\rangle_P = \langle \Phi_R, \Phi_R\rangle_R \quad ,$$

and similar conditions for V' and R'. Let U (resp. U') denote the set P/S (resp. the set P/S'), viewed as a P-set-R by left and right multiplication (resp. as a P-set-R'). Then $V \cong \mathbb{Q}U \otimes_{\mathbb{Q}R} \Phi_R$, and $V' \cong \mathbb{Q}U' \otimes_{\mathbb{Q}R'} \Phi_{R'}$. Hence

$$\begin{aligned} \langle V, V'\rangle_P &= \langle \mathbb{Q}U \otimes_{\mathbb{Q}R} \Phi_R, \mathbb{Q}U' \otimes_{\mathbb{Q}R'} \Phi_{R'}\rangle_P \\ &= \langle \Phi_R, \mathbb{Q}(U^{op} \times_P U') \otimes_{\mathbb{Q}R'} \Phi_{R'}\rangle_R \end{aligned}$$

But $U^{op} \times_P U'$ is the set $S\backslash P/S'$, viewed as an R-set-R' (by left and right multiplication). Hence there is an isomorphism of R-sets-R'

$$U^{op} \times_P U' \cong \bigsqcup_{x\in[T\backslash P/T']} S\backslash TxT'/S' \quad ,$$

where $[T\backslash P/T']$ is a set of representatives of double cosets. For each $x \in [T\backslash P/T']$, the biset $S\backslash TxT'/S'$ is transitive, and the stabilizer L_x of SxS' in $R \times R'$ is the set of pairs $(rS, r'S')$ such that $rSxS'r'^{-1} = SxS'$. Since $rSxS'r'^{-1} = Srxr'^{-1}S'$ for $r \in T$ and $r' \in T'$, this gives

$$L_x = \{(rS, r'S') \in R \times R' \mid r \in SxS'r'x^{-1}\} \quad .$$

The biset $S\backslash TxT'/S'$ is isomorphic to $(R \times R')/L_x$, and this can be decomposed using notation and methods of sections 3.1 and 3.3 of [1] : the projection of L_x on R is equal to

$$p_{1,x} = p_1(L_x) = (SxT'x^{-1} \cap T)/S = S({}^xT' \cap T)/S \quad ,$$

and the projection on R of $L_x \cap (R \times \mathbf{1})$ is equal to

$$k_{1,x} = k_1(L_x) = (SxS'x^{-1} \cap T)/S = S({}^xS' \cap T)/S \quad .$$

Similarly

$$p_{2,x} = p_2(L_x) = S'(T' \cap T^x)/S' \qquad k_{2,x} = k_2(L_x) = S'(T' \cap S^x)/S' \quad .$$

It follows that

$$q_x = p_1(L_x)/k_1(L_x) \cong ({}^xT' \cap T)/({}^xS' \cap T)({}^xT' \cap S)$$

and

$$q'_x = p_2(L_x)/k_2(L_x) \cong (T' \cap T^x)/(S' \cap T^x)(T' \cap S^x) \quad .$$

The canonical isomorphism $q_x \to q'_x$ is the obvious isomorphism induced by conjugation by x. The group q_x is a factor group of the subgroup $p_1(L_x)$ of R. It is isomorphic to R if and only if $S({}^xT' \cap T) = T$ and ${}^xS' \cap T \subseteq S$. In particular q_x is isomorphic to both R and R' if and only if $(T,S) \text{---} {}^x(T',S')$.

These remarks show that $\mathbb{Q}(U^{op} \times_P U') \otimes_{\mathbb{Q}R'} \Phi_{R'}$ is isomorphic to

$$\bigoplus_{x \in [T\backslash P/T']} \mathrm{Ind}_{p_{1,x}}^R \mathrm{Inf}_{q_x}^{p_{1,x}} \mathrm{Iso}_{q'_x}^{q_x} \mathrm{Def}_{q'_x}^{p_{2,x}} \mathrm{Res}_{p_{2,x}}^{R'} \Phi_{R'} \quad ,$$

and this gives

$$\langle V, V' \rangle_P = \sum_{x \in [T\backslash P/T']} \langle \mathrm{Def}_{q_x}^{p_{1,x}} \mathrm{Res}_{p_{1,x}}^R \Phi_R, \mathrm{Iso}_{q'_x}^{q_x} \mathrm{Def}_{q'_x}^{p_{2,x}} \mathrm{Res}_{p_{2,x}}^{R'} \Phi_{R'} \rangle_{q_x} \quad . \qquad \mathbf{(7.12)}$$

Moreover the sum in the right hand side is a sum of non-negative integers.

Let $x \in [T\backslash P/T']$ such that $(T,S) \text{---} {}^x(T',S')$. For such an element x one has that

$$p_{2,x} = R' \quad , \qquad q'_x = R' \quad , \qquad q_x = R \quad , \qquad p_{1,x} = R \quad .$$

It follows that $\mathrm{Def}_{q_x}^{p_{1,x}} \mathrm{Res}_{p_{1,x}}^R \Phi_R = \Phi_R$, and that $\mathrm{Iso}_{q'_x}^{q_x} \mathrm{Def}_{q'_x}^{p_{2,x}} \mathrm{Res}_{p_{2,x}}^{R'} \Phi_{R'}$ is the image of $\Phi_{R'}$ by an isomorphism $R \to R'$. Since Φ_R is the unique faithful rational irreducible representation of R, one has that

$$\mathrm{Iso}_{q'_x}^{q_x} \mathrm{Def}_{q'_x}^{p_{2,x}} \mathrm{Res}_{p_{2,x}}^{R'} \Phi_{R'} \cong \Phi_R \quad .$$

It follows that

$$\langle V, V' \rangle_P \geq \langle (T,S) \mid (T',S') \rangle_{\mathbb{Z}} \langle \Phi_R, \Phi_R \rangle_R \geq 0 \quad .$$

Thus if $V \not\cong V'$, one has that $\langle (T,S) \mid (T',S') \rangle_{\mathbb{Z}} = 0$.

Conversely, suppose that $V \cong V'$. Then

$$\langle V, V' \rangle_P = \langle V, V \rangle_P = \langle \Phi_R, \Phi_R \rangle_R$$

since $R = T/S$ and (T,S) is a genetic section of P for V. Hence

$$\langle \Phi_R, \Phi_R \rangle_R \geq \langle (T,S) \mid (T',S') \rangle_{\mathbb{Z}} \langle \Phi_R, \Phi_R \rangle_R \geq 0 \quad .$$

Thus $\langle (T,S) \mid (T',S') \rangle_{\mathbb{Z}} \in \{0,1\}$.

Moreover $R \cong R'$ by Corollary 5.9. Let $\mathcal{S}$ denote the set of elements $x \in [T\backslash P/T']$ for which the corresponding term s_x in the right hand side of equation 7.12 is non-zero. By Lemma 7.8, if $x \in \mathcal{S}$, then

$$s_x^2 \geq \langle \Phi_R, \Phi_R \rangle_R \langle \Phi_{R'}, \Phi_{R'} \rangle_{R'} = \langle \Phi_R, \Phi_R \rangle_R^2$$

Hence $s_x \geq \langle \Phi_R, \Phi_R \rangle_R$, and $\sum_{x \in \mathcal{S}} s_x = \langle \Phi_R, \Phi_R \rangle_R$. Hence $\mathcal{S}$ consists of a single element x. Moreover by Lemma 7.8, the modules $\mathrm{Def}^{p_{1,x}}_{q_x} \mathrm{Res}^R_{p_{1,x}} \Phi_R$ and $\mathrm{Iso}^{q_x}_{q'_x} \mathrm{Def}^{p_{2,x}}_{q'_x} \mathrm{Res}^{R'}_{p_{2,x}} \Phi_{R'}$ have a unique common irreducible direct summand W, which has type Σ such that $H_R = H_{R'} = H_\Sigma$.

If R (and R') have order at least p^2, it follows from Theorem 5.6 that $R \cong R' \cong \Sigma$. Moreover Σ is a section of q_x, which is a section of R. Thus $q_x \cong R \cong R'$ in this case, hence $(T,S) \,\text{—}\, {}^x(T',S')$. Thus $\langle (T,S) \mid (T',S') \rangle_{\mathbb{Z}}$ is non-zero, hence equal to 1.

Now if R (and R') are trivial, then V and V' are isomorphic to the trivial module, thus $(T,S) = (T',S') = (P,P)$, and

$$\langle (T,S) \mid (T',S') \rangle_{\mathbb{Z}} = 1 \quad .$$

The remaining case is when R and R' both have order p. In this case q_x is trivial or has order p. If it has order p, the conclusion is the same as before, as $q_x \cong R \cong R'$. And if q_x is trivial, then $p_{1,x}$ is trivial or equal to R. It cannot be equal to R, for in that case $\mathrm{Def}^{p_{1,x}}_{q_x} \mathrm{Res}^R_{p_{1,x}} \Phi_R$ is a proper deflation of Φ_R, hence zero by Lemma 3.12. Thus $p_{1,x} = \mathbf{1}$, which means that $T \cap {}^xT' \subseteq S$. Corollary 7.6 shows that S is a basic subgroup of P, and that $T = \tilde{S} = N_P(S)$. Now ${}^xT' \cap N_P(S) \leq S$, and the defining property of basic subgroups implies that $|T'| \leq |S|$. But $\dim_{\mathbb{Q}} V = |P:T|(p-1) = |P:T'|(p-1)$. Hence $|T'| = |T| = p|S|$. This contradiction shows that this case cannot occur, and this completes the proof of the theorem. □

Remark 7.13 This theorem shows that if (T,S) and (T',S') are genetic sections for the same simple $\mathbb{Q}P$-module V, then there exists a unique double coset TxT', for $x \in P$, such that $(T,S) \,\text{—}\, {}^x(T',S')$. Such an element x defines an isomorphism

$$T/S \xrightarrow{\cong} (T \cap {}^xT')/(S \cap {}^xS') \xrightarrow{\cong} (T^x \cap T')/(S^x \cap S') \xrightarrow{\cong} T'/S'$$

which is well defined up to an inner automorphism of T/S (or T'/S').

Theorem 7.14 *Let P be a finite p-group. If $L \in \mathcal{N}_1$, let $\mathcal{S}_L$ be the set of origins Q of P such that $T_Q/S_Q \cong L$, and let $[\mathcal{S}_L]$ be a set of representatives of $\mathcal{S}_L$ modulo the relation $\doteqdot_P$. Then*

$$\forall Q, R \in \mathcal{S}_L, \quad \langle (T_Q,S_Q) \mid (T_R,S_R) \rangle_{\mathbb{Z}} = \begin{cases} 1 & \text{if } Q \doteqdot_P R \\ 0 & \text{otherwise} \end{cases} \quad .$$

The images of the elements (T_Q,S_Q), for $Q \in [\mathcal{S}_L]$ form an orthonormal basis of $S_{L,k}(P) = kB_L(P)/\mathrm{Rad}\langle\ |\ \rangle_k$.

Proof : Let Q and R be origins in P. The corresponding simple modules V_Q and V_R are isomorphic if and only if $Q \doteqdot_P R$. Moreover (T_Q, S_Q) is a genetic section of V_Q, by Proposition 7.4. Now the first assertion follows from Theorem 7.11. The images of the pairs (T_Q, S_Q), for $Q \in [\mathcal{S}_L]$, form an orthonormal set in $kB_L(P)$. The cardinality of this set is equal to the number of equivalence classes of origins Q in P for which $T_Q/S_Q \cong L$, modulo the relation $\doteqdot_P$. This number is equal to the number of simple $\mathbb{Q}P$-modules of type L, i.e. the dimension of $S_{L,k}(P)$ by Theorem 5.12, and this is equal to the rank of the form $\langle\ |\ \rangle_k$. This completes the proof of the theorem. □

There is a similar result for the simple module $S_{\mathbf{1},k}$ and the group $B_{\mathbf{1}}(P)$:

Theorem 7.15 *Let P be a finite p-group. Let $\mathcal{S}_{C_p}$ be the set of origins Q of P for which $T_Q/S_Q \cong C_p$, and let $[\mathcal{S}_{C_p}]$ be a set of representatives of $\mathcal{S}_{C_p}$ modulo the relation $\doteqdot_P$. The elements of $B_{\mathbf{1}}(P)$ defined by*

$$u_{\mathbf{1}} = (P,P) \qquad u_Q = (Q,Q) - \Big(N_P(Q), N_P(Q)\Big) \textit{ for } Q \in \mathcal{S}_{C_p}$$

are such that

$$\forall Q, R \in \{\mathbf{1}\} \sqcup \mathcal{S}_{C_p}, \quad \langle u_Q \mid u_R\rangle_{\mathbb{Z}} = \begin{cases} 1 & \textit{if } Q = R = \mathbf{1} \\ p-1 & \textit{if } Q \doteqdot_P R \neq \mathbf{1} \\ 0 & \textit{if } Q \not\doteqdot_P R \end{cases} \quad .$$

The images of the elements u_X, for $X \in \{\mathbf{1}\} \sqcup [\mathcal{S}_{C_p}]$ form an orthogonal basis of $S_{\mathbf{1},k}(P) = kB_{\mathbf{1}}(P)/\mathrm{Rad}\langle\ |\ \rangle_k$.

Proof : It is clear that

$$\forall R \in \{\mathbf{1}\} \sqcup \mathcal{S}_{C_p}, \quad \langle u_{\mathbf{1}} \mid u_R\rangle_{\mathbb{Z}} = \begin{cases} 1 & \text{if } R = \mathbf{1} \\ 0 & \text{if } R \neq \mathbf{1} \end{cases} \quad ,$$

because for any subgroups X and Y of P, one has that $\langle (X,X) \mid (Y,Y)\rangle_{\mathbb{Z}} = |X\backslash P/Y|$. In other words

$$\langle (X,X) \mid (Y,Y)\rangle_{\mathbb{Z}} = \overline{\langle \mathbb{Q}(P/X), \mathbb{Q}(P/Y)\rangle_P} \quad ,$$

where $\bar{n}$ is the image in k of the integer n.

Now let Q and R be in $\mathcal{S}_{C_p}$. Then

$$\langle u_Q \mid u_R\rangle_{\mathbb{Z}} = \langle \mathbb{Q}(P/Q) - \mathbb{Q}(P/\tilde{Q}), \mathbb{Q}(P/R) - \mathbb{Q}(P/\tilde{R})\rangle_P = \langle V_Q, V_R\rangle_P \quad .$$

This is zero if $Q \not\doteqdot_P R$. Moreover if $Q \doteqdot_P R$, then

$$\langle u_Q \mid u_R\rangle_{\mathbb{Z}} = \langle V_Q, V_R\rangle_P = \langle \Phi_{C_p}, \Phi_{C_p}\rangle_{C_p} = p-1 \quad .$$

Since $p-1=-1$ in k, the images of the elements u_X, for $X\in\{\mathbf{1}\}\sqcup[\mathcal{S}_{C_p}]$, form an orthogonal set of vectors of non-zero length in $kB_{C_p}(P)$. The cardinality of this set is equal to 1 plus the number of equivalence classes of origins Q in P for which $T_Q/S_Q\cong C_p$, modulo the relation $\doteqdot_P$. This number is equal to the number of simple $\mathbb{Q}P$-modules of type $\mathbf{1}$ or C_p, i.e. the dimension of $S_{\mathbf{1},k}(P)$ by Theorem 5.12, and this is equal to the rank of the form $\langle\ |\ \rangle_k$. This completes the proof of the theorem. □

Remark 7.16 If Q is a basic subgroup of P, and if V_Q has type C_p, then Q is an origin. Indeed in this case, one has that

$$\langle V_Q,V_Q\rangle_P=p-1=(p-1)\frac{|N_P(Q):Q|}{p}|\mathcal{I}_P(Q,Q)|\quad .$$

This implies that $N_P(Q)/Q$ has order p, and that $I_P(Q,Q)=N_P(Q)$. Thus $T_Q=N_P(Q)$, and $S_Q=Q$. Now T_Q/S_Q is cyclic of order p, and Q is an origin.

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