Correspondence functors

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Abstract: This is a report on some recent joint work with Jacques Thévenaz, which appears in [1] and [2]. It is an expanded version of a talk given at the RIMS workshop *Cohomology of finite groups and related topics*, February 18-20, 2015.

The first part of this joint work is presented in Thévenaz's report, in these proceedings.

1. Introduction

1.1. This is an exposition of a joint work in progress with Jacques Thévenaz[1], on the *representation theory of finite sets*, by which we mean the following: let $\mathcal{C}$ denote the category in which objects are finite sets. For any two finite sets X and Y, the set of morphisms from X to Y in $\mathcal{C}$ is the set of all *correspondences* from X to Y, i.e. the set of subsets of $Y \times X$. We denote[2] this set by $\mathcal{C}(Y,X)$. A correspondence from X to itself is called a *relation* on X. The composition of correspondences is defined as follows: for finite sets X, Y, Z, for $R \subseteq Y \times X$ and $S \subseteq Z \times Y$

$$S \circ R (= SR) = \{(z,x) \in Z \times X \mid \exists y \in Y,\ (z,y) \in S \text{ and } (y,x) \in R\} \ .$$

The identity morphism of the finite set X is the diagonal

$$\Delta_X = \{(x,x) \mid x \in X\} \subseteq X \times X \ .$$

We now fix a commutative ring k (with identity element 1), and we consider functors from $\mathcal{C}$ to the category k-Mod of k-modules. Equivalently, we first introduce the k-linearization $k\mathcal{C}$ of $\mathcal{C}$, i.e. the category with the same objects as $\mathcal{C}$, but in which the set of morphisms from X to Y is the free k-module $k\mathcal{C}(Y,X)$ on the set $\mathcal{C}(Y,X)$, and composition is k-linearly extended from composition in $\mathcal{C}$. Then we consider *correspondence functors* over k, i.e. k-linear functors from $k\mathcal{C}$ to k-Mod. These functors are the objects of a category $\mathcal{F}_k$, in which morphisms are natural transformations of functors. The category $\mathcal{F}_k$ is an abelian k-linear category.

[1] cf. Jacques Thévenaz's report in these Proceedings.

[2] We emphasize that our notation is opposite to the usual notation $\mathcal{C}(X,Y)$ of category theory.

1.2. Examples : For any finite set E, the representable functor $\mathsf{Y}_{E,k}$ sending a finite set X to the set $\mathrm{Hom}_{k\mathcal{C}}(E,X) = k\mathcal{C}(X,E)$ is a *projective* object of $\mathcal{F}_k$, by the Yoneda Lemma. In particular:

- When $E = \emptyset$, then $\mathsf{Y}_{E,k}(X) \cong k$ for any finite set X, and for any correspondence $U \subseteq Y \times X$ from X to a finite set Y, the map $Y_{E,k}(U) : \mathsf{Y}_{E,k}(X) \to \mathsf{Y}_{E,k}(Y)$ is the identity map of k. In other words, the functor $\mathsf{Y}_{\emptyset,k}$ is *the constant functor* equal to k everywhere.
- When $E = \bullet$ is a set of cardinality one, then for any finite set X, the module $\mathsf{Y}_{E,k}(X)$ is the free k-module with basis the set 2^X of subsets of X. Hence $\mathsf{Y}_{\bullet,k}$ is *the functor of subsets.*
- The Yoneda Lemma implies that $\mathrm{End}_{\mathcal{F}_k}(\mathsf{Y}_{E,k})$ is isomorphic to the algebra $k\mathcal{C}(E,E)$ of all relations on E. In particular, when R is a preorder on E, i.e. R is a reflexive and transitive relation on E, or equivalently $\Delta_E \subseteq R = R^2$, then we get a direct summand $\mathsf{Y}_{E,k}R$ of $\mathsf{Y}_{E,k}$ defined on a finite set X by $\mathsf{Y}_{E,k}R(X) = k\mathcal{C}(X,E)R$. The functor $\mathsf{Y}_{E,k}R$ is a projective object of $\mathcal{F}_k$.

2. Functors associated to lattices

2.1. The previous examples are special cases of a more general construction that we now introduce. Recall that a lattice $T = (T, \vee, \wedge)$ is a poset in which any pair $\{x, y\}$ of elements has an least upper bound $x \vee y$ (called the *join* of x and y) and a greatest lower bound $x \wedge y$ (called the *meet* of x and y). A finite lattice T admits a smallest element 0_T (the meet of all elements of T) and a largest element 1_T (the join of all elements of T).

2.2. Definition : *Let T be a finite lattice.*

- *When X is a finite set, let $F_T(X) = k(T^X)$ denote the free k-module with basis the set T^X of all maps from X to T.*
- *When $U \subseteq Y \times X$ is a correspondence from X to a finite set Y, let $F_T(U) : F_T(X) \to F_T(Y)$ be the k-linear map sending $\varphi : X \to T$ to the map $F_T(U)(\varphi) : Y \to T$, also denoted by $U\varphi$, defined by*

$$\forall y \in Y, \;\; (U\varphi)(y) = \bigvee_{(y,x)\in U} \varphi(x) \;\; .$$

Recall that a lattice T is called *distributive* if $\vee$ is distributive with respect to $\wedge$ or, equivalently, if $\wedge$ is distributive with respect to $\vee$.

2.3. Theorem : *Let T be a finite lattice. Then F_T is a correspondence functor. Moreover F_T is projective in $\mathcal{F}_k$ if and only if T is distributive.*

This result motivates the following definition:

2.4. Definition : *Let $k\mathcal{L}$ denote the following category:*

- *The objects of $k\mathcal{L}$ are the finite lattices.*
- *For two finite lattices T and T', the set of morphisms from T to T' in $k\mathcal{L}$ is the free k-module with basis the set of all maps $f : T \to T'$ which respect the join operation, i.e. such that*

$$\forall A \subseteq T, \quad f(\bigvee_{t\in A} t) = \bigvee_{t\in A} f(t) \quad .$$

- *The composition of morphisms in $k\mathcal{L}$ is the k-linear extension of the composition of maps.*

2.5. Remark : Note that a map from a finite lattice T to a finite lattice T' which respects the join operation need not respect the meet operation. On the other hand, it has to send the smallest element 0_T of T (which is equal to the join $\bigvee_{t\in\emptyset} t$) to the smallest element $0_{T'}$ of T'.

2.6. Theorem : *The assignment $T \mapsto F_T$ is a fully faithful k-linear functor from $k\mathcal{L}$ to $\mathcal{F}_k$.*

2.7. We will conclude this section by introducing a canonical subfunctor H_T of F_T, for any finite lattice T, which will be fundamental in the explicit description of simple correspondence functors.

First recall that an element e of a finite lattice T is called *irreducible* if for any subset A of T, the equality $e = \bigvee_{t\in A} t$ implies that $e \in A$. In other words $e \neq 0_T$, and if $e = x \vee y$ for $x, y \in T$, then $e = x$ or $e = y$. We denote by $\mathrm{Irr}(T)$ the set of irreducible elements of T, viewed as a full subposet of T.

2.8. Definition : *Let T be a finite lattice. For a finite set X, let $H_T(X)$ denote the k-submodule of $F_T(X) = k(T^X)$ generated by all maps $\varphi : X \to T$ such that $\varphi(X) \not\supseteq \mathrm{Irr}(T)$.*

2.9. Lemma :

1. *Let Y, X be finite sets, let $U \in \mathcal{C}(Y, X)$, and let $\varphi : X \to T$. Then $(U\varphi)(Y) \cap \mathrm{Irr}(T) \subseteq \varphi(X) \cap \mathrm{Irr}(T)$.*
2. *The assignment $X \mapsto H_T(X)$ is a subfunctor of F_T.*

Proof : Let $U \in \mathcal{C}(Y, X)$, let $\varphi : X \to T$, let $e \in (U\varphi)(Y) \cap \mathrm{Irr}(T)$, and $y \in Y$ such that $e = (U\varphi)(y)$. Then $e = \bigvee_{(y,x)\in U} \varphi(x)$, so there exists x such that $(y, x) \in U$ and $e = \varphi(x)$. Hence $e \in \varphi(X) \cap \mathrm{Irr}(T)$, proving Assertion 1. Assertion 2 follows trivially. □

3. Simple functors

3.1. Let S be a simple object of $\mathcal{F}_k$, that is, a correspondence functor admitting exactly two subfunctors. Then S is non zero, so there is a set E of minimal cardinality such that $S(E) \neq \{0\}$. As explained in Jacques Thévenaz's report in these proceedings, the evaluation $S(E)$ is a simple module for the algebra $\mathcal{E}_E$ of essential relations on E, defined by

$$\mathcal{E}_E = k\mathcal{C}(E, E) / \sum_{|F|<|E|} k\mathcal{C}(E, F)\mathcal{C}(F, E) \ .$$

It follows from [1] that the simple $\mathcal{E}_E$ modules (up to isomorphism) are parametrized by pairs (R, W) of a partial order R on E and a simple $k\mathrm{Aut}(E, R)$-module W (up to permutation of E), where $\mathrm{Aut}(E, R)$ is the automorphism group of the pair (E, R), i.e. the group of permutations of E which preserve R.

Conversely, if E is a finite set, if R is a partial order on E, and if W is a simple $k\mathrm{Aut}(E, R)$-module, then there is a unique simple correspondence functor $S = S_{E,R,W}$ such that E is minimal with $S(E) \neq \{0\}$ and $S(E) \cong W$ as $\mathcal{E}_E$-modules. This gives the following:

3.2. Theorem : *The simple correspondence functors over k (up to isomorphism) are parametrized by triples (E, R, W) consisting of a finite set E, a partial order R on E, and a simple $k\mathrm{Aut}(E, R)$-module W (up to identification of triples (E, R, W) and (E', R', W') for which there exists an isomorphism of posets $\varphi : (E, R) \to (E', R')$ sending W to W').*

3.3. Examples : Assume that k is a field.

- The representable functor $\mathsf{Y}_{\emptyset,k}$ (see 1.2) is simple, projective, and injective in $\mathcal{F}_k$. The corresponding triple is $(\emptyset, tot, k)$, where tot is the unique (order) relation on $\emptyset$, and k is the unique simple module for $k\mathrm{Aut}(\emptyset, tot) \cong k$.
- The representable functor $\mathsf{Y}_{\bullet,k}$ is not simple, but one can show that it is isomorphic to the direct sum of the previous one $\mathsf{Y}_{\emptyset,k}$ and the simple functor $S_{\bullet,tot,k}$, where tot is the unique order relation on the set $\bullet$, and k is the unique simple module for $k\mathrm{Aut}(\bullet, tot) \cong k$. This functor $S_{\bullet,tot,k}$ is also simple, projective ans injective in $\mathcal{F}_k$.

3.4. The two previous examples deal with a total order on a set of cardinality 0 and 1. We now consider the general case of a total order.

For this, we chose a non negative integer n, and we denote by $\underline{n}$ the totally ordered set $\{0, 1, \ldots, n\}$. Then $\underline{n}$ is a lattice, in which $x \vee y = \mathrm{Max}(x, y)$ and $x \wedge y = \mathrm{Min}(x, y)$. We denote by $[n]$ the set $\mathrm{Irr}(T)$. Clearly $[n] = \underline{n} - \{0\} = \{1, 2, \ldots, n\}$.

3.5. Theorem : *For $n \in \mathbb{N}$, set $\mathbb{S}_{[n]} = F_{\underline{n}}/H_{\underline{n}}$. Then:*

1. *The surjection $F_{\underline{n}} \to \mathbb{S}_{[n]}$ splits. The functor $\mathbb{S}_{[n]}$ is projective.*
2. *If X is a finite set, then $\mathbb{S}_{[n]}(X)$ is a free k-module of rank $\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i+1)^{|X|}$.*
3. $F_{\underline{n}} \cong \bigoplus_{A \subseteq [n]} \mathbb{S}_{[|A|]} \cong \bigoplus_{j=0}^{n} \mathbb{S}_{[j]}^{\oplus \binom{n}{j}}$.
4. $End_{k\mathcal{L}}(\underline{n}) \cong End_{\mathcal{F}_k}(F_{\underline{n}}) \cong \prod_{j=0}^{n} M_{\binom{n}{j}}(k)$.
5. *If k is a field, then $\mathbb{S}_{[n]}$ is simple (and projective, and injective), isomorphic to $S_{[n],tot,k}$.*

3.6. In order to deal with the general case of simple functors, we need to introduce some notation. We start with a finite poset (E, R), and we first choose a finite lattice T with the following two properties:

(1) The poset $\mathrm{Irr}(T)$ is isomorphic to (E, R).

(2) The natural restriction map $\mathrm{Aut}(T) \to \mathrm{Aut}(E, R)$ is an isomorphism.

Using Condition (1), we will identify (E, R) with the subposet $\mathrm{Irr}(T)$ of T. In Condition (2), we denote by $\mathrm{Aut}(T)$ the group of automorphisms of the poset T (one can show that this is equal to the group of *bijections* of T which respect the join operation - see Definition 2.4). An automorphism of T clearly maps an irreducible element to an irreducible element, so we have a restriction map $\mathrm{Aut}(T) \to \mathrm{Aut}\big(\mathrm{Irr}(T)\big)$. This map is injective, because any element t of T is equal to the join $\bigvee_{\substack{e\in \mathrm{Irr}(t)\\ e\leq_T t}} e$ of those irreducible elements smaller that t in T, thus any automorphism of T is determined by its restriction to $\mathrm{Irr}(T)$. So Condition (2) above amounts to requiring that any automorphism of the poset (E, R) can be extended to an automorphism of T.

The poset (E, R) being given, it is always possible to choose a finite lattice T with the above two properties, e.g. the lattice $I_\downarrow(E, R)$ consisting of lower ideals of (E, R) (i.e. subsets A of E such that $(x, y) \in R$ and $y \in A$ implies $x \in A$, for any $x, y \in E$), ordered by inclusion of subsets (the join operation on $I_\downarrow(E, R)$ is union of subsets, and the meet operation is intersection of subsets).

3.7. When T is a finite poset, and $t \in T$, we set

$$r(t) = \bigvee_{\substack{x\in T\\ x<_T t}} x \ \ .$$

Thus $r(t) = t$ if $t \notin \mathrm{Irr}(T)$, and if $t \in \mathrm{Irr}(T)$, then $r(t)$ is the largest element of T strictly smaller than t.

When $A \subseteq T$, we denote by $\gamma_A : E \to T$ the map defined by

$$\forall e \in E,\ \gamma_A(e) = \left\{\begin{array}{ll} e & \text{if } e \notin A\\ r(e) & \text{if } e \in A\end{array}\right. \ \ .$$

We define moreover an element γ of $k(T^E)$ by

$$\gamma = \sum_{A\subseteq E} (-1)^{|A|}\gamma_A \ \ ,$$

and we view $k(T^E)$ as the evaluation at E of the functor $F_{T^{op}}$, where T^{op} is the opposite lattice to T (i.e. the lattice obtained by replacing the order relation on T by its opposite, or equivalently, by switching the join and meet operations of T).

Finally we denote by $\mathbb{S}_{E,R}$ the subfunctor of $F_{T^{op}}$ *generated* by the element γ of $F_{T^{op}}(E)$, i.e. the intersection of all subfunctors M of $F_{T^{op}}$ such that $\gamma \in M(E)$.

3.8. Theorem :

1. *The functor* $\mathbb{S}_{E,R}$ *doesn't depend on the choice of* T*, up to isomorphism.*
2. *There exists a positive integer* $f = f_{E,R}$ *(explicitly computable) such that, for any finite set* X*, the* k*-module* $\mathbb{S}_{E,R}(X)$ *is free of rank*
$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}(i+f)^{|X|} \quad .$$
Moreover $\mathbb{S}_{E,R}(X)$ *is a free right* $k\mathrm{Aut}(E,R)$*-module.*
3. *Let* W *be a* $kAut(E,R)$*-module. For a finite set* X*, define*
$$\mathbb{S}_{E,R,W}(X) = \mathbb{S}_{E,R}(X) \otimes_{k\mathrm{Aut}(E,R)} W \quad .$$
Then the assignment $X \mapsto \mathbb{S}_{E,R,W}(X)$ *is a correspondence functor.*
4. *If* k *is a field and* W *is simple, then* $\mathbb{S}_{E,R,W} \cong S_{E,R,W}$*.*

Proof : (Sketch) • First we introduce a non-degenerate functorial bilinear pairing $F_T \times F_{T^{op}} \to k$, in the following way: if X is a finite set, if $\varphi : X \to T$ and $\psi : X \to T^{op}$, we set
$$(\varphi,\psi)_X = \begin{cases} 1 & \text{if } \phi(x) \leq_T \psi(x)\ \forall x \in X, \\ 0 & \text{otherwise.} \end{cases}$$
This pairing is functorial in the sense that for any correspondence $U \subseteq Y \times X$ from X to a finite set Y, for any $\varphi : X \to Y$ and any $\psi : Y \to T^{op}$, we have that
$$(U\varphi,\psi)_Y = (\varphi, U^{op} \star \psi)_X \quad ,$$
where $U^{op} = \{(x,y) \in X \times Y \mid (y,x) \in U\}$ denotes the opposite correspondence, and $U^{op} \star \psi = F_{T^{op}}(U^{op})(\psi) \in F_{T^{op}}(X)$ is the image of ψ under U^{op}.

This pairing is non degenerate in the strong sense that it induces an isomorphism between $F_T(X)$ and the k-dual of $F_{T^{op}}(X)$, for any finite set X (so it induecs an isomorphism between $F_{T^{op}}$ and the *dual functor* $(F_T)^\natural$).

• We show that there exists a surjective homomorphism of correspondence functors
$$\Theta_T : F_T/H_T \to \mathbb{S}_{E,R^{op}} \quad ,$$
where R^{op} is the opposite partial order to R on E.

• We define a subset G of T, containing E, and invariant under $\mathrm{Aut}(E,R)$, with the property that for any finite set X, the image under $\Theta_{T,X} \circ \pi_{T,X}$ of the set

$$\{\varphi : X \to T \mid E \subseteq \varphi(X) \subseteq G\}$$

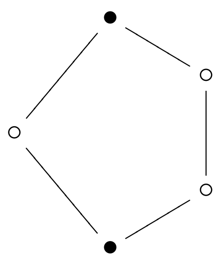
of elements of $F_T(X)$ is a k-basis of $\mathbb{S}_{E,R^{op}}(X)$, where $\pi_T : F_T \to F_T/H_T$ is the quotient morphism. Then the integer $f = f_{E,R}$ appearing in Theorem 3.8 is equal to $|G| - |E|$. □

3.9. Corollary : *Let k be a field. Let (E, R) be a finite poset, and W be a simple $k\mathrm{Aut}(E, R)$-module. Then for any finite set X,*

$$\dim_k S_{E,R,W}(X) = \frac{\dim_k W}{|\mathrm{Aut}(E,R)|} \sum_{i=0}^{|E|} (-1)^{|E|-i} \binom{|E|}{i} (i + f_{E,R})^{|X|} \quad .$$

4. Examples

4.1. Let D denote the following lattice:



where the white dots are the irreducible elements. Then over a field of odd characteristic, the functor F_D is semisimple: its splits as

$$F_D \cong \mathbb{S}_{[0]} \oplus 4\mathbb{S}_{[1]} \oplus 4\mathbb{S}_{[2]} \oplus \mathbb{S}_{[3]} \oplus 2\mathbb{S}_{\bullet\bullet} \oplus \mathbb{S}_{\bullet\,\vdots} \quad ,$$

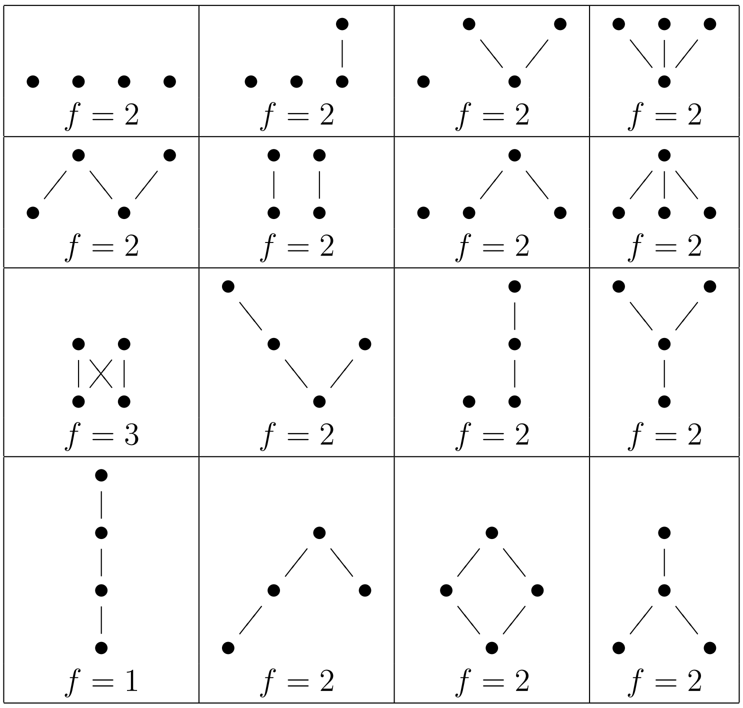
where $\mathbb{S}_{\bullet\bullet}$ denotes the functor $\mathbb{S}_{E,\Delta}$ for a set E of cardinality 2, ordered by the equality relation, and $\mathbb{S}_{\bullet\,\vdots}$ is the functor $\mathbb{S}_{F,R}$ associated to a poset (F, R) of cardinality 3 with 2 connected components.

Observe that for any $i \in \mathbb{N}$, the multiplicity of the functor $\mathbb{S}_{[i]}$ as a summand of F_D is equal to the number of increasing sequences

$$0_D = x_0 < x_1 < \ldots < x_i$$

in D. This statement holds more generally for an abitrary finite lattice T.

4.2. There are 16 posets up to isomorphism on a set of cardinality 4. The following table displays the Hasse diagrams of these posets, together with the corresponding value of the integer f appearing in Theorem 3.8:



The only poset for which $f = 1$ is the total order. This is a general phenomenon: if (E, R) is a finite poset, then $f_{E,R} = 1$ if and only if R is a total order.

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References

[1] S. Bouc and J. Thévenaz. The algebra of essential relations on a finite set. to appear in J. reine angew. Math., 2013.

[2] S. Bouc and J. Thévenaz. The representation theory of finite sets and correspondences. In preparation, 2015.

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