REPRESENTATIONS OF FINITE SETS AND CORRESPONDENCES

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Abstract. This is a report on a long term joint work with Jacques Thévenaz on *Correspondence functors*. It is an expanded version of the mini-course I gave at the workshop of the ICRA2018 conference in Prague in August 2018.

1. Introduction

This report is an expanded version of the mini-course I gave during the workshop of the ICRA2018 conference in Prague in August 2018. It it a survey of the main results obtained so far in this long term joint work with Jacques Thévenaz on *Correspondence functors*, in which we develop the representation theory of finite sets and correspondences.

The representation theory of finite sets has been considered from various points of view. Pirashvili [Pir00] treats the case of pointed sets and maps, while Church, Ellenberg and Farb [CEF15] consider the case where the morphisms are all injective maps. Putman and Sam [PS17] use all k-linear splittable injections between finite-rank free k-modules (where k is a commutative ring). We have chosen instead not to use any kind of maps, but rather all correspondences as morphisms, in order to get a self dual category of finite sets. At first we had no specific application in mind, our main motivation being provided by the fact that finite sets are basic objects in mathematics, and correspondences a natural generalization of maps. Still, it turns out that this functorial approach of sets and correspondences sheds a new light on the representation theory of the algebra of relations on a finite set (Theorem 11.7). The theory has many other surprising results, e.g. the fact that the finitely generated correspondence functors over a field have finite length, and that they are characterized by the exponential behaviour of the dimension of their evaluations (Theorem 7.5).

This survey is organized as follows: in Section 2, we recall some basics on representations of categories. The category of finite sets and correspondences is introduced in Section 3, as well as the algebra of essential relations on a finite set. Sections 4 and 5 describe the simple modules for this algebra, leading to a parametrization of the simple correspondence functors, introduced in Section 6. Sections 7 and 8 are concerned with finiteness properties and stability properties of correspondence functors. In Section 9, correspondence functors associated to finite lattices are introduced, and some of their properties are stated. This leads in Section 10 to the notion of fundamental functor associated to a finite poset. Finally, Section 11 gives a complete description of the simple correspondence functors.

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2. The representation theory of categories

We first recall some standard facts from the representation theory of categories. Let $\mathcal{D}$ be a category and let X and Y be two objects of $\mathcal{D}$. We reverse the usual notation and write $\mathcal{D}(Y,X)$ for the set of all morphisms from X to Y. We assume that $\mathcal{D}$ is skeletally small. This allows us to talk about the *set* of natural transformations between two functors starting from $\mathcal{D}$.

Throughout this paper, k denotes a commutative ring.

2.1. Definition. *The* k-linearization *of a category $\mathcal{D}$, where k is any commutative ring, is defined as follows :*

- *The objects of $k\mathcal{D}$ are the objects of $\mathcal{D}$.*
- *For any two objects X and Y, the set of morphisms from X to Y is the free k-module $k\mathcal{D}(Y,X)$ with basis $\mathcal{D}(Y,X)$.*
- *The composition of morphisms in $k\mathcal{D}$ is the k-bilinear extension*

$$k\mathcal{D}(Z,Y) \times k\mathcal{D}(Y,X) \longrightarrow k\mathcal{D}(Z,X)$$

of the composition in $\mathcal{D}$.

2.2. Definition. *Let $\mathcal{D}$ be a category and k a commutative ring. A* k-representation *of the category $\mathcal{D}$ is a k-linear functor from $k\mathcal{D}$ to the category k-*Mod *of k-modules.*

We could have defined a k-representation of $\mathcal{D}$ as a functor from $\mathcal{D}$ to k-Mod, but it is convenient to linearize first the category $\mathcal{D}$ (just as for group representations, where one can first introduce the group algebra).

If $F : k\mathcal{D} \to k\text{-Mod}$ is a k-representation of $\mathcal{D}$ and if X is an object of $\mathcal{D}$, then $F(X)$ will be called the *evaluation* of F at X. Morphisms in $k\mathcal{D}$ act on the left on the evaluations of F by setting, for every $m \in F(X)$ and for every morphism $\alpha \in k\mathcal{D}(Y,X)$,

$$\alpha \cdot m := F(\alpha)(m) \in F(Y) \; .$$

We often use a dot for this action of morphisms on evaluation of functors.

The category $\mathcal{F}_k(k\mathcal{D}, k\text{-Mod})$ of all k-representations of $\mathcal{D}$ is an abelian category. A sequence of functors

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

is exact if and only if, for every object X, the evaluation sequence

$$0 \longrightarrow F_1(X) \longrightarrow F_2(X) \longrightarrow F_3(X) \longrightarrow 0$$

is exact. Also, a k-representation of $\mathcal{D}$ is called *simple* if it is nonzero and has no proper nonzero subfunctor.

For any object X of $\mathcal{D}$, consider the representable functor $k\mathcal{D}(-,X)$ (which is a projective functor by Yoneda's lemma). Its evaluation at an object Y is the k-module $k\mathcal{D}(Y,X)$, which has a natural structure of a $(k\mathcal{D}(Y,Y), k\mathcal{D}(X,X))$-bimodule by composition.

2.3. Notation. *Let X be an object of $\mathcal{D}$ and let V be a $k\mathcal{D}(X,X)$-module. We define*

$$L_{X,V} := k\mathcal{D}(-,X) \otimes_{k\mathcal{D}(X,X)} V \; .$$

This is a k-representation of $\mathcal{D}$.

This satisfies the following adjunction property.

2.4. Lemma. *Let $\mathcal{F} = \mathcal{F}_k(k\mathcal{D}, k\text{-}\mathrm{Mod})$ be the category of all k-representations of $\mathcal{D}$ and let X be an object of $\mathcal{D}$.*

(a) The functor

$$k\mathcal{D}(X,X)\text{-}\mathrm{Mod} \longrightarrow \mathcal{F}\ , \quad V \mapsto L_{X,V}$$

is left adjoint of the evaluation functor

$$\mathcal{F} \longrightarrow k\mathcal{D}(X,X)\text{-}\mathrm{Mod}\ , \quad F \mapsto F(X)\ .$$

In other words, for any k-representation $F : k\mathcal{D} \to k\text{-}\mathrm{Mod}$ and any $k\mathcal{D}(X,X)$-module V, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{F}}(L_{X,V}, F) \cong \mathrm{Hom}_{k\mathcal{D}(X,X)}\big(V, F(X)\big)\ .$$

Moreover $L_{X,V}(X) \cong V$ as $k\mathcal{D}(X,X)$-modules. In particular, there is a k-algebra isomorphism $\mathrm{End}_{\mathcal{F}}(L_{X,V}) \cong \mathrm{End}_{k\mathcal{D}(X,X)}(V)$.

(b) The functor $k\mathcal{D}(X,X)\text{-}\mathrm{Mod} \longrightarrow \mathcal{F}$ is right exact. It maps projective modules to projective functors, and indecomposable modules to indecomposable functors.

Proof. Part (a) is straightforward and is proved in Section 2 of [Bou96]. Part (b) follows because this functor is left adjoint of an exact functor and satisfies the property $L_{X,V}(X) \cong V$. □

Our next result is a slight extension of the first lemma of [Bou96].

2.5. Lemma. *Let X be an object of $\mathcal{D}$ and let V be a $k\mathcal{D}(X,X)$-module. For any object Y of $\mathcal{D}$, let*

$$J_{X,V}(Y) := \Big\{ \sum_i \phi_i \otimes w_i \in L_{X,V}(Y) \mid \forall \psi \in k\mathcal{D}(X,Y), \sum_i (\psi\phi_i)\cdot w_i = 0 \Big\}\ .$$

(a) $J_{X,V}$ is the largest subfunctor of $L_{X,V}$ which vanishes at X.

(b) If V is a simple module, then $J_{X,V}$ is the unique maximal subfunctor of $L_{X,V}$ and $L_{X,V}/J_{X,V}$ is a simple functor.

Proof. The proof is sketched in Lemma 2.3 of [BST13] in the special case of biset functors for finite groups, but it extends without change to representations of an arbitrary category $\mathcal{D}$. □

Lemma 2.5 is our main tool for dealing with simple functors. We first fix the notation.

2.6. Notation. *Let X be an object of $\mathcal{D}$ and let V be a $k\mathcal{D}(X,X)$-module. We define*

$$S_{X,V} := L_{X,V}/J_{X,V}\ .$$

If V is a simple $k\mathcal{D}(X,X)$-module, then $S_{X,V}$ is a simple functor.

We emphasize that $L_{X,V}$ and $S_{X,V}$ are defined for any $k\mathcal{D}(X,X)$-module V and any commutative ring k. Note that we always have $J_{X,V}(X) = \{0\}$ because if $a = \sum\limits_i \phi_i \otimes w_i \in J_{X,V}(X)$, then $a = \mathrm{id}_X \otimes (\sum\limits_i \phi_i \cdot w_i) = 0$.

Therefore, we have isomorphisms of $k\mathcal{D}(X,X)$-modules

$$L_{X,V}(X) \cong S_{X,V}(X) \cong V\ .$$

2.7. Proposition. *Let S be a simple k-representation of $\mathcal{D}$ and let Y be an object of $\mathcal{D}$ such that $S(Y) \neq 0$.*

(a) $S(Y)$ is a simple $k\mathcal{D}(Y,Y)$-module.
(b) $S \cong S_{Y,S(Y)}$.
(c) S is generated by $S(Y)$, that is, $S(X) = k\mathcal{D}(X,Y)S(Y)$ for all objects X. More precisely, if $0 \neq u \in S(Y)$, then $S(X) = k\mathcal{D}(X,Y) \cdot u$.

Proof. (c) Given $0 \neq u \in S(Y)$, let $S'(X) = k\mathcal{D}(X,Y) \cdot u$ for all objects X. This clearly defines a nonzero subfunctor S' of S, so $S' = S$ by simplicity of S.

(a) This follows from (c).

(b) By the adjunction of Lemma 2.4, the identity $\mathrm{id} : S(Y) \to S(Y)$ corresponds to a non-zero morphism $\theta : L_{Y,S(Y)} \to S$, which must be surjective since S is simple. But $S_{Y,S(Y)}$ is the unique simple quotient of $L_{Y,S(Y)}$, by Lemma 2.5 and Notation 2.6, so $S \cong S_{Y,S(Y)}$. □

2.8. It should be noted that a simple k-representation S has many possible realizations $S \cong S_{Y,V}$ as above, where $V = S(Y) \neq 0$. However, if there is a notion of unique minimal object, then one can parametrize simple functors S by setting $S \cong S_{Y,V}$, where Y is the unique minimal object such that $S(Y) \neq 0$. In this case, the evaluation $V = S(Y)$ is actually a module for the quotient algebra

$$k\widehat{\mathcal{D}}(Y) = k\mathcal{D}(Y,Y)/\sum_{Z} k\mathcal{D}(Y,Z)k\mathcal{D}(Z,Y)$$

where Z runs through objects which are stricly smaller than Y. In the case of the category of finite sets and correspondences, this will be the motivation for considering the *algebra of essential relations* on a finite set (see Definition 3.3).

The following result establishes a link between the simple subquotients of a functor and the simple subquotients of its evaluations:

2.9. Proposition. [[BST13] Proposition 3.5, [BT18b] Proposition 2.8] *Let S be a simple k-representation of $\mathcal{D}$ and let Y be an object of $\mathcal{D}$ such that $S(Y) \neq 0$. Let F be any k-representation of $\mathcal{D}$. Then the following are equivalent:*

(a) S is isomorphic to a subquotient of F.
(b) The simple $k\mathcal{D}(Y,Y)$-module $S(Y)$ is isomorphic to a subquotient of the $k\mathcal{D}(Y,Y)$-module $F(Y)$.

Proof. It is clear that (a) implies (b). Suppose that (b) holds and let W_1, W_2 be submodules of $F(Y)$ such that $W_2 \subset W_1$ and $W_1/W_2 \cong S(Y)$. For $i \in \{1,2\}$, and for any object X of $\mathcal{D}$, set $F_i(X) = k\mathcal{D}(X,Y) \cdot W_i \subseteq F(X)$. Then $F_2 \subseteq F_1$ are subfunctors of F. Moreover $F_i(Y) = W_i$ for $i \in \{1,2\}$, and $(F_1/F_2)(Y) = W_1/W_2 \cong S(Y)$. The isomorphism $S(Y) \to (F_1/F_2)(Y)$ induces, by the adjunction of Lemma 2.4, a nonzero morphism $\theta : L_{Y,S(Y)} \to F_1/F_2$. Since $S(Y)$ is simple, the functor $L_{Y,S(Y)}$ has a unique maximal subfunctor $J_{Y,S(Y)}$, by Lemma 2.5, and $L_{Y,S(Y)}/J_{Y,S(Y)} \cong S_{Y,S(Y)} \cong S$, by Proposition 2.7. Let $F_1' = \theta(L_{Y,S(Y)})$ and $F_2' = \theta(J_{Y,S(Y)})$. Since $\theta \neq 0$, we obtain

$$F_1'/F_2' \cong L_{Y,S(Y)}/J_{Y,S(Y)} \cong S_{Y,S(Y)} \cong S\,,$$

showing that S is isomorphic to a subquotient of F. □

(Actually, as observed by Hida and Yagita in Lemma 3.1 of [HY14], we have an equality $F_1' = F_1$, because both subfunctors are generated by their common evaluation at Y.)

3. Correspondences and relations

3.1. Let X and Y be finite sets. A *correspondence* from X to Y is a subset R of $Y \times X$. In case $X = Y$, we say that R is a *relation* on X.

Correspondences can be composed as follows. If $R \subseteq Y \times X$ and $S \subseteq Z \times Y$, then SR is the correspondence from X to Z defined by

$$SR = \{(z,x) \in Z \times X \mid \exists\, y \in Y \text{ such that } (z,y) \in S \text{ and } (y,x) \in R\}\,.$$

One can check easily that the composition of correspondences is associative. For a finite set X, the *equality relation*

$$\Delta_X = \{(x,x) \mid x \in X\} \subseteq X \times X$$

acts as an identity element for the composition of correspondences. In particular the set of all relations on X is a monoid.

3.2. Definition. *The* category $\mathcal{C}$ of finite sets and correspondences *is defined as follows:*

- *The objects of $\mathcal{C}$ are finite sets.*
- *if X and Y are finite sets, the set of morphisms from X to Y in $\mathcal{C}$ is the set $\mathcal{C}(Y,X)$ of correspondences from X to Y.*
- *The composition of morphisms in $\mathcal{C}$ is the composition of correspondences.*
- *The identity morphism of the finite set X is $\Delta_X \in \mathcal{C}(X,X)$.*

Given a commutative ring k, we will also consider the k-linearization $k\mathcal{C}$ of $\mathcal{C}$, introduced in Section 2, called the k-category of finite sets and correspondences.

When X is a finite set, the endomorphism algebra $\mathcal{R}_X = k\mathcal{C}(X,X)$ of X in the category $k\mathcal{C}$ is called the *algebra of relations* on X. It is the algebra over k of the monoid of relations on X.

3.3. Definition. *A relation R on a finite set X is called* inessential *if there exists a finite set Y with $|Y| < |X|$ and correspondences $U \in \mathcal{C}(X,Y)$ and $V \in \mathcal{C}(Y,X)$ such that $R = UV$. A relation is called* essential *if it is not inessential.*

The subset $\mathcal{I}_X$ of the algebra $\mathcal{R}_X$ consisting of k-linear combinations of inessential relations is a two sided ideal. The quotient algebra $\mathcal{E}_X = \mathcal{R}_X/\mathcal{I}_X$ is called the algebra of essential relations *on X (or the* essential algebra*, for short).*

In other words, an essential relation is a relation of maximal Schein rank, in the sense of Section 1.4 of [Kim82].

The algebra $\mathcal{E}_X$ has a k-basis consisting of essential relations. The product in $\mathcal{E}_X$ of two essential relations R and S on X is equal to RS if RS is essential, and to 0 otherwise.

3.4. Example.

- *Let X be a finite set with $|X| \geq 2$, and A, B be subsets of X. Then the relation $R = A \times B$ on X is inessential: if Y is a set of cardinality 1, then $R = UV$, where $U = A \times Y$ and $V = Y \times B$.*
- *Let $X = \{1,2\}$, and $R = \{(1,1),(1,2),(2,1)\} \in \mathcal{C}(X,X)$. Then one checks easily that R is essential, but $R^2 = X \times X$ is not.*

3.5. Lemma. [[BT16] Lemma 2.1] *Let X be a finite set of cardinality n, and $R \in \mathcal{C}(X,X)$. Then R is inessential if and only if there are subsets U_i and V_i of X, for $1 \leq i \leq n-1$, such that $R = \bigcup\limits_{i=1}^{n-1} (U_i \times V_i)$.*

3.6. Classical definitions When X and Y are finite sets and $R \in \mathcal{C}(Y,X)$, we denote by $R^{op} \in \mathcal{C}(X,Y)$ the *opposite* correspondence, defined by

$$R^{op} = \{(x,y) \in X \times Y \mid (y,x) \in R\}.$$

With this notation, the following classical properties of a relation R on a finite set X can be translated as follows:

$$\begin{aligned}
R \text{ is } \textit{reflexive} &\iff \Delta_X \subseteq R\\
R \text{ is } \textit{transitive} &\iff R^2 \subseteq R\\
R \text{ is } \textit{a preorder} &\iff \Delta_X \subseteq R = R^2\\
R \text{ is } \textit{symmetric} &\iff R = R^{op}\\
R \text{ is } \textit{an equivalence relation} &\iff \Delta_X \subseteq R = R^{op} = R^2\\
R \text{ is } \textit{antisymmetric} &\iff R \cap R^{op} \subseteq \Delta_X\\
R \text{ is } \textit{an order} &\iff R = R^2 \text{ and } R \cap R^{op} = \Delta_X.
\end{aligned}$$

3.7. Remark.

- Note that unless otherwise specified, by *order* we always mean *partial order.*
- It follows from Lemma 3.5 that if R is a preorder on X which is not an order, then R is inessential.
- On the other hand, if R is an order on X and $\Delta_X \subseteq Q \subseteq R$, then Q is essential.

3.8. Let X be a finite set. For an element σ of the group Σ_X of permutations of X, we set

$$\Delta_\sigma = \{\big(\sigma(x), x\big) \mid x \in X\}.$$

We call Δ_σ the *permuted diagonal* associated to σ. One checks easily that the map $\sigma \in \Sigma_X \mapsto \Delta_\sigma \in \mathcal{C}(X,X)$ is a monoid homomorphism, and it follows in particular that Δ_σ is an essential relation on X, for any $\sigma \in \Sigma_X$.

3.9. Theorem. [[BT16] Theorem 3.2] *Let X be a finite set and R be an essential relation on X. Then there exists a permutation σ of X such that $\Delta_\sigma \subseteq R$, i.e. $R = S\Delta_\sigma$, where S is a reflexive relation on X.*

Proof. Let R be an essential relation on X. For any $a \in A$, set

$$R_a = \{x \in X \mid (x,a) \in R\} \text{ and } {}_aR = \{x \in X \mid (a,x) \in R\}.$$

For any subset A of X, define

$$R_A = \{x \in X \mid \exists\, a \in A \text{ such that } (x,a) \in R\} = \bigcup_{a \in A} R_a.$$

Then R decomposes as a union

$$R = \Big(\bigcup_{y \notin A} (R_y \times \{y\})\Big) \bigcup \Big(\bigcup_{x \in R_A} (\{x\} \times {}_xR)\Big).$$

Since R is essential, $\mathrm{Card}(X - A) + \mathrm{Card}(R_A)$ cannot be strictly smaller than $\mathrm{Card}(X)$, by Lemma 3.5. Therefore $\mathrm{Card}(R_A) \geq \mathrm{Card}(A)$, for all subsets A of X, that is

$$\mathrm{Card}\,\big(\bigcup_{a \in A} R_a\big) \geq \mathrm{Card}(A)\,.$$

This is precisely the assumption in a theorem of Philip Hall (see Theorem 5.1.1 in [Hal86], or [Hal35] for the original version which is slightly different). The conclusion is that there exist elements $x_y \in R_y$, where y runs over X, which are all distinct. In other words $\sigma : y \mapsto x_y$ is a permutation and

$$(\sigma(y), y) = (x_y, y) \in R \quad \text{for all } y \in X .$$

This means that R contains Δ_σ, as required. □

4. The algebra of permuted orders

4.1. Let X be a finite set, and S be a reflexive relation on X. Then $\Delta_X \subseteq S$, and for any integer $m \geq 1$

$$\Delta_X \subseteq S \subseteq S^2 \subseteq \ldots \subseteq S^m.$$

Since X is finite, there exists m such that $S^m = S^{m+1}$. Let $\overline{S} = S^m$ denote this limit value. The relation $\overline{S}$ is called the *transitive closure* of S. It is a preorder. There are now two cases, by Remark 3.7:

- either $\overline{S}$ is not an order, and then $\overline{S} = 0$ in $\mathcal{E}_X$.
- or $\overline{S}$ is an order, and then S is essential, since $\Delta_X \subseteq S \subseteq \overline{S}$.

4.2. Theorem. [[BT16] Theorem 5.3] *Let $\mathcal{N}_X$ be the subset of $\mathcal{E}_X$ consisting of k-linear combinations of elements of the form $(S - \overline{S})\Delta_\sigma$, where S is a reflexive relation on X, and $\sigma \in \Sigma_X$.*

(a) $\mathcal{N}_X$ is a two sided nilpotent ideal of $\mathcal{E}_X$.
(b) The quotient algebra $\mathcal{P}_X = \mathcal{E}_X/\mathcal{N}_X$ has a k-basis consisting of elements of the form $S\Delta_\sigma$, where S is an order on X, and $\sigma \in \Sigma_X$.

Proof. (sketch) (a) Let $S \supseteq \Delta_X$, and $m \in \mathbb{N} - \{0\}$ such that $S^m = \overline{S}$.

- Let $Q \supseteq \Delta_X$. Then $Q(S - \overline{S}) = QS - Q\overline{S} = (QS - \overline{QS}) - (Q\overline{S} - \overline{Q\overline{S}})$ since $\overline{QS} = \overline{Q\overline{S}}$. Hence $Q\mathcal{N}_X \subseteq \mathcal{N}_X$, and similarly $\mathcal{N}_X Q \subseteq \mathcal{N}_X$.
- Since S and $\overline{S}$ commute, we have

$$\begin{aligned}(S - \overline{S})^m &= \sum_{i=0}^{m} (-1)^i \binom{m}{i} S^{m-i}\overline{S}^i \\ &= \overline{S} + \sum_{i=1}^{m} (-1)^i \binom{m}{i} \underbrace{S^{m-i}\overline{S}}_{\overline{S}} \\ &= \Big(\sum_{i=0}^{m} (-1)^i \binom{m}{i}\Big)\overline{S} = 0.\end{aligned}$$

Then $\mathcal{N}_X$ is a two sided ideal of the k-algebra $\mathcal{E}_X$, linearly generated by a finite number of nilpotent elements, so it is a nilpotent ideal of $\mathcal{E}_X$ (see Lemma 5.1 of [BT16] for details).

(b) By Theorem 3.9, every essential relation R on X can be written $R = S\Delta_\sigma$, where S is reflexive and $\sigma \in \Sigma_X$. By definition of $\mathcal{N}_X$, we have $S\Delta_\sigma = \overline{S}\Delta_\sigma$ in $\mathcal{P}_X$, and $\overline{S} = 0$ in $\mathcal{E}_X$ if $\overline{S}$ is not an order, by Remark 3.7. Hence $\mathcal{P}_X$ is linearly generated by the elements $S\Delta_\sigma$, where S is an order on X, and one checks easily that these elements are linearly independent. □

Theorem 4.2 is particularly useful to describe the simple $\mathcal{E}_X$-modules: as $\mathcal{N}_X$ acts by zero on every such module, the simple $\mathcal{E}_X$-modules are actually simple $\mathcal{P}_X$-modules.

4.3. A relation R on a finite set X which can be written $R = S\Delta_\sigma$, where S is an order and $\sigma \in \Sigma_X$, is called a *permuted order* on X. One can show that this decomposition of R is unique (in other words, if R is a permuted order, there is a unique $\sigma \in \Sigma_X$ such that $\Delta_\sigma \subseteq R$).

In view of Assertion (b) of Theorem 4.2, the algebra $\mathcal{P}_X$ is called the *algebra of permuted orders* on X. If S and T are orders on X, and if $\sigma, \tau \in \Sigma_X$, the product $S\Delta_\sigma T\Delta_\tau$ in $\mathcal{P}_X$ is equal to $\overline{S.{}^\sigma T}\Delta_{\sigma\tau}$ if $\overline{S.{}^\sigma T}$ is an order, and to 0 otherwise, where ${}^\sigma T = \Delta_\sigma T \Delta_{\sigma^{-1}}$.

In particular $\mathcal{P}_X$ is Σ_X-graded: for $\sigma \in \Sigma_X$, the degree σ part of $\mathcal{P}_X$ is the k-submodule generated by elements $S\Delta_\sigma$, where S is an order on X. The subalgebra $\mathcal{P}_1$ has a k-basis consisting of the set $\mathcal{O}_X$ of orders on X. For $S, T \in \mathcal{O}_X$, the product of ST in $\mathcal{P}_1$ is equal to $\overline{ST} = \overline{S \cup T}$ if this is an order, and to 0 otherwise. In particular $\mathcal{P}_1$ is *commutative*. Moreover, the group Σ_X acts by conjugation on $\mathcal{O}_X$, and the algebra $\mathcal{P}_X$ identifies with the semidirect product $\mathcal{P}_1 \rtimes \Sigma_X$.

If $R \in \mathcal{O}_X$, then $R^2 = R$ in $\mathcal{P}_1$. Moreover if $R, S \in \mathcal{P}_1$, then either $RS = 0$ in $\mathcal{P}_1$, or $RS = \overline{R \cup S}$ if this is an order. We observe that if *we order* $\mathcal{O}_X$ *by inclusion of subsets*, then RS is either 0 or the least upper bound of R and S in the poset $\mathcal{O}_X$. So we have a k-basis $\mathcal{O}_X$ of idempotents of the commutative algebra $\mathcal{P}_1$, which is ordered and contains $1 = \Delta_X$, such that the product of any two of them is either 0 or their supremum in the poset $\mathcal{O}_X$. In such a situation, there is a standard procedure to produce a decomposition of the identity as a sum of orthogonal idempotents.

4.4. Notation. *For $R \in \mathcal{O}_X$, let f_R be the element of $\mathcal{P}_1$ defined by*

$$f_R = \sum_{\substack{S \in \mathcal{O}_X \\ S \supseteq R}} \mu_{\mathcal{O}_X}(R, S) S,$$

where $\mu_{\mathcal{O}_X}$ is the Möbius function of the poset $\mathcal{O}_X$.

4.5. Theorem. [[BT16] Theorem 6.2]

(a) *The elements f_R, for $R \in \mathcal{O}_X$, are orthogonal idempotents of $\mathcal{P}_1$, and $\sum_{R \in \mathcal{O}_X} f_R = 1$.*
(b) *Moreover $\mathcal{P}_1 f_R = k f_R$, for any $R \in \mathcal{O}_X$.*
(c) *The algebra $\mathcal{P}_1$ is isomorphic to the product $\prod_{R \in \mathcal{O}_X} k f_R \cong k^{|\mathcal{O}_X|}$.*

4.6. Notation. *For $R \in \mathcal{O}_X$, let $\Sigma_{X,R}$ denote the stabilizer of R in Σ_X, i.e.*

$$\Sigma_{X,R} = \{\sigma \in \Sigma_X \mid {}^\sigma R = R\}.$$

Let moreover $e_R \in \mathcal{P}_1$ be the orbit sum

$$e_R = \sum_{\sigma \in [\Sigma_X / \Sigma_{X,R}]} f_{{}^\sigma R}.$$

We note that the group $\Sigma_{X,R}$ is isomorphic to the automorphism group $\mathrm{Aut}(X, R)$ of the poset (X, R). We will generally identify those two groups.

4.7. Theorem. [[BT16] Lemma 7.4 and Theorem 7.5]

(a) *The elements e_R, for R in a set $[\Sigma_X \backslash \mathcal{O}_X]$ of representatives of Σ_X-orbits on $\mathcal{O}_X$, are central orthogonal idempotents of $\mathcal{P}_X$, and $\sum_{R \in [\Sigma_X \backslash \mathcal{O}_X]} e_R = 1.$*

(b) The algebra $\mathcal{P}_X$ *is isomorphic to* $\prod_{R\in[\Sigma_X\backslash\mathcal{O}_X]} \mathcal{P}_X e_R$.

(c) For $R \in \mathcal{O}_X$*, the algebra* $\mathcal{P}_X e_R$ *is isomorphic to the full matrix algebra* $M_{|\Sigma_X:\Sigma_{X,R}|}(k\Sigma_{X,R})$.

Proof. (sketch) The proof relies on the fact that the set $\{\Delta_\sigma f_R \mid \sigma \in \Sigma_X,\ R \in \mathcal{O}_X\}$ is a k-basis of $\mathcal{P}_X$, and that the product of two of these basis elements is either zero or another element of the basis (see [BT16] for details). □

5. The simple modules for the essential algebra

5.1. In this section we deal with the simple modules for the essential algebra $\mathcal{E}_X$ and the algebra of permuted orders $\mathcal{P}_X = \mathcal{E}_X/\mathcal{N}_X$ on a finite set X. Theorems 4.2 and 4.7 allow for a description of the simple modules for these algebras.

Recall that Σ_X is the group of permutations of X. The group Σ_X acts on the set $\mathcal{O}_X$ of orders on X. For $R \in \mathcal{O}_X$, we denote by $\Sigma_{X,R}$ the stabilizer of R in Σ_X, and by f_R the idempotent associated to R in $\mathcal{P}_X$ (see Notation 4.6).

5.2. Theorem. [[BT16] Theorem 7.5 and Theorem 8.1] *Let* X *be a finite set.*

(a) The surjection $\mathcal{E}_X \twoheadrightarrow \mathcal{P}_X$ *induces a one to one correspondence between the simple* $\mathcal{E}_X$*-modules and the simple* $\mathcal{P}_X$*-modules.*

(b) Let $R \in \mathcal{O}_X$*. Then* $\mathcal{P}_X f_R$ *has a k-basis* $\{\Delta_\sigma f_R \mid \sigma \in \Sigma_X\}$*, so* $\mathcal{P}_X f_R \cong k\Sigma_X$ *as a k-module. It is an* $(\mathcal{R}_X, k\Sigma_{X,R})$*-bimodule, free as a right* $k\Sigma_{X,R}$*-module.*

(c) The simple $\mathcal{P}_X$*-modules (up to isomorphism) are the modules of the form* $\mathcal{P}_X f_R \otimes_{k\Sigma_{X,R}} W$*, where* W *is a simple* $k\Sigma_{R,X}$*-module (up to isomorphism).*

(d) If k is a field with $\mathrm{char}(k) = 0$ *or* $\mathrm{char}(k) > |X|$*, then* $\mathcal{P}_X$ *is semisimple, and* $\mathcal{N}_X$ *is equal to the Jacobson radical of* $\mathcal{E}_X$.

Proof. (a) follows from Theorem 4.2. Now (b) and (c) follow from Theorem 4.7, and then (d) follows from the semisimplicity of all the group algebras $k\Sigma_{X,R}$, when $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > |X|$. □

As the essential algebra $\mathcal{E}_X$ is a quotient of the algebra $\mathcal{R}_X = k\mathcal{C}(X, X)$, we now get a precise description of some simple $\mathcal{R}_X$-modules.

5.3. Proposition. [[BT16] Theorem 8.1 and Proposition 8.5] *Let* R *be an order on* X*. If* $S \in \mathcal{R}_X$*, define a k-endomorphism* $\beta_R(S)$ *of* $k\Sigma_X$ *by*

$$\beta_R(S) : \sigma \in \Sigma_X \mapsto \begin{cases} \tau\sigma & \text{if } \exists \tau \in \Sigma_X,\ \Delta \subseteq \Delta_{\tau^{-1}} S \subseteq {}^{\sigma}R \\ 0 & \text{if there is no such } \tau. \end{cases}$$

(a) The map $\beta_R(S)$ *is well defined, and* $\beta_R(S) \in \mathrm{End}_{k\Sigma_{X,R}}(k\Sigma_X)$.

(b) The map $S \mapsto \beta_R(S)$ *extends to an algebra homomorphism from* $\mathcal{R}_X$ *to* $\mathrm{End}_{k\Sigma_{X,R}}(k\Sigma_X)$*, which makes* $k\Sigma_X$ *a* $(\mathcal{R}_X, k\Sigma_{X,R})$*-bimodule, isomorphic to* $\mathcal{P}_X f_R$.

(c) For a $k\Sigma_{X,R}$*-module* W*, let* $\Lambda_{R,W}$ *denote the* $\mathcal{R}_X$*-module* $\mathcal{P}_X f_R \otimes_{k\Sigma_{X,R}} W$*. If* W *is a simple* $k\Sigma_{X,R}$*-module, then* $\Lambda_{R,W}$ *is a simple* $\mathcal{R}_X$*-module.*

(d) If (R', W') *is another pair consisting of an order* R' *on* X *and a simple* $k\Sigma_{X,R'}$*-module* W'*, then the* $\mathcal{R}_X$*-modules* $\Lambda_{R,W}$ *and* $\Lambda_{R',W'}$ *are isomorphic if and only if the pairs* (R, W) *and* (R', W') *are conjugate by* Σ_X.

Proof. (sketch) (a) and (b) follow from a precise description of the action of $\mathcal{R}_X$ on $\mathcal{P}_X f_R$, using the basis $\{\Delta_\sigma f_R \mid \sigma \in \Sigma_X\}$ of $\mathcal{P}_X f_R$. Then (*c*) and (*d*) follow from Theorem 4.7. □

5.4. Remark. If G is a finite group, a simple kG-module is actually a $(k/m)G$-module, where m is some maximal ideal of k. It follows that when dealing with simple correspondence functors over k, one could always suppose that k is a field.

5.5. Examples. Let k be a field.

- If $R = \Delta_X$, then $\Sigma_{X,R} = \Sigma_X$, and $\mathcal{R}_X$ maps surjectively to $k\Sigma_X$, by $S \mapsto \sigma$ if $S = \Delta_\sigma$ for $\sigma \in \Sigma_X$, and $S \mapsto 0$ is there is no such $\sigma \in \Sigma_X$.
- If R is a total order, then $\Sigma_{X,R} = \{1\}$, and $\mathcal{P}_X e_R \cong M_{n!}(k)$ (where $n = |X|$). In this case $k\Sigma_X$ becomes a simple $\mathcal{R}_X$-module.

5.6. Remark. Recall that for an order R on a finite set X, the group $\Sigma_{X,R}$ is isomorphic to the group $\mathrm{Aut}(X, R)$ of automorphisms of the poset (X, R). One may ask which finite groups can occur as automorphism group of some finite poset. The answer is quite simple: every finite group can. This was first proved by Birkhoff ([Bir46]), and later simplified by Thornton ([Tho72]) and Barmak-Minian ([BM09])

6. Correspondence functors

6.1. Definition. *Let k be a commutative ring. A* correspondence functor *over k is a k-representation of the category $k\mathcal{C}$ introduced in Definition 3.2. The category of correspondence functors $\mathcal{F}_k(k\mathcal{C}, k\text{-}\mathrm{Mod})$ is denoted by $\mathcal{F}_k$.*

6.2. Remarks. The category $k\mathcal{C}$ has the following special important properties:

- The functor sending a finite set to itself and a correspondence to its opposite (introduced in 3.6) induces a k-linear equivalence from $k\mathcal{C}$ to the opposite category $k\mathcal{C}^{op}$.
- ([[BT18b] Lemma 3.4] Let X and Y be finite sets such that $|X| \leq |Y|$. Let $i : X \hookrightarrow Y$ be an injective map, and set
$$i_* = \big\{\big(i(x), x\big) \mid x \in X\big\} \in \mathcal{C}(Y, X),$$
$$i^* = \big\{\big(x, i(x)\big) \mid x \in X\big\} \in \mathcal{C}(X, Y).$$
Then $i^* i_* = \Delta_X$. It follows that if $F \in \mathcal{F}_k$, then $F(i^*)F(i_*) = \mathrm{id}_{F(X)}$, so $F(X)$ is isomorphic to a direct summand of $F(Y)$. In particular $F(X) \neq 0$ implies $F(Y) \neq 0$.

6.3. Examples.

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X, E)$, where E is a fixed finite set, e.g.
  - * $E = \emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X, \emptyset) \cong k$, for any finite set X. The functor $\mathsf{Y}_{\emptyset,k}$ is the constant functor with value k.
  - * $E = \{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X) = k\mathcal{C}(X, \bullet) \cong k(2^X)$, for any finite set X.

  Since $\mathrm{Hom}_{\mathcal{F}_k}(Y_{E,k}, M) \cong M(E)$ by the Yoneda Lemma, for any $M \in \mathcal{F}_k$, the functor $\mathsf{Y}_{E,k}$ is a *projective* object of $\mathcal{F}_k$, for any E.
- In particular $\mathrm{End}_{\mathcal{F}_k}(\mathsf{Y}_{E,k}) \cong k\mathcal{C}(E, E)$. Let R be a *preorder* on E, i.e. $R \in \mathcal{C}(E, E)$ such that $\Delta_E \subseteq R = R^2$. Then $\mathsf{Y}_{E,k}R : X \mapsto k\mathcal{C}(X, E)R$ is a projective object of $\mathcal{F}_k$.

6.4. Lemma. [[BT18b] Corollary 3.5 and Corollary 3.6] *Let X and Y be finite sets with $|X| \leq |Y|$.*

(a) The representable functor $k\mathcal{C}(-,X)$ is isomorphic to a direct summand of the representable functor $k\mathcal{C}(-,Y)$.
(b) The left $k\mathcal{C}(Y,Y)$-module $k\mathcal{C}(Y,X)$ is projective.

Proof. (a) In the category $k\mathcal{C}$, the object X is a retract of Y, by Remark 6.2.
(b) In particular $k\mathcal{C}(Y,X)$ is isomorphic to a direct summand of the free module $k\mathcal{C}(Y,Y)$, so it is projective. □

6.5. Definition. *For $F \in \mathcal{F}_k$, the dual $F^\natural$ of F is defined by $F^\natural(X) = \mathrm{Hom}_k\big(F(X),k\big)$ for a finite set X, and by $F^\natural(S) = {}^tF(S^{op})$ for a correspondence S from X to a finite set Y.*

6.6. The results of Sections 2 and 5 yield a parametrization of simple correspondence functors, as well as a rough description of their evaluations, which will be enough for stating finiteness properties of correspondence functors in the next section. We will give a more complete description of simple correspondence functors in Section 11.

Let S be a simple correspondence functor (i.e. a simple object of $\mathcal{F}_k$). If E is a set of minimal cardinality such that $V = S(E) \neq 0$, then by Proposition 2.7 and the comments in 2.8, the functor S is isomorphic to $S_{E,V}$, and V is a simple module for the essential algebra $\mathcal{E}_E$. So there exists a unique pair (R,W) consisting of an order R on E and a simple $k\,\mathrm{Aut}(E,R)$-module W such that V is isomorphic to the module $\Lambda_{R,W} = \mathcal{P}_E f_R \otimes_{k\,\mathrm{Aut}(E,R)} W$ of Proposition 5.3.

Conversely, if (E,R,W) is a triple consisting of a finite set E, an order R on E, and a simple $k\,\mathrm{Aut}(E,R)$-module W, we denote by $S_{E,R,W}$ the simple correspondence functor $S_{E,\Lambda_{R,W}}$. The assignment $(E,R,W) \mapsto S_{E,R,W}$ is a parametrization of the simple correspondence functors over k.

6.7. Theorem. [[BT18b] Theorem 4.7] *Assume that k is a field. The set of isomorphism classes of simple correspondence functors over k is parametrized by the set of isomorphism classes of triples (E,R,W), where E is a finite set, R is an order on E, and W is a simple $k\,\mathrm{Aut}(E,R)$-module.*

6.8. Examples. Let k be a field.

- The constant functor $Y_{\emptyset,k}$ is simple, isomorphic to $S_{\emptyset,tot,k}$: here *tot* denotes the unique (total) order on the empty set, and k is the trivial module for the trivial group $\mathrm{Aut}(\emptyset,tot)$.
- The Yoneda functor $Y_{\bullet,k}$ is not simple, but splits as a direct sum of the previous functor $S_{\emptyset,tot,k}$ and the simple functor $S_{\bullet,tot,k}$, where *tot* is the unique (total) order on a set $\bullet$ of cardinality 1, and k is the trivial module for the trivial group $\mathrm{Aut}(\bullet,tot)$.

As the Yoneda functors are projective, the simple functors $S_{\emptyset,tot,k}$ and $S_{\bullet,tot,k}$ are also projective. These are two special cases of Theorem 11.2.

The following result is the first step in the description of simple correspondence functors. It gives lower and upper estimates on the dimension of their evaluations.

6.9. Theorem. [[BT18b] Theorem 8.2] *Suppose that k is a field and let $S_{E,R,W}$ be a simple correspondence functor, where E is a finite set, R is an order on E, and W is a simple $k\,\mathrm{Aut}(E,R)$-module. There exists a positive integer N and a* *positive real number c such that, for any finite set X of cardinality at least N, we have*

$$c|E|^{|X|} \leq \dim\big(S_{E,R,W}(X)\big) \leq \big(2^{|E|}\big)^{|X|}\ .$$

Proof. (sketch) The main tool is the fact that the simple functor $S_{E,R,W}$ is a quotient of the representable functor $k\mathcal{C}(-,E)$. This readily gives the upper bound, since $\dim_k k\mathcal{C}(X,E) = 2^{|E||X|}$ for any finite set X.

The proof of the lower bound is harder. Let A be a set of representatives of the action of the group Σ_E on the set of surjections $\varphi : X \to E$. For $\varphi \in A$, set

$$\Lambda_\varphi = \{(x,e) \in X \times E \mid \big(\varphi(x), e\big) \in R\} \in \mathcal{C}(X,E).$$

Then one can show that the images under the surjection $k\mathcal{C}(X,E) \to S_{E,R,W}(X)$ of the elements Λ_φ, for $\varphi \in A$, are linearly independent. The lower bound now follows from an estimate of the cardinality of A (see [BT18b] for details). □

7. Finiteness properties

The notion of *finitely generated module* over an algebra splits in two different notions in the realm of correspondence functors. Recall that k is a commutative ring, and that $\mathcal{F}_k$ denotes the category of correspondence functors over k.

7.1. Definition. *Let $M \in \mathcal{F}_k$.*

- *(a) Let $(E_i)_{i\in I}$ be a family of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is* generated by $(m_i)_{i\in I}$, *and write $M = \langle m_i\rangle_{i\in I}$, if for any finite set X and any $m \in M(X)$, there exists a finite subfamily $J \subseteq I$ and elements $\alpha_j \in k\mathcal{C}(X,E_j)$, for $j \in J$, such that $m = \sum\limits_{j\in J} M(\alpha_j)(m_j)$.*
- *(b) M is* finitely generated *if $M = \langle m_i\rangle_{i\in I}$, where I is finite.*
- *(c) M has* bounded type *if there is a finite set E such that $M = \langle M(E)\rangle$.*

7.2. Examples.

- The Yoneda functor $Y_{E,k}$ is finitely generated (by the single element Δ_E of $Y_{E,k}(E) = k\mathcal{C}(E,E)$), for any finite set E.
- When E is a finite set, and V a $k\mathcal{C}(E,E)$-module, the functor $L_{E,V}$ (see Notation 2.3) has bounded type, generated by $L_{E,V}(E) \cong V$. It is finitely generated if and only if V is finitely generated.

7.3. Proposition. [[BT18b] Proposition 6.4] *Let k be a commutative ring, and M be a correspondence functor over k. The following are equivalent:*

- *(a) M is finitely generated.*
- *(b) M is isomorphic to a quotient of a finite direct sum $\mathop{\oplus}\limits_{i=1}^{n} Y_{E_i,k}$.*
- *(c) M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*
- *(d) There exists a finite set E and a finite subset B of $M(E)$ such that M is generated by B. In particular M has bounded type.*

Proof. (a)⇒(b). If M is generated by the set $\{m_i \mid i = 1,\ldots,n\}$, where $m_i \in M(E_i)$, then each m_i yields a morphism $Y_{E_i,k} \to M$, by Yoneda's lemma. The direct sum of these morphisms is surjective.

(b)⇒(c). If M is a quotient of a finite direct sum $\mathop{\oplus}\limits_{i=1}^{n} Y_{E_i,k}$. If E is the largest of the sets E_i, then each $Y_{E_i,k}$ is a direct summand, hence a quotient, of $Y_{E,k}$, so M is a quotient of $(Y_{E,k})^{\oplus n}$.

(c)⇒(d). Let M be a quotient of $\big(k\mathcal{C}(-,E)\big)^{\oplus n}$. Since $k\mathcal{C}(-,E)$ is generated by $\Delta_E \in k\mathcal{C}(E,E)$, the functor M is generated by the images in $M(E)$ of the elements Δ_E of each of the n components of the direct sum.

(d)⇒(a). This is obvious. □

7.4. Lemma. [[BT18b] Lemma 9.1] *Let k be a field and let M be a finitely generated correspondence functor over k. Then M has a maximal subfunctor.*

Proof. Since M is finitely generated, it is generated by $M(E)$ for some finite set E, by Proposition 7.3. Moreover $M(E)$ is a finitely generated $\mathcal{R}_E$-module. Let N be a maximal submodule of $M(E)$ as an $\mathcal{R}_E$-module. Then $M(E)/N$ is a simple $\mathcal{R}_E$-module. By Proposition 2.9, there exist two subfunctors $F \subseteq G \subseteq M$ such that G/F is simple, $G(E) = M(E)$, and $F(E) = N$. Since M is generated by $M(E)$ and $G(E) = M(E)$, we have $G = M$. Therefore, F is a maximal subfunctor of M. □

The following result gives a characterization of the finitely generated correspondence functors over a field. It shows that a correspondence functor over a field is finitely generated if and only if it has finite length (i.e. it admits a finite composition series). It should be noted that this equivalence is false in general for other categories of functors (e.g. biset functors).

7.5. Theorem. [[BT18b] Theorem 8.7 and Theorem 9.2] *Let k be a field, and M be a correspondence functor over k. The following conditions are equivalent:*

- *(a) M is finitely generated.*
- *(b) there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
- *(c) M has finite length.*

Proof. (sketch) (a)⇒(b). If M is finitely generated, it is a quotient of a finite direct sum $\big(k\mathcal{C}(-,E)\big)^{\oplus n}$, by Proposition 7.3. Now $\dim_k M(X) \leq n \dim_k k\mathcal{C}(X,E) = n2^{|X||E|}$, for any finite set X, so (b) holds.

The proof that (b) implies (a) is harder. We first use Theorem 6.9 to show that if $S_{E,R,W}$ is a simple subquotient of M, then the cardinality of E is bounded above by the number b appearing in (b). Then one builds a morphism π from a finite direct sum L of representable functors to M, which is surjective on evaluations at any set X of cardinality at most b. If now F is a finite set of minimal cardinality such that $\mathrm{Coker}\,\pi(F)$ is non-zero, then $|F| > b$. Moreover $\mathrm{Coker}\,\pi(F)$ is a non-zero module for the essential algebra $\mathcal{E}_F$, hence it admits a simple submodule V. By Proposition 2.9, it follows that $\mathrm{Coker}\,\pi$, hence M, admits a simple subquotient isomorphic to $S_{F,V}$. But then $|F| \leq b$, a contradiction proving that π is surjective, and that (b) implies (a). So (a) and (b) are equivalent, and in particular, any subfunctor of a finitely generated correspondence functor over k is finitely generated.

(c)⇒(b). This follows from Theorem 6.9.

(b)⇒(c). By Lemma 7.4, we can build a filtration $M = M_0 \supset M_1 \supset M_2 \ldots$ in which every subquotient M_i/M_{i+1} is simple, isomorphic to S_{E_i,R_i,W_i} for some triple (E_i, R_i, W_i). Now (b) implies by Theorem 6.9 that $|E_i| \leq b$ for any i. In particular, there is only a finite number of simple quotients S_{E_i,R_i,W_i} in the filtration, up to isomorphism. But since the dimension of $M(X)$ is finite for any finite set X, each of the simple quotients S_{E_i,R_i,W_i} can only occur a finite number of times in the filtration. This shows that the filtration is finite, so M has finite length. □

The equivalence of (a) and (c) in Theorem 7.5 has been proved independently by Gitlin [Git18] (for an infinite field k), using results of Wiltshire-Gordon on categories of dimension 0 [WG19].

The following is a consequence of the adjunction of Lemma 2.4.

7.6. Proposition. [[BT18b] Lemma 7.3] *Let M be a correspondence functor over k, and E be a finite set.*

(a) If M is projective in $\mathcal{F}_k$, and M is generated by $M(E)$, then $M \cong L_{F,M(F)}$ for any finite set F with $|F| \geq |E|$, and $M(F)$ is a projective $\mathcal{R}_F$-module.
(b) The functor $L_{E,V}$ is projective (resp. indecomposable) if and only if V is a projective (resp. indecomposable) $\mathcal{R}_E$-module.

We introduce next the notion of *residue* of a correspondence functor at a finite set, which is an important tool to study finiteness conditions.

7.7. Definition. *Let M be a correspondence functor over k, and E be a finite set. The* residue *of M at E is the k-module defined by*

$$\overline{M}(E) = M(E)/\sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

7.8. Theorem. [[BT18b] Theorem 11.4 and Corollary 11.5] *Let k be a commutative noetherian ring. Let M be a subfunctor of a correspondence functor L over k, and let E and F be finite sets.*

(a) If L is generated by $L(F)$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.
(b) If L is generated by $L(F)$ and $|E| \geq 2^{|F|}$, then M is generated by $M(E)$.
(c) If L has bounded type, then M has bounded type.
(d) If L is finitely generated, then M is finitely generated.

Proof. (sketch) (a) is proved by localization and Artin-Rees Lemma (Theorem 8.5 in [Mat89]). Then (b), (c), and (d) follow easily. □

7.9. Corollary. *Functors of bounded type form an abelian subcategory $\mathcal{F}_k^b$ of $\mathcal{F}_k$. Finitely generated functors form an abelian subcategory $\mathcal{F}_k^f$ of $\mathcal{F}_k^b$.*

7.10. Proposition. [[BT18b] Proposition 6.6] *Let k be a noetherian ring.*

(a) For any a finitely generated correspondence functor M over k, the algebra $\mathrm{End}_{\mathcal{F}_k}(M)$ is a finitely generated k-module.
(b) For any two finitely generated correspondence functors M and N over k, the k-module $\mathrm{Hom}_{\mathcal{F}_k}(M,N)$ is finitely generated.
(c) If k is a field, then the Krull-Remak-Schmidt theorem holds for finitely generated correspondence functors over k.

Proof. (a) M is a quotient of a projective functor $\overset{n}{\underset{i=1}{\oplus}} k\mathcal{C}(-,E)$, so $\mathrm{End}_{\mathcal{F}_k}(M)$ is a quotient of a k-submodule of the finitely generated k-module $M_n\big(k\mathcal{C}(E,E)\big)$.
(b) $\mathrm{Hom}_{\mathcal{F}_k}(M,N)$ is a direct summand of $\mathrm{End}_{\mathcal{F}_k}(M \oplus N)$.
(c) Splitting $M \in \mathcal{F}_k$ amounts to splitting the identity as a sum of orthogonal idempotents in the finite dimensional k-algebra $\mathrm{End}_{\mathcal{F}_k}(M)$. □

The following three results show that when k is a field, the category $\mathcal{F}_k^f$ shares many properties of the category of modules over a finite group algebra.

7.11. Theorem. [[BT18b] Theorem 10.2 and Corollary 10.3] *Let E be a finite set.*

(a) The representable functor $Y_{E,k} = k\mathcal{C}(-, E)$ is isomorphic to its dual.
(b) Let $\mathcal{R}_E = k\mathcal{C}(E, E)$ be the algebra of relations on E. Then $\mathcal{R}_E$ is a symmetric k-algebra, in the sense of [Bro09]. *More precisely, let $t_E : \mathcal{R}_E \to k$ be defined by*

$$\forall R \in \mathcal{C}(E, E),\ t_E(R) = \begin{cases} 1 & \textit{if } R \cap \Delta_E = \emptyset, \\ 0 & \textit{otherwise.} \end{cases}$$

Then t_E is a symmetrizing form for $\mathcal{R}_E$.
(c) If k is a field (or more generally if k is self injective), any finitely generated projective correspondence functor over k is also injective. In particular $\mathcal{F}_k^f$ has infinite global dimension.

Proof. (b) Let $R, S \in \mathcal{C}(E, E)$. Then $t_E(RS) = 1 \Leftrightarrow R \subseteq (E\times E) - S^{op}$. The matrix $\big(t_E(RS)\big)_{R,S\in\mathcal{C}(E,E)}$ is the product of a permutation matrix $(S \mapsto (E \times E) - S^{op})$ with the matrix of an order ($\subseteq$), hence it is invertible (over $\mathbb{Z}$).
(a) follows from a similar argument, and (c) is an easy consequence of (a). □

7.12. Theorem. [[BT18b] Theorem 10.6] *Let k be a field, and M be a finitely generated correspondence functor over k. The following are equivalent:*

(a) M is projective and indecomposable.
(b) M is projective and admits a unique maximal proper subfunctor.
(c) M is projective and admits a unique minimal non-zero subfunctor.
(d) M is injective and indecomposable.
(e) M is injective and admits a unique maximal proper subfunctor.
(f) M is injective and admits a unique minimal non-zero subfunctor.

Proof. We first note that if M is finitely generated, it has finite length by Theorem 7.5, hence (if it is non zero) it admits a maximal subfunctor and a minimal subfunctor.

(a)⇒(b). If M is projective and indecomposable, and generated by its evaluation at some finite set E, we can assume $M = L_{E,V}$, where V is an indecomposable projective $\mathcal{R}_E$-module, by Proposition 7.6. Since $\mathcal{R}_E$ is a finite dimensional k-algebra, the module V has a unique maximal submodule W. If N is a subfunctor of M, then $N(E)$ is a submodule of $M(E) = V$. If $N(E) = V$, then $N = M$, as M is generated by $M(E) = V$. Hence $N(E) \subseteq W$ if $N \neq M$. So if N and N' are distinct maximal subfunctors of M, then $N + N' = M$, which implies $N(E) + N'(E) = V$, a contradiction since $N(E)$ and $N'(E)$ are both contained in W.

(b)⇒(a). If M has a unique maximal subfunctor and $M = M_1 \oplus M_2$ for non zero subfunctors M_1 and M_2, then M_1 and M_2 are finitely generated by Theorem 7.5, so admit maximal subfunctors N_1 and N_2, respectively. Then $M_1 \oplus N_2$ and $N_1 \oplus M_2$ are distinct maximal subfunctors of M.

(a)⇒(d). If M is a finitely generated indecomposable projective functor, then M is injective (and indecomposable) by Theorem 7.11.

(d)⇒(a). If M is finitely generated and injective, then $M^\natural$ is projective and finitely generated (as M and $M^\natural$ both have finite length), hence injective, so $M \cong (M^\natural)^\natural$ is projective.

(a)⇒(c). For a finitely generated functor M, the duality between M and $M^\natural$ induces an order reversing bijection between the subfunctors of M and the subfunctors of $M^\natural$. If M is projective and indecomposable, then so is $M^\natural$, that is, (a) holds for $M^\natural$. Thus (b) holds for $M^\natural$ and the functor $M^\natural$ has a unique maximal subfunctor. Hence M has a unique minimal subfunctor.

(c)⇒(a). If M is projective and admits a unique minimal subfunctor, then M is also injective, and its dual $M^\natural$ is projective and admits a unique maximal subfunctor. Hence $M^\natural$ is indecomposable, so M is indecomposable.

It is now clear that (e) and (f) are both equivalent to (a), (b), (c) and (d). □

7.13. Theorem. [[BT18b] Theorem 10.7] *Let k be a field.*

(a) Let M be a finitely generated projective correspondence functor. Then

$$M/\operatorname{Rad}(M) \cong \operatorname{Soc}(M).$$

(b) Let M and N be finitely generated correspondence functors over k. If M is projective, then

$$\dim_k \operatorname{Hom}_{\mathcal{F}_k}(M,N) = \dim_k \operatorname{Hom}_{\mathcal{F}_k}(N,M) < +\infty.$$

Proof. (sketch) The proof is a generalization of the proof of the similar result for modules over a symmetric algebra ([Ben91] Theorem 1.6.3). It relies mainly on Theorem 7.11 (see [BT18b] for details). □

8. Stability

When k is noetherian, the Hom-sets and more generally the Ext-groups from correspondence functors of bounded type over k to arbitrary correspondence functors over k are detected on their evaluations at large enough finite sets. More precisely:

8.1. Theorem. [[BT18b] Theorem 12.3] *Let k be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let E, F be finite sets.*

(a) If M is generated by $M(E)$, then for $|F| \geq 2^{|E|}$, the evaluation map

$$\operatorname{Hom}_{\mathcal{F}_k}(M,N) \to \operatorname{Hom}_{\mathcal{R}_F}\big(M(F), N(F)\big)$$

is an isomorphism.

(b) If M has bounded type, then for any $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such that if $|F| \geq n_i$, the map

$$\operatorname{Ext}^i_{\mathcal{F}_k}(M,N) \to \operatorname{Ext}^i_{\mathcal{R}_F}\big(M(F), N(F)\big)$$

is an isomorphism.

When k is noetherian, this leads to the following alternative description of $\mathcal{F}_k^b$, up to equivalence of categories.

8.2. Definition. *Let $\mathcal{G}_k$ be the following category:*

- *the objects of $\mathcal{G}_k$ are pairs (E,U), where E is a finite set, and U is an $\mathcal{R}_E$-module.*
- *a morphism $(E,U) \to (F,V)$ in $\mathcal{G}_k$ is a morphism of $\mathcal{R}_E$-modules*

$$U \to k\mathcal{C}(E,F) \otimes_{\mathcal{R}_F} V.$$

- *the composition in $\mathcal{G}_k$ of $f : (E,U) \to (F,V)$ and $g : (F,V) \to (G,W)$ given respectively by*

$$f : U \to k\mathcal{C}(E,F) \otimes_{\mathcal{R}_F} V \;\text{ and }\; g : V \to k\mathcal{C}(F,G) \otimes_{\mathcal{R}_G} W$$

is the morphism $(E,U) \to (G,W)$ given by the composition

$$U \xrightarrow{f} k\mathcal{C}(E,F) \otimes_{\mathcal{R}_F} V \xrightarrow{h} k\mathcal{C}(E,F) \otimes_{\mathcal{R}_F} k\mathcal{C}(F,G) \otimes_{\mathcal{R}_G} W \xrightarrow{l} k\mathcal{C}(E,G) \otimes_{\mathcal{R}_G} W$$

where $\left\{ \begin{array}{l} h = \mathrm{id}_{k\mathcal{C}(E,F)} \otimes_{\mathcal{R}_F} g \\ l = \mu \otimes_{\mathcal{R}_G} \mathrm{id}_W \end{array} \right.$ *, and* $\mu : k\mathcal{C}(E,F) \otimes_{\mathcal{R}_F} k\mathcal{C}(F,G) \to k\mathcal{C}(E,G)$ *is the composition in* $k\mathcal{C}$*.*

- *the identity morphism of* (E,U) *is* $U \xrightarrow{\cong} k\mathcal{C}(E,E) \otimes_{\mathcal{R}_E} U$*.*

8.3. Theorem. [[BT18b] Theorem 12.7]

(a) *The assignment* $(E,U) \mapsto L_{E,U}$ *is a fully faithful* k*-linear functor* $\mathcal{G}_k \to \mathcal{F}_k^b$*.*
(b) *When* k *is noetherian, it is an equivalence of categories. In particular* $\mathcal{G}_k$ *is abelian.*

9. Correspondence functors and lattices

9.1. In this section, we consider finite lattices, and build important examples of correspondence functors from them. Recall that a lattice $(T, \vee, \wedge)$ is a poset in which any pair $\{x, y\}$ of elements admits a least upper bound $x \vee y$, called the *join* of x and y, and a largest lower bound $x \wedge y$, called the *meet* of x and y.

If T is a lattice, and T^{op} is the opposite poset, then $(T^{op}, \wedge, \vee)$ is a lattice. A lattice T is called *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for any x, y, and z in T. One can show that T is distributive if and only if T^{op} is, that is, if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any x, y, and z in T.

A *finite* lattice T always admits a *largest element* (the join of all elements of T), denoted by 1_T, and a *smallest element* (the meet of all elements of T), denoted by 0_T.

9.2. Notation. *Let* T *be a finite lattice.*

- *For a finite set* X*, we denote by* $F_T(X) = k(T^X)$ *the free* k*-module with basis the set* T^X *of all maps from* X *to* T*.*
- *For a finite set* Y *and a correspondence* $S \in \mathcal{C}(Y,X)$*, we define a* k*-linear map* $F_T(S) : F_T(X) \to F_T(Y)$ *by sending* $\varphi : X \to T$ *to* $S\varphi : Y \to T$ *defined by*
$$\forall y \in Y, \ \ (S\varphi)(y) = \bigvee_{(y,x)\in S} \varphi(x).$$

9.3. Theorem. [[BT19a] Proposition 4.2 and Theorem 4.12] *Let* k *be a commutative ring, and* T *be finite lattice.*

(a) F_T *is a correspondence functor over* k*.*
(b) F_T *is* projective *in* $\mathcal{F}_k$ *if and only if* T *is* distributive.

In order to study the functorial aspects of the construction $T \mapsto F_T$, we introduce the following category $\mathcal{L}$ and its k-linearization (see Definition 2.1):

9.4. Definition.

- *Let* T *and* T' *be finite lattices. A map* $f : T \to T'$ *is called a* join morphism *if* $f(\bigvee_{t\in A} t) = \bigvee_{t\in A} f(t)$*, for any subset* A *of* T*.*
- *Let* $\mathcal{L}$ *denote the following category:*
  - *the objects of* $\mathcal{L}$ *are the finite lattices.*
  - *for finite lattices* T *and* T'*, the set* $\mathrm{Hom}_{\mathcal{L}}(T,T')$ *is the set of join morphisms from* T *to* T'*.*
  - *the composition of morphisms in* $\mathcal{L}$ *is the composition of maps, and the identity morphism of a finite lattice* T *is the identity map of* T*.*
- *Let* $k\mathcal{L}$ *be the* k*-linearization of* $\mathcal{L}$*.*

Note that for finite lattices T and T', a join morphism $f : T \to T'$ always maps 0_T to $0_{T'}$ (this follows from the case $A = \emptyset$ in the definition). Conversely, if f is a map from T to T' such that $f(0_T) = 0_{T'}$ and $f(a \vee b) = f(a) \vee f(b)$ for any a and b in T, then f is a join morphism.

For our next theorem, we need some notation. Let $f : T \to T'$ be a morphism in the category $\mathcal{L}$. For a finite set X, let $F_{f,X} : F_T(X) \to F_{T'}(X)$ be the k-linear map sending the function $\varphi : X \to T$ to the function $f \circ \varphi : X \to T'$.

9.5. Theorem. [[BT19a] Theorem 4.8]

(a) Let $f : T \to T'$ be a morphism in the category $\mathcal{L}$. Then the collection of maps $F_{f,X} : F_T(X) \to F_{T'}(X)$, for all finite sets X, yields a natural transformation $F_f : F_T \to F_{T'}$ of correspondence functors.

(b) The assignment sending a lattice T to F_T, and a morphism $f : T \to T'$ in $\mathcal{L}$ to $F_f : F_T \to F_{T'}$, yields a functor $\mathcal{L} \to \mathcal{F}_k$. This functor extends uniquely to a k-linear functor

$$F_? : k\mathcal{L} \longrightarrow \mathcal{F}_k \ .$$

(c) The functor $F_?$ is fully faithful.

The functor $F_?$ is also well behaved with the duality $F \mapsto F^\natural$ of functors introduced in Definition 6.5. To state this, we need the following notation:

9.6. Notation. *For a morphism $f : T \to T'$ in $\mathcal{L}$, let $f^{op} : T' \to T$ be the map defined by*

$$\forall t' \in T',\ \ f^{op}(t') = \bigvee_{\substack{x \in T\\ f(x) \leq t'}} x,$$

where the join is taken in T.

9.7. Theorem. [[BT19a] Lemma 8.1 and Theorem 8.9]

(a) Let T and T' be finite lattices. If $f : T \to T'$ is a morphism in $\mathcal{L}$, then $f^{op} : T'^{op} \to T^{op}$ is a morphism in $\mathcal{L}$.

(b) The assignment $T \mapsto T^{op}$ extends to an equivalence $\mathcal{L} \to \mathcal{L}^{op}$ and a k-linear equivalence $k\mathcal{L} \to k\mathcal{L}^{op}$.

(c) The functors $T \mapsto (F_T)^\natural$ and $T \mapsto F_{T^{op}}$ from $k\mathcal{L}$ to $\mathcal{F}_k^{op}$ are naturally isomorphic.

10. Fundamental functors

10.1. Let (E, R) be a finite poset. In Proposition 5.3, we have introduced, for any $k\,\mathrm{Aut}(E, R)$-module W, an $\mathcal{R}_E$-module $\Lambda_{R,W} = \mathcal{P}_E f_R \otimes_{k\,\mathrm{Aut}(E,R)} W$. So far we have used this construction only in the case where W is simple, in order to parametrize the simple correspondence functors over k, by setting $S_{E,R,W} = S_{E,\Lambda_{R,W}}$, where $S_{E,\Lambda_{R,W}}$ is the functor $S_{X,V}$ introduced in Notation 2.6, in the case of the object $X = E$ of the category $\mathcal{D} = \mathcal{C}$, and the $k\mathcal{D}(X, X) = \mathcal{R}_E$-module $V = \Lambda_{R,W}$.

We will now consider the correspondence functor $S_{E,\Lambda_{R,W}}$ over k associated to the *free module* $W = k\,\mathrm{Aut}(E, R)$. Then $\Lambda_{R,W} = \mathcal{P}_E f_R \otimes_{k\,\mathrm{Aut}(E,R)} k\,\mathrm{Aut}(E, R) \cong \mathcal{P}_E f_R$.

10.2. Definition. *Let (E,R) be a finite poset. The* fundamental correspondence functor *over k associated to (E,R) is the functor*

$$\mathbb{S}_{E,R} = S_{E,\mathcal{P}_E f_R}.$$

We will see that the fundamental functors associated to finite posets allow for a precise description of all the simple correspondence functors.

Our first observation is that $\mathbb{S}_{E,R}$ is by definition a quotient of the functor $L_{E,\mathcal{P}_E f_R} = k\mathcal{C}(-,E)\otimes_{\mathcal{R}_E}\mathcal{P}_E f_R$, which is itself a quotient of the representable functor $k\mathcal{C}(-,E)$, since $L_{E,\mathcal{P}_E f_R}$ is generated by the element $\Delta_E\otimes_{\mathcal{R}_E} f_R$ of its evaluation $L_{E,\mathcal{P}_E f_R}(E) = k\mathcal{C}(E,E)\otimes_{\mathcal{R}_E}\mathcal{P}_E f_R \cong \mathcal{P}_E f_R$.

Since $R = \sum\limits_{\substack{S\in\mathcal{O}_E\\ S\supseteq R}} f_S$ in $\mathcal{P}_E$ by definition of the idempotents f_S, we have $Rf_R = f_R$ in $\mathcal{P}_E$. Thus $R(\Delta_E\otimes_{\mathcal{R}_E} f_R) = R\otimes_{\mathcal{R}_E} f_R = \Delta_E\otimes_{\mathcal{R}_E} Rf_R = \Delta_E\otimes_{\mathcal{R}_E} f_R$, and it follows that the direct summand $k\mathcal{C}(-,E)R$ of $k\mathcal{C}(-,E)$ maps surjectively onto $\mathbb{S}_{E,R}$, so we get a canonical surjective morphism

$$\pi_{E,R} : k\mathcal{C}(-,E)R \twoheadrightarrow \mathbb{S}_{E,R}.$$

Before going further, we need to recall some classical definitions and results on finite lattices.

10.3. Definition.

- *Let T be a finite lattice. An element e of T is called* join-irreducible *if whenever e can be written $e = \bigvee\limits_{a\in A} a$ for some subset A of T, then $e\in A$. We denote by* $\mathrm{Irr}(T)$ *the set of join-irreducible elements of T, viewed as a full subposet of T.*
- *Let (E,R) be a finite poset. For $e\in E$, set*

  $$]\cdot,e]_{E,R} = R_e = \{x\in E \mid (x,e)\in R\}.$$

  A lower ideal *of (E,R) is a* closed downwards *subset of E, i.e. a subset A such that $]\cdot,x]_{E,R}\subseteq A$ whenever $x\in A$. Let $I_\downarrow(E,R)$ be the set of lower ideals of (E,R), ordered by inclusion of subsets.*

The poset $I_\downarrow(E,R)$ is a distributive lattice, the join operation being union of subsets, and the meet operation being intersection of subsets. It is easily seen moreover that

$$\mathrm{Irr}(I_\downarrow(E,R)) = \{]\cdot,e]_{E,R} \mid e\in E\}.$$

The map $e\in E\mapsto]\cdot,e]_{E,R}$ is an isomorphism of posets from (E,R) to $\mathrm{Irr}(I_\downarrow(E,R))$. We will freely abuse notation and identify these two posets.

When T is a finite lattice, and $(E,R)=\mathrm{Irr}(T)$, the map

$$s_T : A\in I_\downarrow(E,R)\mapsto \bigvee_{a\in A} a\in T$$

is a surjective join-morphism of lattices. By standard theorems (see [Sta12] Theorem 3.4.1 and Proposition 3.4.2, and also [Rom08] Theorem 6.2), it is a bijection if and only if T is distributive, and then it is an isomorphism of lattices.

10.4. Proposition. [[BT19a] Proposition 4.5 and Theorem 6.5] *Let T be a finite lattice, and $(E,R)=\mathrm{Irr}(T)$.*

(a) There is a canonical isomorphism of functors $F_{I_\downarrow(E,R)}\cong k\mathcal{C}(-,E)R^{op}$.

(b) The morphism $F_{I_\downarrow(E,R)}\cong k\mathcal{C}(-,E)R^{op}\xrightarrow{\pi_{R,T}}\mathbb{S}_{E,R^{op}}$ factors as

$$F_{I_\downarrow(E,R)}\xrightarrow{F_{s_T}} F_T\xrightarrow{\Theta_T}\mathbb{S}_{E,R^{op}}.$$

10.5. Definition. *Let T be a finite lattice, and $(E,R)=\mathrm{Irr}(T)$.*

- *For $t\in T$, let $r(t)=\bigvee_{\substack{e\in E\\ e<t}} e$ and $\sigma(t)=\bigwedge_{\substack{e\in E\\ e>t}} e$. Define inductively $r^n(t)$ and $\sigma^n(t)$ by $r^n(t)=r\big(r^{n-1}(t)\big)$ and $\sigma^n(t)=\sigma\big(\sigma^{n-1}(t)\big)$.*
  Then $t\geq r(t)\geq\ldots\geq r^n(t)=r^{n+1}(t)$ for some n. Set $r^\infty(t)=r^n(t)$. Similarly $t\leq\sigma(t)\leq\ldots\sigma^m(t)=\sigma^{m+1}(t)$ for some m. Set $\sigma^\infty(t)=\sigma^m(t)$.
- *Let $G_T=E\sqcup\{t\in T\mid t=r^\infty\sigma^\infty(t)\}$, and $G(E,R)=G_{I_\downarrow(E,R)}$.*

Note that the union in the definition of G_T is disjoint because $r(t)<t$ if (and only if) $t\in E$.

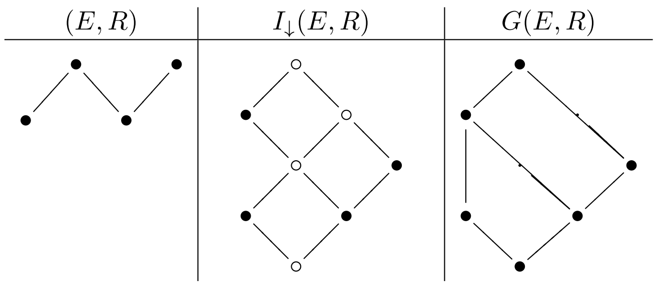
10.6. Theorem. [[BT18a] Theorem 5.6] *Let (E,R) be a finite poset, and T be a finite lattice such that $\mathrm{Irr}(T)\cong(E,R)$.*

(a) Let $\overline{G}(E,R)$ denote the image of $G(E,R)$ by $s_T: I_\downarrow(E,R)\twoheadrightarrow T$.

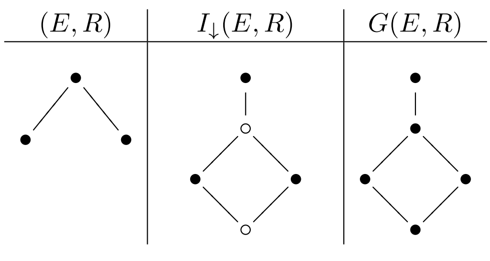
- *The restriction of the map s_T to $G(E,R)$ is an isomorphism of posets $G(E,R)\to\overline{G}(E,R)$. Moreover $\overline{G}(E,R)=G_T$. In particular G_T only depends on the poset $\mathrm{Irr}(T)$, up to isomorphism.*
- *For any finite set X, the set*
  $$\{\varphi: X\to T\mid E\subseteq\varphi(X)\subseteq\overline{G}(E,R)\}\subseteq F_T(X)$$
  is mapped by $\Theta_{T,X}: F_T(X)\to\mathbb{S}_{E,R^{op}}(X)$ to a k-basis of $\mathbb{S}_{E,R^{op}}(X)$.

(b) For any finite set X, the k-module $\mathbb{S}_{E,R}(X)$ is free of rank
$$rk_k\,\mathbb{S}_{E,R}(X)=\sum_{i=0}^{e}(-1)^i\binom{e}{i}(g-i)^{|X|},$$
where $e=|E|$ and $g=|G(E,R)|$.

10.7. Examples. (The black nodes of $I_\downarrow(E,R)$ are the join-irreducible elements)



$$rk_k\mathbb{S}_{E,R}(X)=6^x-4.5^x+6.4^x-4.3^x+2^x,\ \text{where } x=|X|.$$



$$rk_k\mathbb{S}_{E,R}(X)=5^x-3.4^x+3.3^x-2^x,\ \text{where } x=|X|.$$

11. The simple correspondence functors

11.1. Theorem. [[BT18a] Theorem 6.6 and Theorem 6.10] *Let (E, R) be a finite poset, and X be a finite set.*

(a) $\mathbb{S}_{E,R}(X)$ *is a free right* $k\operatorname{Aut}(E, R)$*-module.*
(b) Let k be a field, and W be a simple $k\operatorname{Aut}(E, R)$*-module. Then*

$$S_{E,R,W}(X) \cong \mathbb{S}_{E,R}(X) \otimes_{k\operatorname{Aut}(E,R)} W.$$

In particular

$$\dim_k S_{E,R,W}(X) = \frac{\dim_k W}{|\operatorname{Aut}(E, R)|} \sum_{i=0}^{e} (-1)^i \binom{e}{i} (g-i)^{|X|},$$

where $e = |E|$ and $g = |G(E, R)|$.

In the case of a totally ordered poset, the previous result takes the following form. Let $n \in \mathbb{N}$. Set $\underline{n} = \{0 < 1 < \ldots < n\}$, and $[n] = \operatorname{Irr}(\underline{n}) = \underline{n} - \{0\}$. Set $\mathbb{S}_n = \mathbb{S}_{[n],tot}$, where tot is the natural total order on $[n]$. Observe that $([n], tot) \cong ([n], tot^{op})$

11.2. Theorem. [[BT19a] Proposition 10.2, Theorem 11.1, Theorem 11.6 and Theorem 11.8]

(a) The surjection $F_{\underline{n}} \to \mathbb{S}_n$ splits. The functor $\mathbb{S}_n$ is projective.
(b) If X is a finite set, then $\mathbb{S}_n(X)$ is a free k-module of rank

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i+1)^{|X|}.$$

(c) $F_{\underline{n}} \cong \bigoplus_{A \subseteq [n]} \mathbb{S}_{|A|} \cong \bigoplus_{j=0}^{n} (\mathbb{S}_j)^{\oplus \binom{n}{j}}$.
(d) $\operatorname{End}_{k\mathcal{L}}(\underline{n}) \cong \operatorname{End}_{\mathcal{F}_k}(F_{\underline{n}}) \cong \prod_{j=0}^{n} M_{\binom{n}{j}}(k)$.
(e) If k is a field, then $\mathbb{S}_n$ is simple (and projective, and injective), isomorphic to $S_{[n],tot,k}$.

The case of a total order raises the question of determining all simple projective correspondence functors. The answer is as follows.

11.3. Definition.

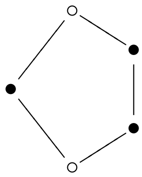
- *If X and Y are posets, let $X * Y$ denote the set $X \sqcup Y$, ordered by the orders of X and Y, and moreover $x \leq y$ for $x \in X$ and $y \in Y$.*
- *A* pole poset *is a poset of the form $E_1 * E_2 * \ldots E_n$, where E_i is a set of cardinality 1 or 2, ordered by equality, for $i = 1, \ldots, n$.*

11.4. Theorem. [[BT19b] Theorem 4.5] *Let k be a field, and $S_{E,R,W}$ be a simple correspondence functor, where (E, R) is a finite poset, and W is a simple* $k\operatorname{Aut}(E, R)$*-module. The following conditions are equivalent:*

(a) $S_{E,R,W}$ *is projective.*
(b) (E, R) *is a pole poset, and W is a projective* $k\operatorname{Aut}(E, R)$*-module.*
(c) Either (E, R) is totally ordered or (E, R) is a pole poset and $\operatorname{char}(k) \neq 2$.

11.5. Examples. [[BT18a] Example 8.9 and Example 8.10]

- Let D be the following lattice:

$D =$ 

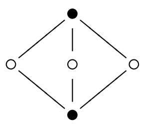
One can show that the functor F_D splits as

$$F_D \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 4\mathbb{S}_2 \oplus \mathbb{S}_3 \oplus 2\mathbb{S}_{\bullet\bullet} \oplus \mathbb{S}_{\bullet\,\vdots}\ ,$$

where $\bullet\bullet$ is a set of cardinality 2, ordered by equality, and $\bullet\,\vdots$ is a poset of cardinality 3 with 2 connected components.

In particular, when k is a field of characteristic different from 2, the functor F_D is semisimple. Since D is not distributive, the functor F_D is not projective, by Theorem 9.3. The only non projective direct summand in the above decomposition is $\mathbb{S}_{\bullet\,\vdots}$.

- Let T be the following lattice :

$T =$ 

One can show that F_T splits as

$$F_T \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 3\mathbb{S}_2 \oplus 3\mathbb{S}_{\bullet\bullet} \oplus \mathbb{S}_{\bullet\bullet\bullet}\ ,$$

where $\bullet\bullet\bullet$ is a set of cardinality 3 ordered by equality. All the summands in this decomposition of F_T, except possibly $\mathbb{S}_{\bullet\bullet\bullet}$, are projective functors. Since the lattice T is not distributive, the functor F_T is not projective, thus $\mathbb{S}_{\bullet\bullet\bullet}$ is actually not projective either.

11.6. Remark. The sequence of multiplicities of $\mathbb{S}_n$ as a direct summand of F_D is 1, 4, 4, 1, 0 for $n = 0, 1, 2, 3, 4$, and the number of chains $0_D = x_0 < x_1 < \ldots < x_n$ in D is equal to 1, 4, 4, 1, 0 for $n = 0, 1, 2, 3, 4$. Similarly, the sequence of multiplicities of $\mathbb{S}_n$ as a direct summand of F_T is 1, 4, 3, 0 for $n = 0, 1, 2, 3$, and the number of chains $0_T = x_0 < x_1 < \ldots < x_n$ in T is equal to 1, 4, 3, 0 for $n = 0, 1, 2, 3$. This is *not* a coincidence... (see [BT19a] Section 10 and Section 11 for details).

11.7. Theorem. [[BT18a] Theorem 7.1 and Theorem 7.2] *Let X be a finite set, and $\mathcal{R}_X = k\mathcal{C}(X, X)$, where k is a field.*

(a) *The set of isomorphism classes of simple $\mathcal{R}_X$-modules is parametrized by the set of isomorphism classes of triples (E, R, W), where E is a finite set with $|E| \leq |X|$, R is an order relation on E, and W is a simple $k\,\mathrm{Aut}(E, R)$-module.*

(b) *The simple $\mathcal{R}_X$-module parametrized by the triple (E, R, W) is $S_{E,R,W}(X)$.*

(c) *In particular, the dimension of this simple module is equal to*

$$\dim_k S_{E,R,W}(X) = \frac{\dim_k W}{|\,\mathrm{Aut}(E, R)|} \sum_{i=0}^{|E|} (-1)^i \binom{|E|}{i} \big(|G(E, R)| - i\big)^{|X|}.$$

11.8. Corollary. [[BT18a] Theorem 7.4] *Let k be a field of characteristic 0. Let X be a finite set, and $J(\mathcal{R}_X)$ denote the Jacobson radical of $\mathcal{R}_X = k\mathcal{C}(X,X)$. Set $n = |X|$. Then*

$$\dim_k J(\mathcal{R}_X) = 2^{n^2} - \sum_{e=0}^{n} \sum_{R} \frac{1}{|\operatorname{Aut}(E,R)|} \Big(\sum_{i=0}^{e} (-1)^i \binom{e}{i} \big(|G(E,R)| - i\big)^n \Big)^2,$$

where R runs through a set of Σ_E-conjugacy classes of order relations on $E = \{1,\ldots,e\}$, and $E = \emptyset$ if $e = 0$.

For small values of n, the dimension of $J(\mathcal{R}_X)$ is given in the following table. For $n \leq 2$, this dimension is 0, so the algebra $k\mathcal{R}_X$ is semisimple. For $n = 3$, the dimension of $J(\mathcal{R}_X)$ is 42, and follows from a computer calculation in [Bre14]. This value can be confirmed by hand using the above formula. This value has also been confirmed by a direct computation using the computer software GAP [GAP15]. For $n \geq 4$, a direct computation using the algebra $\mathcal{R}_X$ is not possible on current computers, because the dimension 2^{n^2} of this algebra is too big. The table below has been completed via a computer calculation using GAP and the above formula.

n	$\dim_k J(\mathcal{R}_X)$
≤ 2	0
3	42
4	$32,616$
5	$29,446,050$
6	$67,860,904,320$
7	$562,649,705,679,642$
8	$18,446,568,932,288,588,616$

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