Biset functors and genetic sections for p-groups

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Abstract: In this note I show that if F is a biset functor defined over finite p-groups, then for each finite p-group P, there is a direct summand of $F(P)$ admitting a natural direct sum decomposition indexed by the irreducible rational representations of P, or equivalently, by the equivalence classes of origins in P, or also by equivalence classes of genetic sections of P. This leads to a description of the torsion part of the group of relative syzygies in the Dade group of P, and to a conjecture on the structure of the torsion part of the whole Dade group of P.

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1. Basic subgroups and origins

1.1. In Sections 1 to 3 of this paper I recall some notation and definitions which are used in the statement of the main theorem (Theorem 3.2). Section 4 is devoted to some properties of origins and genetic sections. Section 5 is the proof of Theorem 3.2, and Section 6 is an application to the Dade group. In Section 7, I use the notion of *rational* biset functor to give a description of the torsion part of the functor of relative syzygies in the Dade group.

1.2. Throughout this paper, the letter p denotes a fixed prime number. If P is a finite p-group, denote by $R_{\mathbb{Q}}(P)$ the Grothendieck group of finitely generated $\mathbb{Q}P$-modules. There is a natural bilinear form on $R_{\mathbb{Q}}(P)$, with values in $\mathbb{Z}$, defined by

$$\langle V, W\rangle_P = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}P}(V, W)$$

for $\mathbb{Q}P$-modules V and W.

Recall some definitions from [1] :

1.3. Definition : [**[1] 2.3**]*Let P be a finite p-group. A subgroup Q of P is called basic if the following two conditions hold :*

1. *The quotient $N_P(Q)/Q$ is cyclic or generalized quaternion.*
2. *If R is any subgroup of P such that $R\cap N_P(Q) \subseteq Q$, then $|R| \leq |Q|$.*

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1.4. Associated simple modules. If Q is a proper basic subgroup of P, then there is a unique subgroup $\tilde{Q} \supset Q$ of P with $|\tilde{Q} : Q| = p$, and the kernel of the projection map

$$\mathbb{Q}P/Q \to \mathbb{Q}P/\tilde{Q}$$

is an irreducible $\mathbb{Q}P$-module, denoted by V_Q.

The group P itself is a basic subgroup of P, and by convention V_P is the trivial $\mathbb{Q}P$-module $\mathbb{Q}$. With this notation, any irreducible $\mathbb{Q}P$-module is isomorphic to V_Q, for some basic subgroup Q of P.

If Q and Q' are basic subgroups of P, then the $\mathbb{Q}P$-modules V_Q and $V_{Q'}$ are isomorphic if and only if $Q \doteqdot_P Q'$, where $\doteqdot_P$ is the relation defined in [1] 2.7 by

$$Q \doteqdot_P Q' \Leftrightarrow \left\{ \begin{array}{l} |Q| = |Q'| \\ \exists x \in P,\ Q^x \cap N_P(Q') \subseteq Q' \end{array} \right. .$$

1.5. Notation and Definition : **[[1] 2.4]** *If Q and R are subgroups of P, set*

$$I_P(Q,R) = \{x \in P \mid Q^x \cap N_P(R) \subseteq R\} \qquad \mathcal{I}_P(Q,R) = Q\backslash I_P(Q,R)/N_P(R) .$$

If V is a simple $\mathbb{Q}P$-module, and W is a finitely generated $\mathbb{Q}P$-module, denote by $m(V,W)$ the multiplicity of V as a direct summand of W.

If Q is a basic subgroup of P, and R is any subgroup of P, then (see [1] 2.5)

$$m(V_Q, \mathbb{Q}P/R) = \frac{|\mathcal{I}_P(R,Q)|}{|\mathcal{I}_P(Q,Q)|} .$$

In particular, if this is non-zero, then $|R| \leq |Q|$.

1.6. Definition : **[[1] 7.3]** *If Q is a basic subgroup of P, set*

$$T_Q = \langle I_P(Q,Q) \rangle \qquad S_Q = \bigcap_{x \in T_Q} {}^xQ .$$

The basic subgroup Q of P is called an origin (in P) if T_Q/S_Q has normal p-rank 1, i.e. if it is cyclic, generalized quaternion, semi-dihedral, or dihedral of order at least 16.

If V is a simple $\mathbb{Q}P$-module, there exists an origin Q in P such that $V \cong V_Q$ (see [1] 7.5). Then Q is called an origin of V, and V is said to have *type* T_Q/S_Q. One can show that the type of V is well defined up to isomorphism (see [1] 7.5). If T_Q/S_Q is cyclic or generalized quaternion, then $S_Q = Q$ and $T_Q = N_P(Q)$ (see [1] 7.6). If T_Q/S_Q is dihedral or semi-dihedral, then $|Q : S_Q| = 2$ (see [1] 7.4).

If R is a finite p-group of normal p-rank 1, then there is a unique faithful irreducible $\mathbb{Q}R$-module Φ_R, up to isomorphism (see [1] 3.7).

1.7. Definition : *A pair (T,S) of subgroups of P with $S \trianglelefteq T$ is called a section of P. Two sections (T,S) and (T',S') are said to be linked (notation $(T,S) \,\text{—}\, (T',S')$) if*

$$T \cap S' = S \cap T' \text{ and } T.S' = S.T' .$$

They are linked modulo P (notation $(T,S) \,\text{—}_P\, (T',S')$) if there exists $x \in P$ such that $(T,S) \,\text{—}\, ({}^xT', {}^xS')$.

The section (T,S) is called genetic if T/S has normal p-rank 1, and if setting

$$V = \mathrm{Ind}_T^P \mathrm{Inf}_{T/S}^T \Phi_{T/S} ,$$

there is a $\mathbb{Q}$-algebra isomorphism

$$\mathrm{End}_{\mathbb{Q}P}(V) \cong \mathrm{End}_{\mathbb{Q}T/S}(\Phi_{T/S}) \quad .$$

In particular, it follows that $\mathrm{End}_{\mathbb{Q}P}(V)$ is a (skew) field, hence V is an irreducible $\mathbb{Q}P$-module, called the irreducible $\mathbb{Q}P$-module *associated* to (T,S). The pair (T,S) is called an genetic section of P for V, or associated to V. If Q is an origin in P, then (T_Q,S_Q) is an genetic section of P associated to V_Q (see [1] 7.4).

The section (P,P) of P is called the *trivial genetic section* of P. It is the unique genetic section (T,S) of P for which $T=S$: indeed $\Phi_{\mathbf{1}}=\mathbb{Q}$, and $\mathrm{Ind}_T^P\mathrm{Inf}_{\mathbf{1}}^T\Phi_{\mathbf{1}}=\mathbb{Q}P/T$ is simple if and only if $T=P$, since $\mathbb{Q}$ is always a direct summand of $\mathbb{Q}P/T$.

If (T,S) and (T',S') are genetic sections of P, and if V and V' are the associated simple $\mathbb{Q}P$-modules, then $V\cong V'$ if and only if $(T,S) \,—_P\, (T',S')$ (see [1] 7.11). In particular, the relation $—_P$ is an equivalence relation on the set of genetic sections of P. Recall moreover that if $(T,S) \,—_P\, (T',S')$, then there exists a unique double coset TxT' in P such that $(T,S) \,—\, {}^x(T',S')$.

2. Biset functors

2.1. Notation and Definition : *Let k be a commutative ring. Denote by $\mathcal{C}_{p,k}$ the following category :*

- *The objects of $\mathcal{C}_{p,k}$ are the finite p-groups.*
- *If P and Q are finite p-groups, then $\mathrm{Hom}_{\mathcal{C}_{p,k}}(P,Q)=k\otimes_{\mathbb{Z}}B(Q\times P^{op})$, where $B(Q\times P^{op})$ is the Burnside group of finite (Q,P)-bisets.*
- *The composition of morphisms is k-bilinear, and if P, Q, R are finite p-groups, if U is a finite (Q,P)-biset, and V is a finite (R,Q)-biset, then the composition of (the isomorphism classes of) V and U is the (isomorphism class) of $V\times_Q U$. The identity morphism Id_P of the p-group P is the class of the set P, with left and right action by multiplication.*

*Let $\mathcal{F}_{p,k}$ denote the category of k-linear functors from $\mathcal{C}_{p,k}$ to the category k-*Mod *of k-modules. An object of $\mathcal{F}_{p,k}$ is called a biset functor (defined over p-groups, with values in k-*Mod*).*

If F is an object of $\mathcal{F}_{p,k}$, if P and Q are finite p-groups, and if $\varphi\in\mathrm{Hom}_{\mathcal{C}_{p,k}}(P,Q)$, then the image of $w\in F(P)$ by the map $F(\varphi)$ will generally be denoted by $\varphi(w)$. The composition $\psi\circ\varphi$ of morphisms $\varphi\in\mathrm{Hom}_{\mathcal{C}_{p,k}}(P,Q)$ and $\psi\in\mathrm{Hom}_{\mathcal{C}_{p,k}}(Q,R)$ will also be denoted by $\psi\times_Q\varphi$.

2.2. Examples. Recall that this formalism of bisets gives a single framework for the usual operations of induction, restriction, inflation, deflation, and transport by isomorphism via the following correspondences :

- If Q is a subgroup of P, then let $\mathrm{Ind}_Q^P\in\mathrm{Hom}_{\mathcal{C}_{p,k}}(Q,P)$ denote the set P, with left action of P and right action of Q by multiplication.
- If Q is a subgroup of P, then let $\mathrm{Res}_Q^P\in\mathrm{Hom}_{\mathcal{C}_{p,k}}(P,Q)$ denote the set P, with left action of Q and right action of P by multiplication.

- If $N \trianglelefteq P$, and $Q = P/N$, then let $\mathrm{Inf}_Q^P \in \mathrm{Hom}_{\mathcal{C}_{p,k}}(Q,P)$ denote the set Q, with left action of P by projection and multiplication, and right action of Q by multiplication.
- If $N \trianglelefteq P$, and $Q = P/N$, then let $\mathrm{Def}_Q^P \in \mathrm{Hom}_{\mathcal{C}_{p,k}}(P,Q)$ denote the set Q, with left action of Q by multiplication, and right action of P by projection and multiplication.
- If $\varphi : P \to Q$ is a group isomorphism, then let $\mathrm{Iso}_P^Q = \mathrm{Iso}_P^Q(\varphi) \in \mathrm{Hom}_{\mathcal{C}_{p,k}}(P,Q)$ denote the set Q, with left action of Q by multiplication, and right action of P by taking image by φ, and then multiplying in Q.

2.3. Sections and isomorphisms. Let (T,S) and (T',S') be sections of the group P, and let $x \in P$ be an element such that $(T,S) \,\text{—}\, ({}^xT', {}^xS')$. Let U denote the set $S\backslash TxT'/S'$, viewed as a $(T/S, T'/S')$-biset. If $t' \in T'$, then there exists $t \in T \cap {}^xT'$ and $s' \in S'$ such that $t' = t^x s'$. The correspondence $u : t'S' \mapsto tS$ is well defined, and it is a group isomorphism from T'/S' to T/S. Moreover, the correspondence

$$\forall (t,t') \in T \times T',\ Stxt'S' \in S\backslash TxT'/S' \mapsto tS.u(t'S') \in T/S$$

is an isomorphism of $(T/S, T'/S')$-bisets between U and $\mathrm{Iso}_{T'/S'}^{T/S}(u)$.

2.4. Opposite bisets. If P and Q are finite p-groups, and if U is a finite (Q,P)-biset, then let U^{op} denote the opposite biset : as a set, it is equal to U, and it is a (P,Q)-biset for the following action

$$\forall h \in Q, \forall u \in U, \forall g \in P,\ g.u.h \text{ (in } U^{op}) = h^{-1}ug^{-1} \text{ (in } U) \quad .$$

This definition can be extended by linearity, to give an isomorphism

$$\varphi \mapsto \varphi^{op} : \mathrm{Hom}_{\mathcal{C}_{p,k}}(P,Q) \to \mathrm{Hom}_{\mathcal{C}_{p,k}}(Q,P) \quad .$$

It is easy to check that $(\varphi \circ \psi)^{op} = \psi^{op} \circ \varphi^{op}$, with obvious notation, and the functor

$$\begin{cases} P \mapsto P \\ \varphi \mapsto \varphi^{op} \end{cases}$$

is an equivalence of categories from $\mathcal{C}_{p,k}$ to the dual category.

2.5. The functor $P \mapsto R_{\mathbb{Q}}(P)$. If P and Q are finite p-groups, if V is a finitely generated $\mathbb{Q}P$-module, if U is a finite (Q,P)-biset, then $\mathbb{Q}U$ is a $(\mathbb{Q}Q, \mathbb{Q}P)$-bimodule, and $\mathbb{Q}U \otimes_{\mathbb{Q}P} V$ is a finitely generated $\mathbb{Q}Q$-module. This definition can be extended by linearity, and gives a structure of biset functor (with values in $\mathbb{Z}$-Mod) to the correspondence $P \mapsto R_{\mathbb{Q}}(P)$.

This functorial structure is well-behaved with respect to the bilinear forms $\langle\ ,\ \rangle_P$, in the following sense : if P and Q are finite p-groups, if $a \in R_{\mathbb{Q}}(P)$ and $b \in R_{\mathbb{Q}}(Q)$, if $\varphi \in \mathrm{Hom}_{\mathcal{C}_{p,k}}(P,Q)$, then

$$\langle b, \varphi(a) \rangle_Q = \langle \varphi^{op}(b), a \rangle_P \quad .$$

2.6. Some idempotents in $\mathrm{End}_{\mathcal{C}_{p,k}}(P)$. Let P be a finite p-group, and let $N \trianglelefteq P$. Then it is clear from the definitions that

$$\mathrm{Def}^P_{P/N} \circ \mathrm{Inf}^P_{P/N} = (P/N) \times_P (P/N) = \mathrm{Id}_{P/N} \quad .$$

It follows that the composition

$$e^P_N = \mathrm{Inf}^P_{P/N} \circ \mathrm{Def}^P_{P/N}$$

is an idempotent in $\mathrm{End}_{\mathcal{C}_{p,k}}(P)$. It is the class of the set P/N, viewed as a (P,P)-biset by left and right action given by projection on P/N, and multiplication in P/N. Moreover, if M and N are normal subgroups of P, then

$$e_N \circ e_M = (P/N) \times_P (P/M) \quad ,$$

which is easily seen to be isomorphic to P/NM as (P,P)-biset. In other words

$$e^P_N \circ e^P_M = e^P_{NM} \quad .$$

Moreover $e^P_1 = \mathrm{Id}_P$. The following Lemma is well-known (and can be proved for instance using Lemme 8 of [3]) :

2.7. Lemma : *If $N \trianglelefteq P$, define $f^P_N \in \mathrm{End}_{\mathcal{C}_{p,k}}(P)$ by*

$$f^P_N = \sum_{\substack{M \trianglelefteq P \\ N \subseteq M}} \mu_{\trianglelefteq P}(N,M) e^P_M \quad ,$$

where $\mu_{\trianglelefteq P}$ denotes the Möbius function of the poset of normal subgroups of P. Then the elements f^P_N, for $N \trianglelefteq P$, are orthogonal idempotents of $\mathrm{End}_{\mathcal{C}_{p,k}}(P)$, and their sum is equal to Id_P.

Now if $N \trianglelefteq P$, the poset $]\mathbf{1}, N[^P$ of normal subgroups of P which are non-trivial and properly contained in N is contractible, unless the action of P on N is semi-simple. Since P is a p-group, this implies that if $\mu_{\trianglelefteq P}(\mathbf{1}, N) \neq 0$, then N is central in P. In this case moreover $\mu_{\trianglelefteq P}(\mathbf{1}, N)$ is equal to the Möbius function $\mu(\mathbf{1}, N)$ of the poset of all proper non-trivial subgroups of N. This is zero if N is not elementary abelian. This gives finally

$$f^P_{\mathbf{1}} = \sum_{N \subseteq \Omega_1 Z(P)} \mu(\mathbf{1}, N) P/N \quad ,$$

where $\Omega_1 Z(P)$ is the subgroup of the center of P consisting of elements of order at most p.

2.8. Faithful elements for biset functors. If F is an object of $\mathcal{F}_{p,k}$, and if P is a finite p-group, then the idempotent $f^P_{\mathbf{1}}$ of $\mathrm{End}_{\mathcal{C}_{p,k}}(P)$ acts on $F(P)$. Its image

$$\partial F(P) = F(f^P_{\mathbf{1}})F(P)$$

is a direct summand of $F(P)$ as k-module : it will be called the set of *faithful* elements of $F(P)$, because any element $u \in F(P)$ which is inflated from a proper quotient of P is such that $F(f^P_{\mathbf{1}})u = 0$. In the case $k = \mathbb{Z}$ and $F = R_{\mathbb{Q}}$, the module $\partial F(P)$ has a basis over $\mathbb{Z}$ consisting of the faithful irreducible rational representations of P.

3. Bisets and genetic sections

3.1. Notation : *Let P be a finite p-group, and (T,S) be an genetic section of P, with $S \neq T$. Then (T/S) has a unique central subgroup of order p, and this subgroup is equal to $\hat{S}/S$, for some well defined subgroup $\hat{S}$ of T. Then P/S and $P/\hat{S}$ are $(P,T/S)$-bisets, and I denote by $a_{T,S}$ the element of $\mathrm{Hom}_{\mathcal{C}_{p,k}}(T/S,P)$ defined by*

$$a_{T,S} = P/S - P/\hat{S} \quad .$$

I denote by $b_{T,S}$ the element of $\mathrm{Hom}_{\mathcal{C}_{p,k}}(P,T/S)$ corresponding to the opposite bisets, i.e.

$$b_{T,S} = S\backslash P - \hat{S}\backslash P \quad .$$

If $(T,S) = (P,P)$ is the trivial genetic section of P, I denote by $a_{T,S}$ the image in $\mathrm{Hom}_{\mathcal{C}_{p,k}}(\mathbf{1},P)$ of a $(P,\mathbf{1})$-biset of cardinality 1, and by $b_{T,S}$ the image in $\mathrm{Hom}_{\mathcal{C}_{p,k}}(P,\mathbf{1})$ of a $(\mathbf{1},P)$-biset of cardinality 1.

The main theorem can now be stated :

3.2. Theorem : *Let F be a biset functor defined over p-groups, with values in k-Mod. Then for any finite p-group P, and any set $\mathcal{S}$ of representatives of genetic sections of P modulo the relation $\frac{\quad}{}_P$, the map*

$$\mathcal{I}_{\mathcal{S}} = \mathop{\oplus}_{(T,S)\in\mathcal{S}} F(a_{T,S}) : \mathop{\oplus}_{(T,S)\in\mathcal{S}} \partial F(T/S) \to F(P) \quad ,$$

is a split injection of k-modules, with left inverse

$$\mathcal{D}_{\mathcal{S}} = \mathop{\oplus}_{(T,S)\in\mathcal{S}} F(b_{T,S}) : F(P) \to \mathop{\oplus}_{(T,S)\in\mathcal{S}} \partial F(T/S) \quad .$$

4. Some combinatorial properties of origins and genetic sections

4.1. Lemma : *Let P be a finite p-group, and Q be an origin in P, with associated genetic section (T_Q, S_Q), and corresponding simple $\mathbb{Q}P$-module V_Q. Then if R is any subgroup of P, the multiplicity $m(V_Q, \mathbb{Q}P/R)$ of V_Q in the permutation module $\mathbb{Q}P/R$ is equal to*

$$m(V_Q, \mathbb{Q}P/R) = |Q : S_Q|\, a + b \quad ,$$

where

$$\begin{aligned} a &= \mathrm{card}\{x \in T_Q\backslash P/R \mid T_Q \cap {}^xR \subseteq S_Q\} \\ b &= \mathrm{card}\{x \in T_Q\backslash P/R \mid |(T_Q \cap {}^xR)S_Q/S_Q| = p,\ (T_Q \cap {}^xR)S_Q/S_Q \not\subseteq Z(T_Q/S_Q)\} \quad . \end{aligned}$$

Proof: First, the result holds if $Q = P$, for in this case $V_Q = \mathbb{Q}$ and $T_Q = S_Q = P$, thus $m(V_Q, \mathbb{Q}P/R) = 1$, and moreover $a = 1$ and $b = 0$.

Suppose now that Q is a proper subgroup of P. Set $(T,S)=(T_Q,S_Q)$, and $D=T/S$. Then :

$$\begin{aligned}\langle V_Q,\mathbb{Q}P/R\rangle_P &= \langle \mathrm{Ind}_T^P\mathrm{Inf}_D^T\Phi_D,\mathbb{Q}P/R\rangle_P\\ &= \langle \Phi_D,\mathrm{Def}_D^T\mathrm{Res}_T^P\mathbb{Q}P/R\rangle_D \quad .\end{aligned}$$

Now $\mathrm{Def}_D^T\mathrm{Res}_T^P\mathbb{Q}P/R$ is isomorphic to $\mathbb{Q}S\backslash P/R$ as $\mathbb{Q}D$-module. In other words

$$\mathrm{Def}_D^T\mathrm{Res}_T^P\mathbb{Q}P/R \cong \bigoplus_{x\in T\backslash P/R}\mathbb{Q}S\backslash TxR/R \quad ,$$

and the $\mathbb{Q}D$-module $\mathbb{Q}S\backslash TxR/R$ is isomorphic to $\mathbb{Q}D/E_x$, where $E_x=(T\cap {}^xR)S/S$, for each $x\in P$: indeed $S\backslash TxR/R$ is a transitive D-set, and the stabilizer of SxR in D is equal to E_x. It follows that

$$\langle V_Q,\mathbb{Q}P/R\rangle_P=\sum_{x\in T\backslash P/R}\langle \Phi_D,\mathbb{Q}D/E_x\rangle_D \quad .$$

Now there are two cases : if E_x contains the unique central subgroup $\hat{S}/S$ of order p of D, then the permutation module $\mathbb{Q}D/E_x$ is inflated from a proper quotient of D. Hence it is a direct sum of non-faithful irreducible $\mathbb{Q}D$-modules. Hence

$$\langle \Phi_D,\mathbb{Q}D/E_x\rangle_D=0$$

in this case. And if E_x does not contain $\hat{S}/S$, then it is either trivial, or non-central of order p (this can only happen of course if P is dihedral or semi-dihedral, hence if $p=2$).

If E_x is trivial, i.e. if $T\cap {}^xR\subseteq S$, then

$$\langle \Phi_D,\mathbb{Q}D/E_x\rangle_D=\dim_{\mathbb{Q}}\Phi_D \quad ,$$

And if E_x is non-central of order p in D (hence if $p=2$ and D is dihedral or semi-dihedral), then

$$\langle \Phi_D,\mathbb{Q}D/E_x\rangle_D=\langle \mathbb{Q}D/E_x,\mathbb{Q}D/E_x\rangle_D=\langle \Phi_D,\Phi_D\rangle_D m(\Phi_D,\mathbb{Q}D/E_x) \quad .$$

But in this case $E=E_x$ is a basic subgroup of D, and Φ_D is the corresponding simple $\mathbb{Q}D$-module V_E (see [1] 3.7). By assertion 2 of Proposition 2.5 of [1]

$$m(V_E,\mathbb{Q}D/E)=\frac{|\mathcal{I}_D(E,E)|}{|\mathcal{I}_D(E,E)|}=1 \quad ,$$

thus

$$\langle \Phi_D,\mathbb{Q}D/E_x\rangle_D=\langle \Phi_D,\Phi_D\rangle_D \quad .$$

This gives finally

$$\langle V_Q,\mathbb{Q}P/R\rangle_P=a\dim_{\mathbb{Q}}\Phi_D+b\langle \Phi_D,\Phi_D\rangle_D$$

where

$$\begin{aligned}a &= \mathrm{card}\{x\in T\backslash P/R \mid E_x=1\}\\ b &= \mathrm{card}\{x\in T\backslash P/R \mid |E_x|=2,\ E_x\not\subseteq Z(D)\}\end{aligned}$$

On the other hand

$$\langle V_Q, \mathbb{Q}P/R\rangle_P = \langle V_Q, V_Q\rangle_P m(V_Q, \mathbb{Q}P/R) = \langle \Phi_D, \Phi_D\rangle_D m(V_Q, \mathbb{Q}P/R) \quad ,$$

hence

$$m(V_Q, \mathbb{Q}P/R) = \frac{\dim_{\mathbb{Q}} \Phi_D}{\langle \Phi_D, \Phi_D\rangle_D}\, a + b \quad .$$

Now there are two cases : if D is cyclic or generalized quaternion, then

$$\dim_{\mathbb{Q}} \Phi_D = (p-1)|D|/p = \langle \Phi_D, \Phi_D\rangle_D$$

(see [1] 3.7 and 5.5), and $Q = S_Q$ in this case. Lemma 4.1 holds in this case. And if D is dihedral or semi-dihedral, then $\dim_{\mathbb{Q}} \Phi_D = |D|/4$ and $\langle \Phi_D, \Phi_D\rangle_D = |D|/8$, thus $\frac{\dim_{\mathbb{Q}} \Phi_D}{\langle \Phi_D, \Phi_D\rangle_D} = 2$. But $|Q : S_Q| = 2$ in this case, and this completes the proof. □

4.2. Lemma : *Let Q be an origin in P, with associated section (T_Q, S_Q) and corresponding simple $\mathbb{Q}P$-module V_Q. Then the multiplicity $m(V_Q, \mathbb{Q}P/S_Q)$ is equal to $|Q : S_Q|$, and the kernel of the projection map $\mathbb{Q}P/S_Q \to \mathbb{Q}P/\hat{S}_Q$ is isomorphic to a direct sum of $|Q : S_Q|$ copies of V_Q.*

Proof: Let $D = T_Q/S_Q$. If D is cyclic or generalized quaternion, then $Q = S_Q$, $\tilde{Q} = \hat{S}_Q$, and $m(V_Q, \mathbb{Q}P/Q) = \frac{|\mathcal{I}_P(Q,Q)|}{|\mathcal{I}_P(Q,Q)|} = 1$, and the result holds in this case, since V_Q is the kernel of the projection map $\mathbb{Q}P/Q \to \mathbb{Q}P/\tilde{Q}$.

Suppose that D is dihedral or semi-dihedral (hence $p = 2$), and let Z denote the unique central subgroup of order 2 of D. There is a unique faithful irreducible $\mathbb{Q}D$-module Φ_D, such that $\mathrm{Def}^D_{D/Z}\Phi_D = 0$ (see [1] 3.12). Any other irreducible $\mathbb{Q}D$-module W is inflated from D/Z, i.e. such that $W = \mathrm{Inf}^D_{D/Z}\mathrm{Def}^D_{D/Z}W$. Now $\mathbb{Q}D/\mathbf{1}$ splits as

$$\mathbb{Q}D/\mathbf{1} = m\Phi_D \oplus D' \quad ,$$

for some $m \in \mathbb{N}$, where D' is a sum of simple modules which are inflated from D/Z. The above considerations show that $D' = \mathrm{Inf}^D_{D/Z}\mathrm{Def}^D_{D/Z}\mathbb{Q}D = \mathbb{Q}D/Z$. Taking dimensions in the previous equality gives

$$|D| = m \dim_{\mathbb{Q}} \Phi_D + |D|/2 = m|D|/4 + |D|/2 \quad ,$$

and it follows that $m = 2$, thus

$$\mathbb{Q}D/\mathbf{1} = 2\Phi_D \oplus \mathbb{Q}D/Z \quad .$$

Taking inflation from D to T_Q, and then induction to P gives

$$\mathbb{Q}P/S_Q = 2V_Q \oplus \mathbb{Q}P/\hat{S}_Q \quad ,$$

thus

$$m(V_Q, \mathbb{Q}P/S_Q) = 2 + m(V_Q, \mathbb{Q}P/\hat{S}_Q) \quad .$$

Since Q/S_Q is a non-central subgroup of order 2 of T_Q/S_Q, it follows that $Q \not\subseteq \hat{S}_Q$, and $|Q.\hat{S}_Q| > |Q|$ (actually $|Q.\hat{S}_Q| = 2|Q|$ since $Q \cap \hat{S}_Q = S_Q$). Since Q is basic, it follows that $I_P\big((Q.\hat{S}_Q), Q\big) = \emptyset$, hence that $m\big(V_Q, \mathbb{Q}P/(Q.\hat{S}_Q)\big) = 0$.

Now if $m(V_Q, \mathbb{Q}P/\hat{S}_Q) \neq 0$, it follows that V_Q appears as a direct summand of the kernel K_Q of the projection map

$$\mathbb{Q}P/\hat{S}_Q \to \mathbb{Q}P/(Q.\hat{S}_Q) \quad .$$

The dimension of K_Q is equal to

$$\dim_{\mathbb{Q}} K_Q = \frac{|P:S_Q|}{2} - \frac{|P:S_Q|}{4} = \frac{|P:S_Q|}{4} = \dim_{\mathbb{Q}} V_Q \quad .$$

Hence $K_Q \cong V_Q$ is an irreducible $\mathbb{Q}P$-module. By Proposition 4 of [4], it follows that $\hat{S}_Q$ is a basic subgroup of P, and in particular the group $W = N_P(\hat{S}_Q)/\hat{S}_Q$ is cyclic or generalized quaternion. But W contains the group $T_Q/\hat{S}_Q \cong D/Z$, which is dihedral. This cannot happen, and it follows that $m(V_Q, \mathbb{Q}P/\hat{S}_Q) = 0$, thus $m(V_Q, \mathbb{Q}P/S_Q) = 2$ in this case, as was to be shown.

Now the kernel L_Q of the projection map $\mathbb{Q}P/S_Q \to \mathbb{Q}P/\hat{S}_Q$ has a direct summand isomorphic to a direct sum of 2 copies of V_Q. But

$$\dim_{\mathbb{Q}} L_Q = |P:S_Q| - \frac{1}{2}|P:S_Q| = 2\dim_{\mathbb{Q}} V_Q \quad ,$$

hence $L_Q \cong 2V_Q$, completing the proof. □

4.3. Corollary : *Let (T, S) be a non-trivial genetic section of P, and let $x \in P$.*

1. *If $\hat{S} \cap {}^xS \subseteq S$, then $x \in T$. In particular $T = N_P(S)$.*
2. *The group $(T \cap {}^xS)S/S$ cannot be a non-central subgroup of order p of T/S.*

Proof: There exists an origin Q in P such that $(T, S) = (T_Q, S_Q)$. With the notation of the two previous lemmas, one has that

$$|Q:S|a + b = |Q:S| \quad ,$$

where

$$\begin{aligned} a &= \operatorname{card}\{x \in T\backslash P/S \mid T \cap {}^xS \subseteq S\} \\ b &= \operatorname{card}\{x \in T\backslash P/S \mid |(T \cap {}^xS)S/S| = p,\ (T \cap {}^xS)S/S \not\subseteq Z(T/S)\} \end{aligned}$$

Thus $a \geq 1$, and $b \geq 0$. Since $|Q:S|(a-1) + b = 0$, it follows that $b = 0$ and $a = 1$. Assertion 2 follows from $b = 0$. For Assertion 1, observe that the equality $a = 1$ means that

$$\forall x \in P,\ T \cap {}^xS \subseteq S \implies x \in T \quad .$$

Set $L = (T \cap {}^xS)S$. Then L/S is a subgroup of T/S. If $\hat{S} \cap {}^xS \subseteq S$, then $L/S \cap \hat{S}/S = \mathbf{1}$. Since L/S cannot be non-central of order 2 in T/S, it follows that $L/S = \mathbf{1}$. Thus $T \cap {}^xS \subseteq S$, and $x \in T$. □

Corollary 4.3 leads to the following combinatorial characterization of genetic sections :

4.4. Proposition : *Let P be a finite p-group, and let (T, S) be a section of P. Let $Z_P(S)$ denote the subgroup of P defined by*

$$Z_P(S)/S = Z\big(N_P(S)/S\big) \quad .$$

Then the following conditions are equivalent :

1. *The section (T,S) is an genetic section of P.*
2. *The group $N_P(S)/S$ has normal p-rank 1, the group T is equal to $N_P(S)$, and if $x \in P$ is such that $S^x \cap Z_P(S) \subseteq S$, then $x \in N_P(S)$.*

Proof: Suppose first that (T,S) is an genetic section of P. Then either $T = S = P$, and Assertion 2 holds trivially, or (T,S) is a non-trivial genetic section. In this case $T = N_P(S)$ by Corollary 4.3, thus $N_P(S)/S = T/S$ has normal p-rank 1. Let $x \in P$. Since $\hat{S}/S = \Omega_1\big(Z_P(S)/S\big)$, the hypotheses $S^x \cap \hat{S} \subseteq S$ and $S^x \cap Z_P(S) \subseteq S$ are equivalent, and Assertion 2 holds by Corollary 4.3.

Conversely, suppose that Assertion 2 holds. Then either $S = T = P$, and (T,S) is an genetic section of P, or S is a proper subgroup of P, thus also a proper subgroup of $T = N_P(S)$. Let $\hat{S}$ be the subgroup of P defined by $\hat{S}/S = \Omega_1\big(Z(T/S)\big)$.

Let Q/S be a basic subgroup of T/S, such that $Q/S \cap Z(T/S) = \mathbf{1}$. Such a subgroup exists because T/S has a faithful irreducible rational representation. Then in particular $\hat{S} \subseteq N_P(Q)$ and $Q \cap \hat{S} = S$. Now if $x \in I_P(Q,Q)$, then

$$S^x \cap \hat{S} \subseteq Q^x \cap N_P(Q) \subseteq Q \cap \hat{S} = S \quad ,$$

hence $S^x = S$. Thus $I_P(Q,Q) \subseteq N_P(S) = T$, and in particular $N_P(Q) = N_T(Q)$. Let $\bar{I}$ denote the set of elements $x \in T$ such that $xS \in I_{T/S}(Q/S, Q/S)$. If $x \in \bar{I}$, then

$$Q^x \cap N_P(Q) = Q^x \cap N_T(Q) \subseteq Q \quad ,$$

thus $x \in I_P(Q,Q)$. Moreover $\bar{I}$ generates T, since $I_{T/S}(Q/S, Q/S)$ generates T/S by Lemma 7.2 of [1]. It follows that the group T_Q generated by $I_P(Q,Q)$ contains T, and is contained in $N_P(S)$. Hence $T_Q = T = N_P(S)$.

Now S_Q is the largest normal subgroup of T contained in Q, hence it is equal to S, since Q/S intersects the center of T/S trivially. Hence $(T_Q, S_Q) = (T,S)$, and T_Q/S_Q has normal p-rank 1. Thus Q is an origin in P, and $(T,S) = (T_Q, S_Q)$ is an genetic section of P by Proposition 7.4 of [1]. □

4.5. Lemma : *Let (T,S) and (T',S') be non-trivial genetic sections of P. Suppose that there exists x and y in P such that*

$$\hat{S} \cap {}^xS' \subseteq S \qquad \textit{and} \qquad \hat{S'} \cap {}^yS \subseteq S' \quad .$$

Then $(T,S) \,\text{—}_P\, (T',S')$.

Proof: Choose origins R and R' in P such that $(T,S) = (T_R, S_R)$ and $(T',S') = (T_{R'}, S_{R'})$. Consider the subgroup $L = (T \cap {}^xS').S/S$ of T/S. Since

$$L \cap (\hat{S}/S) = \big((T \cap {}^xS').S \cap \hat{S}\big)/S = (T \cap {}^xS' \cap \hat{S}).S/S = (\hat{S} \cap {}^xS').S/S = \mathbf{1} \quad ,$$

the group L is trivial or non-central of order 2. By Lemma 4.1, the multiplicity $m(V_R, \mathbb{Q}P/S')$ is non-zero. Since R is basic, it follows that $|S'| \leq |R|$. By symmetry $m(V_{R'}, \mathbb{Q}P/S) \neq 0$ and $|S| \leq |R'|$.

Suppose that $|R'| < |R|$. Then $|R'| \leq |R|/p \leq |S| \leq |R'|$. Thus $|R'| = |R|/p = |S|$ in this case, and this can only happen if $p = 2$ and T/S is dihedral or semi-dihedral.

Since $|R| \neq |R'|$, the modules V_R and $V_{R'}$ are not isomorphic. Hence there is a decomposition

$$\mathbb{Q}P/S = \mathbb{Q} \oplus 2V_R \oplus V_{R'} \oplus W \quad ,$$

for some $\mathbb{Q}P$-module W, since $m(V_R, \mathbb{Q}P/S) = 2$ by Lemma 4.2. It follows that

$$|P:S| > 2\frac{|P:S|}{4} + \frac{|P:R'|}{2} = \frac{|P:S|}{2} + \frac{|P:R'|}{2} \quad ,$$

hence $|R'| > |S| = |R|/2$. It follows that $|R'| \geq |R|$, contradicting the assumption $|R'| < |R|$.

By symmetry, it follows that $|R| = |R'|$. Suppose that $V_R \not\cong V_{R'}$. If T/S and T'/S' are both cyclic or generalized quaternion, then $R = S$ and $R' = S'$. Since $\mathbb{Q} \oplus V_R \oplus V_{R'}$ is a direct summand of $\mathbb{Q}P/S = \mathbb{Q}P/R$, it follows that

$$|P:R| > (1 - \frac{1}{p})|P:R| + (1 - \frac{1}{p})|P:R'| = (2 - \frac{2}{p})|P:R| \geq |P:R|$$

since $p \geq 2$. This is impossible.

Hence $p = 2$, and at least one of T/S or T'/S', say T/S, is dihedral or semi-dihedral. This implies in particular that $|R:S| = 2$.

If $m(V_{R'}, \mathbb{Q}P/R) \neq 0$, then $\mathbb{Q} \oplus V_R \oplus V_{R'}$ is a direct summand of $\mathbb{Q}P/R$, and

$$|P:R| > \frac{1}{2}|P:R| + \frac{1}{2}|P:R'| = |P:R| \quad ,$$

which is impossible. Thus $m(V_{R'}, \mathbb{Q}P/R) = 0$.

Now there are decompositions

$$\begin{aligned} \mathbb{Q}P/S &= \mathbb{Q} \oplus 2V_R \oplus V_{R'} \oplus W \\ \mathbb{Q}P/R &= \mathbb{Q} \oplus V_R \oplus W' \quad , \end{aligned}$$

for some $\mathbb{Q}P$-modules W and W', with $m(V_R, W') = m(V_{R'}, W') = 0$. Hence the kernel K of the projection map

$$\mathbb{Q}P/S \to \mathbb{Q}P/R$$

has a direct summand isomorphic to $V_R \oplus V_{R'}$. But

$$\dim_{\mathbb{Q}} K = 2|P:R| - |P:R| = |P:R| \quad ,$$

whereas

$$\dim_{\mathbb{Q}}(V_R \oplus V_{R'}) = \frac{1}{2}|P:R| + \frac{1}{2}|P:R'| = |P:R| \quad .$$

Hence $K \cong V_R \oplus V_{R'}$, and

$$\mathbb{Q}P/S \cong V_R \oplus V_{R'} \oplus \mathbb{Q}P/R \cong 2V_R \oplus \mathbb{Q}P/\hat{S}$$

by Lemma 4.2. This shows that $V_{R'}$ appears as a direct summand of $\mathbb{Q}P/\hat{S}$. On the other hand $V_{R'}$ cannot appear as a direct summand of $\mathbb{Q}P/(R.\hat{S})$, since R' is basic, and $|R.\hat{S}| > |R'|$. Hence $V_{R'}$ is a direct summand of the kernel L of the projection map $\mathbb{Q}P/\hat{S} \to \mathbb{Q}P/(R.\hat{S})$. Since

$$\dim_{\mathbb{Q}} L = \frac{1}{2}|P:S| - \frac{1}{4}|P:S| = \frac{1}{4}|P:S| = \frac{1}{2}|P:R| = \frac{1}{2}|P:R'| = \dim_{\mathbb{Q}} V_{R'} \quad ,$$

it follows that $L \cong V_{R'}$.

Since L is a simple $\mathbb{Q}P$-module, it follows that $\hat{S}$ is a basic subgroup of P, hence that $N_P(\hat{S})/\hat{S}$ is cyclic or generalized quaternion. But this group contains $T/\hat{S}$, which is dihedral. This cannot happen, and this contradiction shows that $V_R \cong V_{R'}$, or equivalently $(T,S) \longrightarrow_P (T',S')$, completing the proof. □

4.6. Theorem : *Let P be a finite p-group. If $\sigma=(T,S)$ and $\sigma'=(T',S')$ are genetic sections of P, set*

$$\pi^{\sigma}_{\sigma'} = b_{T,S}\circ a_{T',S'} \in \mathrm{Hom}_{\mathcal{C}_{p,k}}(T'/S',T/S) \quad .$$

If $\pi^{\sigma}_{\sigma'}\neq 0$, then $\sigma \,\textbf{—}_P\, \sigma'$.

(Recall that in this case, the simple $\mathbb{Q}P$-modules corresponding to σ and σ' are isomorphic).

Proof: Suppose first that σ is the trivial genetic section. If σ' is also trivial, there is nothing to prove since then $\sigma'=\sigma$, hence $\sigma' \,\textbf{—}_P\, \sigma$. And if σ' is non-trivial, then

$$b_{T,S}\circ a_{T',S'} = P\backslash P\times_P (P/S' - P/\hat{S}') = P\backslash P/S - P\backslash P/\hat{S}' = 0 \quad .$$

A similar argument, or the fact that $(\pi^{\sigma}_{\sigma'})^{op}=\pi^{\sigma'}_{\sigma}$ shows that $\pi^{\sigma}_{\sigma'}=0$ if σ' is trivial and σ is not.

Hence I can suppose that σ and σ' are both non-trivial. Then

$$\pi^{\sigma}_{\sigma'} = (S\backslash P - \hat{S}\backslash P)\times_P (P/S'-P/\hat{S}') \quad .$$

Claim : *Let (B,A) be a section of P such that $A\backslash P\circ(P/S-P/\hat{S})$ is a non-zero element of $\mathrm{Hom}_{\mathcal{C}_{p,k}}(T/S,B/A)$. Then there exists an element $x\in P$ such that $\hat{S}\cap A^x\subseteq S$.*

Indeed the hypothesis means that $A\backslash P/S\neq A\backslash P/\hat{S}$ in $\mathrm{Hom}_{\mathcal{C}_{p,k}}(T/S,B/A)$. Hence there exists $x\in P$ such that $AxS\neq Ax\hat{S}$, or equivalently $x\hat{S}\not\subseteq AxS$, i.e. $\hat{S}\not\subseteq A^x.S$. Since $|\hat{S}:S|=p$, this means that $\hat{S}\cap A^x\subseteq S$, proving the claim.

Thus if $\pi^{\sigma}_{\sigma'}\neq 0$ in $\mathrm{Hom}_{\mathcal{C}_{p,k}}(T'/S',T/S)$, then at least one of the elements

$$u = S\backslash P\circ(P/S'-P/\hat{S}') \qquad \text{or} \qquad v=\hat{S}\backslash P\circ(P/S'-P/\hat{S}')$$

is non-zero. If $u\neq 0$, then there exists $y\in P$ such that $\hat{S}'\cap {}^yS\subseteq S'$. And if $v\neq 0$, then there exists $y\in P$ such that $\hat{S}'\cap {}^y\hat{S}\subseteq S'$, which implies $\hat{S}'\cap {}^yS\subseteq S'$. Similarly, since $(\pi^{\sigma}_{\sigma'})^{op}=\pi^{\sigma'}_{\sigma}\neq 0$, it follows that there exists $x\in P$ such that $\hat{S}\cap {}^xS'\subseteq S$. By Lemma 4.5, it follows that $(T,S)\,\textbf{—}_P\,(T',S')$. ∎

4.7. Lemma : *Let (T,S) and (T',S') be genetic sections of P, and suppose that $(T,S)\,\textbf{—}_P\,(T',S')$. Then :*

1. *There exists a unique double coset SxT' in P such that $S^x\cap T'\subseteq S'$. In this case $(T,S)\,\textbf{—}\,({}^xT',{}^xS')$.*
2. *If $x\in P$, the group $(S^x\cap T')S'/S'$ is not a non-central subgroup of order p of T'/S'.*

Proof: If (T,S) is trivial, then so is (T',S'), and there is nothing to prove. Hence I can suppose that (T,S) and (T',S') are both non-trivial, and choose origins R and R' in P such that $(T_R,S_R)=(T,S)$ and $(T_{R'},S_{R'})=(T',S')$.

There exists a unique double coset TxT' such that $(T,S)\,\textbf{—}\,{}^x(T',S')$. Since the simple $\mathbb{Q}P$-modules V_R and $V_{R'}$ associated to (T,S) and (T',S') are isomorphic, it follows that $|R|=|R'|$. Moreover $R/S\cong R'/S'$, and $T/S\cong T'/S'$.

By Lemmas 4.1 and 4.2, one has that

$$m(V_R,\mathbb{Q}P/S') = m(V_{R'},\mathbb{Q}P/S') = |R':S'| = |R:S| = |R:S|a+b \quad ,$$

where

$$\begin{array}{rcl} a & = & \mathrm{card}\{x \in T'\backslash P/S \mid T' \cap {}^xS \subseteq S'\} \\ b & = & \mathrm{card}\{x \in T'\backslash P/S \mid |(T' \cap {}^xS)S'/S'| = p,\ (T' \cap {}^xS)S'/S' \not\subseteq Z(T'/S')\} \quad . \end{array}$$

Since $a \geq 1$ if $(T,S) \longrightarrow_P (T',S')$, it follows that $a = 1$ and $b = 0$. Assertion 2 follows from $b = 0$. Since $a = 1$, there is a unique double coset SxT' in P such that $S^x \cap T' \subseteq S'$. This is the case in particular if $(T,S) \longrightarrow {}^x(T',S')$. Thus $SxT' = TxT'$ is also the unique double coset TxT in P such that $(T,S) \longrightarrow {}^x(T',S')$. □

4.8. Lemma : *Let (T,S) be an genetic section of P. Then*

$$a_{T,S} = a_{T,S} f_{\mathbf{1}}^{T/S} \qquad b_{T,S} = f_{\mathbf{1}}^{T/S} b_{T,S} \quad .$$

Proof: Since $b_{T,S} = (a_{T,S})^{op}$ and $f_{\mathbf{1}}^{T/S} = (f_{\mathbf{1}}^{T/S})^{op}$, the two assertions are equivalent. Moreover if (T,S) is the trivial genetic section, the result is trivial. And if (T,S) is non-trivial, then $\Omega_1 Z(T/S) = \hat{S}/S$, thus

$$f_{\mathbf{1}}^{T/S} = (T/S)/(S/S) - (T/S)/(\hat{S}/S) = T/S - T/\hat{S} \quad ,$$

It follows that

$$(P/S) \times_{T/S} f_{\mathbf{1}}^{T/S} = P/S - P/\hat{S} = a_{T,S} \quad ,$$

hence $a_{T,S} = a_{T,S} f_{\mathbf{1}}^{T/S}$, since $f_{\mathbf{1}}^{T/S}$ is an idempotent. □

4.9. Theorem : *Let $\sigma = (T,S)$ and $\sigma' = (T',S')$ be genetic sections of P, and suppose that $(T,S) \longrightarrow_P (T',S')$. Let TxT' denote the unique double coset in P such that $(T,S) \longrightarrow {}^x(T',S')$. Denote by $\varphi_{\sigma'}^{\sigma}$ the $(T/S, T'/S')$-biset $S\backslash TxT'/S'$, viewed as an element of $\mathrm{Hom}_{\mathcal{C}_{p,k}}(T'/S', T/S)$. Then*

$$\pi_{\sigma'}^{\sigma} = b_{T,S} \circ a_{T',S'} = f_{\mathbf{1}}^{T/S} \varphi_{\sigma'}^{\sigma} = \varphi_{\sigma'}^{\sigma} f_{\mathbf{1}}^{T'/S'}$$

in $\mathrm{Hom}_{\mathcal{C}_{p,k}}(T'/S', T/S)$.

Proof: If σ is trivial, so is σ', and the result is trivial in this case. Suppose then that σ and σ' are non-trivial, and denote by R and R' origins of P such that $(T,S) = (T_R, S_R)$ and $(T',S') = (T_{R'}, S_{R'})$. Here again $|R| = |R'|$, the groups R/S and R'/S' are isomorphic, as well as the groups T/S and T'/S'. In particular $|S| = |S'|$.

There are two steps in the proof :

Step 1 : **Claim :** *The composition $\hat{S}\backslash P \circ (P/S' - P/\hat{S}')$ is zero in* $\mathrm{Hom}_{\mathcal{C}_{p,k}}(T'/S', T/S)$: indeed otherwise, by the Claim of the proof of Theorem 4.6, there exists an element x in P such that $\hat{S}^x \cap \hat{S}' \subseteq S'$. Then the subgroup $L = (\hat{S}^x \cap T').S'/S'$ of T'/S' is such that $L \cap \hat{S}'/S = \mathbf{1}$. Thus L is either trivial, or non-central of order p in T'/S'. By Lemma 4.1, the multiplicity $m(V_{R'}, \mathbb{Q}P/\hat{S})$ is non-zero. Since R' is basic, this implies

$$|R'| \leq |\hat{S}| = |\hat{S}'| \leq |R| \quad ,$$

hence $|R'| = |\hat{S}'| = |R|$. Hence $|R' : S'| = p$, and this can occur only if $p = 2$ and T/S and T'/S' are dihedral or semi-dihedral. Moreover $|R.\hat{S}| = 2|R| > |R'|$, thus

$m\big(V_{R'}, \mathbb{Q}P/(R.\hat{S})\big) = 0$. Hence $V_{R'}$ appears as a direct summand of the kernel K of the projection map $\mathbb{Q}P/\hat{S} \to \mathbb{Q}P/(R.\hat{S})$. This kernel has dimension

$$\dim_{\mathbb{Q}} K = \frac{1}{2}|P:S| - \frac{1}{4}|P:S| = \frac{1}{4}|P:S| = \dim_{\mathbb{Q}} V_{R'} \quad ,$$

hence $K \cong V_{R'}$ is irreducible. Thus $\hat{S}$ is basic, hence $N_P(\hat{S})/\hat{S}$ is cyclic or generalized quaternion. But this group contains $T/\hat{S}$, which is dihedral. This contradiction proves the claim.

Step 2 : It follows from Step 1 that

$$\begin{aligned} \pi^{\sigma}_{\sigma'} &= S\backslash P \times_P (P/S' - P/\hat{S}') \\ &= \sum_{x \in T\backslash P/T'} (S\backslash TxT'/S' - S\backslash TxT'/\hat{S}') \quad \text{in } \mathrm{Hom}_{\mathcal{C}_{p,k}}(T'/S', T/S) \quad . \end{aligned}$$

Moreover for $x \in P$

$$SxS' \neq Sx\hat{S}' \Leftrightarrow \hat{S}' \neq (S^x \cap \hat{S}').S' \Leftrightarrow S^x \cap \hat{S}' \subseteq S' \quad .$$

In that case the subgroup $L = (S^x \cap T').S'/S'$ of T'/S' is trivial or non-central of order 2, hence trivial by Lemma 4.7. There is a single double coset Tx_0T' of such elements x in P, and

$$\pi^{\sigma}_{\sigma'} = \left(\sum_{x \in T\backslash (P - Tx_0T')/T'} (S\backslash TxT'/S' - S\backslash TxT'/\hat{S}') \right) + (S\backslash Tx_0T'/S' - S\backslash Tx_0T'/\hat{S}') \quad .$$

If $x \in P - Tx_0T'$, then $SyS' = Sy\hat{S}'$ for all $y \in TxT'$. It follows that

$$S\backslash TxT'/S' = S\backslash TxT'/\hat{S}'$$

as $(T/S, T'/S')$-bisets, and finally

$$\pi^{\sigma}_{\sigma'} = S\backslash Tx_0T'/S' - S\backslash Tx_0T'/\hat{S}' \quad .$$

Now recall that $f_{\mathbf{1}}^{T'/S'} = T'/S' - T'/\hat{S}'$. It clearly follows that

$$\pi^{\sigma}_{\sigma'} = (S\backslash Tx_0T'/S') \times_{T'/S'} f_{\mathbf{1}}^{T'/S'} = \varphi^{\sigma}_{\sigma'} f_{\mathbf{1}}^{T'/S'} \quad .$$

Exchanging σ and σ' now gives

$$\pi^{\sigma'}_{\sigma} = \varphi^{\sigma'}_{\sigma} f_{\mathbf{1}}^{T/S} \quad ,$$

and taking opposite bisets gives

$$\pi^{\sigma}_{\sigma'} = f_{\mathbf{1}}^{T/S} \varphi^{\sigma}_{\sigma'} \quad ,$$

as was to be shown. □

5. Proof of Theorem 3.2

Let F be a biset functor defined over p-groups, with values in k-Mod. Let P be a finite p-group, and let $\mathcal{S}$ be a set of representatives of genetic sections of P modulo the relation $\frac{\quad}{\quad}_P$. Denote by $\mathcal{I}_\mathcal{S}$ the map

$$\mathcal{I}_\mathcal{S} = \mathop{\oplus}_{(T,S)\in\mathcal{S}} F(a_{T,S}) : \mathop{\oplus}_{(T,S)\in\mathcal{S}} \partial F(T/S) \to F(P) \quad .$$

It follows from Lemma 4.8 that the image of the map $F(b_{T,S})$ is contained in $\partial F(T/S)$, and one can define a map $\mathcal{D}_\mathcal{S}$ by

$$\mathcal{D}_\mathcal{S} = \mathop{\oplus}_{(T,S)\in\mathcal{S}} F(b_{T,S}) : F(P) \to \mathop{\oplus}_{(T,S)\in\mathcal{S}} \partial F(T/S) \quad .$$

Choose $\sigma = (T,S) \in \mathcal{S}$. Then for any $u \in \partial F(T/S)$, and for any (T',S') in $\mathcal{S}$, different from (T,S), one has that

$$F(b_{T',S'})F(a_{T,S})(u) = 0$$

by Theorem 4.6. Moreover, by Theorem 4.9

$$F(b_{T,S})F(a_{T,S})(u) = F(\varphi_\sigma^\sigma)F(f_{\mathbf{1}}^{T/S})(u) = u \quad ,$$

since $F(f_{\mathbf{1}}^{T/S})(u) = u$ for $u \in \partial F(T/S)$, and since moreover

$$\varphi_\sigma^\sigma = S\backslash T/S = T/S = \mathrm{Id}_{T/S} \quad .$$

It follows that $\mathcal{D}_\mathcal{S} \circ \mathcal{I}_\mathcal{S}$ is the identity map, and Theorem 3.2 follows. □

6. An application to the Dade group

The definitions and notation concerning the Dade group refer to [6] (in particular, see [6] Corollary 3.10 for the definition of the map $D(U)$ between Dade groups associated to a biset U).

6.1. Theorem : *Let P be a finite p-group. Denote by $D^t(P)$ the torsion subgroup of the Dade group of P, and by $T^t(P)$ the subgroup of $D^t(P)$ formed by the images of the torsion endo-trivial modules.*

1. *If Q is a non-trivial p-group of normal p-rank one, then $T^t(Q)$ is equal to the kernel of the map $\mathrm{Def}^Q_{Q/Z} : D^t(Q) \to D^t(Q/Z)$, where Z is the unique central subgroup of order p in Q. It follows that $T^t(Q) = D(f_{\mathbf{1}}^Q)D^t(Q)$.*
2. *Let $\mathcal{S}$ be a set of representatives of genetic sections of P, modulo the relation $\frac{\quad}{\quad}_P$. Then the map*

$$\mathcal{I}_\mathcal{S} = \mathop{\oplus}_{(R,S)\in\mathcal{S}} \mathrm{Teninf}^P_{R/S} : \mathop{\oplus}_{(R,S)\in\mathcal{S}} T^t(R/S) \to D^t(P) \quad ,$$

is a split injection.

Proof: Assertion 1 follows from Dade's Theorem on the structure of the Dade group of cyclic groups (see [9], [10]), and of Lemma 10.2 of [8] for the other cases.

Assertion 2 is not a direct consequence of Theorem 3.2, for the Dade group is *not* a biset functor over p-groups, because of the extra phenomenon of "Galois torsion", described in Section 3 of [6]. However, the proof of Theorem 3.2 can be adapted to take this Galois torsion into account, in the following way :

First observe that since the Dade group of the trivial group is trivial, one can always replace $\mathcal{S}$ by $\mathcal{S}-\{\mathbf{1}\}$ in the statement of the theorem, i.e. suppose that the genetic sections considered here are non-trivial ones.

It follows from Assertion 1 that if Q is a non-trivial p-group of normal p-rank one, and if $u\in T^t(Q)$, then $\mathrm{Def}^Q_{Q/Z}u=0$.

This means that the map $\mathcal{I}_\mathcal{S}$ can also be defined by

$$\mathcal{I}_\mathcal{S}=\mathop{\oplus}_{(R,S)\in\mathcal{S}} D(a_{R,S})\quad,$$

since $D(P/\hat{S})=\mathrm{Teninf}^P_{R/\hat{S}}\circ\mathrm{Def}^{R/S}_{R/\hat{S}}$.

Moreover, if $v\in D^t(P)$, then the element $u=D(b_{R,S})(v)$ is an element of $D^t(R/S)$ in the kernel of the deflation map $\mathrm{Def}^{R/S}_{R/\hat{S}}$, hence $u\in T^t(R/S)$. This allows to define a map

$$\mathcal{D}_\mathcal{S}:D^t(P)\to\mathop{\oplus}_{(R,S)\in\mathcal{S}} T^t(R/S)\quad,$$

and this map will be a left inverse to $\mathcal{I}_\mathcal{S}$.

To see this, let (R',S') be an element of $\mathcal{S}$, and let (R,S) be any section of P. If u is an element of $D^t(R'/S')$, one has that

$$\begin{aligned}D(S\backslash P)D(P/S')(u)&=\mathrm{Defres}^P_{R/S}\mathrm{Teninf}^P_{R'/S'}u\\&=\mathrm{Def}^R_{R/S}\mathrm{Res}^P_R\mathrm{Ten}^P_{R'}\mathrm{Inf}^{R'}_{R'/S'}u\\&=\sum_{x\in R\backslash P/R'}\gamma_{|S:S\cap{}^xR'|}D(S\backslash RxR'/S')(u)\quad.\end{aligned}$$

Similarly

$$D(S\backslash P)D(P/\hat{S}')(u)=\sum_{x\in R\backslash P/R'}\gamma_{|S:S\cap{}^xR'|}D(S\backslash RxR'/\hat{S}')(u)$$

by Proposition 3.10 of [6]. Hence

$$D(S\backslash P)D(a_{R',S'})(u)=\sum_{x\in R\backslash P/R'}\gamma_{|S:S\cap{}^xR'|}\big(D(S\backslash RxR'/S')(u)-D(S\backslash RxR'/\hat{S}')(u)\big)\quad.$$

If this is non-zero, then there exists $y\in P$ such that $Sy\hat{S}'\neq SyS'$, or equivalently $\hat{S}'\cap S^y\subseteq S'$.

Suppose now that $(R,S)\in\mathcal{S}$. The same argument shows that if $D(\hat{S}\backslash P)D(a_{R,S})(u)$ is non-zero, then there exists $y\in P$ such that $\hat{S}'\cap\hat{S}^y\subseteq S'$, and this implies $\hat{S}'\cap S^y\subseteq S'$.

Since

$$D(b_{R,S})=D(S\backslash P)-D(\hat{S}\backslash P)\quad,$$

it follows that if $D(b_{R,S})D(a_{R',S'})(u)$ is non-zero, then there exists $y\in P$ such that $\hat{S}'\cap S^y\subseteq S'$.

Recall that $D(P/\hat{S}')(u) = 0$ for $u \in D^t(R'/S')$. Hence

$$D(b_{R,S})D(a_{R,S})(u) = D(b_{R,S})D(P/S)(u) \quad .$$

By the above computation, this is equal to

$$\sum_{x \in R\backslash P/R'} \big(\gamma_{|S:S\cap {}^xR'|} D(S\backslash RxR'/S')(u) - \gamma_{|\hat{S}:\hat{S}\cap {}^xR'|} D(\hat{S}\backslash RxR'/S')(u)\big) \quad .$$

If this is non-zero, then at least one of the following holds :

- There exists $x \in P$ such that $|S : S \cap {}^xR'| \neq |\hat{S} : \hat{S} \cap {}^xR'|$, or equivalently
$$|\hat{S} : S| = p \neq |\hat{S} \cap {}^xR' : S \cap {}^xR'| \quad ,$$
hence $\hat{S} \cap {}^xR' \subseteq S$. This implies $\hat{S} \cap {}^xS' \subseteq S$.
- There exists $x \in P$ such that $\hat{S}xS' \neq SxS'$, or equivalently $\hat{S} \cap {}^xS' \subseteq S$.

In both cases, there exists $x \in P$ with $\hat{S} \cap {}^xS' \subseteq S$.

Now Lemma 4.5 shows that if $D(b_{R,S})D(a_{R',S'})(u) \neq 0$, then $(R,S) \;\text{—}_P\; (R',S')$, hence $(R,S) = (R',S')$ since both pairs are in $\mathcal{S}$.

In this case by Corollary 4.3, there is a unique double coset RxR in P such that $\hat{S} \cap {}^xS \subseteq S$, namely the coset R. It follows that

$$\begin{aligned} D(b_{R,S})D(a_{R,S})(u) &= \gamma_{|S:S\cap R|} D(S\backslash R/S)(u) - \gamma_{|\hat{S}:\hat{S}\cap R|} D(\hat{S}\backslash R/S)(u) \\ &= D(R/S)(u) - \gamma_{|\hat{S}:\hat{S}\cap R|} D(R/\hat{S})(u) \\ &= u \end{aligned}$$

since $R/S = \mathrm{Id}_{R/S}$ and $D(R/\hat{S})(u) = 0$ for $u \in D^t(R/S)$. It follows that $\mathcal{D}_\mathcal{S} \circ \mathcal{I}_\mathcal{S}$ is the identity map, and this completes the proof of the theorem. □

6.2. Conjecture : *The map $\mathcal{I}_\mathcal{S}$ of Theorem 6.1 is an isomorphism, for any finite p-group P, and any set $\mathcal{S}$ of representatives of genetic sections of P modulo the relation —_P .*

By [8], and by Theorem 6.1, this conjecture is equivalent to the following :

6.3. Conjecture : *The torsion part $D^t(P)$ of the Dade group of P is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n_P} \oplus (\mathbb{Z}/4\mathbb{Z})^{m_P}$, where m_P is equal to the number of rational irreducible representations of P of generalized quaternion type, and n_P is equal to the number of rational irreducible representations of P whose type is*

- *cyclic of order at least 3, or semi-dihedral, or generalized quaternion, if the ground field contains cubic roots of unity.*
- *cyclic of order at least 3, or semi-dihedral, or generalized quaternion of order at least 16, if the ground field does not contain cubic roots of unity.*

6.4. Remark. Conjectures 6.2 and 6.3 are known to be true in the following cases :

- For $p \neq 2$, by a result of J. Carlson and J. Thévenaz ([7] Theorem 13.3).
- For $p \neq 2$ and metacyclic p-groups, by a result of N. Mazza, independent of Carlson and Thévenaz's result ([11]).
- For cyclic, generalized quaternion, dihedral or semi-dihedral 2-groups, by [8].
- For some other 2-groups, such as $D_8 * C_4$, $D_8 * D_8$, or $D_8 * Q_8$, by results of J. Thévenaz, and more generally for all (almost) extraspecial p-groups, by a result of N. Mazza and the author ([5]).

7. Rational biset functors and the torsion part of D^Ω

The results of previous sections lead to the following definition :

7.1. Definition : *Let F be a biset functor defined over p-groups, with values in k-Mod. The functor F is called rational if for any finite p-group P, there exists a set $\mathcal{S}$ of representatives of genetic sections of P modulo the relation $\frac{\ \ }{\ \ }_P$, such that the map*

$$\mathcal{I}_\mathcal{S} = \mathop{\oplus}_{(T,S)\in\mathcal{S}} F(a_{T,S}) : \mathop{\oplus}_{(T,S)\in\mathcal{S}} \partial F(T/S) \to F(P) \quad ,$$

is a k-module isomorphism.

7.2. Example : *The word "rational" in the previous definition comes from the example $k = \mathbb{Z}$, and $F = R_\mathbb{Q}$. Then for any p-group Q with normal p-rank 1, there is a unique faithful irreducible rational representation Φ_Q, and $\partial F(Q) \cong \mathbb{Z}$, with basis Φ_Q. Moreover, any rational irreducible representation V of P is isomorphic to $\mathrm{Ind}_T^P\mathrm{Inf}_{T/S}^T\Phi_{T/S}$ for a suitable genetic section (T,S) of P, which can be chosen in a given set $\mathcal{S}$ of representatives of such sections. Thus $V = R_\mathbb{Q}(a_{T,S})(\Phi_{T/S})$, and the map $\mathcal{I}_\mathcal{S}$ is surjective for any p-group, and any set of representatives $\mathcal{S}$. Hence it is an isomorphism, and $R_\mathbb{Q}$ is rational.*

The following lemma shows that if F is rational, then the map $\mathcal{I}_\mathcal{S}$ is an isomorphism for *any* set $\mathcal{S}$ of representatives of equivalence classes of genetic sections of P.

7.3. Lemma : *Let F be a biset functor and P be a p-group. Let $\mathcal{S}$ and $\mathcal{S}'$ be sets of representatives of genetic sections of P for the relation $\frac{\ \ }{\ \ }_P$. If the map $\mathcal{I}_\mathcal{S}$ is an isomorphism, then the map $\mathcal{I}_{\mathcal{S}'}$ is an isomorphism.*

Proof: Showing that the map $\mathcal{I}_{\mathcal{S}'}$ is an isomorphism is equivalent to showing that the map $\mathcal{D}_{\mathcal{S}'}$ is injective, since this map is split surjective by Theorem 3.2. Equivalently, since $\mathcal{I}_\mathcal{S}$ is an isomorphism by hypothesis, this amounts to showing that the map $f = \mathcal{D}_{\mathcal{S}'} \circ \mathcal{I}_\mathcal{S}$ is injective. Let $u = (u_{T,S})_{(T,S)\in\mathcal{S}}$ an element of $\mathrm{Ker}(f)$, with $u_{T,S} \in \partial F(T/S)$. If $(T', S') \in \mathcal{S}'$, then the component of $f(u)$ in $\partial F(T'/S')$ is equal to

$$b_{T',S'}\big(\sum_{(T,S)\in\mathcal{S}} a_{T,S}(u_{T,S})\big) \quad .$$

Now $b_{T',S'} \circ a_{T,S}$ is equal to zero by Theorem 4.6, unless $(T', S') \frac{\ \ }{\ \ }_P (T,S)$. There is a unique such $(T,S) \in \mathcal{S}$, and for this one

$$b_{T',S'} \circ a_{T,S}(u_{T,S}) = \varphi_{T,S}^{T',S'}(u_{T,S}) \quad ,$$

by Theorem 4.9, since $f_{\mathbf{1}}^{T/S}(u_{T,S}) = u_{T,S}$. It follows that this is equal to the component of $f(u)$ in $\partial F(T'/S')$. Hence this is zero, and $u_{T,S} = 0$ since $\varphi_{T,S}^{T',S'}$ is an isomorphism. Since for any $(T,S) \in \mathcal{S}$, there is a unique $(T',S') \in \mathcal{S}'$ such that $(T',S') \frac{\ \ }{\ \ }_P (T,S)$, thus $u_{T,S} = 0$ for any $(T,S) \in \mathcal{S}$, and $u = 0$. □

7.4. Proposition : *Let F be a biset functor over p-groups, with values in k-Mod, let F' be a subfunctor of F, and let M be any k-module.*

1. *The functor F is rational if and only if F' and F/F' are rational.*
2. *If F is rational, then* $\mathrm{Hom}(F, M)$ *is rational.*

The first assertion means that the class of rational biset functors is a Serre subclass of all biset functors. In the second assertion, recall that $\mathrm{Hom}(F,M)$ is defined by $\mathrm{Hom}(F,M)(P)=\mathrm{Hom}_k\big(F(P),M\big)$ for any p-group P, and

$$\mathrm{Hom}(F,M)(\varphi)(\alpha)=\alpha\circ F(\varphi^{op})$$

for $\varphi\in\mathrm{Hom}_{\mathcal{C}_{p,k}}(P,Q)$ and $\alpha\in\mathrm{Hom}(F,M)(P)$.

Proof: Let P be a finite p-group, and $\mathcal{S}$ be a set of representatives of equivalence classes of genetic sections of P. Let $F''=F/F'$. The diagram

$$\begin{array}{ccc}
0 & & 0\\
\downarrow & & \downarrow\\
\mathop{\oplus}\limits_{(T,S)\in\mathcal{S}}\partial F'(T/S) & \stackrel{i'}{\to} & F'(P)\\
\downarrow & & \downarrow\\
\mathop{\oplus}\limits_{(T,S)\in\mathcal{S}}\partial F(T/S) & \stackrel{i}{\to} & F(P)\\
\downarrow & & \downarrow\\
\mathop{\oplus}\limits_{(T,S)\in\mathcal{S}}\partial F''(T/S) & \stackrel{i''}{\to} & F''(P)\\
\downarrow & & \downarrow\\
0 & & 0
\end{array}$$

is commutative, where the horizontal maps are the maps $\mathcal{I}_{\mathcal{S}}$ for the corresponding functors. By the snake's lemma, and since i', i', and i'' are (split) injective, there is an exact sequence

$$0\to\mathrm{Coker}(i')\to\mathrm{Coker}(i)\to\mathrm{Coker}(i'')\to 0\quad .$$

Now saying that F is rational is equivalent to saying that $\mathrm{Coker}(i)=0$ for any P and $\mathcal{S}$. Thus $\mathrm{Coker}(i')=\mathrm{Coker}(i'')=0$, and F' and F'' are rational. Conversely, if F' and F'' are rational, then $\mathrm{Coker}(i')=\mathrm{Coker}(i'')=0$, and $\mathrm{Coker}(f)=0$, thus F is rational.

For the second assertion, denote by $\hat{F}$ the functor $\mathrm{Hom}(F,M)$, and observe that for any genetic section (T,S) of P, one has that $b_{T,S}=a_{T,S}^{op}$, and that $(f_{\mathbf{1}}^{T/S})^{op}=f_{\mathbf{1}}^{T/S}$. Applying the functor $\mathrm{Hom}_k(-,M)$ to the isomorphism

$$\mathcal{I}_{\mathcal{S}}=\mathop{\oplus}_{(T,S)\in\mathcal{S}}F(a_{T,S}):\mathop{\oplus}_{(T,S)\in\mathcal{S}}\partial F(T/S)\to F(P)\quad ,$$

gives the isomorphism

$$\mathcal{D}_{\mathcal{S}}=\mathop{\oplus}_{(T,S)\in\mathcal{S}}\hat{F}(b_{T,S}):\hat{F}(P)\to\mathop{\oplus}_{(T,S)\in\mathcal{S}}\partial\hat{F}(T/S)\quad ,$$

for the functor $\hat{F}$. Thus $\hat{F}$ is rational. □

7.5. Corollary : *The functor D^{Ω}_{tors} is rational.*

Proof: This follows from the fact that, by [2], the image of the subfunctor $R_{\mathbb{Q}}^*=\mathrm{Hom}(R_{\mathbb{Q}},\mathbb{Z})$ of the dual Burnside functor B^* is equal to the torsion subfunctor D^{Ω}_{tors} of D^{Ω}. Hence D^{Ω}_{tors} is a quotient of $\mathrm{Hom}(R_{\mathbb{Q}},\mathbb{Z})$. Since $R_{\mathbb{Q}}$ is a rational $\mathbb{Z}$-valued biset functor, so is its dual, and any quotient of it. □

7.6. Corollary : *Let P be a finite p-group. Then*

$$D^{\Omega}_{tors}(P) \cong (\mathbb{Z}/4\mathbb{Z})^{a_P} \oplus (\mathbb{Z}/2\mathbb{Z})^{b_P} \quad ,$$

where a_P is equal to the number of isomorphism classes of rational irreducible representations of P whose type is generalized quaternion, and b_P is the number of isomorphism classes of rational irreducible representations of P whose type is cyclic of order at least 3 or semi-dihedral.

Proof: This follows from the structure of the Dade group of groups of normal p-rank 1 (see [8] Theorems 5.4, 6.3, 7.1). □

Note that Corollary 7.5 is actually more precise, since it gives a generating set for $D^{\Omega}_{tors}(P) \cong (\mathbb{Z}/4\mathbb{Z})^{a_P} \oplus (\mathbb{Z}/2\mathbb{Z})^{b_P}$: if Q has normal p-rank 1, then $\partial D^{\Omega}_{tors}(Q)$ is a subgroup of the endo-trivial subgroup of $D(Q)$. In particular $D^{\Omega}(a_{T,S})(u) = \mathrm{Teninf}_{T/S}^{P}(u)$ for any genetic section (T,S) of P, and any $u \in \partial D^{\Omega}_{tors}(T/S)$.

Now if $Q = T/S$ is cyclic or generalized quaternion, then $\partial D^{\Omega}_{tors}(Q)$ is generated by $u_{T,S} = \Omega_{Q/\mathbf{1}}$, and if Q is semi-dihedral, then $\partial D^{\Omega}_{tors}$ is generated by $u_{T,S} = \Omega_{Q/\mathbf{1}} + \Omega_{Q/R}$, where R is a non-central subgroup of Q of order 2. The elements $v_{T,S} = \mathrm{Teninf}_{T/S}^{P}(u_{T,S})$, for (T,S) in a set $\mathcal{S}$ of representatives of equivalence classes of genetic sections of P for which T/S is cyclic of order at least 3, or semi-dihedral, or generalized quaternion, are a set of generators of $D^{\Omega}_{tors}(P)$, and these elements are as linearly independent as they can be : the linear combination $\sum_{(T,S)\in\mathcal{S}} \lambda_{T,S} v_{T,S}$ is zero if and only if for any $(T,S) \in \mathcal{S}$, the integer $\lambda_{T,S}$ is a multiple of the order of $v_{T,S}$, which is equal to 4 if T/S is generalized quaternion, and 2 otherwise.

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