Tensor induction of relative syzygies

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Abstract: The main result of this paper states a formula expressing the effect of tensor induction on relative syzygies in the Dade group of a p-group. This provides tools to study the structure of this group, and a new proof of theorems A and B of [5]. Finally, it is possible to say when a relative syzygy is a torsion element of the Dade group.

1. Introduction

Let p be a prime number, and k be a field of characteristic p. If P is a finite p-group, a finitely generated kP-module M is called an *endo-permutation* module if $\mathrm{End}_k(M)$ is a permutation module, i.e. admits a P-invariant k-basis. Those modules appears in many different places in the modular representation theory of finite groups, and they have been studied intensively.

In [8],[9], E.C. Dade defined a group structure on the set of equivalence classes of endo-permutation kP-modules, which is now called the Dade group of P, and denoted by $D_k(P)$. This group has natural functorial properties: when Q is a subgroup of P, there is an operation of restriction Res_Q^P from $D_k(P)$ to $D_k(Q)$. Similarly, if R is a quotient of P, there is an operation of inflation Inf_R^P from $D_k(R)$ to $D_k(P)$, and an operation Def_R^P from $D_k(P)$ to $D_k(R)$, called *deflation* in [5], which corresponds to Dade's *slash* construction. These three operations are already defined and used in Dade's papers.

It was observed by L. Puig ([13]) that when Q is a subgroup of P, tensor induction of modules gives an operation Ten_Q^P from $D_k(Q)$ to $D_k(P)$. There is moreover an obvious operation $\mathrm{Iso}_P^{P'}$ associated to a group isomorphism from P to a group P'. Recently, in a joint work with J. Thévenaz ([5]), we showed that these five operations can be described via a single formalism, using bisets.

Typical examples of endo-permutation modules are the syzygies of the trivial module. Following a suggestion of L. Puig, J.L. Alperin proved more generally that the syzygies of the trivial module relative to any permutation module are endo-permutation modules. These are defined as the kernel of the augmentation map $kX \to k$ sending each basis element x of the P-set X to $1 \in k$, and denoted by Ω_X.

The aim of this paper is to describe the effect of the five functorial operations on these relative syzygies, i.e. to state formulas giving $F_P^{P'}(\Omega_X)$ in terms of elements of the Dade group of P', for a P-set X, and one of the above operations $F_P^{P'}$. Actually, the only difficult case is the case of tensor induction, and this will be the central question of this work.

The paper is organized as follows: in sections 1 to 3, I recall some basic definitions and facts about relative projectivity and the Dade group. Section 4 describes the easy cases of the functorial aspect of relative syzygies. Section 5 is devoted to the proof of the formula for tensor induction.

This formula is used in section 6 to show that the subgroup of $D_k(P)$ generated by relative syzygies is actually a subfunctor of D_k, i.e. that it is invariant by the five operations.

In section 7, some information on the structure of the Dade group is given: in particular, the subgroup generated by relative syzygies and by the intersection of the kernels of all restriction-deflation maps to elementary abelian sections has finite index in the whole Dade group, and the exponent of the quotient group divides the order of P. Roughly speaking, one can say that an element of $D_k(P)$ can be recovered up to a power of p and up to a torsion element from its restriction-deflations to elementary abelian sections of P. This provides also an alternative proof of Theorem A and Theorem B of [5].

Finally in section 8, I state a theorem telling when a relative syzygy can be a torsion element in the Dade group. The case of $\Omega_{P/Q}$ was originally a question of J. Thévenaz.

2. Relative syzygies

When studying endo-permutation modules, one considers usually the case where the ground ring is a field of characteristic p (or a discrete valuation ring of characteristic 0 with residue field of characteristic p), and the group is a p-group. However, the first few results can be stated for an arbitrary commutative ring $\mathcal{O}$, and arbitrary finite groups.

The symbol $\otimes$ denotes tensor product over $\mathcal{O}$. If G is a group and M and N are $\mathcal{O}G$-modules, their tensor product $M \otimes N$ is always viewed as an $\mathcal{O}G$-module via diagonal action. Similarly the module $\mathrm{Hom}_{\mathcal{O}}(M, N)$ is viewed as an $\mathcal{O}G$-module with G-action given by $(g\varphi)(m) = g\varphi(g^{-1}m)$ for $g \in G$, $\varphi \in \mathrm{Hom}_{\mathcal{O}}(M, N)$, and $m \in M$. I denote by M^* the $\mathcal{O}$-dual of M, i.e. $M^* = \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$.

2.1. X-split morphisms

(2.1.1) **Definition:** *Let G be a finite group.*

- *A morphism of $\mathcal{O}G$-modules $f : M \to N$ is said to be split if there exists a morphism of $\mathcal{O}G$-modules $a : N \to M$ such that $f = f \circ a \circ f$.*
- *Let X be a finite G-set. A morphism of $\mathcal{O}G$-modules $f : M \to N$ is said to be X-split if the morphism $\mathcal{O}X \otimes f : \mathcal{O}X \otimes M \to \mathcal{O}X \otimes N$ is split.*
- *A short exact sequence of $\mathcal{O}G$-modules*

$$0 \longrightarrow L \xrightarrow{a} M \xrightarrow{b} N \longrightarrow 0$$

is said to be X-split if a is X-split, or equivalently, if b is X-split.

Note indeed that if f is a surjective (resp. injective) morphism of $\mathcal{O}G$-modules, then f is split if and only if it admits a right (resp. left) inverse.

If $f : M \to N$ is a morphism of $\mathcal{O}G$-modules, I denote by $Sec(f)$ the set of $\mathcal{O}$-linear morphisms $a : N \to M$ such that $f = f \circ a \circ f$. The group G acts on $Sec(f)$ by conjugation: if $g \in G$ and $a \in Sec(f)$, the morphism $g.a : N \to M$ defined for $n \in N$ by

$$(g.a)(n) = ga(g^{-1}n)$$

is clearly in $Sec(f)$.

(2.1.2) **Lemma:** *Let G be a finite group, let X be a finite G-set, and $f : M \to N$ be a morphism of $\mathcal{O}G$-modules. Then the following are equivalent:*

1. *The morphism f is X-split.*
2. *For any $x \in X$, the stabilizer G_x of x in G admits a fixed point on $Sec(f)$.*
3. *There exists a morphism of G-sets from X to $Sec(f)$.*

Proof: Let $a : \mathcal{O}X \otimes N \to \mathcal{O}X \otimes N$ be an $\mathcal{O}$-linear morphism. Then a is defined for $x \in X$ and $n \in N$ by

$$a(x \otimes n) = \sum_{y \in X} y \otimes a_{y,x}(n) \tag{2.1.3}$$

where $a_{y,x} \in \mathrm{Hom}_{\mathcal{O}}(N, M)$. Then a is a morphism of $\mathcal{O}G$-modules if and only if for any $g \in G$, any $x, y \in X$ and $n \in N$

$$a_{gy,gx}(n) = g a_{y,x}(g^{-1}n)$$

or equivalently $g.a_{y,x} = a_{gy,gx}$. Now for any $x \in X$ and $m \in M$

$$\begin{aligned}(\mathcal{O}X \otimes f) \circ a \circ (\mathcal{O}X \otimes f)(x \otimes n) &= (\mathcal{O}X \otimes f) \circ a\Big(x \otimes f(m)\Big)\\ &= (\mathcal{O}X \otimes f)\Big(\sum_{y \in X} y \otimes a_{y,x} f(m)\Big)\\ &= \sum_{y \in X} y \otimes f a_{y,x} f(m)\end{aligned}$$

The condition $\mathcal{O}X \otimes f = (\mathcal{O}X \otimes f) \circ a \circ (\mathcal{O}X \otimes f)$ is equivalent to

$$f a_{y,x} f(m) = \delta_{y,x} f(m) \tag{2.1.4}$$

for all $x, y \in X$ and $m \in M$, the symbol $\delta_{y,x}$ being a Kronecker symbol.

In particular, if f is X-split, there exists such an a. Now for any $x \in X$, the element $a_{x,x}$ is in $Sec(f)$, and it is invariant by the action of G_x. Thus condition 2) holds.

Clearly condition 2) implies condition 3) (just choose a G_x-invariant element α_x of $Sec(x)$ for a set of representatives $[G\backslash X]$ of $G\backslash X$, and map $g.x$ to $g.\alpha_x$ for any $g \in G$ and $x \in [G\backslash X]$).

Finally if 3) holds, let α be a morphism of G-sets from X to $Sec(f)$. For $x, y \in X$, define $a_{y,x} : N \to M$ by

$$a_{y,x} = \delta_{y,x} \alpha(x)$$

This clearly defines (by equation 2.1.3) a morphism of $\mathcal{O}G$-modules from $\mathcal{O}X \otimes N$ to $\mathcal{O}X \otimes M$, which is in $Sec(\mathcal{O}X \otimes f)$ by equation 2.1.4. □

Assertion 3) of the lemma gives the following

(2.1.5) **Corollary:** *With the same notation*

1. *If $f : M \to N$ is X-split, then so is $f \otimes L : M \otimes L \to N \otimes L$, for any $\mathcal{O}G$-module L.*
2. *Let Y be a finite G-set such that $\mathrm{Hom}_{G-set}(Y, X) \neq \emptyset$. Then if $f : M \to N$ is X-split, it is also Y-split.*

3. If $f : M \to N$ is X-split, then $f^* : N^* \to M^*$ is X-split.

Proof: If $\varphi : X \to Sec(f)$ is a morphism of G-sets, then the map $x \mapsto \varphi(x) \otimes Id_L$ is a morphism of G-sets from X to $Sec(f \otimes Id_L)$. This proves 1). Assertion 2) is obvious. Assertion 3) uses the fact that since X is finite, the module $\mathcal{O}X$ is free and finitely generated over $\mathcal{O}$. Hence $(\mathcal{O}X \otimes M)^* \simeq \mathcal{O}X \otimes M^*$ (see [7] prop 2.29). □

2.2. Relative projectivity

The notion of relative projectivity appears under various forms in representation theory and category theory, with successive generalizations: projectivity of an $\mathcal{O}G$-module with respect to a subgroup of the group G (see D.G. Higman [12], J.A. Green [11]), with respect to a G-set (see [1] def. 3.6.13), with respect to a module (see [6]), projectivity of an object relative to a functor (see [4] def 4.1), projectivity of an object with respect to a cotriple ([16] def. 8.6.5). The definition I will use here is the second one:

(2.2.1) **Definition:** *Let G be a finite group, and X be a finite G-set. An $\mathcal{O}G$-module M is projective relative to X, or X-projective, if it satisfies one of the following equivalent conditions:*

1. *There exists an $\mathcal{O}G$-module N such that M is a direct summand of $\mathcal{O}X \otimes N$.*
2. *The morphism from $\mathrm{End}_{\mathcal{O}}(\mathcal{O}X) \otimes M$ to M mapping $f \otimes m$ to $tr(f)m$ is a split epimorphism of $\mathcal{O}G$-modules.*
3. *The morphism from $\mathcal{O}X \otimes M$ to M mapping $x \otimes m$ to m (for $x \in X$ and $m \in M$) is a split epimorphism of $\mathcal{O}G$-modules.*
4. *For any diagram*

$$\begin{array}{ccc} & & M \\ & & \big\downarrow{\scriptstyle \alpha} \\ L & \xrightarrow{\beta} & N \end{array}$$

such that the morphism $\mathcal{O}X \otimes \beta$ is a split epimorphism, there exists a morphism of $\mathcal{O}G$-modules $\gamma : M \to L$ such that $\beta \circ \gamma = \alpha$.

(2.2.2) **Remarks:** In assertion 2), I denote by $tr(f)$ the trace of f, which is well defined since $\mathcal{O}X$ is $\mathcal{O}$-free and finitely generated, since X is finite. The module $\mathcal{O}X$ is its own dual over $\mathcal{O}$, and the functor $M \mapsto \mathcal{O}X \otimes M$ from the category of $\mathcal{O}G$-modules to itself, is its own left and right adjoint. The morphism in condition 2) is the counit of this adjunction. Now the equivalence of conditions 1), 2) and 4) is standard (see [16] for details). The equivalence of 2) and 3) is straightforward.

It leads to the following version of Higman's criterion:

(2.2.3) **Lemma:** *Let G be a finite group, and X be a finite G-set. If M is an $\mathcal{O}G$-module, the following are equivalent:*

1. *The module M is X-projective.*
2. *There exists a map of G-sets φ from X to $\mathrm{End}_{\mathcal{O}}(M)$ such that*

$$\sum_{x \in X} \varphi(x) = Id_M$$

Proof: An $\mathcal{O}$-linear map $\alpha : M \to \mathcal{O}X \otimes M$ is defined by

$$\alpha(m) = \sum_{x \in X} x \otimes \alpha_x(m)$$

where $\alpha_x \in \mathrm{End}_{\mathcal{O}}(M)$. This defines a morphism of $\mathcal{O}G$-modules if and only if $g.\alpha_x = \alpha_{gx}$ for all $x \in X$ and $g \in G$. Moreover α is a section of ε if and only if $\sum_x \alpha_x$ is the identity of M. □

(2.2.4) **Corollary:** *With the same notation*

1. *If M is an X-projective $\mathcal{O}G$-module, then so is $M \otimes N$, for any $\mathcal{O}G$-module N.*
2. *If Y is a finite G-set such that $\mathrm{Hom}_{G-set}(X, Y) \neq \emptyset$, and if M is an X-projective $\mathcal{O}G$-module, then M is Y-projective.*
3. *If M is an X-projective $\mathcal{O}G$-module, then so is M^*.*

Proof: Consider a map of G-sets from X to $\mathrm{End}_{\mathcal{O}}(M)$ such that $\sum_{x \in X} \varphi(x) = Id_M$. For the first assertion, consider the map from X to $\mathrm{End}_{\mathcal{O}}(M \otimes N)$ sending $x \in X$ to $\varphi(x) \otimes Id_N$. For the second one, if $f : X \to Y$ is a map of G-sets, define a map from Y to $\mathrm{End}_{\mathcal{O}}(M)$ by $\psi(y) = \sum_{x \in f^{-1}(y)} \varphi(x)$. For the third assertion, note that if M is a direct summand of $\mathcal{O}X \otimes N$, then M^* is a direct summand of $(\mathcal{O}X \otimes N)^*$, which is isomorphic to $\mathcal{O}X \otimes N^*$ since X is finite. □

2.3. Relative Shanuel's lemma

The previous definitions lead to the following relative version of Shanuel's lemma:

(2.3.1) **Proposition:** *Let G be a finite group, and X be a finite G-set. If*

$$0 \to L \to N \to M \to 0$$

$$0 \to L' \to N' \to M \to 0$$

are X-split exact sequences of $\mathcal{O}G$-modules such that N and N' are projective relative to X, then there is an isomorphism of $\mathcal{O}G$-modules

$$L \oplus N' \simeq L' \oplus N$$

Proof: see [1], Lemma 3.9.1 or Lemma 1.5.3. □

(2.3.2) **Notation:** *If X is a G-set, then let $\Omega_X = \Omega_X(\mathcal{O})$ denote the kernel of the $\mathcal{O}$-linear augmentation map $\varepsilon : \mathcal{O}X \to \mathcal{O}$ sending each $x \in X$ to $1 \in \mathcal{O}$.*

More generally, if M is an $\mathcal{O}G$-module, then $\Omega_X(M)$ denotes the kernel of the map $\varepsilon \otimes Id_M : \mathcal{O}X \otimes M \to \mathcal{O} \otimes M \simeq M$. The module $\Omega_X(M)$ is called the syzygy of M relative to X. Thus if M is $\mathcal{O}$-flat, then $\Omega_X(M)$ is isomorphic to $\Omega_X \otimes M$.

Thus Ω_X is the set of linear combinations $\sum_{x \in X} r_x x$ of elements of X with coefficients in $\mathcal{O}$, such that $\sum_x r_x = 0$. If $x_0 \in X$ is given, then Ω_X admits a basis over $\mathcal{O}$, formed of the elements $x - x_0$, for $x \in X - \{x_0\}$, hence it is a free $\mathcal{O}$-module.

(2.3.3) **Lemma:** [Alperin] *Let G be a finite group, and X be a non-empty finite G-set. Then there is an isomorphism of $\mathcal{O}G$-modules*

$$(\Omega_X \otimes \Omega_X^*) \oplus \mathcal{O}X \oplus \mathcal{O}X \simeq \mathcal{O} \oplus (\mathcal{O}X \otimes \mathcal{O}X)$$

Proof: Clearly the module $\mathcal{O}X$ is X-projective, by condition 1) of definition 2.2.1. Moreover, the morphism $\varepsilon : \mathcal{O}X \to \mathcal{O}$ is X-split: the diagonal map $x \mapsto x \otimes x$ is a section of $\mathcal{O}X \otimes \varepsilon$.

The module Ω_X is $\mathcal{O}$-free and finitely generated, so Ω_X^* is also $\mathcal{O}$-free and finitely generated. Tensoring with the ($\mathcal{O}$-free, hence $\mathcal{O}$-flat) module Ω_X^* the exact sequence

$$0 \to \Omega_X \to \mathcal{O}X \to \mathcal{O} \to 0 \tag{2.3.4}$$

gives the exact sequence

$$0 \to \Omega_X \otimes \Omega_X^* \to \mathcal{O}X \otimes \Omega_X^* \to \Omega_X^* \to 0 \tag{2.3.5}$$

The module $\mathcal{O}X \otimes \Omega_X^*$ is X-projective, and the map $\mathcal{O}X \otimes \Omega_X^* \to \Omega_X^*$ is X-split since ε is.

On the other hand, dualizing the sequence 2.3.4 gives the sequence

$$0 \to \mathcal{O} \to \mathcal{O}X \to \Omega_X^* \to 0 \tag{2.3.6}$$

This sequence is exact since $\mathcal{O}$ is $\mathcal{O}$-free. The morphism $\mathcal{O}X \to \Omega_X^*$ is also X-split, by corollary 2.1.5. Now using Shanuel's lemma and sequences 2.3.5 and 2.3.6 gives the isomorphism

$$(\Omega_X \otimes \Omega_X^*) \oplus \mathcal{O}X \simeq \mathcal{O} \oplus (\mathcal{O}X \otimes \Omega_X^*)$$

Adding $\mathcal{O}X$ on both sides gives the isomorphism

$$(\Omega_X \otimes \Omega_X^*) \oplus \mathcal{O}X \oplus \mathcal{O}X \simeq \mathcal{O} \oplus (\mathcal{O}X \otimes \Omega_X^*) \oplus \mathcal{O}X$$

Since the sequence

$$0 \to \mathcal{O}X \to \mathcal{O}X \otimes \mathcal{O}X \to \mathcal{O}X \otimes \Omega_X^* \to 0$$

is split, the direct sum $(\mathcal{O}X \otimes \Omega_X^*) \oplus \mathcal{O}X$ is isomorphic to $\mathcal{O}X \otimes \mathcal{O}X$. Finally, I get the required isomorphism

$$(\Omega_X \otimes \Omega_X^*) \oplus \mathcal{O}X \oplus \mathcal{O}X \simeq \mathcal{O} \oplus (\mathcal{O}X \otimes \mathcal{O}X)$$

This shows in particular that $\Omega_X \otimes \Omega_X^*$, which is isomorphic to $\mathrm{End}_{\mathcal{O}}(\Omega_X)$ since Ω_X is $\mathcal{O}$-free and finitely generated, is a direct summand of a permutation $\mathcal{O}G$-module. □

One can also view this lemma from the point of view of Rickard's *endo-split* permutation resolutions (see [14]): a complex C of $\mathcal{O}G$-modules is an endo-split permutation resolution of the $\mathcal{O}G$-module M if it is a bounded complex of permutation $\mathcal{O}G$-modules with homology concentrated in one degree, isomorphic to M, and such that the complex $\mathrm{End}_{\mathcal{O}}(C)$ is a split complex of $\mathcal{O}G$-modules. With this definition:

(2.3.7) **Lemma:** *The complex* $C : 0 \to \mathcal{O}X \stackrel{\varepsilon}{\to} \mathcal{O} \to 0$ *is an endo-split permutation resolution of* Ω_X.

Proof: Clearly C has homology concentrated in one degree, and equal to Ω_X. Moreover, the complex $\mathrm{End}_{\mathcal{O}}(C)$ is

$$0 \to \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{O}X) \to \mathrm{End}_{\mathcal{O}}(\mathcal{O}X) \oplus \mathrm{End}_{\mathcal{O}}(\mathcal{O}) \to \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}X, \mathcal{O}) \to 0$$

Hence it is isomorphic to the complex

$$0 \to \mathcal{O}X \xrightarrow{a} \mathrm{End}_{\mathcal{O}}(\mathcal{O}X) \oplus \mathcal{O} \xrightarrow{b} \mathcal{O}X \to 0$$

where the map a is defined by $a(x) = (c_x, 1) \in \mathrm{End}_{\mathcal{O}}(\mathcal{O}X) \oplus \mathcal{O}$ for $x \in X$, the endomorphism c_x of $\mathcal{O}X$ being given by $c_x(y) = x$ for $y \in X$. The map b is defined by $b(f) = \sum_{x,y\in X} v_{x,y}x$ for $f \in \mathrm{End}_{\mathcal{O}}(\mathcal{O}X)$ with matrix $f_{x,y}$ in basis X, and $b(1) = -\sum_{x\in X} x$.

Since C is a complex of free $\mathcal{O}$-modules, the homology of $\mathrm{End}_{\mathcal{O}}(C)$ is concentrated in the middle term, and isomorphic to $\mathrm{End}_{\mathcal{O}}(\Omega_X)$.

Now the map a has a section s, defined by $s(f) = \sum_{x\in X} f_{x,x}x$ for $f \in \mathrm{End}_{\mathcal{O}}(\mathcal{O}X)$, and $s = 0$ on $\mathcal{O}$. The map b has a section t defined by $t(x) = (\delta_x, 0) \in \mathrm{End}_{\mathcal{O}}(X) \oplus \mathcal{O}$, where δ_x is the endomorphism of $\mathcal{O}X$ defined by $\delta_x(y) = \delta_{x,y}x$ for $y \in X$. Hence C is an endo-split permutation resolution of Ω_X. □

(2.3.8) **Corollary:** *Let* M *be an* $\mathcal{O}$*-projective finitely generated* $\mathcal{O}G$*-module. If* $\mathrm{End}_{\mathcal{O}}(M)$ *is a direct summand of a permutation* $\mathcal{O}G$*-module, then* $\mathrm{End}_{\mathcal{O}}\Big(\Omega_X(M)\Big)$ *is a direct summand of a permutation* $\mathcal{O}G$*-module.*

Proof: As I already noted above $\Omega_X(M) \simeq \Omega_X \otimes M$ since M is flat over $\mathcal{O}$. Moreover since Ω_X and M are $\mathcal{O}$-projective and finitely generated, so is $\Omega_X(M)$. Now for any finitely generated projective $\mathcal{O}$-module M, and any $\mathcal{O}G$-module N, the linear map from $M \otimes N^*$ to $\mathrm{Hom}_{\mathcal{O}}(N, M)$ sending $m \otimes f$, for $m \in M$ and $f \in N^*$ to the homomorphism from N to M defined by $u \mapsto f(u)m$, for $u \in M$, is an isomorphism of abelian groups (see [7] prop 2.29). In case M and N are $\mathcal{O}G$-modules, this map is an isomorphism of $\mathcal{O}G$-modules. It follows that $\Omega_X(M)^* \simeq \Omega_X^* \otimes M^*$, and also that

$$\begin{aligned} \mathrm{End}_{\mathcal{O}}\Big(\Omega_X(M)\Big) &\simeq \Omega_X(M) \otimes \Omega_X(M)^* \\ &\simeq \Omega_X \otimes \Omega_X^* \otimes M \otimes M^* \\ &\simeq \mathrm{End}_{\mathcal{O}}(\Omega_X) \otimes \mathrm{End}_{\mathcal{O}}(M) \end{aligned}$$

is the tensor product of two direct summands of permutation modules. Hence it is a direct summand of permutation module. □

3. The Dade group

From now on, following [15], I will suppose that the ring $\mathcal{O}$ is a complete noetherian local ring with residue field k of characteristic $p > 0$. Any direct summand of an $\mathcal{O}$-free finitely generated $\mathcal{O}$-module is $\mathcal{O}$-free. If G is a finite group, then an $\mathcal{O}G$-lattice is by definition an $\mathcal{O}$-free finitely generated $\mathcal{O}G$-module. Krull-Schmidt theorem holds for $\mathcal{O}G$-lattices. If M is an indecomposable $\mathcal{O}G$-lattice, then a *vertex* of M is a minimal subgroup P of G such that M is G/P-projective. It is unique up to conjugation in G, and it is a p-group.

3.1. Capped modules and permutation algebras

I will recall first some definitions and facts from Dade's paper [9]:

(3.1.1) **Definition:** *Let P be a p-group. An $\mathcal{O}P$-lattice M is called an endo-permutation lattice if* $\mathrm{End}_{\mathcal{O}}(M)$ *is a permutation $\mathcal{O}P$-module, i.e. if it admits a P-invariant $\mathcal{O}$-basis.*

An endo-permutation $\mathcal{O}P$-lattice M is said to be capped if it has some non-zero indecomposable direct summand with vertex P. Such a direct summand is unique up to isomorphism, and is called a cap of M, and denoted by cap(M).

Two capped endo-permutation $\mathcal{O}G$-lattices M and N are said to be equivalent if cap(M) $\simeq$ cap(N). This is equivalent to saying that $\mathcal{O}$ is a direct summand of $M \otimes N^*$.

Tensor product over $\mathcal{O}$ induces a group structure on the set of equivalence classes of capped endo-permutation lattices called *the Dade group of capped endo-permutation $\mathcal{O}P$-lattices*, denoted by $D'_{\mathcal{O}}(P)$ (see [15] Remark 5.29.6).

If M is an endo-permutation $\mathcal{O}P$-lattice, then $A = \mathrm{End}_{\mathcal{O}}(M)$ is an $\mathcal{O}$-simple algebra, i.e. it is isomorphic to a matrix algebra $M_n(\mathcal{O})$ for some n (see [15] 1.7). Thus it is a permutation $\mathcal{O}$-simple P-algebra. Moreover M is capped if and only if the Brauer quotient $A[P]$ is non-zero (see [15] 5.28). This leads to the following definition:

(3.1.2) **Definition:** [see [15] 5.29] *An $\mathcal{O}$-simple permutation P-algebra such that $A[P] \neq 0$ is called a Dade P-algebra (over $\mathcal{O}$).*

A Dade P-algebra is called neutral if there exists a permutation $\mathcal{O}P$-lattice M having $\mathcal{O}$ as a direct summand, such that $A \simeq \mathrm{End}_{\mathcal{O}}(M)$ as P-algebra.

Two Dade P-algebras A and B are called similar if $A \otimes B^{op}$ is neutral.

Tensor product of algebras induces a group structure on the set of equivalence classes of Dade P-algebras for this relation. This group is *the Dade group of P*, and is denoted by $D_{\mathcal{O}}(P)$. When the residue field k is algebraically closed, the natural group homomorphism $D'_{\mathcal{O}}(P) \to D_{\mathcal{O}}(P)$ mapping a capped $\mathcal{O}P$-lattice to its endomorphism algebra is surjective (and even split see [15] Remark 5.29.6). The kernel is isomorphic to the group of one-dimensional characters of P (i.e. group homomorphisms from P to the multiplicative group of units of $\mathcal{O}$). In particular if $\mathcal{O} = k$ is algebraically closed, then the groups $D'_{\mathcal{O}}(P)$ and $D_{\mathcal{O}}(P)$ are isomorphic.

3.2. Relative syzygies in the Dade group

Let P be a p-group. If X is a non-empty finite P-set, lemma 2.3.3 shows that there is an isomorphism of $\mathcal{O}P$-lattices

$$(\Omega_X \otimes \Omega_X^*) \oplus \mathcal{O}X \oplus \mathcal{O}X \simeq \mathcal{O} \oplus (\mathcal{O}X \otimes \mathcal{O}X)$$

Taking Brauer quotients gives

$$(\Omega_X \otimes \Omega_X^*)[P] \oplus k(X^P) \oplus k(X^P) \simeq k \oplus \Big(k(X^P) \otimes k(X^P)\Big) \tag{3.2.1}$$

It follows that the dimension over k of $(\Omega_X \otimes \Omega_X^*)[P]$ is

$$|X^P|^2 - 2|X^P| + 1 = (|X^P| - 1)^2$$

This formula leads to the following:

(3.2.2) **Definition:** *I will say that a non-empty finite P-set X is a Dade P-set if $\mathrm{End}_{\mathcal{O}}(\Omega_X)$ is a Dade P-algebra, or equivalently, if P does not admit a single fixed point on X (i.e. if X^P is empty or if $|X^P| > 1$).*

(3.2.3) **Notation:** *If P is a p-group, and X is a Dade P-set, I will denote by Ω_X the image of $\mathrm{End}_{\mathcal{O}}\Big(\Omega_X(\mathcal{O})\Big)$ in the Dade group $D_{\mathcal{O}}(P)$.*

The following is elementary:

(3.2.4) **Lemma:** *Let P be a p-group, and X be a Dade P-set.*

1. *If $X^P \neq \emptyset$, then $\Omega_X = 0$ in $D_{\mathcal{O}}(P)$.*
2. *If $p \neq 2$ and if $\Omega_X = 0$ in $D_{\mathcal{O}}(P)$, then $X^P \neq \emptyset$.*

Proof: If P has a fixed point on X, then the sequence

$$0 \to \Omega_X \to \mathcal{O}X \xrightarrow{\varepsilon} \mathcal{O} \to 0$$

is split (mapping $1 \in \mathcal{O}$ to $x \in \mathcal{O}X$, for $x \in X^P$, is a section of ε). Thus Ω_X is a direct summand of $\mathcal{O}X$, hence it is a permutation module, since P is a p-group. Hence $\Omega_X = 0$ in $D_{\mathcal{O}}(P)$.

Conversely, if $\Omega_X = 0$ in $D_{\mathcal{O}}(P)$, then there is a finite P-set Y such that $Y^P \neq \emptyset$ and $\Omega_X(\mathcal{O}) \otimes \Omega_X(\mathcal{O})^* \simeq \mathcal{O}Y \otimes \mathcal{O}Y$ as P-algebras. It follows in particular that $(|X|-1)^2 = |Y|^2$, for dimension reasons. Thus $|X| = 1 + |Y|$, since X is non-empty. Now if $X^P = \emptyset$, equation 3.2.1 gives

$$k(Y^P \times Y^P) \simeq k$$

hence $|Y^P| = 1$. It follows that $|Y|$ is congruent to 1 modulo p, thus $|X|$ is congruent to 2 modulo p. Thus $2 \equiv |X| \equiv |X^P| = 0$ modulo p, hence $p = 2$. □

(3.2.5) **Remark:** Assertion 2) is false if $p = 2$: the Dade group $D_{\mathcal{O}}(P)$ of the cyclic group P of order 2 is trivial, thus $\Omega_{P/1} = 0$, but P has no fixed points on $P/1$.

The first assertion of the previous lemma leads to the following

(3.2.6) **Convention:** If X is a finite P-set such that $X^P \neq \emptyset$, I will set $\Omega_X = 0$ in $D_{\mathcal{O}}(P)$. With this convention, the element Ω_X of $D_{\mathcal{O}}(P)$ is defined for any non-empty finite P-set X.

One could also define $\Omega_\emptyset = 0$ in $D_{\mathcal{O}}(P)$, since it is the only reasonable value for it. However, to avoid too many special cases in the proofs, I will not define $\Omega_\emptyset$ in this paper.

(3.2.7) **Lemma:** *Let P be a p-group. If X and Y are non-empty finite P-sets such that for any subgroup Q of P, the set X^Q is non-empty if and only if Y^Q is non-empty, then $\Omega_X = \Omega_Y$ in $D_{\mathcal{O}}(P)$.*

Proof: The hypothesis on X and Y is equivalent to say that there is a morphism of P-sets from X to Y and a morphism of P-sets from Y to X. Now if $X^P \neq \emptyset$, then $Y^P \neq \emptyset$, and then $\Omega_X = \Omega_Y = 0$ in $D_{\mathcal{O}}(P)$. And if $X^P = Y^P = \emptyset$, in the sequence

$$0 \to \Omega_Y \to \mathcal{O}Y \to \mathcal{O} \to 0$$

the middle term is $\mathcal{O}X$-projective by corollary 2.2.4, because is it $\mathcal{O}Y$-projective, and the right hand side morphism is $\mathcal{O}X$-split by corollary 2.1.5, because it is $\mathcal{O}Y$-split. By Shanuel's lemma, there is an isomorphism of $\mathcal{O}P$-modules

$$\Omega_X \oplus \mathcal{O}Y \simeq \Omega_Y \oplus \mathcal{O}X$$

As $X^P = Y^P = \emptyset$, the cap of Ω_X must be a direct summand of Ω_Y, since $\mathcal{O}X$ has no direct summand with vertex P. Hence $cap(\Omega_X) = cap(\Omega_Y)$ in this case, and $\Omega_X = \Omega_Y$ in $D_{\mathcal{O}}(P)$. □

(3.2.8) **Lemma:** *Let P be a p-group, let X be a non-empty finite P-set, and*

$$0 \to W \to \mathcal{O}X \to V \to 0$$

be an X-split exact sequence of $\mathcal{O}P$-lattices. Then:

1. *The lattice V is an endo-permutation $\mathcal{O}P$-lattice if and only if W is an endo-permutation $\mathcal{O}P$-lattice.*
2. *If $X^P = \emptyset$, then V is a capped endo-permutation $\mathcal{O}P$-lattice if and only if W is a capped endo-permutation $\mathcal{O}P$-lattice.*
3. *If V and W are capped endo-permutation $\mathcal{O}P$-lattices, then*

$$W = \Omega_X + V \quad in \quad D_{\mathcal{O}}(P)$$

Proof: Tensoring the exact sequence of the lemma with W^* gives an exact sequence

$$0 \to W \otimes W^* \to \mathcal{O}X \otimes W^* \to V \otimes W^* \to 0$$

Dualizing the sequence of the lemma, and tensoring next with V, gives the exact sequence

$$0 \to V \otimes V^* \to V \otimes \mathcal{O}X \to V \otimes W^* \to 0$$

Now by Shanuel's lemma

$$(W \otimes W^*) \oplus (V \otimes \mathcal{O}X) \simeq (V \otimes V^*) \oplus (\mathcal{O}X \otimes W^*) \tag{3.2.9}$$

Since the sequence of the lemma is X-split, there is moreover an isomorphism

$$\mathcal{O}X \otimes \mathcal{O}X \simeq (\mathcal{O}X \otimes W) \oplus (\mathcal{O}X \otimes V)$$

which gives also, since $(\mathcal{O}X \otimes W)^* \simeq \mathcal{O}X \otimes W^*$

$$\mathcal{O}X \otimes \mathcal{O}X \simeq (\mathcal{O}X \otimes W^*) \oplus (\mathcal{O}X \otimes V^*)$$

It follows that

$$(W \otimes W^*) \oplus (\mathcal{O}X \otimes V) \oplus (\mathcal{O}X \otimes V^*) \simeq (V \otimes V^*) \oplus (\mathcal{O}X \otimes \mathcal{O}X)$$

Hence $W \otimes W^*$ is a permutation lattice if $V \otimes V^*$ is. Similarly

$$(W \otimes W^*) \oplus (\mathcal{O}X \otimes \mathcal{O}X) \simeq (V \otimes V^*) \oplus (\mathcal{O}X \otimes W^*) \oplus (\mathcal{O}X \otimes W)$$

and $V \otimes V^*$ is a permutation lattice if $W \otimes W^*$ is. This proves assertion 1).

If $X^P = \emptyset$, then the lattices $\mathcal{O}X$, $\mathcal{O}X \otimes W^*$ and $\mathcal{O}X \otimes V$ have no direct summand with vertex P. Equation 3.2.9 shows that

$$(W \otimes W^*)[P] \simeq (V \otimes V^*)[P]$$

Thus V has a direct summand of vertex P if and only if W does. This proves 2).

If V and W are capped, there are two cases: suppose first that $X^P \neq \emptyset$. In this case V is a direct summand of $\mathcal{O}X \otimes V$, hence of $\mathcal{O}X \otimes \mathcal{O}X$. If V is capped, then $cap(V)$ is a permutation lattice of vertex P, hence isomorphic to $\mathcal{O}$. Now W is also a direct summand of $\mathcal{O}X \otimes W$, hence of $\mathcal{O}X \otimes \mathcal{O}X$. It follows that $V = W = \Omega_X = 0$ in $D_\mathcal{O}(P)$, and the formula of assertion 3) holds.

And if $X^P = \emptyset$, the exact sequence

$$0 \to \Omega_X \otimes V \to \mathcal{O}X \otimes V \to V \to 0$$

and Shanuel's lemma give

$$W \oplus (\mathcal{O}X \otimes V) \simeq (\Omega_X \otimes V) \oplus \mathcal{O}X$$

This shows that the cap of W is a direct summand of $\Omega_X \otimes V$, since $\mathcal{O}X$ has no direct summand with vertex P. Hence $W = \Omega_X \otimes V$ in the Dade group of P, as was to be shown. □

4. Functorial properties of relative syzygies

4.1. Restriction, inflation, isomorphisms

(4.1.1) **Lemma:** *Let P and Q be p-groups, and let $f : P \to Q$ be a group homomorphism. If M is an $\mathcal{O}Q$-lattice (resp. if X is a Q-set), denote by $\mathrm{Res}_f M$ (resp. $\mathrm{Res}_f X$) the $\mathcal{O}P$-lattice obtained by restriction along f. Then if X is a non-empty finite Q-set, there is an isomorphism of $\mathcal{O}P$-lattices*

$$\mathrm{Res}_f \Omega_X \simeq \Omega_{\mathrm{Res}_f X}$$

Proof: Since $\mathrm{Res}_f \mathcal{O}X \simeq \mathcal{O}\mathrm{Res}_f X$, the restriction along f of the exact sequence

$$0 \to \Omega_X \to \mathcal{O}X \xrightarrow{\varepsilon} \mathcal{O} \to 0$$

is the exact sequence

$$0 \to \mathrm{Res}_f \Omega_X \to \mathcal{O}\mathrm{Res}_f X \xrightarrow{\varepsilon} \mathcal{O} \to 0$$

and the lemma follows. □

(4.1.2) **Corollary:**

1. *Let $Q \subseteq P$ be p-groups. If X is a non-empty finite P-set, then there is an isomorphism of $\mathcal{O}Q$-lattices*

$$\mathrm{Res}^P_Q \Omega_X \simeq \Omega_{\mathrm{Res}^P_Q X}$$

2. *Let $Q \trianglelefteq P$ be p-groups. If X is a non-empty finite P/Q-set, then there is an isomorphism of $\mathcal{O}P$-lattices*

$$\mathrm{Inf}^P_{P/Q} \Omega_X \simeq \Omega_{\mathrm{Inf}^P_{P/Q} X}$$

3. *Let $\varphi : P \to Q$ be a group isomorphism. If X is a non-empty finite P-set, then there is an isomorphism of $\mathcal{O}Q$-modules*

$$(\Omega_X)_\varphi = \Omega_{X_\varphi}$$

where $(\Omega_X)_\varphi$ (resp. X_φ) denotes the module Ω_X (resp. the set X) on which Q acts via φ^{-1}.

4.2. Deflation

Let $Q \trianglelefteq P$ be p-groups. One can define a deflation operation $\mathrm{Def}^P_{P/Q} : D_{\mathcal{O}}(P) \to D_k(P/Q)$ (called *slash* construction by Dade), in the following way: if M is a capped endo-permutation $\mathcal{O}P$-module, then the Brauer quotient $\mathrm{End}_{\mathcal{O}}(M)[Q]$ is a Dade P/Q-algebra over k, associated to an endo-permutation $k(P/Q)$-module $\mathrm{Def}^P_{P/Q}M$. This construction is compatible with the equivalence relation defining the Dade group.

Comparison of equality in lemma 2.3.3 and equation 3.2.1 suggests the following:

(4.2.1) **Lemma:** *Let $Q \trianglelefteq P$ be p-groups, and X be a non-empty finite P-set.*

1. *If $X^Q = \emptyset$, then $\mathrm{Def}^P_{P/Q}\Omega_X = 0$ in $D_k(P/Q)$.*
2. *If $X^Q \neq \emptyset$, then $\mathrm{Def}^P_{P/Q}\Omega_X = \Omega_{X^Q}$ in $D_k(P/Q)$*

Proof: Note first that Ω_X in the left hand side is in $D_{\mathcal{O}}(P)$, whereas Ω_{X^Q} in the right hand side is in $D_k(P/Q)$. Actually for this lemma, I can suppose $\mathcal{O} = k$.

If $X^P \neq \emptyset$, then $(X^Q)^{P/Q} \neq \emptyset$, and there is nothing to prove. Now if $X^Q = \emptyset$, lemma 2.3.3 shows that $(\Omega_X \otimes \Omega_X^*)[Q] \simeq k$ as P/Q-algebras, and assertion 1) follows.

Finally if $X^P = \emptyset$, by lemma 2.3.7, the complex

$$0 \to kX \xrightarrow{a} \mathrm{End}_k(kX) \oplus k \xrightarrow{b} kX \to 0 \qquad (4.2.2)$$

is a split complex of kP-modules. Now $(kX)[Q] \simeq k(X^Q)$ and since kX is a permutation module

$$\Big(\mathrm{End}_k(kX)\Big)[Q] \simeq \mathrm{End}_k\Big(k(X^Q)\Big)$$

The Brauer quotient at Q of 4.2.2 is the split complex of $k(P/Q)$-modules

$$0 \to k(X^Q) \xrightarrow{a[Q]} \mathrm{End}_k\Big(k(X^Q)\Big) \oplus k \xrightarrow{b[Q]} k(X^Q) \to 0$$

with homology concentrated in the middle term. This homology can be viewed either as $\Big(\mathrm{End}_k(\Omega_X)\Big)[Q]$ or as $\mathrm{End}_k(\Omega_{X^Q})$. Hence those two algebras are isomorphic as (P/Q)-algebras, and the lemma follows. □

5. Tensor induction

5.1. The formula for tensor induction

The previous lemmas describe the action of four of the five functorial operations on the Dade group. The only missing one is tensor induction, and this section will be devoted to the proof of the corresponding theorem. First I need some notation:

(5.1.1) **Notation:** *Let P be a group. I denote by s_P the set of conjugacy classes of subgroups of P. It is ordered by the following relation: if C and C' are elements of s_P, then $C \leq_P C'$ if there exists $Q \in C$ and $Q' \in C'$ such that $Q \subseteq Q'$. I denote by μ_P the Möbius function of the poset s_P.*

If Q is a subgroup of P, I denote by $\overline{Q}$ its conjugacy class in P, viewed as an element of s_P. If Q and Q' are subgroups of P, the notation $Q \leq_P Q'$ means that some conjugate of Q in P is contained in Q', or equivalently, that $\overline{Q} \leq_P \overline{Q'}$. In this case, I will write $\mu_P(Q,Q')$ instead of $\mu_P(\overline{Q},\overline{Q'})$.

Finally, I denote by $[s_P]$ a set of representatives of s_P.

(5.1.2) **Theorem:** *Let $Q \subseteq P$ be p-groups. Let X be a non-empty finite Q-set. Then in the Dade group $D_{\mathcal{O}}(P)$*

$$\mathrm{Ten}_Q^P \Omega_X = \sum_{\substack{U,V \in [s_P] \\ U \leq_P V}} \mu_P(U,V)\ |\{a \in V\backslash P/Q \mid X^{V^a \cap Q} \neq \emptyset\}|\ \Omega_{P/U}$$

I will prove this theorem by induction on $|P:Q|$. There will be three steps: first the case $P = Q$, then the inductive step when $|P:Q| \geq p^2$, and finally the case $|P:Q| = p$.

5.2. The case $P = Q$

This case is an easy consequence of the following lemma, due to J. Thévenaz:

(5.2.1) **Lemma:** [Thévenaz] *Let P be a p-group. If X and Y are non-empty finite P-sets, then*

$$\Omega_{X \sqcup Y} + \Omega_{X \times Y} = \Omega_X + \Omega_Y \quad in \quad D_{\mathcal{O}}(P)$$

Proof: Note first that the formula holds if $X^P \neq \emptyset$: in this case $\Omega_X = 0$, and $\Omega_{X \sqcup Y} = 0$. Moreover there exists a morphism of P-sets from $X \times Y$ to Y (the projection), and from Y to $X \times Y$ (mapping $y \in Y$ to (x,y), for a given $x \in X^P$). Thus $\Omega_Y = \Omega_{Y \times X}$, and the formula holds. Hence I can assume $X^P = Y^P = \emptyset$ by symmetry.

Let C denote the complex $0 \to \mathcal{O}X \to \mathcal{O} \to 0$, and similarly let D denote the complex $0 \to \mathcal{O}Y \to \mathcal{O} \to 0$. The tensor product of those complexes is the complex

$$0 \to \mathcal{O}X \otimes \mathcal{O}Y \to \mathcal{O}X \oplus \mathcal{O}Y \to \mathcal{O} \to 0$$

which is isomorphic to the complex

$$0 \longrightarrow \mathcal{O}(X \times Y) \xrightarrow{a} \mathcal{O}(X \sqcup Y) \xrightarrow{b} \mathcal{O} \longrightarrow 0$$

where a is defined by $a(x,y) = x - y \in \mathcal{O}(X \sqcup Y)$ for $x \in X$ and $y \in Y$, and b is defined by $b(x) = b(y) = 1$ for $x \in X$ and $y \in Y$.

Since C and D are complexes of free $\mathcal{O}$-modules, with homology concentrated in one degree, equal to Ω_X and Ω_Y respectively, the only non-zero homology group of $C \otimes D$ is the kernel of a, and it is isomorphic to $\Omega_X \otimes \Omega_Y$. Now the kernel of b is $\Omega_{X \sqcup Y}$, and there is an exact sequence

(5.2.2)
$$0 \longrightarrow \Omega_X \otimes \Omega_Y \longrightarrow \mathcal{O}(X \times Y) \xrightarrow{a} \Omega_{X \sqcup Y} \longrightarrow 0$$

Now the map a is $(X \times Y)$-split: if (x_0, y_0) is given in $X \times Y$, define a map σ_{x_0,y_0} from $\Omega_{X \sqcup Y}$ to $\mathcal{O}(X \times Y)$ by

$$\sigma_{x_0,y_0}(\sum_{x \in X} \alpha_x x + \sum_{y \in Y} \beta_y y) = \sum_{x \in X} \alpha_x (x, y_0) - \sum_{y \in Y} \beta_y (x_0, y) - \sum_{x \in X} \alpha_x (x_0, y_0)$$

The element $\sum_{x \in X} \alpha_x x + \sum_{y \in Y} \beta_y y$ is in $\Omega_{X \sqcup Y}$ if and only if $\sum_{x \in X} \alpha_x + \sum_{y \in Y} \beta_y = 0$ Now

$$a \circ \sigma_{x_0,y_0}(\sum_{x \in X} \alpha_x x + \sum_{y \in Y} \beta_y y) = \sum_{x \in X} \alpha_x x + \sum_{y \in Y} \beta_y y - \sum_{x \in X} \alpha_x y_0 - \sum_{y \in Y} \beta_y x_0 - \sum_{x \in X} \alpha_x (x_0 - y_0)$$

Hence σ_{x_0,y_0} is a section of a, and the map $(x_0, y_0) \mapsto \sigma_{x_0,y_0}$ is clearly a map of P-sets from $X \times Y$ to $Sec(a)$.

Now it follows from lemma 3.2.8 that in $D_{\mathcal{O}}(P)$

$$\Omega_X + \Omega_Y = \Omega_{X\times Y} + \Omega_{X\sqcup Y}$$

as was to be shown. □

In the case $P = Q$, the equality to prove in theorem 5.1.2 takes the form of the following lemma:

(5.2.3) **Lemma:** *Let P be a p-group and X be a non-empty finite P-set. Then in* $D_{\mathcal{O}}(P)$

(5.2.4)
$$\Omega_X = \sum_{\substack{U,V\in[s_P]\\ U\leq_P V\\ X^V\neq\emptyset}} \mu_P(U,V)\Omega_{P/U}$$

Proof: Suppose first that X is isomorphic to a disjoint union of copies of a single transitive P-set, isomorphic to P/S for some subgroup S of P. Then X^V is non-empty if and only if $V \leq_P S$. The sum in the right hand side of 5.2.4 is

$$\sum_{\substack{U,V\in[s_P]\\ U\leq_P V\leq_P S}} \mu_P(U,V)\Omega_{P/U}$$

Now for a given $\overline{U} \in s_P$, the sum for $\overline{V}$ in the "interval" $[\overline{U}, \overline{S}]$ of s_P of $\mu_P(U,V)$ is zero if $\overline{U} \neq \overline{S}$, by definition of the Möbius function on s_P, and it is equal to 1 otherwise. The only term left in the previous sum is obtained for $\overline{U} = \overline{S}$, hence the sum is equal to $\Omega_{P/S}$. On the other hand $\Omega_X = \Omega_{P/S}$ in this case, by lemma 3.2.7. Thus formula 5.2.4 holds in this case.

For the general case, denote by Ω'_X the right hand side of 5.2.4. Clearly Ω'_X only depends on the set

$$F(X) = \{Q \in [s_P] \mid X^Q \neq \emptyset\}$$

and by lemma 3.2.7 the same is true for Ω_X. I will prove that $\Omega_X = \Omega'_X$ by induction on the cardinality of $F(X)$.

If this cardinality is equal to 1, then P acts freely on X, hence X is a union of copies of $P/1$, and $\Omega_X = \Omega'_X$ in this case.

If $|F(X)| > 1$, choose a maximal element Q of $F(X)$. Then up to isomorphism, the set X can be written as $Y \sqcup Z$, where Z is a non-empty union of copies of P/Q, and Y is a set such that $Y^Q = \emptyset$.

If $Y = \emptyset$, then $X = Z$ is a union of copies of P/Q, and the formula holds in this case.

Now if $Y \neq \emptyset$, it is clear that $|F(Y)| < |F(X)|$ since $F(Y) \subseteq F(X)$ and $Q \in F(X)–F(Y)$. Similarly $F(Y\times Z) \subseteq F(Y)$ and $|F(Y\times Z)| < |F(X)|$. Hence formula 5.2.4 holds for Y and $Y \times Z$ by induction hypothesis. It also holds for Z since Z is isomorphic to a union of copies of P/Q. Now by lemma 5.2.1

$$\Omega_X = \Omega_Y + \Omega_Z - \Omega_{Y\times Z}$$

It follows that

$$\Omega_X = \Omega'_Y + \Omega'_Z - \Omega'_{Y\times Z} \tag{5.2.5}$$

For a subgroup V of P set

$$\Sigma_V = \sum_{\substack{U\in[s_P]\\ U\leq_P V}} \mu_P(U,V)\Omega_{P/U}$$

With this notation

$$\Omega'_X = \sum_{V\in F(X)} \Sigma_V$$

and equality 5.2.5 becomes

$$\Omega_X = \sum_{V\in F(Y)} \Sigma_V + \sum_{V\in F(Z)} \Sigma_V - \sum_{V\in F(Y\times Z)} \Sigma_V$$

Since $F(Y\times Z) = F(Y)\cap F(Z)$, this is also

$$\Omega_X = \sum_{V\in F(Y)\cup F(Z)} \Sigma_V$$

and since $F(Y)\cup F(Z) = F(Y\sqcup Z) = F(X)$, this gives finally

$$\Omega_X = \sum_{V\in F(X)} \Sigma_V = \Omega'_X$$

This completes the proof of lemma 5.2.3. □

5.3. The inductive step $|P:Q|\geq p^2$

Suppose that formula of theorem 5.1.2 holds for all p-groups $Q'\subseteq P'$ with index $|P':Q'|<|P:Q|$. Choose a subgroup S such that $Q\subset S\subset P$, the inclusions being proper ones. This is possible if $|P:Q|\geq p^2$.

By induction hypothesis

$$\mathrm{Ten}_Q^S\Omega_X = \sum_{\substack{U,V\in[s_S]\\ U\leq_S V}} \mu_S(U,V)\ |\{a\in V\backslash S/Q \mid X^{V^a\cap Q}\neq\emptyset\}|\ \Omega_{S/U}$$

in $D_{\mathcal{O}}(S)$. Taking tensor induction up to P gives

$$\mathrm{Ten}_Q^P\Omega_X = \sum_{\substack{U,V\in[s_S]\\ U\leq_S V}} \mu_S(U,V)\ |\{a\in V\backslash S/Q \mid X^{V^a\cap Q}\neq\emptyset\}|\ \mathrm{Ten}_S^P\Omega_{S/U}$$

since tensor induction is transitive. Since $|P:S|<|P:Q|$, by induction hypothesis

$$\mathrm{Ten}_S^P\Omega_{S/U} = \sum_{\substack{A,B\in[s_P]\\ A\leq_P B}} \mu_P(A,B)\ |\{b\in B\backslash P/S \mid (S/U)^{B^b\cap S}\neq\emptyset\}|\ \Omega_{P/A}$$

Thus

$$\mathrm{Ten}_Q^P \Omega_X = \sum_{\substack{U,V\in[s_S]\\ U\leq_S V}} \mu_S(U,V) \sum_{\substack{a\in V\backslash S/Q\\ X^{V^a\cap Q}\neq\emptyset}} \sum_{\substack{A,B\in[s_P]\\ A\leq_P B}} \mu_P(A,B) \sum_{\substack{b\in B\backslash P/S\\ B^b\cap S\leq_S U}} \Omega_{P/A}$$

Changing the order of summations gives

$$\mathrm{Ten}_Q^P \Omega_X = \sum_{\substack{A,B\in[s_P]\\ A\leq_P B}} \mu_P(A,B) \left(\sum_{b\in B\backslash P/S} \sum_{V\in[s_S]} \sum_{\substack{a\in V\backslash S/Q\\ X^{V^a\cap Q}\neq\emptyset}} \sum_{\substack{U\in[s_S]\\ B^b\cap S\leq_S U\leq_S V}} \mu_S(U,V) \right) \Omega_{P/A}$$

The inner summation on U is zero unless $V =_S B^b \cap S$. It follows that

$$\mathrm{Ten}_Q^P \Omega_X = \sum_{\substack{A,B\in[s_P]\\ A\leq_P B}} \mu_P(A,B) \left(\sum_{b\in B\backslash P/S} \sum_{\substack{a\in B^b\cap S\backslash S/Q\\ X^{(B^b\cap S)^a\cap Q}\neq\emptyset}} 1 \right) \Omega_{P/A}$$

Now $(B^b \cap S)^a \cap Q = B^{ba} \cap S \cap Q = B^{ba} \cap Q$. Moreover, when b runs through $B\backslash P/S$ and a runs through $B^b \cap S\backslash S/Q$, the element $c = ba$ runs through $B\backslash P/Q$. Hence

$$\mathrm{Ten}_Q^P \Omega_X = \sum_{\substack{A,B\in[s_P]\\ A\leq_P B}} \mu_P(A,B) (\sum_{\substack{c\in B\backslash P/Q\\ X^{B^c\cap Q}\neq\emptyset}} 1) \Omega_{P/A}$$

and finally

$$\mathrm{Ten}_Q^P \Omega_X = \sum_{\substack{A,B\in[s_P]\\ A\leq_P B}} \mu_P(A,B)\, |\{c \in B\backslash P/Q \mid X^{B^c\cap Q} \neq \emptyset\}|\, \Omega_{P/A}$$

which completes the proof of the inductive step. □

5.4. The case $|P : Q| = p$

(a) *The case* $X^Q \neq \emptyset$. If $X^Q \neq \emptyset$, the formula to check is the following

$$0 = \sum_{\substack{U,V\in[s_P]\\ U\leq_P V}} \mu_P(U,V)\, |P : VQ|\, \Omega_{P/U}$$

The right hand side is also

$$S = \sum_{\substack{U,V\in[s_P]\\ U\leq V\leq Q}} \mu_P(U,V) p \Omega_{P/U} + \sum_{\substack{U,V\in[s_P]\\ U\leq V\not\leq Q}} \mu_P(U,V) \Omega_{P/U}$$

since either $V \subseteq Q$ or $VQ = P$. As $\Omega_{P/P} = 0$, I can fix a subgroup $U \neq P$ and look at the sum on V

$$\sum_{\substack{V \in [s_P] \\ U \leq_P V \leq P}} \mu_P(U,V) = 0 = \sum_{\substack{V \in [s_P] \\ U \leq_P V \leq Q}} \mu_P(U,V) + \sum_{\substack{V \in [s_P] \\ U \leq_P V \not\leq Q}} \mu_P(U,V)$$

It follows that

$$S = \sum_{\substack{U \in [s_P] \\ U \neq P}} \sum_{\substack{V \in [s_P] \\ U \leq_P V \leq Q}} (p-1)\mu_P(U,V)\Omega_{P/U}$$

The sum on V is zero if $U \neq Q$. Finally

$$S = (p-1)\Omega_{P/Q}$$

Then either p is odd, and this is zero since $\Omega_{P/Q} = \mathrm{Inf}_{P/Q}^{P}\Omega_{(P/Q)/(Q/Q)}$ has order 2 in $D_{\mathcal{O}}(P)$, or $p = 2$ and $\Omega_{P/Q} = 0$, since it is inflated from $D_{\mathcal{O}}(P/Q) = 0$. In both cases, the formula holds.

(b) *The case* $X^Q = \emptyset$. I denote by C the complex $0 \to \mathcal{O}X \to \mathcal{O} \to 0$, where $\mathcal{O}$ is in degree 0 and $\mathcal{O}X$ is in degree 1. It is a complex of $\mathcal{O}$-free $\mathcal{O}Q$-modules. I consider next the tensor induced complex $T = \mathrm{Ten}_Q^P C$ (see [2] section 4.1): for this, I choose a total ordering of a set $[P/Q]$ of representatives of P/Q. Now the module T_j is isomorphic to

$$T_j = \bigoplus_{\substack{A \subseteq [P/Q] \\ |A| = j}} (\mathcal{O}X)^{\otimes j}$$

It can also be viewed as the free $\mathcal{O}$-module with basis the set of pairs (A, φ), where A is a subset of cardinality j of $[P/Q]$ and φ is a map from A to X, or equivalently as the free $\mathcal{O}$-module with basis the pairs (A, φ), where A is a subset of P such that $AQ = A$ and $|A/Q| = j$, and φ is a Q-invariant map from A to X.

The action of $g \in P$ on T_j is given by

$$g.(A, \varphi) = \varepsilon(A, g)(gA, g\varphi)$$

where $g\varphi$ is the function from gA to X defined by

$$(g\varphi)(a) = \varphi(g^{-1}a)$$

and $\varepsilon(A, g)$ is a sign, defined by

$$\varepsilon(A, g) = (-1)^{|\{(a,b) \in [A/Q]^2 \mid a < b,\, ga > gb\}|}$$

where $[A/Q]$ is the set of elements of $[P/Q]$ representing elements of A, and the notation $ga > gb$ means that this inequality holds between the representatives of ga and gb in $[P/Q]$.

Therefore the module T_j is isomorphic as an $\mathcal{O}P$-module to

$$\bigoplus_{\substack{A \subseteq P \\ AQ = A, |A/Q| = j \\ A \bmod. P}} \mathrm{Ind}_{P_A}^{P} \mathcal{O}^{\varepsilon} \mathrm{Hom}_Q(A^{op}, X)$$

where P_A denotes the stabilizer of A in P, and $\mathcal{O}^\varepsilon \mathrm{Hom}_Q(A^{op}, X)$ is the free $\mathcal{O}$-module with basis the set of Q-invariant maps from A to X, on which the element $g \in P_A$ acts by

$$g.(A, \varphi) = \varepsilon(A, g)(A, g\varphi)$$

and $\varepsilon(A, g)$ is the signature of g acting on A/Q. Note that if p is odd, this signature is always $+1$, since cycles of odd length are even. If $p = 2$ and $\mathcal{O}$ has characteristic 2, this sign is also +1=-1. Finally, if $j \neq p$, then P_A is a proper subgroup of P, containing Q, hence equal to Q, and P_A acts trivially on A/Q since $Q \trianglelefteq P$. The only case where signs do appear is the case $j = p = 2$ and $\mathcal{O}$ has characteristic different from 2. All other modules T_j are permutation modules.

The differential $d_j : T_j \to T_{j-1}$ is given on the element (A, φ), for $A = a_1Q \sqcup \ldots \sqcup a_jQ$ with $a_i \in [P/Q]$ for $1 \leq i \leq j$ such that $a_1 < a_2 < \ldots < a_j$, by

$$d_j\Big((A, \varphi)\Big) = \sum_{l=1}^{j} (-1)^{l-1}(A - a_lQ, \varphi_{|A-a_lQ})$$

Since C is a complex of free $\mathcal{O}$-modules, with homology concentrated in degree one, isomorphic to Ω_X, the complex T has its homology concentrated in degree p. This homology group is 'almost' isomorphic to $\mathrm{Ten}_Q^P \Omega_X$: if $p = 2$ and $\mathcal{O}$ has characteristic different from 2, one has to take the sign into account: the element g of P acts on the module $T_p = (\mathcal{O}X)^{\otimes p}$ as it acts on $\mathrm{Ten}_Q^P \mathcal{O}X$, up to a sign which is the signature of the permutation induced by g on the cosets P/Q, equal to 1 if $g \in Q$ and to -1 if $g \notin Q$.

Let j be an integer, with $1 \leq j \leq p$. I will denote by Z_j the set of pairs (A, φ), where A is a subset of P such that $AQ = A$ and $|A/Q| = j$, and $\varphi \in \mathrm{Hom}_Q(A^{op}, X)$. The set Z_j is a P-set, and the previous remarks show that T_i is isomorphic to the permutation module $\mathcal{O}Z_i$ if $i < p$. In particular the module T_i is Z_i projective.

(5.4.1) **Lemma:** *The differential $d_i : T_i \to T_{i-1}$ is Z_j-split if $i < p$ and $i \leq j$.*

Proof: I will fix $(A, \varphi) \in Z_j$, where the set A is written as

$$A = a_1Q \sqcup \ldots \sqcup a_jQ \quad \text{with} \quad \forall l,\ 1 \leq l \leq j,\ a_l \in [P/Q],\ a_1 < a_2 < \ldots < a_j$$

I define a map θ_i from T_{i-1} to T_i in the following way: if $(B, \psi) \in Z_{i-1}$, where B can be written as

$$B = b_1Q \sqcup \ldots \sqcup b_{i-1}Q \quad \text{with} \quad \forall l,\ 1 \leq l \leq i-1,\ b_l \in [P/Q],\ b_1 < b_2 < \ldots < b_{i-1}$$

denote by l_B the least integer l such that $a_l \notin B$. Such an integer exists since $i - 1 < j$. Set $I(B) = \{a_1, \ldots, a_{l_B - 1}\}$, and denote by r_B the least integer r such that $a_{l_B} < b_r$ (or $r_B = i$ if $a_{l_B} > b_r$ for all r). Now I set

$$\theta_i\Big((B, \psi)\Big) = \sum_{U \subseteq I(B)} (-1)^{|U| + r_B - 1}(B \sqcup a_{l_B}Q, [\varphi, \psi]_U)$$

where $[\varphi, \psi]_U$ is the map from $B \sqcup a_{l_B}Q$ to X defined by

$$[\varphi, \psi]_U(c) = \begin{cases} \psi(c) & \text{if } c \in B - UQ \\ \varphi(c) & \text{if } c \in UQ \sqcup a_{l_B}Q \end{cases}$$

The map θ_i in invariant under the action of the stabilizer $P_{(A,\varphi)}$ of (A, φ) in P: indeed, since $i < p$, the stabilizer of A is equal to Q, and $P_{(A,\varphi)}$ is the set of elements q of Q such that

$$\varphi(a) = \varphi(q^{-1}a) = \varphi\Big(a(q^{-1})^a\Big) = q^a\varphi(a)$$

for all $a \in A$. In other words

$$P_{(A,\varphi)} = \cap_{a\in A} {}^aQ_{\varphi(a)}$$

It acts trivially on P/Q. Thus if $q \in P_{(A,\varphi)}$, then $q^{-1}C = C$ for any subset C of P such that $CQ = C$, and

$$\begin{aligned} q\theta_i\Big((q^{-1}B, q^{-1}\psi)\Big) &= \sum_{U\subseteq I(B)} (-1)^{|U|+r_B-1} q(B \sqcup a_{l_B}Q, [\varphi, q^{-1}\psi]_U) \\ &= \sum_{U\subseteq I(B)} (-1)^{|U|+r_B-1} (B \sqcup a_{l_B}Q, q[\varphi, q^{-1}\psi]_U) \end{aligned}$$

Moreover

$$q[\varphi, q^{-1}\psi]_U(c) = [\varphi, q^{-1}\psi]_U(q^{-1}c) = \begin{cases} (q^{-1}\psi)(q^{-1}c) = \psi(c) & \text{if } c \in B-UQ \\ \varphi(q^{-1}c) = \varphi(c) & \text{if } c \in UQ \sqcup a_{l_B}Q \end{cases}$$

It follows that θ_i is $P_{(A,\varphi)}$-invariant.

I will show that $d_i\theta_i + \theta_{i-1}d_{i-1} = Id_{T_{i-1}}$. First

$$d_i\theta_i\Big((B,\psi)\Big) = \sum_{U\subseteq I(B)} (-1)^{|U|+r_B-1} d_i(B \sqcup a_{l_B}Q, [\varphi, \psi]_U)$$

and moreover

$$\begin{aligned} d_i(B \sqcup a_{l_B}Q, [\varphi,\psi]_U) = \sum_{m=1}^{r_B-1} (-1)^{m-1} \Big((B \sqcup a_{l_B}Q) - b_mQ, f_{m,U}\Big) + \\ +(-1)^{r_B-1}(B, f_U) + \\ + \sum_{m=r_B}^{i-1} (-1)^{m-1+1} \Big((B \sqcup a_{l_B}Q) - b_mQ, f_{m,U}\Big) \end{aligned}$$

where $f_{m,U}$ denotes the restriction of $[\varphi,\psi]_U$ to $B \sqcup a_{l_B}Q - b_mQ$, and f_U is the map equal to ψ on $B-UQ$ and to φ on UQ.

Finally this gives

$$d_i\theta_i\Big((B,\psi)\Big) = S_1 + S_2 + S_3$$

where S_1, S_2 and S_3 are given by

$$S_1 = \sum_{U\subseteq I(B)} \sum_{m=1}^{r_B-1} (-1)^{|U|+r_B+m} \Big((B \sqcup a_{l_B}Q) - b_mQ, f_{m,U}\Big)$$

$$S_2 = \sum_{U\subseteq I(B)} (-1)^{|U|+r_B+r_B} (B, f_U)$$

$$S_3 = \sum_{U\subseteq I(B)} \sum_{m=r_B}^{i-1} (-1)^{|U|+r_B+m-1} \Big((B \sqcup a_{l_B}Q) - b_mQ, f_{m,U}\Big)$$

On the other hand

$$\theta_{i-1}d_{i-1}\Big((B,\psi)\Big) = \theta_{i-1}\Big(\sum_{m=1}^{i-1}(-1)^{m-1}(B-b_mQ,\psi_{|B-b_mQ})\Big)$$

To compute $\theta_{i-1}\Big((B-b_mQ,\psi_{|B-b_mQ})\Big)$, I must distinguish the case $b_m \in A$ and $m \leq r_B - 1$, the case $b_m \notin A$ and $m \leq r_B - 1$, and the case $m \geq r_B$.

If $m \leq r_B - 1$ and $b_m = a_n$, then $l_{B-b_mQ} = n$ and $I(B-b_mQ) = \{a_1,\ldots,a_{n-1}\}$. Moreover $r_{B-b_mQ} = m$ since b_{m+1} is the least element of $(B-b_mQ)/Q$ greater than a_n, and the m^{th} element of $(B-b_mQ)/Q$. In this case $(B-b_mQ)\sqcup a_nQ = B$ and

$$\theta_{i-1}\Big((B-b_mQ,\psi_{|B-b_mQ})\Big) = \sum_{U\subseteq\{a_1,\ldots,a_{n-1}\}}(-1)^{|U|+m-1}(B,g_{n,U})$$

where $g_{n,U}$ is equal to ψ on $B-(UQ\sqcup a_nQ)$ and to φ on $UQ\sqcup a_nQ$.

If $m \leq r_B - 1$ and $b_m \notin A$, then $l_{B-b_mQ} = l_B$ and $I(B-b_mQ) = I(B)$. Moreover $r_{B-b_mQ} = r_B - 1$. In this case

$$\theta_{i-1}\Big((B-b_mQ,\psi_{|B-b_mQ})\Big) = \sum_{U\subseteq I(B)}(-1)^{|U|+r_B}\Big((B\sqcup a_{l_B}Q)-b_mQ,h_{m,U}\Big)$$

where $h_{m,U}$ is the map equal to ψ on $(B\sqcup a_{l_B}Q)-(b_mQ\sqcup UQ)$ and to φ on $a_{l_B}Q\sqcup UQ$.

If $m \geq r_B$, then again $l_{B-b_mQ} = l_B$ and $I(B-b_mQ) = I(B)$. Moreover $r_{B-b_mQ} = r_B$, and

$$\theta_{i-1}\Big((B-b_mQ,\psi_{|B-b_mQ})\Big) = \sum_{U\subseteq I(B)}(-1)^{|U|+r_B-1}\Big((B\sqcup a_{l_B}Q)-b_mQ,h_{m,U}\Big)$$

with the same notation.

Finally, this gives

$$\theta_{i-1}d_{i-1}\Big((B,\psi)\Big) = S'_1 + S'_2 + S'_3$$

where S'_1, S'_2 and S'_3 are given by

$$S'_1 = \sum_{\substack{1\leq m\leq r_B-1\\ b_m=a_n\in A}}\ \sum_{U\subseteq\{a_1,\ldots,a_{n-1}\}}(-1)^{m-1+|U|+m-1}(B,g_{n,U})$$

$$S'_2 = \sum_{\substack{1\leq m\leq r_B-1\\ b_m\notin A}}\ \sum_{U\subseteq I(B)}(-1)^{m-1+|U|+r_B}\Big((B\sqcup a_{l_B}Q)-b_mQ,h_{m,U}\Big)$$

$$S'_3 = \sum_{r_B\leq m\leq i-1}\ \sum_{U\subseteq I(B)}(-1)^{m-1+|U|+r_B-1}\Big((B\sqcup a_{l_B}Q)-b_mQ,h_{m,U}\Big)$$

Replacing U by $U\sqcup\{a_n\}$, the double summation in S'_1 can also be written

$$S'_1 = \sum_{\emptyset\neq U\subseteq I(B)}(-1)^{|U|-1}(B,f_U)$$

It follows in particular that the only non-zero term in $S_2 + S'_1$ is obtained in S_2 for $U = \emptyset$, thus

$$S_2 + S'_1 = (-1)^0(B,f_\emptyset) = (B,\psi)$$

Now the double summation in S_1 looks like the double summation in S_2': both are indexed by a subset U of $I(B)$ and an integer m such that $1 \leq m \leq r_B - 1$. In S_2', I require moreover that $b_m \notin A$. Apart from this, the general term of S_1 is

$$(-1)^{|U|+r_B+m}\Big((B \sqcup a_{l_B}Q) - b_m Q, f_{m,U}\Big)$$

whereas the general term of S_2' is

$$(-1)^{m-1+|U|+r_B}\Big((B \sqcup a_{l_B}Q) - b_m Q, h_{m,U}\Big)$$

Both maps $f_{m,U}$ and $h_{m,U}$ are equal to ψ on the set $(B \sqcup a_{l_B}Q) - (b_m Q \sqcup UQ)$ and to φ on the set $(a_{l_B}Q \sqcup UQ) - b_m Q$. Hence the corresponding terms vanish. The only left terms in $S_1 + S_2'$ are the terms of S_1 for which $b_m \in A$. Thus

$$S_1 + S_2' = \sum_{U \subseteq I(B)} \sum_{\substack{1 \leq m \leq r_B - 1 \\ b_m \in A}} (-1)^{|U|+r_B+m}\Big((B \sqcup a_{l_B}Q) - b_m Q, f_{m,U}\Big)$$

Switching the summation gives

$$S_1 + S_2' = \sum_{\substack{1 \leq m \leq r_B - 1 \\ b_m \in A}} \sum_{U \subseteq I(B)} (-1)^{|U|+r_B+m}\Big((B \sqcup a_{l_B}Q) - b_m Q, f_{m,U}\Big)$$

Now for a given m and a given $b_m \in A$, the summation on U can be split as

$$\sum_{\substack{U \subseteq I(B) \\ b_m \in U}} (-1)^{|U|+r_B+m}\Big((B \sqcup a_{l_B}Q) - b_m Q, f_{m,U}\Big) + \\ + \sum_{\substack{U \subseteq I(B) \\ b_m \notin U}} (-1)^{|U|+r_B+m}\Big((B \sqcup a_{l_B}Q) - b_m Q, f_{m,U}\Big)$$

Now every $U \ni b_m$ can be written as $U' \sqcup \{b_m\}$, where $U' \not\ni b_m$. Moreover the map $f_{m,U'}$ is equal to ψ on

$$(B \sqcup a_{l_B}Q) - (b_m Q \cup UQ) = (B \sqcup a_{l_B}Q) - (b_m Q \sqcup U'Q)$$

and to φ on $(a_{l_B}Q \sqcup U'Q) - b_m Q = (a_{l_B}Q \sqcup UQ) - b_m Q$. Since moreover $|U| = |U'| + 1$, the signs are opposite, and $S_1 + S_2' = 0$.

Finally the double summation in S_3 looks like the summation in S_3': both are indexed by a subset U of $I(B)$ and an integer $m \geq r_B$. The general term of S_3 is

$$(-1)^{|U|+r_B+m-1}\Big((B \sqcup a_{l_B}Q) - b_m Q, f_{m,U}\Big)$$

whereas the general term of S_3' is

$$(-1)^{m-1+|U|+r_B-1}\Big((B \sqcup a_{l_B}Q) - b_m Q, h_{m,U}\Big)$$

Here again, the maps $f_{m,U}$ and $h_{m,U}$ are the same. As the signs are opposite, the sum $S_3 + S_3'$ is zero.

This shows finally that

$$(d_i\theta_i + \theta_{i-1}d_{i-1})\Big((B,\psi)\Big) = S_1 + S_2 + S_3 + S_1' + S_2' + S_3' = (B,\psi)$$

and $d_i\theta_i + \theta_{i-1}d_{i-1}$ is the identity map of T_{i-1}. It follows that $d_i = d_i \circ \theta_i \circ d_i$, as claimed. ∎

Now for $i < p$, there is an exact sequence

$$0 \longrightarrow \mathrm{Ker}d_i \longrightarrow T_i \simeq \mathcal{O}Z_i \xrightarrow{d_i} \mathrm{Ker}d_{i-1} \longrightarrow 0$$

and for every $(A,\varphi) \in Z_j$, for $j \geq i$, the restriction of θ_i to $\mathrm{Ker}d_{i-1}$ is a section of d_i, which is invariant under the action of $P_{(A,\varphi)}$. By lemma 2.1.2, this means that this sequence is Z_j-split.

Moreover $T_i \simeq \mathcal{O}Z_i$ is Z_i-projective. Since the stabilizer of (A,φ) is contained in Q, the group P has no fixed points on Z_i. An easy induction using lemma 3.2.8, starting from the sequence

$$0 \to \mathrm{Ker}d_1 \to T_1 \to T_0 \simeq \mathcal{O} \to 0$$

shows that $\mathrm{Ker}d_i$ is a capped endo-permutation $\mathcal{O}P$-lattice, and that

(5.4.2) $$\mathrm{Ker}d_i = \Omega_{Z_i} + \mathrm{Ker}d_{i-1}$$

in $D_{\mathcal{O}}(P)$.

At the last stage $i = p$, there is an exact sequence

(5.4.3) $$0 \to \mathcal{O}^\varepsilon \otimes \mathrm{Ten}_Q^P\Omega_X \to T_p \simeq \mathcal{O}^\varepsilon \otimes \mathrm{Ten}_Q^P\mathcal{O}X \to \mathrm{Ker}d_{p-1} \to 0$$

where $\mathcal{O}^\varepsilon$ is the module $\mathcal{O}$ on which $g \in P$ acts by the sign of the permutation it induces on P/Q. The module $\mathrm{Ten}_Q^P\mathcal{O}X$ is the permutation module $\mathcal{O}Z_p$, and Z_p identifies with the set of Q-equivariant maps from P to X. In particular T_p is Z_p-projective.

The submodule $T_P' = \mathcal{O}^\varepsilon \otimes \mathrm{Ten}_Q^P\Omega_X$ of T_p can be viewed as the set of linear combinations

$$\sum_{\varphi\in Z_p} \lambda_\varphi \varphi$$

such that for all $a \in [P/Q]$ and all Q-invariant maps ρ from $P - aQ$ to X, the sum of the coefficients λ_φ corresponding to those φ's for which $\varphi_{|P-aQ} = \rho$ is zero.

Now the inclusion $T_p' \to T_p$ is Z_p-split: if $\varphi \in Z_p$ is given, define a map θ from T_p to T_p' by

$$\theta(\psi) = \sum_{U\subseteq[P/Q]} (-1)^{|U|}[\varphi,\psi]_U$$

where $[\varphi,\psi]_U$ is the map equal to ψ on $P - UQ$, and to φ on UQ. The element $\theta(\psi)$ is in T_p', for if $a \in [P/Q]$ is given, and if $a \notin U$, the maps $[\varphi,\psi]_U$ and $[\varphi,\psi]_{U\sqcup\{a\}}$ have the same restriction to $P - aQ$, and their coefficients in the expression of $\theta(\psi)$ are opposite.

Let g be an element of the stabilizer of φ in P, and let ε_g be the signature of the permutation it induces on P/Q. Then g maps the element ψ of T_p to $\varepsilon_g(g.\psi)$, and

$$\theta(\varepsilon_g g\psi) = \sum_{U\subseteq[P/Q]} (-1)^{|U|}\varepsilon_g[\varphi, g\psi]_U$$

It is clear moreover that

$$[\varphi, g.\psi]_U = [g\varphi, g.\psi]_U = g([\varphi,\psi]_{g^{-1}U})$$

so that replacing U by $g^{-1}U$ in the summation gives

$$\theta(\varepsilon_g g\psi) = \sum_{U\subseteq[P/Q]} (-1)^{|U|}\varepsilon_g g([\varphi,\psi]_{g^{-1}U}) = g \sum_{U\subseteq[P/Q]} (-1)^{|U|}[\varphi,\psi]_U = g\theta(\psi)$$

Hence θ is invariant by the stabilizer of φ. Moreover, it is a section of the inclusion of T'_p into T_p: indeed if $f = \sum_{\psi\in Z_p}\lambda_\psi\psi$ is an element of T'_p, then

$$\theta(f) = \sum_{\psi\in Z_p}\lambda_\psi \sum_{U\subseteq[P/Q]} (-1)^{|U|}[\varphi,\psi]_U$$

The coefficient μ_h of a given $h\in Z_p$ in this summation is equal to

$$\mu_h = \sum_{U\subseteq S}(-1)^{|U|}\sum_{\psi\in R_U}\lambda_\psi$$

where S is the set of elements a of $[P/Q]$ for which $\varphi(a) = h(a)$, and R_U is the set of elements ψ of Z_p which coincide with h on $P-UQ$.

Now if $U\neq\emptyset$, the sum of coefficients λ_ψ for ψ in R_U is zero: indeed, if $a\in U$ is given, this sum is equal to

$$\sum_{g\in E_{a,U,h}} \sum_{\substack{\psi\in Z_p\\ \psi_{|P-aQ}=g_{|P-aQ}}} \lambda_\psi$$

where $E_{a,U,h}$ is the set of Q-equivariant maps from $P-aQ$ to X which coincide with h on $P-UQ$. The inner sum is zero if $f\in T'_p$. Thus in the expression of μ_h, the only terms left correspond to $U=\emptyset$, and $R_\emptyset = \{h\}$. Hence $\mu_h = \lambda_h$ for all h, and $\theta(f) = f$ if $f\in T'_p$.

This shows that the inclusion of T'_p into T_p is Z_p-split, and completes the proof of lemma 5.4.1. □

(c) *End of the proof of theorem 5.1.2 in the case $|P:Q| = p$.* An element $\psi\in Z_p$ is fixed by P if

$$\psi(gah) = h^{-1}\psi(a)$$

for all $g, a\in P$ and $h\in Q$. Since P is transitive on P, this implies $\psi(a) = \psi(1)$ for all $a\in P$, and finding such an element is equivalent to choosing $\psi(1)\in X^Q$. As this set is empty by assumption, it follows that $Z_p^P = \emptyset$.

Now tensoring sequence 5.4.3 with $\mathcal{O}^\varepsilon$ gives the sequence

$$0\to \mathrm{Ten}_Q^P\Omega_X \to \mathrm{Ten}_Q^P\mathcal{O}Z_p \to \mathcal{O}^\varepsilon\otimes\mathrm{Ker}d_{p-1}\to 0$$

This is still a Z_p-split exact sequence. Lemma 3.2.8 shows that $\mathrm{Ten}_Q^P\Omega_X$ is a capped $\mathcal{O}P$-lattice, and that

$$\mathrm{Ten}_Q^P\Omega_X = \Omega_{Z_p} + \mathrm{Ker}d_{p-1} \tag{5.4.4}$$

in $D_\mathcal{O}(P)$, since $\mathrm{Ker}d_{p-1}$ and $\mathcal{O}^\varepsilon\otimes\mathrm{Ker}d_{p-1}$ have the same image in that group.

Summarizing the successive equalities 5.4.2 and 5.4.4 gives finally the following expression for $\mathrm{Ten}_Q^P\Omega_X$ in $D_\mathcal{O}(P)$

$$\mathrm{Ten}_Q^P\Omega_X = \Omega_{Z_p} + \Omega_{Z_{p-1}} + \ldots + \Omega_{Z_1}$$

Now by lemma 5.2.3, for each $i \in \{1, \ldots, p\}$

$$\Omega_{Z_i} = \sum_{\substack{U,V \in [s_P] \\ U \leq_P V \\ Z_i^V \neq \emptyset}} \mu_P(U,V)\Omega_{P/U}$$

The set Z_i^V is non-empty if and only if there exists a subset A of P such that $VAQ = A$ and $|A/Q| = i$, and a map $\psi : A \to X$ such that

$$\psi(vaq) = q^{-1}\psi(a)$$

for all $v \in V$, $a \in A$, and $q \in Q$. If $i < p$, this implies $V \subseteq Q$ (otherwise $VQ = P$, and $A = VAQ = AVQ = AP = P$), and it is equivalent to find an element $\psi(a) \in X^{V^a}$ for each $a \in A/Q$. Thus Z_i^V is non-empty if $V \subseteq Q$ and if there are at least i cosets aQ in P/Q such that $X^{V^a} \neq \emptyset$. Note that $aQ = VaQ$ and $P/Q = V\backslash P/Q$ in this case. Similarly $V^a = V^a \cap Q$. Hence there are at least i double cosets VaQ such that $X^{V^a \cap Q} \neq \emptyset$.

If $i = p$, then Z_p^V is non-empty if there is a map ψ from P to X such that $\psi(vaq) = q^{-1}\psi(a)$ for $v \in V$, $a \in P$, and $q \in Q$. If $V \subseteq Q$, the result is the same as before: such an element ψ exists if and only if for all $a \in P$, the set X^{V^a} is non-empty, or equivalently, if there are p double cosets VaQ such that $X^{V^a \cap Q} \neq \emptyset$.

And if $V \not\subseteq Q$, then $VQ = P$, and such a ψ is defined by $\psi(1)$, which must be such that $q^{-1}\psi(1) = \psi(1)$ for $q \in Q$ whenever there is an element $v \in V$ with $vq = 1$. In other words $\psi(1) \in X^{V \cap Q}$, and the set Z_p^V is non-empty if and only if $X^{V \cap Q}$ is. Note that $|V\backslash P/Q| = 1$ in this case.

Those remarks show that in $D_{\mathcal{O}}(P)$

$$\mathrm{Ten}_Q^P \Omega_X = \sum_{i=1}^{p} \sum_{\substack{U,V \in [s_P] \\ U \leq_P V \leq Q \\ |E(V,Q,X)| \geq i}} \mu_P(U,V)\Omega_{P/U} + \sum_{\substack{U,V \in [s_P] \\ U \leq_P V \not\leq Q \\ |E(V,Q,X)| = 1}} \mu_P(U,V)\Omega_{P/U}$$

where $E(V,Q,X)$ is the set of double cosets VaQ in P such that $X^{V^a \cap Q} \neq \emptyset$. Switching the summations on i and U, V gives

$$\mathrm{Ten}_Q^P \Omega_X = \sum_{\substack{U,V \in [s_P] \\ U \leq_P V \leq Q}} |E(V,Q,X)| \mu_P(U,V)\Omega_{P/U} + \sum_{\substack{U,V \in [s_P] \\ U \leq_P V \not\leq Q \\ |E(V,Q,X)| = 1}} \mu_P(U,V)\Omega_{P/U}$$

This second sum is also equal to

$$\sum_{\substack{U,V \in [s_P] \\ U \leq_P V \not\leq Q}} |E(V,Q,X)| \mu_P(U,V)\Omega_{P/U}$$

since when $V \not\leq Q$, there is only one double coset VaQ. Finally, this gives

$$\mathrm{Ten}_Q^P \Omega_X = \sum_{\substack{U,V \in [s_P] \\ U \leq_P V}} |E(V,Q,X)| \mu_P(U,V)\Omega_{P/U}$$

and this completes the proof of the case $|P : Q| = p$, and the proof of theorem 5.1.2. $\square$

6. Some elements in the Dade group

(6.0.1) **Notation:** *I denote by* $D^{\Omega}_{\mathcal{O}}(P)$ *the subgroup of* $D_{\mathcal{O}}(P)$ *generated by the relative syzygies* Ω_X*, for non-empty finite* P*-sets* X.

Note that the reduction morphism $D_{\mathcal{O}}(P) \to D_k(P)$ restricts to an isomorphism $D^{\Omega}_{\mathcal{O}}(P) \simeq D^{\Omega}_k(P)$, since it is always injective, and since the element Ω_X of $D^{\Omega}_k(P)$ is the reduction of the element Ω_X of $D^{\Omega}_{\mathcal{O}}(P)$. Moreover corollary 4.1.2 and theorem 5.1.2 show that the operations of restriction, inflation, group isomorphism, and tensor induction preserve $D^{\Omega}_{\mathcal{O}}$. Lemma 4.2.1 shows that deflation maps $D^{\Omega}_{\mathcal{O}}$ to D^{Ω}_k. This could be used to define a deflation map from $D^{\Omega}_{\mathcal{O}}(P)$ to $D^{\Omega}_{\mathcal{O}}(P/Q))$.

6.1. Linear relations in the Dade group

(6.1.1) **Proposition:** *Let* P *be a* p*-group. Then for any subgroup* Q *of* P*, for any finite* P*-set* X*, and any finite* Q*-set* Y*, the following equalities hold in* $D_{\mathcal{O}}(P)$

(6.1.2)
$$0 = \sum_{\substack{U,V\in[s_P]\\ U\leq_P V}} \mu_P(U,V)\, |V\backslash P/Q|\, \Omega_{P/U}$$

(6.1.3)
$$\mathrm{Ten}_Q^P\Omega_Y = -\sum_{\substack{U,V\in[s_P]\\ U\leq_P V}} \mu_P(U,V)\, |\{a\in V\backslash P/Q \mid Y^{V^a\cap Q}=\emptyset\}|\, \Omega_{P/U}$$

(6.1.4)
$$\Omega_X = -\sum_{\substack{U,V\in[s_P]\\ U\leq_P V\\ X^V=\emptyset}} \mu_P(U,V)\Omega_{P/U}$$

Proof: Equation 6.1.2 follows from the special case $X^Q \neq \emptyset$ of theorem 5.1.2, which gives

$$\mathrm{Ten}_Q^P\Omega_X = 0 = \sum_{\substack{U,V\in[s_P]\\ U\leq_P V}} \mu_P(U,V)\, |V\backslash P/Q|\, \Omega_{P/U}$$

Now if Y is any Q-set, this formula and theorem 5.1.2 show that $\mathrm{Ten}_Q^P\Omega_Y$ can also be written as

$$\mathrm{Ten}_Q^P\Omega_Y = -\sum_{\substack{U,V\in[s_P]\\ U\leq_P V}} \mu_P(U,V)\, |\{a\in V\backslash P/Q \mid Y^{V^a\cap Q}=\emptyset\}|\, \Omega_{P/U}$$

and in the special case $P = Q$, this gives for any P-set X

$$\Omega_X = -\sum_{\substack{U,V\in[s_P]\\ U\leq_P V\\ X^V=\emptyset}} \mu_P(U,V)\Omega_{P/U}$$

completing the proof. □

6.2. The element $\Delta(P)$

(6.2.1) **Notation:** *If P is a p-group, I denote by $M(P)$ the disjoint union of the P-sets P/Q, when Q runs through the set of maximal proper subgroups of P, i.e. the set of subgroups of P of index p. I denote by*

$$\Delta(P) = \Omega_{M(P)}$$

the corresponding relative syzygy in $D_{\mathcal{O}}(P)$.

I denote by $D_{\mathcal{O}}^{\Delta}(P)$ the subgroup of $D_{\mathcal{O}}(P)$ generated by all the elements $\mathrm{Ten}_Q^P\Delta(Q)$, when Q runs through the subgroups of P.

(6.2.2) **Remark:** It is clear by construction that every proper subgroup of P has fixed points on $M(P)$. By lemma 3.2.4 and corollary 4.1.2, this shows that the restriction of $\Delta(P)$ to any proper subgroup of P is zero.

Similarly, if Q is a normal subgroup of P, then the set $M(P)^Q$ identifies with the set $M(P/Q)$. By lemma 4.2.1, this shows that $\mathrm{Def}_{P/Q}^P\Delta(P) = \Delta(P/Q)$.

Finally, it is clear that if $\phi : P \to P'$ is a group isomorphism, then the image of $\Delta(P)$ by ϕ in $D_{\mathcal{O}}(P)$ is equal to $\Delta(P')$, since ϕ induces an isomorphism from $M(P)$ to $M(P')$.

(6.2.3) **Proposition:** *Let P be a p-group. Then*

$$\Delta(P) = -\sum_{U \subseteq P} \mu(U,P)\Omega_{P/U}$$

where $\mu(U,P)$ is the Möbius function of the poset of subgroups of P.

Proof: Formula 6.1.4 gives

$$\Delta(P) = -\sum_{\substack{U,V\in[s_P] \\ U\leq_P V \\ M(P)^V=\emptyset}} \mu_P(U,V)\Omega_{P/U}$$

Now $M(P)^V$ is empty if and only if $V = P$, and

$$\Delta(P) = -\sum_{U\in[s_P]} \mu_P(U,P)\Omega_{P/U}$$

But $\mu_P(U,P)$ is the reduced Euler-Poincaré characteristic of the "interval" $]\overline{U},\overline{P}[_P$ of classes of proper subgroups of P which are strictly greater than $\overline{U}$ in s_P. If U does not contain the Frattini subgroup $\Phi(P)$ of P, this set is contractible, via the maps

$$\overline{V} \mapsto \overline{V.\Phi(P)} \mapsto \overline{U.\Phi(P)}$$

Now it suffices to sum over the classes of subgroups U containing $\Phi(P)$. Such subgroups are normal in P, and $]\overline{U},\overline{P}[_P$ is isomorphic to the ordinary poset $]U,P[$ of proper subgroups of P containing U as a proper subgroup. This poset has Euler-Poincaré characteristic $\mu(U,P)$ equal to $(-1)^n p^{\binom{n}{2}}$ if P/U is of order p^n. If $U \not\supseteq \Phi(P)$, the poset $]U,P[$ is contractible, by the same argument as above. Finally

$$\Delta(P) = -\sum_{U \subseteq P} \mu(U,P)\Omega_{P/U}$$

as was to be shown. ∎

(6.2.4) **Remark:** It is clear from the definition that the Frattini subgroup of P acts trivially on $M(P)$, and the element $\Delta(P)$ should be inflated from the quotient $P/\Phi(P)$. This is indeed the case, since if $\mu(U,P) \neq 0$, then $U \supseteq \Phi(P)$, and $\Omega_{P/U}$ is inflated from $P/\Phi(P)$. This shows in particular that if $p = 2$ and P is cyclic, then $\Delta(P) = 0$, since $D_{\mathcal{O}}(\mathbb{Z}/2\mathbb{Z}) = 0$.

(6.2.5) **Corollary:** *The element $\Delta(P)$ has infinite order in $D_{\mathcal{O}}(P)$ if P is not cyclic, it has order 2 if p is odd and P is cyclic and non-trivial, and it is zero if P is trivial or cyclic of even order.*

Proof: Since the natural reduction morphism $D_{\mathcal{O}}(P) \to D_k(P)$ is injective (see [15] Proposition (29.4)), and since the reduction of $\Omega_X \in D_{\mathcal{O}}(P)$ is $\Omega_X \in D_k(P)$, it suffices to consider the case $\mathcal{O} = k$. Now $\Delta(P) = \mathrm{Inf}_{P/\Phi(P)}^P \Delta\Big(P/\Phi(P)\Big)$, and deflation is a section of inflation. In particular, inflation is injective, and it suffices to consider the case where P is elementary abelian.

In this case, the elements $\Omega_{P/Q}$, for P/Q not cyclic, are linearly independent in $D_k(P)$, by a theorem of Dade ([10]). In particular if P is not cyclic, the element $\Omega_{P/1}$ has infinite order, and it has order 2 if P is cyclic of order $p > 2$.

Now the element $2\Delta(P)$ is a linear combination of elements $\Omega_{P/Q}$, for P/Q non-cyclic, and the coefficient of $\Omega_{P/1}$ is equal to $2\mu(1,P)$, which is non-zero. Thus if P is not cyclic, the element $\Delta(P)$ has infinite order.

And if P is cyclic of order $p \geq 3$, then $\Delta(P) = \Omega_{P/1}$ has order 2. Finally, if $|P| \leq 2$, then $D_k(P)$ is the trivial group. ∎

6.3. Characterization

The element $\Delta(P)$ is always inflated from $P/\Phi(P)$. When P is elementary abelian, it can be characterized as follows:

(6.3.1) **Proposition:** *Let P be an elementary abelian p-group. Then:*

1. *There is a unique element $\tilde{\Delta}(P)$ in $D_{\mathcal{O}}^{\Omega}(P)$ such that*

$$\Delta(P) = p\tilde{\Delta}(P)$$

2. *Let $\underline{D_{\mathcal{O}}}(P)$ denote the subgroup of $D_{\mathcal{O}}(P)$ defined by*

$$\underline{D_{\mathcal{O}}}(P) = \bigcap_{Q \subset P} \mathrm{Ker}\mathrm{Res}_Q^P$$

where Q runs through the proper subgroups of P. Then $\underline{D_{\mathcal{O}}}(P)$ is the subgroup of $D_{\mathcal{O}}(P)$ generated by $\tilde{\Delta}(P)$.

Proof: Assertion 1) is a consequence of the expression of $\Delta(P)$ given in proposition 6.2.3:

$$\Delta(P) = -\sum_{U \subseteq P} \mu(U,P)\Omega_{P/U}$$

Since $\Omega_{P/P}$ is zero, the summation runs through proper subgroups U of P. If U has index p^n in P, with $n \geq 2$, then the coefficient $\mu(U,P) = (-1)^n p^{\binom{n}{2}}$ of $\Omega_{P/U}$ is a multiple of p. And if $|P:U|=p$, then the element $\Omega_{P/U}$ has order 2 in $D_{\mathcal{O}}(P)$ if p is odd, or it is zero if $p=2$. In any case it is equal to $p\Omega_{P/U}$. The existence of $\tilde{\Delta}(P)$ follows. The unicity is also clear, since $D_{\mathcal{O}}(P)$ has no p-torsion if P is elementary abelian (even for $p=2$, since $D_{\mathcal{O}}(P)$ is torsion free in this case). Assertion 1) follows.

Now for assertion 2), first observe that $\tilde{\Delta}(P)$ is an element of $\underline{D_{\mathcal{O}}}(P)$: indeed, by remark 6.2.2, for any proper subgroup Q of P

$$p\mathrm{Res}_Q^P\tilde{\Delta}(P)=\mathrm{Res}_Q^P\Delta(P)=0$$

Since $D_{\mathcal{O}}(Q)$ has no p-torsion by Dade's theorem, it follows that

$$\mathrm{Res}_Q^P\tilde{\Delta}(P)=0$$

as required.

I will prove that $\tilde{\Delta}(P)$ is a generator of $\underline{D_{\mathcal{O}}}(P)$ by induction on $|P|$. First observe that if Q and R are subgroups of P, then by corollary 4.1.2

$$\mathrm{Res}_R^P\Omega_{P/Q}=\Omega_{\mathrm{Res}_R^P P/Q}$$

Since moreover $\mathrm{Res}_R^P P/Q$ is a union of $|P:QR|$ copies of $R/R\cap Q$, it follows that

$$\mathrm{Res}_R^P\Omega_{P/Q}=\Omega_{R/R\cap Q}$$

Now by Dade's theorem (see [10]), the elements $\Omega_{P/Q}$ for $Q\subset P$ generate $D_{\mathcal{O}}(P)$. An element $u=\sum_{Q\subset P}n_Q\Omega_{P/Q}$ is in $\underline{D_{\mathcal{O}}}(P)$ if and only if for any proper subgroup R of P

$$0=\sum_{Q\subset P}n_Q\Omega_{R/R\cap Q}$$

Moreover the elements $\Omega_{R/S}$, for R/S non-cyclic, are linearly independent in $D_{\mathcal{O}}(P)$, and the elements $\Omega_{R/S}$, for $|R/S|=p$, are elements of order 2 if p is odd, and independent over $\mathbb{F}_2$. If $p=2$, these elements are equal to zero.

It follows that $u\in\underline{D_{\mathcal{O}}}(P)$ if and only if for any pair of subgroups $S\subset R\subset P$ (with proper inclusions)

$$\text{(6.3.2)}\qquad \sum_{\substack{Q\subset P\\ Q\cap R=S}}n_Q\begin{cases}=0 & \text{if}\ \ |R/S|\geq p^2\\ \equiv 0\mod. 2 & \text{if}\ \ |R/S|=p>2\end{cases}$$

If $|P|\leq 2$, then $D_{\mathcal{O}}(P)=0$ and there is nothing to prove. If $|P|=p>2$, then $\underline{D_{\mathcal{O}}}(P)=D_{\mathcal{O}}(P)=\mathbb{Z}/2\mathbb{Z}$, and there is nothing to prove. If $|P|=4$, then $\underline{D_{\mathcal{O}}}(P)=D_{\mathcal{O}}(P)$ is isomorphic to $\mathbb{Z}$, generated by

$$\Omega_{P/1}=-\frac{1}{2}\Delta(P)=-\tilde{\Delta}(P)$$

If $|P|=p^2>4$, then condition 6.3.2 for a given subgroup R of index p in P and $S=\{1\}$ gives

$$n_1+\sum_{\substack{|Q|=p\\ Q\neq R}}n_Q\equiv 0\mod. 2$$

which is equivalent to

$$n_R\equiv n_1+\sum_{|Q|=p}n_Q\mod. 2$$

Hence n_R is independent of R modulo 2, equal to m, say. Therefore

$$m \equiv n_1 + (p+1)m \equiv n_1 \mod. \ 2$$

Finally u is in $\underline{D}_{\mathcal{O}}(P)$ if and only if it can be written

$$u = n_1(\Omega_{P/1} + \sum_{|P:Q|=p} \Omega_{P/Q}) = n_1(\Omega_{P/1} - \sum_{|P:Q|=p} \Omega_{P/Q})$$

But

$$\tilde{\Delta}(P) = -\Omega_{P/1} + \sum_{|P:Q|=p} \Omega_{P/Q}$$

hence assertion 2) holds in this case.

The last special case is $|P| = p^3$. For any subgroup R of index p in P, condition 6.3.2 for $S = 1$ gives

$$n_1 + \sum_{\substack{|Q|=p \\ Q \not\subseteq R}} n_Q = 0 \tag{6.3.3}$$

Summing these relations for subgroups R containing a given subgroup Q_0 of order p of P gives

$$(p+1)n_1 + \sum_{\substack{|Q|=p \\ Q \neq Q_0}} pn_Q = 0$$

Adding n_{Q_0} to both sides gives

$$(p+1)n_1 + \sum_{|Q|=p} pn_Q = n_{Q_0}$$

The left hand side is independent of Q_0, thus n_{Q_0} is independent of Q_0, and equal to m, say. Equation 6.3.3 gives

$$n_1 + p^2 m = 0$$

Thus $n_1 = -p^2 m$. If $p = 2$, then

$$u = m(-p^2\Omega_{P/1} + \sum_{|Q|=p} \Omega_{P/Q}) = -m\tilde{\Delta}(P)$$

since $\Omega_{P/R} = 0$ for $|P : R| = 2$. Thus I can suppose p odd.

In this case for a subgroup R of index p in P, containing a subgroup S of order p, condition 6.3.2 gives

$$n_S + \sum_{\substack{|Q|=p^2 \\ Q \supset S \\ Q \neq R}} n_Q \equiv 0 \mod. \ 2$$

Adding n_R to both sides gives also, since $n_S = m$

$$m + \sum_{\substack{|Q|=p^2 \\ Q \supset S}} n_Q \equiv n_R \mod. \ 2 \tag{6.3.4}$$

It follows that $l_S = \sum_{\substack{|Q|=p^2 \\ Q\supset S}} n_Q$ is independent modulo 2 of S, since for two distinct subgroups S and S' of order p

$$l_S \equiv n_{S.S'} - m \equiv l'_S \mod. \ 2$$

Let l denote this common value. Equation 6.3.4 shows then that n_R is independent of R modulo 2, and that

$$m + (p+1)l \equiv m \equiv n_R \mod. \ 2$$

thus also $n_R \equiv -m$ modulo 2. Finally

$$u = -p^2 m\Omega_{P/1} + m \sum_{|Q|=p} \Omega_{P/Q} - m \sum_{|R|=p^2} \Omega_{P/R} = -m\tilde{\Delta}(P)$$

and assertion 2) holds.

For the general case $|P| = p^d \geq p^4$, I observe first that if Q and R are subgroups of P, then by lemma 4.2.1 $\mathrm{Def}^P_{P/R}\Omega_{P/Q}$ is equal to zero if $R \not\subseteq Q$, and to $\Omega_{(P/R)/(Q/R)}$ if $R \subseteq Q$. Since moreover

$$\mathrm{Def}^Q_{Q/R}\mathrm{Res}^P_Q = \mathrm{Res}^{P/R}_{Q/R}\mathrm{Def}^P_{P/R}$$

it follows that $\mathrm{Def}^P_{P/R}u \in \underline{D}_k(P/R)$ if $u \in \underline{D}_{\mathcal{O}}(P)$. But if $u = \sum_{Q\subset P} n_Q\Omega_{P/Q}$, then

$$\mathrm{Def}^P_{P/R}u = \sum_{R\subseteq Q\subset P} n_Q\Omega_{(P/R)/(Q/R)}$$

If R is non-trivial, then by induction hypothesis, there is an integer m_R depending only on R such that $\mathrm{Def}^P_{P/R}u = \frac{m_R}{p}\Delta(P/R)$, i.e. after inflating to P

$$\sum_{R\subseteq Q\subset P} n_Q\Omega_{P/Q} = \frac{m_R}{p} \sum_{R\subseteq Q\subset P} \mu(Q,P)\Omega_{P/Q}$$

It follows that if $|P/Q| > p^2$, then $n_Q = \frac{m_R}{p}\mu(Q,P)$, and this equality is only a congruence modulo 2 if $|P/Q| = p > 2$. This must hold whenever R is a non-trivial subgroup of Q. Thus if $R \subseteq R'$ are non-trivial subgroups of index at least p^2 in P, then

$$n_{R'} = \frac{m_R}{p}\mu(R',P) = \frac{m_{R'}}{p}\mu(R',P)$$

thus $m_R = m_{R'}$ since $\mu(R',P) \neq 0$. Since the poset of non-trivial subgroups of index at least p^2 is connected if $|P| \geq p^4$, the value of m_R is constant on this poset, equal to m, say. Moreover, if Q is a subgroup of index $p > 2$ in P, containing a non-trivial subgroup R of index at least p^2, then

$$n_Q \equiv \frac{m}{p}\mu(Q,P) \mod. \ 2$$

Hence there is an integer m such that for any non-trivial subgroup Q of P

$$n_Q = \frac{m}{p}\mu(Q,P)$$

this equality being replaced by a congruence modulo 2 if $|P:Q| = p > 2$.

Now fix a subgroup R of index p in P. Condition 6.3.2 for $S = \{1\}$ means that

$$n_1 + \sum_{\substack{|Q|=p \\ Q \not\subseteq R}} n_Q = 0$$

Thus

$$n_1 = -\sum_{\substack{|Q|=p \\ Q \not\subseteq R}} \frac{m}{p}\mu(Q,P)$$

Now if P has order p^d, there are p^{d-1} subgroups Q of order p not contained in R, and for each of them $\mu(Q,P) = (-1)^{d-1}p^{\binom{d-1}{2}}$. It follows that

$$n_1 = -\frac{m}{p}(-1)^{d-1}p^{d-1}p^{\binom{d-1}{2}} = (-1)^d\frac{m}{p}p^{\binom{d}{2}} = \frac{m}{p}\mu(1,P)$$

Thus $n_Q = \frac{m}{p}\mu(Q,P)$ for any subgroup Q of P, and

$$u = \sum_{Q\subset P} \frac{m}{p}\mu(Q,P)\Omega_{P/Q} = -m\tilde{\Delta}(P)$$

This completes the proof. □

It is convenient to extend the definition of $\tilde{\Delta}(P)$ to arbitrary p-groups, as follows:

(6.3.5) **Notation:** *If P is a (non necessarily elementary abelian) p-group, I denote by $\tilde{\Delta}(P)$ the element of $D^{\Omega}_{\mathcal{O}}(P)$ defined by*

$$\tilde{\Delta}(P) = \mathrm{Inf}_{P/\Phi(P)}^{P}\tilde{\Delta}\Big(P/\Phi(P)\Big)$$

6.4. Linear independence

The elements $\mathrm{Ten}_Q^P\Delta(Q)$ are almost linearly independent in the Dade group:

(6.4.1) **Proposition:** *Let P be a p-group. For $Q \in [s_P]$, let $n_Q \in \mathbb{Z}$, such that*

(6.4.2)
$$\sum_{Q\in[s_P]} n_Q \mathrm{Ten}_Q^P\Delta(Q) = 0$$

in $D_{\mathcal{O}}(P)$. Then

1. *If Q is not cyclic, then $n_Q = 0$.*
2. *If Q is cyclic, non-trivial, and if p is odd, then $n_Q \in 2\mathbb{Z}$.*

Proof: Let R be a maximal element of $[s_P]$ such that $n_R \neq 0$. By Mackey's formula, the restriction of 6.4.2 to R is

$$\sum_{Q\in[s_P]} \sum_{x\in R\backslash P/Q} n_Q \mathrm{Ten}_{R\cap {}^xQ}^{R}\, {}^x\mathrm{Res}_{R^x\cap Q}^{Q}\Delta(Q) = 0$$

Since the restriction of $\Delta(Q)$ to any proper subgroup of Q is zero, the only terms left correspond to $R^x \subseteq Q$. But if Q is not equal to R^x, then $n_Q = 0$ by maximality of R. Finally, this gives

$$n_R|N_P(R):R|\Delta(R) = 0$$

Thus $\Delta(R)$ is a torsion element in $D_{\mathcal{O}}(R)$. This can only happen if R is cyclic, by corollary 6.2.5. This shows 1).

Now $n_Q = 0$ if Q is not cyclic. Let R be a maximal subgroup of P such that $n_R \notin 2\mathbb{Z}$. The same argument shows that

$$n_R|N_P(R):R| \in 2\mathbb{Z}$$

If p is odd, this gives $n_R \in 2\mathbb{Z}$ as required. □

(6.4.3) **Remark:** Assertion 2) is definitely wrong if $p = 2$: recall that $\Delta(Q) = 0$ if Q is cyclic and $p = 2$!

(6.4.4) **Corollary:** *Let P be a p-group. Denote by $nc(P)$ the number of conjugacy classes of non-cyclic subgroups of P, and by $\tilde{c}(P)$ the number of conjugacy classes of non-trivial cyclic subgroups of P. Then:*

1. *If p is odd*
$$D_{\mathcal{O}}^{\Delta}(P) \simeq \mathbb{Z}^{nc(P)} \oplus (\mathbb{Z}/2\mathbb{Z})^{\tilde{c}(P)}$$
2. *If $p = 2$*
$$D_{\mathcal{O}}^{\Delta}(P) \simeq \mathbb{Z}^{nc(P)}$$

6.5. Generation

The following proposition shows that up to a power of p, any relative syzygy is a linear combination of elements $\mathrm{Ten}_R^P\Delta(R)$, for subgroups R of P:

(6.5.1) **Proposition:** *Let P be a p-group, and X be a non-empty P-set. Then*

$$|P|\Omega_X = -\sum_{\substack{R\subseteq Q\subseteq P\\ X^R\neq\emptyset}} |R|\mu(R,Q)\mathrm{Ten}_Q^P\Delta(Q)$$

Proof: Let E_X denote the right hand side of the equality of the proposition, i.e.

$$E_X = -\sum_{\substack{R\subseteq Q\subseteq P\\ X^R\neq\emptyset}} |R|\mu(R,Q)\mathrm{Ten}_Q^P\Delta(Q)$$

Let Q be a subgroup of P. Then by formula 6.1.3

$$\mathrm{Ten}_Q^P\Delta(Q) = -\sum_{\substack{U,V\in[s_P]\\ U\leq_P V}} \mu_P(U,V)\ |\{a\in V\backslash P/Q \mid M(Q)^{V^a\cap Q} = \emptyset\}|\ \Omega_{P/U}$$

Now $M(Q)^{V^a\cap Q}=\emptyset$ is equivalent to $Q\subseteq V^a$, and in this case $V.a.Q=V.^aQ.a=V.a$. Hence, summing now for $a\in P/V$ instead of $a\in V\backslash P$ gives

(6.5.2) $$\mathrm{Ten}_Q^P\Delta(Q)=-\sum_{\substack{U,V\in[s_P]\\ U\leq_P V\\ a\in P/V,\ Q^a\subseteq V}}\mu_P(U,V)\Omega_{P/U}$$

Suppose first that X is isomorphic to a disjoint union of a single transitive P-set P/S for some subgroup S of P. Then X^R is non-empty if and only if $(P/S)^R$ is, and

$$E_X=E_{P/S}=-\sum_{\substack{R\subseteq Q\subseteq P\\ (P/S)^R\neq\emptyset}}|R|\mu(R,Q)\mathrm{Ten}_Q^P\Delta(Q)$$

Now $(P/S)^R\neq\emptyset$ is equivalent to $R\leq_P S$. Using equation 6.5.2

$$E_X=\sum_{\substack{R\subseteq Q\subseteq P\\ R\leq_P S}}|R|\mu(R,Q)\sum_{\substack{U,V\in[s_P]\\ U\leq_P V\\ a\in P/V,\ Q\subseteq^a V}}\mu_P(U,V)\Omega_{P/U}$$

Changing the order of summations gives

$$E_X=\sum_{\substack{U,V\in[s_P]\\ U\leq_P V}}\mu_P(U,V)\sum_{a\in P/V}\sum_{\substack{R\subseteq P\\ R\leq_P S}}|R|\left(\sum_{R\subseteq Q\subseteq^a V}\mu(R,Q)\right)\Omega_{P/U}$$

The inner summation on Q is zero unless $R={}^aV$. Since $R\leq_P S$, this gives

$$E_X=\sum_{\substack{U,V\in[s_P]\\ U\leq_P V\leq_P S}}\mu_P(U,V)|P/V||V|\Omega_{P/U}=|P|\sum_{\substack{U,V\in[s_P]\\ U\leq_P V\leq_P S}}\mu_P(U,V)\Omega_{P/U}$$

Now the summation on V for a given U is zero, unless $U=_P S$, hence

$$E_X=|P|\Omega_{P/S}$$

Moreover $\Omega_X=\Omega_{P/S}$ in this case, by lemma 3.2.7. Hence the formula of the proposition holds if X is isomorphic to a union of copies of a single transitive P-set.

As in the proof of lemma 5.2.3, the proof of the general case of the proposition can be completed by induction on the cardinality of the set

$$F(X)=\{Q\in[s_P]\mid X^Q\neq\emptyset\}$$

Clearly, both sides of the equality of the proposition depend only on $F(X)$.

If $|F(X)|=1$, then P acts freely on X, and X is a disjoint union of copies of $P/1$. The formula holds in this case. And if $|F(X)|>1$, choose a maximal element S of $F(X)$. Then X can be written as a disjoint union $Y\sqcup Z$, where Z is a disjoint union of copies of P/S, and $Y^S=\emptyset$.

If $Y=\emptyset$, then $X=Z$ is a union of copies of P/S, and the formula holds. And if $Y\neq\emptyset$, then by lemma 5.2.1

$$\Omega_X=\Omega_Y+\Omega_Z-\Omega_{Y\times Z}$$

But $|F(Y)| < |F(X)|$ since $F(Y) \subseteq F(X)$ and $S \in F(X) - F(Y)$. Similarly since $F(Y \times Z) \subseteq F(Y)$, I have $|F(Y \times Z)| \leq |F(Y)| < |F(X)|$. By induction hypothesis, the proposition holds for Y and $Y \times Z$. It also holds for Z since Z is a union of copies of P/Q. Thus

$$|P|\Omega_X = E_Y + E_Z - E_{Y\times Z}$$

But as at the end of proof of lemma 5.2.3, the right hand side is equal to $E_{Y \sqcup Z} = E_X$, and this completes the proof. □

The following special case seems of interest (see remark (4.14) of [5]):
(6.5.3) **Corollary:** *Let P be a p-group. Then in $D_{\mathcal{O}}(P)$*

$$|P|\Omega_{P/1} = -\sum_{V \in a_p(P)} |V|\mu(V)\mathrm{Ten}_V^P \Delta(V)$$

where $a_p(P)$ is the set of non-trivial elementary abelian p-groups of P.
Proof: This is just the case $X = P/1$ of the proposition. Then $X^U \neq \emptyset$ implies $U = \{1\}$, and then $\mu(1, V) \neq 0$ implies that V is elementary abelian. □

7. Structure of $D_{\mathcal{O}}(P)$: partial results

7.1. Elementary abelian sections

The elements $\mathrm{Ten}_Q^P\Delta(Q)$ of section 6.2 provide tools to study the structure of $D_{\mathcal{O}}(P)$ as abelian group. Unfortunately, they are not enough to elucidate this structure completely. The main problem here is the same problem we already encountered with J. Thévenaz in [5]: we have almost no information on the subgroup $K_{\mathcal{O}}(P)$ of $D_{\mathcal{O}}(P)$ defined hereafter:
(7.1.1) **Notation:** *Let P be a p-group. If $R \trianglelefteq Q$ are subgroups of P, I will say that Q/R is a section of P. I will denote by $\mathrm{Defres}_{Q/R}^P$ the map from $D_{\mathcal{O}}(P)$ to $D_k(Q/R)$ defined by*

$$\mathrm{Defres}_{Q/R}^P = \mathrm{Def}_{Q/R}^Q \circ \mathrm{Res}_Q^P$$

Similarly, I will denote by $\mathrm{Teninf}_{Q/R}^P$ the map from $D_{\mathcal{O}}(Q/R)$ to $D_{\mathcal{O}}(P)$ defined by

$$\mathrm{Teninf}_{Q/R}^P = \mathrm{Ten}_Q^R \circ \mathrm{Inf}_{Q/R}^Q$$

I denote by $K_{\mathcal{O}}(P)$ the intersection of the kernel of the maps Defres_E^P associated to elementary abelian sections E of P.

Recall from [5] that $K_k(P)$ is a finite subgroup of $D_k(P)$. Clearly for any elementary abelian section E of P, the diagram

$$\begin{array}{ccc} D_{\mathcal{O}}(P) & \xrightarrow{\rho} & D_k(P) \\ \mathrm{Defres}_E^P \downarrow & & \downarrow \mathrm{Defres}_E^P \\ D_k(E) & \xrightarrow[Id]{} & D_k(E) \end{array}$$

where the horizontal map ρ is the reduction morphism, is commutative. Hence ρ maps $K_{\mathcal{O}}(P)$ in $K_k(P)$. Since ρ is injective, it follows that $K_{\mathcal{O}}(P)$ is a subgroup of $K_k(P)$, hence it is a finite group. Actually $K_k(P)$ is conjecturally zero if p is odd (and it is known to be non-zero in some cases if $p = 2$).

7.2. Connection with the Burnside ring

In [5], we showed that there is an exact sequence

$$0 \mapsto \mathbb{Q}D_k(P) \to \mathbb{Q}B(P) \to \mathbb{Q}R_{\mathbb{Q}}(P) \to 0$$

where $B(P)$ is the Burnside ring of P, and $R_{\mathbb{Q}}(P)$ the ring of rational representations of P. This sequence is moreover functorial in P, in the sense of [3]. In particular the inclusion $\mathbb{Q}D_k(P) \to \mathbb{Q}B(P)$ is compatible with restriction to subgroups, and changes tensor induction into ordinary induction of P-sets. The image of this inclusion is the subspace generated by the idempotents e_Q^P of $B(P)$ indexed by non-cyclic subgroups Q of P.

It is easy to see that the subspace of elements $u \in \mathbb{Q}B(P)$ such that $\mathrm{Res}_Q^P u = 0$ for any proper subgroup Q of P is generated by the idempotent e_P^P. Thus the image of $\Delta(P)$ in $\mathbb{Q}B(P)$ should be proportional to e_P^P.

Now the expression of e_P^P is

$$e_P^P = \frac{1}{|P|}\sum_{Q\subseteq P} |Q|\mu(Q,P)P/Q$$

(7.2.1) **Lemma:** *The element pe_P^P lies in $B(P)$, i.e. for all $Q \subseteq P$, the expression $p\dfrac{\mu(Q,P)}{|P:Q|}$ is an integer.*

Proof: Indeed, in the expression of e_P^P, the only non-zero terms correspond to subgroups Q containing $\Phi(P)$. Now if $Q \supseteq \Phi(P)$ is such that $|P:Q| = p^k$

$$\frac{|Q|}{|P|}\mu(Q,P) = \frac{\mu(P,Q)}{|P:Q|} = (-1)^k p^{\binom{k}{2}-k}$$

If $k = 0$ or $k \geq 3$, then $\binom{k}{2} - k \geq 0$, and if $k = 1$ or $k = 2$, it is equal to -1. Thus $p\frac{|Q|}{|P|}\mu(Q,P)$ is always an integer, and the lemma follows. □

Since multiplication by P/Q in $B(P)$ is restriction Res_Q^P followed by induction Ind_Q^P, this lemma shows that it should be of interest to consider the following operation:

(7.2.2) **Notation:** *Let P be a p-group. I denote by τ_P the endomorphism of $D_{\mathcal{O}}(P)$ defined by*

$$\tau_P = \sum_{Q\subseteq P} \frac{p\mu(Q,P)}{|P:Q|}\mathrm{Ten}_Q^P\mathrm{Res}_Q^P$$

If R is a subgroup of P, I denote by $\alpha(R,P)$ the integer defined by

$$\alpha(R,P) = \sum_{S\subseteq R} \frac{p\mu(S,P)}{|P:S|}$$

Note that $\alpha(R,P) = 0$ if $R \not\supseteq \Phi(P)$, and that $\alpha(R,P) = \alpha\Big(R/\Phi(P), P/\Phi(P)\Big)$ otherwise.

(7.2.3) **Lemma:** *Let P be a p-group.*

1. *If R is a subgroup of P, then*

$$\tau_P(\Omega_{P/R}) = -\alpha(R,P)\Delta(P)$$

2. *The image of $D^{\Omega}_{\mathcal{O}}(P)$ by τ_P is equal to the subgroup of $D_{\mathcal{O}}(P)$ generated by $\Delta(P)$.*

Proof: If R is a subgroup of P, then

$$\tau_P(\Omega_{P/R}) = \sum_{Q\subseteq P} \frac{p\mu(Q,P)}{|P:Q|}\mathrm{Ten}_Q^P\mathrm{Res}_Q^P\Omega_{P/R}$$

By corollary 4.1.2 and theorem 5.1.2

$$\mathrm{Ten}_Q^P\mathrm{Res}_Q^P\Omega_{P/R} = \sum_{\substack{U,V\in[s_P]\\ U\leq_P V}} \mu_P(U,V)\ |\{a\in V\backslash P/Q \mid (P/R)^{V^a\cap Q}\neq\emptyset\}|\ \Omega_{P/U}$$

Thus $\tau_P(\Omega_{P/R})$ is expressed as a double summation. Summing first on U and V, next on Q gives

$$\tau_P(\Omega_{P/R}) = \sum_{\substack{U,V\in[s_P]\\ U\leq_P V}} \mu_P(U,V)\beta(R,V)\Omega_{P/U}$$

where $\beta(R,V)$ is defined by

$$\beta(R,V) = \sum_{Q\subseteq P} \frac{p\mu(Q,P)}{|P:Q|}\ |\{a\in V\backslash P/Q \mid V^a\cap Q\leq_P R\}|$$

This is also equal to

$$\beta(R,V) = \sum_{Q\subseteq P} \frac{p\mu(Q,P)}{|P:Q|} \sum_{\substack{a\in P\\ Q\cap V^a\leq_P R}} \frac{|Q\cap V^a|}{|Q||V|}$$

Thus

$$\beta(R,V) = \sum_{\substack{a\in P\\ S\leq_P R}} \frac{p|S|}{|P||V|} \sum_{\substack{Q\subseteq P\\ Q\cap V^a=S}} \mu(Q,P)$$

It is well known that the sum on Q is zero if $V^a\neq P$, i.e. if $V\neq P$. Moreover if $V=P$

$$\beta(R,P) = \sum_{\substack{a\in P\\ S\leq_P R}} \frac{p|S|}{|P|^2}\mu(S,P) = \sum_{S\leq_P R} \frac{p\mu(S,P)}{|P:S|}$$

Note that $\mu(S,P)$ is zero if $S\not\supseteq\Phi(P)$. And if $S\supseteq\Phi(P)$, then $S\trianglelefteq P$, and $S\leq_P R$ is equivalent to $S\subseteq R$. This gives finally

$$\beta(R,P) = \sum_{S\subseteq R} \frac{p\mu(S,P)}{|P:S|} = \alpha(R,P)$$

Moreover

$$\tau_P(\Omega_{P/R}) = \alpha(R,P)\sum_{U\in[s_P]} \mu_P(U,P)\Omega_{P/U}$$

But as I already mentioned in the proof of proposition 6.2.3, the coefficient $\mu_P(U,P)$ is equal to $\mu(U,P)$. Thus

$$\tau_P(\Omega_{P/R}) = -\alpha(R,P)\Delta(P)$$

and assertion 1) follows.

It remains to check that $\Delta(P)$ is in $\tau_P\Big(D_{\mathcal{O}}^{\Omega}(P)\Big)$: but the element $\tilde{\Delta}(P)$ is in $D_{\mathcal{O}}^{\Omega}(P)$ and

$$\tau_P\Big(\tilde{\Delta}(P)\Big) = p\sum_{Q\subseteq P}\frac{\mu(Q,P)}{|P:Q|}\mathrm{Ten}_Q^P\mathrm{Res}_Q^P\tilde{\Delta}(P) = p\tilde{\Delta}(P) = \Delta(P) \tag{7.2.4}$$

since the restriction of $\tilde{\Delta}(P)$ to any proper subgroup of P is zero. Assertion 2) follows. □

(7.2.5) **Corollary:** *Let P be an elementary abelian p-group. There exists a well defined linear form λ_P from $D_{\mathcal{O}}(P)$ to $\mathbb{Z}$ if $|P|\geq p^2$, or to $\mathbb{Z}/2\mathbb{Z}$ if $|P|=p>2$, such that for any element u of $D_{\mathcal{O}}(P)$*

$$\tau_P(u) = \lambda_P(u)\Delta(P)$$

Proof: This is clear since if P is abelian, then $D_{\mathcal{O}}(P) = D_{\mathcal{O}}^{\Omega}(P)$, and since $\Delta(P)$ has infinite order if P is non-cyclic, and order 2 if P is cyclic of order $p>2$. □

With this definition, lemma 7.2.3 shows that $\lambda_P(\Omega_{P/R}) = -\alpha(R,P)$. More generally:

(7.2.6) **Proposition:** *Let P be an elementary abelian p-group (of order at least 3) and X be a non-empty finite P-set. Then*

$$\lambda_P(\Omega_X) = -\sum_{\substack{Q\subseteq P\\ X^Q\neq\emptyset}}\frac{p\mu(Q,P)}{|P:Q|}$$

Proof: By lemma 5.2.3, and since P is abelian

$$\Omega_X = \sum_{\substack{U\subseteq V\subseteq P\\ X^V\neq\emptyset}}\mu(U,V)\Omega_{P/U}$$

Thus

$$\begin{aligned}\lambda_P(\Omega_X) &= -\sum_{\substack{U\subseteq V\subseteq P\\ X^V\neq\emptyset}}\mu(U,V)\sum_{W\subseteq U}\frac{p\mu(W,P)}{|P:W|}\\ &= -\sum_{\substack{W\subseteq V\subseteq P\\ X^V\neq\emptyset}}\frac{p\mu(W,P)}{|P:W|}\left(\sum_{W\subseteq U\subseteq V}\mu(U,V)\right)\end{aligned}$$

The sum on U is zero, unless $W=V$, and the proposition follows. □

Though it is not really necessary, I would like to indicate how to compute the integers $\alpha(R,P)$. By the above remarks, it suffices to consider the case where P is elementary abelian:

(7.2.7) **Lemma:** *Let P be an elementary abelian p-group, and R be a subgroup of P. Then*

$$\alpha(R,P) = \frac{p\mu(R,P)}{|P:R|} m_{P,R}$$

where $m_{P,R}$ is defined as in lemme 16 of [3] by

$$m_{P,R} = \frac{1}{|P|} \sum_{X.R=P} |X|\mu(X,P)$$

Let $|P| = p^d$ and $|R| = p^r$.

- *If $d = 1$, then $\alpha(R,P) = -1$ if $r = 0$ and $\alpha(R,P) = p - 1$ if $r = 1$.*
- *If $d \geq 2$, then $\alpha(R,P) = 0$ if $r \geq d - 1$ and*

$$\alpha(R,P) = (-1)^{d-r} p^{\binom{d-r-1}{2}} (1 - p^{d-2})(1 - p^{d-3}) \dots (1 - p^{d-r-1})$$

  if $r \leq d - 2$.

Proof: By definition

$$\alpha(R,P) = \frac{p}{|P|} \sum_{S \subseteq R} |S|\mu(S,P)$$

By classical formulae, if $S \subseteq R$, then $\mu(S,P)$ can be computed as

$$\mu(S,P) = \sum_{\substack{R' \cap R = S \\ R'.R = P}} \mu(S,R')\mu(R',P)$$

This gives

$$\alpha(R,P) = \frac{p}{|P|} \sum_{R'.R=P} \mu(R,P)\mu(R',P)|R \cap R'| = (\frac{p\mu(R,P)}{|P:R|}) \frac{1}{|P|} \sum_{R'.R=P} |R'|\mu(R',P)$$

And the first formula holds. The other ones are a consequence of the computation of $m_{P,R}$ (see formula (4.8) of [5]):

- If $r = 0$, then $m_{P,R} = 1$.
- If $d = r = 1$, then $m_{P,R} = 1 - 1/p$.
- If $d \geq 2$, then $m_{P,R} = 0$ if $r \geq d - 1$, and

$$m_{P,R} = (1 - p^{d-2})(1 - p^{d-3}) \dots (1 - p^{d-r-1})$$

  if $r \leq d - 2$.

Moreover if $|P| = p^d$ and $|R| = p^r$, then $\frac{p\mu(R,P)}{|P:R|} = (-1)^{d-r} p^{\binom{d-r-1}{2}}$. □

(7.2.8) **Remark:** Note that in the case $d = 1$ the value $\lambda_P(\Omega_{P/R})$ is the image of $\alpha(R,P)$ in $\mathbb{Z}/2\mathbb{Z}$. Hence it is equal to 1 if $R = 1$, and to 0 if $R = P$ since p is odd in this case. This is consistent with the equality $\Omega_{P/P} = 0$.

(7.2.9) **Lemma:** *An element u of $D_{\mathcal{O}}(P)$ lies in $K_{\mathcal{O}}(P)$ if and only if for any subgroup Q of P*

$$\tau_{Q/\Phi(Q)}(\mathrm{Defres}^P_{Q/\Phi(Q)}u) = 0$$

Proof: One way is obvious. Now if $\tau_{Q/\Phi(Q)}(\mathrm{Defres}^P_{Q/\Phi(Q)}u) = 0$ for any subgroup Q of P, I will prove by induction on $|Q|$ that $\mathrm{Defres}^P_{Q/\Phi(Q)}u = 0$, and the lemma will follow, since for any elementary abelian section $E = Q/R$ of P

$$\mathrm{Defres}^P_E = \mathrm{Def}^{Q/\Phi(Q)}_{Q/R} \circ \mathrm{Defres}^P_{Q/\Phi(Q)}$$

The result is clear if $|Q| = 1$. If it holds for groups of order less than $|Q|$, set $v = \mathrm{Defres}^P_{Q/\Phi(Q)}u$. Let R be a subgroup of Q of index p. Then $R \supseteq \Phi(Q)$, and

$$\mathrm{Res}^{Q/\Phi(Q)}_{R/\Phi(Q)}v = \mathrm{Def}^{R/\Phi(R)}_{R/\Phi(Q)}\mathrm{Defres}^P_{R/\Phi(R)}u = 0$$

since $|R| < |Q|$. This shows that $v \in \underline{D}_{\mathcal{O}}\Big(Q/\Phi(Q)\Big)$. Hence there exists an integer m such that $v = m\tilde{\Delta}\Big(Q/\Phi(Q)\Big)$. Now

$$0 = \tau_{Q/\Phi(Q)}(\mathrm{Defres}^P_{Q/\Phi(Q)}u) = \tau_{Q/\Phi(Q)}\Big(m\tilde{\Delta}\Big(Q/\Phi(Q)\Big)\Big) = m\Delta\Big(Q/\Phi(Q)\Big) = pv$$

Thus $v = 0$ as required, since $D_{\mathcal{O}}\Big(Q/\Phi(Q)\Big)$ has no p-torsion. □

7.3. Tensor induction and τ_P

In the Burnside ring, there is a Frobenius formula

$$X.\mathrm{Ind}^P_Q Y = \mathrm{Ind}^P_Q\Big((\mathrm{Res}^P_Q X).Y\Big)$$

In particular in $\mathbb{Q}B(P)$, the product of e^P_P with any element induced from a proper subgroup of P is zero. Hence it should be possible to prove the following:

(7.3.1) **Lemma:** *Let P be a p-group, and Q be a proper subgroup of P. Then*

$$\tau_P \circ \mathrm{Ten}^P_Q = 0$$

Proof: Let $u \in D_{\mathcal{O}}(Q)$. Then

$$\tau_P(\mathrm{Ten}^P_Q u) = \sum_{R \subseteq P} \frac{p\mu(R,P)}{|P:R|}\mathrm{Ten}^P_R \mathrm{Res}^P_R \mathrm{Ten}^P_Q u$$

Applying Mackey's formula gives

$$\tau_P(\mathrm{Ten}^P_Q u) = \sum_{R \subseteq P} \frac{p\mu(R,P)}{|P:R|} \sum_{x \in R\backslash P/Q} \mathrm{Ten}^P_{R\cap {}^xQ}\, {}^x\mathrm{Res}^Q_{R^x \cap Q} u$$

Moreover

$$\mathrm{Ten}_{R\cap {}^xQ}^P {}^x\mathrm{Res}_{R^x\cap Q}^Q u = {}^x\mathrm{Ten}_{R^x\cap Q}^P \mathrm{Res}_{R^x\cap Q}^Q u = \mathrm{Ten}_{R^x\cap Q}^P \mathrm{Res}_{R^x\cap Q}^Q u$$

since P acts trivially on $D_{\mathcal{O}}(P)$. Moreover if $\mu(R,P)\neq 0$ then $R\supseteq P$ and $R\trianglelefteq P$. Thus

$$\begin{aligned}\tau_P(\mathrm{Ten}_Q^P u) &= \sum_{R\subseteq P}\frac{p\mu(R,P)}{|P:R|}|P:R.Q|\mathrm{Ten}_{R\cap Q}^P\mathrm{Res}_{R\cap Q}^Q u\\ &= \sum_{R\subseteq P}\frac{p\mu(R,P)|R\cap Q|}{|Q|}\mathrm{Ten}_{R\cap Q}^P\mathrm{Res}_{R\cap Q}^Q u\end{aligned}$$

Grouping together the terms for which $R\cap Q$ is a given subgroup S of Q gives

$$\tau_P(\mathrm{Ten}_Q^P u) = \sum_{S\subseteq Q}\frac{p|S|}{|Q|}\Big(\sum_{\substack{R\subseteq P\\ R\cap Q=S}}\mu(R,P)\Big)\mathrm{Ten}_S^P\mathrm{Res}_S^Q u$$

If Q is a proper subgroup of P, then the sum on R is zero, and the lemma follows. □

7.4. Some subgroups of $D_{\mathcal{O}}(P)$

Lemma 7.2.9 leads to the following:

(7.4.1) **Lemma:** *Let P be a p-group and u be an element of $D_{\mathcal{O}}(P)$. Then u lies in $D_{\mathcal{O}}^{\Delta}(P)+K_{\mathcal{O}}(P)$ if and only if for any subgroup R of P there is an integer n_R such that*

$$\tau_{R/\Phi(R)}(\mathrm{Defres}_{R/\Phi(R)}u) = n_R\, p\,|N_P(R):R|\,\Delta\Big(R/\Phi(R)\Big)$$

Proof: The element u lies in $D_{\mathcal{O}}^{\Delta}(P)+K_{\mathcal{O}}(P)$ if and only if there exists integers n_Q for subgroups Q of P such that

$$u-\sum_{Q\in[s_P]} n_Q\mathrm{Ten}_Q^P\Delta(Q)\in K_{\mathcal{O}}(P)$$

It is equivalent to require that for any subgroup R of P

$$\text{(7.4.2)}\qquad \mathrm{Defres}_{R/\Phi(R)}^P u = \mathrm{Defres}_{R/\Phi(R)}^P\left(\sum_{Q\in[s_P]} n_Q\mathrm{Ten}_Q^P\Delta(Q)\in K_{\mathcal{O}}(P)\right)$$

Let S be any normal subgroup of R. Then by Mackey's formula

$$\begin{aligned}\mathrm{Defres}_{R/S}^P\mathrm{Ten}_Q^P\Delta(Q) &= \mathrm{Def}_{R/S}^R\mathrm{Res}_R^P\mathrm{Ten}_Q^P\Delta(Q)\\ &= \sum_{x\in R\backslash P/Q}\mathrm{Def}_{R/S}^R\mathrm{Ten}_{R\cap {}^xQ}^R {}^x\mathrm{Res}_{R^x\cap Q}^Q\Delta(Q)\end{aligned}$$

Since the restriction of $\Delta(Q)$ to any proper subgroup of Q is zero, this is also

$$\mathrm{Defres}_{R/S}^P\mathrm{Ten}_Q^P\Delta(Q) = \sum_{\substack{y\in P/R\\ Q^y\subseteq R}}\mathrm{Def}_{R/S}^R\mathrm{Ten}_{Q^y}^R\Delta(Q^y)$$

since obviously $\Delta(Q)^y = \Delta(Q^y)$. Now the element $\Delta(Q^y)$ is inflated from $Q^y/\Phi(Q^y)$, thus by corollary (3.11) of [5]

$$\mathrm{Def}^R_{R/S}\mathrm{Ten}^R_{Q^y}\Delta(Q^y) = \mathrm{Ten}^{R/S}_{Q^y.S/S}\mathrm{Iso}^{Q^y.S/S}_{Q^y/Q^y\cap S}\mathrm{Def}^{Q^y}_{Q^y/Q^y\cap S}\Delta(Q^y)$$

(Actually corollary (3.11) of [5] only deals with the case $\mathcal{O} = k$, but clearly one can reduce everything from $\mathcal{O}$ to k before taking deflations.) Finally by remark 6.2.2 this gives

$$\mathrm{Defres}^P_{R/S}\mathrm{Ten}^P_Q\Delta(Q) = \sum_{\substack{y\in P/R \\ Q^y\subseteq R}} \mathrm{Ten}^{R/S}_{Q^y.S/S}\Delta(Q^y.S/S)$$

Suppose now that R/S is elementary abelian. Applying $\tau_{R/S}$ gives by lemma 7.3.1 and equation 7.2.4

$$\tau_{R/S}\mathrm{Defres}^P_{R/S}\mathrm{Ten}^P_Q\Delta(Q) = \sum_{\substack{y\in P/R \\ Q^y.S=R}} \tau_{R/S}\Big(\Delta(R/S)\Big) = p\sum_{\substack{y\in P/R \\ Q^y.S=R}} \Delta(R/S)$$

If moreover $S = \Phi(R)$, this is zero unless Q is a conjugate of R. Finally

$$\tau_{R/\Phi(R)}\mathrm{Defres}^P_{R/\Phi(R)}\mathrm{Ten}^P_R\Delta(R) = p\,|N_P(R):R|\,\Delta\Big(R/\Phi(R)\Big) \tag{7.4.3}$$

Applying this in equation 7.4.2 gives

$$\tau_{R/\Phi(R)}\mathrm{Defres}^P_{R/\Phi(R)}(u) = n_R\,p\,|N_P(R):R|\,\Delta\Big(R/\Phi(R)\Big)$$

as was to be shown. □

Of course, this condition is void if $|R/\Phi(R)| \leq 2$. Suppose now that $u = |P|v$, for $v \in D_{\mathcal{O}}(P)$. Since the image of $\tau_{R/\Phi(R)}$ is generated by $\Delta\Big(R/\Phi(R)\Big)$, and since $\dfrac{|P|}{p|N_P(R):R|}$ is an integer if R is non-trivial, there is an integer n_R such that

$$\frac{|P|}{p|N_P(R):R|}\tau_{R/\Phi(R)}\mathrm{Defres}^P_{R/\Phi(R)}(v) = n_R\Delta\Big(R/\Phi(R)\Big)$$

More precisely, this integer is equal to

$$n_R = \frac{|P|}{p|N_P(R):R|}\lambda_{R/\Phi(R)}(\mathrm{Defres}^P_{R/\Phi(R)}v)$$

(this equality being a congruence modulo 2 if R is non-trivial and cyclic of odd order). These remarks lead to the following notation:

(7.4.4) **Notation:** *Let P be a p-group. I denote by s'_P the set of conjugacy classes of subgroups R of P such that $|R/\Phi(R)| \geq 3$, and by $[s'_P]$ the set $[s_P]\cap s'_P$. I denote by ψ_P the endomorphism of $D_{\mathcal{O}}(P)$ defined by*

$$\psi_P(v) = \sum_{R\in[s'_P]} \frac{|P|}{p|N_P(R):R|}\lambda_{R/\Phi(R)}(\mathrm{Defres}^P_{R/\Phi(R)}v)\mathrm{Ten}^P_R\Delta(R)$$

Note that by definition of $\tilde{\Delta}(Q)$, this is also

$$\psi_P(v) = \sum_{R\in[s'_P]} \frac{|P|}{|N_P(R):R|}\lambda_{R/\Phi(R)}(\mathrm{Defres}^P_{R/\Phi(R)}v)\mathrm{Ten}^P_R\tilde{\Delta}(R) \tag{7.4.5}$$

Now by definition of the maps $\lambda_{R/\Phi(R)}$ and of the elements $\Delta(R)$, this is also

$$\psi_P(v) = \sum_{R\in[s_P]} \frac{|P|}{p|N_P(R):R|}\mathrm{Teninf}^P_{R/\Phi(R)}\tau_{R/\Phi(R)}\mathrm{Defres}^P_{R/\Phi(R)}(v)$$

(where the summation may run over the set $[s_P]-\{1\}$). Finally, the map ψ_P is equal to

$$\psi_P = \sum_{\substack{R\in[s_P]\\ \Phi(R)\subseteq Q\subseteq R}} \frac{|P|\mu(Q,R)}{|N_P(R):Q|}\mathrm{Teninf}^P_{Q/\Phi(R)}\mathrm{Defres}^P_{Q/\Phi(R)} \tag{7.4.6}$$

$$\psi_P = \sum_{\Phi(R)\subseteq Q\subseteq R} |Q|\mu(Q,R)\mathrm{Teninf}^P_{Q/\Phi(R)}\mathrm{Defres}^P_{Q/\Phi(R)} \tag{7.4.7}$$

(7.4.8) **Proposition:** *Let P be a p-group. Then*

1. *The image of ψ_P is contained in $D^{\Delta}_{\mathcal{O}}(P)$.*
2. *The kernel of ψ_P is equal to $K_{\mathcal{O}}(P)$.*
3. *The image of $\psi_P - |P|Id$ is contained in $K_{\mathcal{O}}(P)$.*
4. *The kernel of $\psi_P - |P|Id$ contains $D^{\Omega}_{\mathcal{O}}(P)$.*

Proof: Assertion 1) follows from the definition of ψ_P. Assertion 2) follows from lemma 7.2.9, and from proposition 6.4.1 and corollary 7.2.5. Assertion 3) follows from the proof of lemma 7.4.1 and from the construction of ψ_P.

For assertion 4), by linearity, it is enough to prove that if X is a non-empty P-set, then

$$\psi_P(\Omega_X) = |P|\Omega_X$$

Now by expression 7.4.5

$$\psi_P(\Omega_X) = \sum_{R\in[s'_P]} \frac{|P|}{|N_P(R):R|}\lambda_{R/\Phi(R)}(\mathrm{Defres}^P_{R/\Phi(R)}\Omega_X)\mathrm{Ten}^P_R\tilde{\Delta}(R)$$

By corollary 4.1.2 and proposition 7.2.6

$$\lambda_{R/\Phi(R)}(\mathrm{Defres}^P_{R/\Phi(R)}\Omega_X) = -\sum_{\substack{\Phi(R)\subseteq S\subseteq R\\ X^S\neq\emptyset}} \frac{p\mu(S,R)}{|R:S|}$$

It follows that

$$\begin{aligned}\psi_P(\Omega_X) &= -\sum_{R\in[s'_P]}\frac{|P|}{|N_P(R):R|}\sum_{\substack{\Phi(R)\subseteq S\subseteq R\\ X^S\neq\emptyset}}\frac{p\mu(S,R)}{|R:S|}\mathrm{Ten}_R^P\tilde{\Delta}(R)\\ &= -\sum_{\substack{R\in[s'_P]\\ \Phi(R)\subseteq S\subseteq R\\ X^S\neq\emptyset}}\frac{|P|}{|N_P(R)|}|S|p\mu(S,R)\mathrm{Ten}_R^P\tilde{\Delta}(R)\\ &= -\sum_{\substack{S\subseteq R\subseteq P\\ \overline{R}\in s'_P\\ X^S\neq\emptyset}}|S|\mu(S,R)\mathrm{Ten}_R^P\Delta(R)\end{aligned}$$

since $p\tilde{\Delta}(P)=\Delta(P)$, and since $\mu(S,R)=0$ if $S\not\supseteq\Phi(R)$. Now assertion 4) follows from Proposition 6.5.1, and from the fact that $\Delta(R)=0$ if $\overline{R}\in s_P-s'_P$. □

(7.4.9) **Proposition:** *Let P be a p-group. Then:*

1. *The subgroup $D^\Omega_\mathcal{O}(P)$ is generated by the elements $\Omega_{P/Q}$, for $Q\subset P$.*
2. *One has $|P|D^\Omega_\mathcal{O}(P)\subseteq D^\Delta_\mathcal{O}(P)\subseteq D^\Omega_\mathcal{O}(P)$. The quotient group $D^\Omega_\mathcal{O}(P)/D^\Delta_\mathcal{O}(P)$ is finite, with exponent dividing the order of P.*
3. *One has $|P|D_\mathcal{O}(P)\subseteq D^\Delta_\mathcal{O}(P)+K_\mathcal{O}(P)\subseteq D^\Omega_\mathcal{O}(P)+K_\mathcal{O}(P)\subseteq D_\mathcal{O}(P)$. The quotient groups $D_\mathcal{O}(P)/\Big(D^\Delta_\mathcal{O}(P)+K_\mathcal{O}(P)\Big)$ and $D_\mathcal{O}(P)/\Big(D^\Omega_\mathcal{O}(P)+K_\mathcal{O}(P)\Big)$ are finite, with exponent dividing $|P|$.*
4. *The intersection $D^\Delta_\mathcal{O}(P)\cap K_\mathcal{O}(P)$ is zero.*
5. *The intersection $D^\Omega_\mathcal{O}(P)\cap K_\mathcal{O}(P)$ is finite, with exponent dividing the order of P.*
6. *The torsion part of $D_\mathcal{O}(P)$ is equal to*
$$D^{tors}_\mathcal{O}(P)=(D^\Delta_\mathcal{O})^{tors}(P)\oplus K_\mathcal{O}(P)$$
7. *The torsion part of $D^\Omega_\mathcal{O}(P)$ has exponent dividing $2|P|$.*
8. *There is an isomorphism*
$$D_\mathcal{O}(P)\simeq D^\Delta_\mathcal{O}(P)\oplus K_\mathcal{O}(P)$$

Proof: Assertion 1) is a trivial consequence of lemma 5.2.3. Assertion 2) follows from proposition 6.5.1, and from the fact that $D^\Omega_\mathcal{O}(P)$ is finitely generated.

Assertion 3) follows from assertions 1) and 3) of proposition 7.4.8.

Now if $u\in D^\Omega_\mathcal{O}(P)\cap K_\mathcal{O}(P)$, then $\psi_P(u)=0=|P|u$, by assertions 2) and 4) of proposition 7.4.8. This proves 5), and 4) follows, since moreover $D^\Delta_\mathcal{O}(P)$ has no p-torsion by corollary 6.4.4.

The restriction of ψ_P to the torsion subgroup of $D_\mathcal{O}(P)$ gives an exact sequence

$$0\longrightarrow K_\mathcal{O}(P)\longrightarrow D^{tors}_\mathcal{O}(P)\xrightarrow{\psi_P}(D^\Delta_\mathcal{O})^{tors}(P)$$

Moreover, the group $(D^\Delta_\mathcal{O})^{tors}(P)$ is a 2-group, which is trivial if $p=2$. Now the restriction of ψ_P to this subgroup is equal to $|P|Id$, and this is always the identity of $(D^\Delta_\mathcal{O})^{tors}(P)$. Hence the previous exact sequence is split, and this proves 6).

If u is a torsion element of $D_{\mathcal{O}}(P)$, then $2u$ is an element of $K_{\mathcal{O}}(P)$, since $2D_{\mathcal{O}}(Q)$ is a free group if Q is elementary abelian. Thus $2u \in D_{\mathcal{O}}^{\Omega}(P) \cap K_{\mathcal{O}}(P)$, and $2|P|u = 0$.

Finally, since the restriction of ψ_P to $(D_{\mathcal{O}}^{\Delta})^{tors}(P)$ is equal to the identity, the map ψ_P induces an injection

$$i_P : D_{\mathcal{O}}(P)/\big((D_{\mathcal{O}}^{\Delta})^{tors}(P) \oplus K_{\mathcal{O}}(P)\big) \hookrightarrow D_{\mathcal{O}}^{\Delta}(P)/(D_{\mathcal{O}}^{\Delta})^{tors}(P)$$

and the quotient group is finite. This shows that the image of i_P is a free $\mathbb{Z}$-module of rank equal to the number $nc(P)$ of conjugacy classes of non-cyclic subgroups of P. Thus

$$D_{\mathcal{O}}(P) \simeq \mathbb{Z}^{nc(P)} \oplus (D_{\mathcal{O}}^{\Delta})^{tors}(P) \oplus K_{\mathcal{O}}(P)$$

proving assertion 8). □

Proposition 7.4.9 gives a new proof of theorems A and B of [5]:

(7.4.10) **Corollary:** *Let P be a p-group. Then*

1. *The dimension over $\mathbb{Q}$ of $\mathbb{Q}D_{\mathcal{O}}(P)$ is equal to the number of conjugacy classes of non-cyclic subgroups of P.*
2. *If p is odd, the quotient of the torsion part of $D_{\mathcal{O}}(P)$ by $K_{\mathcal{O}}(P)$ is an $\mathbb{F}_2$-vector space of dimension equal to the number of conjugacy classes of non-trivial cyclic subgroups of P.*

Proof: Both assertions are clear, since tensoring by $\mathbb{Q}$ kills all finite groups. □

(7.4.11) **Remark:** Actually, the previous results also provide bases for those vector spaces: the elements $\mathrm{Ten}_Q^P\Delta(Q)$, for non-cyclic elements Q of $[s_P]$, form a $\mathbb{Q}$-basis of $\mathbb{Q}D_{\mathcal{O}}(P)$. Similarly, the elements $\mathrm{Ten}_Q^P\Delta(Q)$, for non-trivial cyclic elements Q of $[s_P]$, form a basis of the $\mathbb{F}_2$-vector space of assertion 2).

7.5. Inverting p

(7.5.1) **Notation:** *If R is a commutative ring, and P is a p-group, I denote by $RD_{\mathcal{O}}(P)$ the tensor product*

$$RD_{\mathcal{O}}(P) = R \otimes_{\mathbb{Z}} D_{\mathcal{O}}(P)$$

and similarly I set $RD_{\mathcal{O}}^{\Omega}(P) = R \otimes_{\mathbb{Z}} D_{\mathcal{O}}^{\Omega}(P)$, $RD_{\mathcal{O}}^{\Delta}(P) = R \otimes_{\mathbb{Z}} D_{\mathcal{O}}^{\Delta}(P)$, and $RK_{\mathcal{O}}(P) = R \otimes_{\mathbb{Z}} K_{\mathcal{O}}(P)$. If $u \in D_{\mathcal{O}}(P)$, I denote by u the element $1 \otimes u$ of $RD_{\mathcal{O}}(P)$, and I still denote by Res_Q^P, Ten_Q^P, $\mathrm{Inf}_{P/R}^P$, $\mathrm{Def}_{P/R}^P$, $\mathrm{Iso}_P^{P'}$ the R-linear extensions to $RD_{\mathcal{O}}$ of the corresponding operations defined on $D_{\mathcal{O}}$.

(7.5.2) **Proposition:** *Let R be a commutative ring in which p is invertible. Let $\pi_P = \frac{1}{|P|}\psi_P$. Then for $u \in D_{\mathcal{O}}(P)$*

$$\pi_P(u) = \sum_{Q \in [s'_P]} \frac{1}{|N_P(Q):Q|} \lambda_{Q/\Phi(Q)}(\mathrm{Defres}_{Q/\Phi(Q)}^P u)\mathrm{Ten}_Q^P\tilde{\Delta}(Q)$$

The map π_P is an idempotent, with image $RD_{\mathcal{O}}^{\Delta}(P)$ and kernel $RK_{\mathcal{O}}(P)$. Thus

$$RD_{\mathcal{O}}(P) \simeq RD_{\mathcal{O}}^{\Delta}(P) \oplus RK_{\mathcal{O}}(P)$$

and this decomposition is functorial in P, in that π_P commutes with restriction, tensor induction, inflation, deflation, and group isomorphisms.

Proof: This is just a reformulation of proposition 7.4.9: tensoring with R kills all finite p-groups since p in invertible. Thus

$$RD_{\mathcal{O}}(P) = RD_{\mathcal{O}}^{\Delta}(P) \oplus RK_{\mathcal{O}}(P)$$

Now $\pi_P = \frac{1}{|P|}\psi_P$. Since the image of ψ_P is contained in the kernel of $\psi_P - |P|Id$ by proposition 7.4.8, it follows that $\psi_P^2 = |P|\psi_P$, hence π_P is an idempotent, with image $RD_{\mathcal{O}}^{\Delta}(P)$ and kernel $RK_{\mathcal{O}}(P)$.

It it easy to see that if $F : D_{\mathcal{O}}(P) \to D_{\mathcal{O}'}(P')$ is one of the operations of the proposition (note that the ring may change from $\mathcal{O}$ to $\mathcal{O}' = k$ in the case of deflation), then

$$F\Big(RK_{\mathcal{O}}(P)\Big) \subseteq RK_{\mathcal{O}}(P') \quad \text{and} \quad F\Big(RD_{\mathcal{O}}^{\Delta}(P)\Big) \subseteq RD_{\mathcal{O}'}^{\Delta}(P')$$

this last inclusion coming from the equality $RD_{\mathcal{O}}^{\Delta}(P) = RD_{\mathcal{O}}^{\Omega}(P)$, and from corollary 4.1.2 and from theorem 5.1.2. Now if $u \in RD_{\mathcal{O}}(P)$, then

$$u = \pi_P(u) + \Big(1 - \pi_P(u)\Big)$$

is the decomposition of u as sum of an element of $RD_{\mathcal{O}}^{\Delta}(P)$ and an element of $RK_{\mathcal{O}}(P)$. Thus

$$F(u) = F\Big(\pi_P(u)\Big) + F\Big(1 - \pi_P(u)\Big)$$

This is the decomposition of $F(u)$ in $D_{\mathcal{O}'}(P')$ as sum of an element of $RD_{\mathcal{O}'}^{\Delta}(P')$ and an element of $RK_{\mathcal{O}'}(P')$. Hence in particular

$$\pi_{P'}\Big(F(u)\Big) = F\Big(\pi_P(u)\Big)$$

and F commutes with π_P. □

So there should be formulas expressing the effect of the natural operations on the elements $\Delta(P)$. The effect of restriction, tensor induction, group isomorphism and deflation can be deduced from the previous computations. The effect of inflation follows from the following proposition:

(7.5.3) **Proposition:** *Let P be a p-group and R be a normal subgroup of P. Then*

$$|P|\mathrm{Inf}_{P/R}^{P}\Delta(P/R) = \sum_{\substack{U \subseteq V \subseteq P \\ U.R = P}} |U|\mu(U,V)\mathrm{Ten}_V^P\Delta(V)$$

in $D_{\mathcal{O}}(P)$.

Proof: Since

$$\mathrm{Inf}_{P/R}^{P}\Delta(P/R) = \sum_{R \subseteq Q \subseteq P} \mu(Q,P)\Omega_{P/Q}$$

it follows from proposition 6.5.1 that

$$|P|\mathrm{Inf}_{P/R}^{P}\Delta(P/R) = \sum_{R \subseteq Q \subseteq P} \mu(Q,P) \sum_{\substack{U \subseteq V \subseteq P \\ U \leq_P Q}} |U|\mu(U,V)\mathrm{Ten}_V^P\Delta(V)$$

Now if $\mu(Q,P)\neq 0$, then $Q\supseteq\Phi(P)$, and Q is a normal subgroup of P. Thus $U\leq_P Q$ is equivalent to $U\subseteq Q$, and

$$|P|\mathrm{Inf}_{P/R}^P\Delta(P/R)=\sum_{\substack{R\subseteq Q\subseteq P\\ U\subseteq V\subseteq P\\ U\subseteq Q}}|U|\mu(Q,P)\mu(U,V)\mathrm{Ten}_V^P\Delta(V)$$

Now the sum of $\mu(Q,P)$ for $U.R\subseteq Q\subseteq P$ is zero, unless $U.R=P$. Thus

$$|P|\mathrm{Inf}_{P/R}^P\Delta(P/R)=\sum_{\substack{U\subseteq V\subseteq P\\ U.R=P}}|U|\mu(U,V)\mathrm{Ten}_V^P\Delta(V)$$

as was to be shown. □

(7.5.4) **Remark:** Proposition 7.5.3 shows that the decomposition

$$D_{\mathcal{O}}^{tors}(P)=(D_{\mathcal{O}}^{\Delta})^{tors}(P)\oplus K_{\mathcal{O}}(P)$$

of proposition 7.4.9 is functorial in P, i.e. that the torsion part of $D_{\mathcal{O}}^{\Delta}(P)$ is mapped to itself by tensor induction, restriction, inflation, deflation, and group isomorphism. If $p=2$, there is nothing to prove, since $(D_{\mathcal{O}}^{\Delta})^{tors}(P)=0$ in this case. And if p is odd, the only non trivial fact to check here is that if P/R is cyclic (and non trivial), then $\mathrm{Inf}_{P/R}^P\Delta(P/R)$ can be expressed as a linear combination of elements $\mathrm{Ten}_V^P\Delta(V)$ corresponding to *cyclic* subgroups V of P.

Since $\Delta(P/R)$ has order 2 in this case, proposition 7.5.3 gives

$$\mathrm{Inf}_{P/R}^P\Delta(P/R)=\sum_{\substack{U\subseteq V\subseteq P\\ U.R=P}}|U|\mu(U,V)\mathrm{Ten}_V^P\Delta(V)$$

The coefficient of $\mathrm{Ten}_V^P\Delta(V)$ in the right hand side is zero if $V.R\neq P$, and otherwise it is equal to

$$\sum_{\substack{U\subseteq V\\ U.(V\cap R)=P}}|U|\mu(U,V)=|V|m_{V,V\cap R}$$

But if $V.R=P$, the group $V/V\cap R$ is cyclic, isomorphic to P/R. Then the constant $m_{V,V\cap R}$ is zero if V is non cyclic, as I already recalled in the proof of lemma 7.2.7. And if V is cyclic (hence non trivial since $V.R=P$), then $m_{V,V\cap R}=1$ if $V\cap R\neq V$, and $m_{V,V}=1-1/p$. Since moreover $\mathrm{Ten}_V^P\Delta(V)$ has order 2 in this case, this gives

$$\mathrm{Inf}_{P/R}^P\Delta(P/R)=\sum_{\substack{V\subseteq P\\ V\ cyclic,\ V\neq P\\ V.R=P}}\mathrm{Ten}_V^P\Delta(V)$$

as claimed.

7.6. Back to the Burnside ring

The linear forms λ_Q lead to a more explicit version of the functorial morphism $\mathbb{Q}D_k\to\mathbb{Q}B$. Recall the following notation from [3] page 704, or lemma 4.7 of [5]:

(7.6.1) **Notation:** *If P is a finite group, there is a constant $m(P) \in \mathbb{Q}$ such that if N is a maximal normal subgroup such that $m_{P,N} \neq 0$, then $m_{P,N} = m(P)$. Moreover, if M is a normal subgroup of P such that $m_{P,M} \neq 0$, then*

$$m_{P,M} = \frac{m(P)}{m(P/M)}$$

In the case of a p-group P, the constant $m(P)$ can be computed as follows:

- If $P = \{1\}$, then $m(P) = 1$.
- If P is cyclic and non-trivial, then $m(P) = 1 - 1/p$.
- If $|P/\Phi(P)| = p^d$, with $d \geq 2$, then

$$m(P) = (1-p)(1-p^2)\dots(1-p^{d-2})$$

(7.6.2) **Proposition:** *Let P be a p-group. Define a morphism α_P from $\mathbb{Q}D_k(P)$ to $\mathbb{Q}B(P)$ by*

$$\alpha_P(u) = \sum_{\substack{Q \in [s_P] \\ Q\ non-cyclic}} \lambda_{Q/\Phi(Q)}(\mathrm{Defres}^P_{Q/\Phi(Q)} u)\frac{1}{m(Q)} e^P_Q$$

Then there is an exact sequence

$$0 \longrightarrow \mathbb{Q}D_k(P) \xrightarrow{\alpha_P} QB(P) \xrightarrow{\chi_P} \mathbb{Q}R_{\mathbb{Q}}(P) \longrightarrow 0$$

(where χ_P is the character map), which is functorial with respect to P, i.e. it is compatible with restriction, induction, deflation, inflation, and group isomorphisms.

Proof: We showed in [5] (theorem D) that such an exact sequence of functors exists. Moreover

$$\mathbb{Q}D_{\mathcal{O}}(P) \simeq \mathbb{Q}D^{\Delta}_{\mathcal{O}}(P) \simeq \mathbb{Q}D^{\Delta}_k(P)$$

has a $\mathbb{Q}$-basis consisting of the elements $\mathrm{Ten}^P_Q\Delta(Q)$, for non-cyclic $Q \in [s_P]$.

Since $\Delta(P)$ is the only element, up to a scalar, in the intersection of the kernels of the maps Res^P_Q for $Q \subset P$ $(Q \neq P)$, the map α_P must send $\Delta(P)$ to some multiple $\gamma_P e^P_P$ of e^P_P. Then by functoriality, it must send $\mathrm{Ten}^P_Q\Delta(Q)$ to

$$\gamma_Q \mathrm{Ind}^P_Q e^Q_Q = \gamma_Q |N_P(Q) : Q| e^P_Q$$

By functoriality again, it must map $\mathrm{Def}^P_{P/R}\Delta(P) = \Delta(P/R)$ to

$$\mathrm{Def}^P_{P/R}(\gamma_P e^P_P) = m_{P,R}\gamma_P e^{P/R}_{P/R}$$

for any normal subgroup R of P. It follows that $\gamma_{P/R} = m_{P,R}\gamma_P$. Now if P is non-cyclic the constant $m_{P,R}$ is non-zero if and only if P/R is non-cyclic, and in this case $m_{P,R} = m(P)/m(P/R)$. This shows that $\gamma_P m(P)$ is independent of the non-cyclic group P, and I can suppose it is equal to p. Now the proposition follows from assertion 3) of proposition 7.4.8, since $\mathbb{Q}K_{\mathcal{O}}(P) = 0$. □

8. Some torsion elements in $D^{\Omega}_{\mathcal{O}}(P)$

8.1. Which relative syzygies are torsion elements

The following proposition characterizes the P-sets X such that Ω_X is a torsion element in $D_{\mathcal{O}}(P)$. In the case of a transitive P-set X, implication 1) $\Rightarrow$ 2) was originally a question of J. Thévenaz:

(8.1.1) **Proposition:** *Let P be a p-group, and X be a P-set. Then the following are equivalent:*

1. *The element Ω_X is a torsion element in $D_{\mathcal{O}}(P)$.*
2. *There exists a normal subgroup N of P such that P/N is cyclic or generalized quaternion, and such that $N \supseteq P_x$ for all $x \in X$, with equality for some $x \in X$. Moreover in this case $\Omega_X = \Omega_{P/N}$.*

Before proving this proposition, let me state an equivalent condition to assertion 1. First some notation:

(8.1.2) **Notation:** *If X is a P-set, I denote by F_X the set of elements $g \in P$ such that $X^g \neq \emptyset$.*

If A is a subset of P, I denote by $\sqrt[p]{A}$ the set of elements $g \in P$ such that $g^p \in A$.

(8.1.3) **Lemma:** *Let P be a p-group, and X be a P-set. Then the following are equivalent:*

1. *The element Ω_X is a torsion element in $D_{\mathcal{O}}(P)$.*
2. *Let W be any subgroup of P. Then*
   - *If $W \subseteq F_X$, the set X^W is non-empty.*
   - *If $W \not\subseteq F_X$, then $p|W \cap F_X| = |W \cap \sqrt[p]{F_X}|$*

Proof: By proposition 6.5.1

$$|P|\Omega_X = -\sum_{\substack{R\subseteq Q\subseteq P\\ X^R\neq\emptyset}} |R|\mu(R,Q)\mathrm{Ten}_Q^P\Delta(Q)$$

Now Ω_X is a torsion element if and only if $|P|\Omega_X$ is. Since $\Delta(Q)$ is a torsion element if Q is cyclic, this is equivalent to saying that there exists a positive integer n such that

$$0 = n \sum_{\substack{R\subseteq Q\subseteq P\\ X^R\neq\emptyset\\ Q\ non-cyclic}} |R|\mu(R,Q)\mathrm{Ten}_Q^P\Delta(Q)$$

By proposition 6.4.1, this is equivalent to requiring that for any non-cyclic subgroup Q of P, the integer

$$n_Q = \sum_{\substack{R\subseteq Q\\ X^R\neq\emptyset}} |R|\mu(R,Q)$$

is zero. On the other hand, if Q is cyclic, then the set of subgroups of Q is totally ordered, and there is a biggest subgroup R_0 of Q such that $X^{R_0} \neq \emptyset$. There are three cases:

- Either $R_0 = Q$, or equivalently $X^Q \neq \emptyset$, then

$$n_Q = \sum_{R \subseteq Q} |R|\mu(R,Q) = \sum_{\substack{x \in Q \\ <x> \subseteq R \subseteq Q}} \mu(R,Q) = \varphi(|Q|)$$

where φ is the Euler function.

- Either $R_0 = \Phi(Q) \neq Q$, and in this case

$$n_Q = |R_0|\mu(R_0,Q) = -|R_0| = -|Q|/p$$

- Either R_0 is a proper subgroup of $\Phi(Q)$, and $n_Q = 0$ in this case.

Let W be an arbitrary subgroup of P. Then, by Möbius inversion, requiring that the n_Q's have some prescribed values, depending on Q, is equivalent to requiring that the sum of n_Q's for subgroups Q of W have prescribed values, depending on W. Now summing the n_Q's for $Q \subseteq W$ gives

$$\sum_{Q \subseteq W} n_Q = \sum_{\substack{R \subseteq Q \subseteq W \\ X^R \neq \emptyset}} |R|\mu(R,Q)$$

and the sum on Q is equal to zero if $R \neq W$, and equal to 1 otherwise. Thus

$$\sum_{Q \subseteq W} n_Q = \begin{cases} 0 & \text{if } X^W = \emptyset \\ |W| & \text{otherwise} \end{cases}$$

But this is also the sum of n_Q's for cyclic subgroups Q of W, i.e.

$$\sum_{\substack{Q \subseteq W \\ Q \text{ cyclic} \\ X^Q \neq \emptyset}} \varphi(|Q|) - \sum_{\substack{Q \subseteq W \\ Q \text{ cyclic} \\ X^Q = \emptyset \neq X^{\Phi(Q)}}} |Q|/p$$

Since $\varphi(|Q|)$ is the number of generators of Q, the first sum is also

$$\sum_{\substack{g \in W \\ X^g \neq \emptyset}} 1 = |W \cap F_X|$$

where F_X is the set of elements $g \in P$ such that $X^g \neq \emptyset$.

Now a cyclic subgroup Q of W such that $X^Q = \emptyset \neq X^{\Phi(Q)}$ is a subgroup generated by an element g of W such that $g \notin F_X$, but $g^p \in F_X$, and Q admits $|g| - |g|/p$ such generators. Hence the second sum is equal to

$$\sum_{g \in (W \cap \sqrt[p]{F_X}) - (W \cap F_X)} \frac{|g|/p}{|g| - |g|/p} = \frac{|W \cap \sqrt[p]{F_X}| - |W \cap F_X|}{p-1}$$

This gives

$$\sum_{Q \subseteq W} n_Q = |W \cap F_X| - \frac{|W \cap \sqrt[p]{F_X}| - |W \cap F_X|}{p-1} = \frac{p|W \cap F_X| - |W \cap \sqrt[p]{F_X}|}{p-1}$$

Finally, the element Ω_X is a torsion element if and only if for any subgroup W of P

$$(8.1.4)\qquad \frac{p|W\cap F_X| - |W\cap \sqrt[p]{F_X}|}{p-1} = \begin{cases} 0 & \text{if } X^W = \emptyset \\ |W| & \text{otherwise} \end{cases}$$

If $W \subseteq F_X$, then $W \subseteq \sqrt[p]{F_X}$, and the left hand side of equation 8.1.4 is equal to $|W|$. In particular it is non-zero, and X^W must be non-empty. Thus

$$(8.1.5)\qquad W \subseteq F_X \Rightarrow \exists x \in X,\ W \subseteq P_x$$

On the other hand, if $W \not\subseteq F_X$, then in particular $X^W = \emptyset$. Hence

$$(8.1.6)\qquad W \not\subseteq F_X \Rightarrow p|W\cap F_X| = |W \cap \sqrt[p]{F_X}|$$

This completes the proof of the lemma. □

Condition 8.1.6 alone implies that F_X is a normal subgroup of P:

(8.1.7) **Lemma:** *Let P be a p-group. Let A be a normal subset of P, containing 1, such that for any subgroup W of P*

$$W \not\subseteq A \Rightarrow |W\cap \sqrt[p]{A}| = p|W\cap A|$$

Then A is a normal subgroup of P.

Proof: I first prove by induction on $|W|$ that if W is a cyclic subgroup of P, then $W\cap A$ is a subgroup of W. The result is true if $W=\{1\}$, since $1\in A$. Thus I can suppose that W is a non-trivial cyclic subgroup of P, and by induction $\Phi(W)\cap A$ is a subgroup of $\Phi(W)$. Moreover

$$W\cap \sqrt[p]{A} = \{g\in W \mid g^p\in A\} = \{g\in W\mid g^p\in \Phi(W)\cap A\}$$

since $g^p\in\Phi(W)$ for all $g\in P$. But the set of elements g of W such that g^p belongs to the proper subgroup $\Phi(W)\cap A$ of W is a subgroup of W, of order $p|\Phi(W)\cap A|$. Then either $W\subseteq A$, and there is nothing to prove, or by assumption

$$|W\cap\sqrt[p]{A}| = p|W\cap A| = p|\Phi(W)\cap A|$$

Since $W\cap A\supseteq \Phi(W)\cap A$, it follows that $W\cap A = \Phi(W)\cap A$ is a subgroup of W, as required.

It follows that if $a\in A$, then the subgroup generated by a is contained in A. In particular if $a\in A$, then $a^p\in A$. Equivalently $A\subseteq \sqrt[p]{A}$.

Now I will prove the lemma by induction on $|P|$. If $P=\{1\}$, the result holds since $1\in A$. Suppose that the result holds for all proper subgroups of P. If P' is such a subgroup, set $A' = P'\cap A$. This is a normal subset of P', containing 1. Moreover, if W' is a subgroup of P', not contained in A', then W' is a subgroup of P, not contained in A. Denote by ${}^{\prime}\sqrt[p]{A'}$ the set of elements g' of P' such that $g'^p\in A'$. Then

$${}^{\prime}\sqrt[p]{A'} = P'\cap\sqrt[p]{A}$$

Thus

$$|W'\cap {}^{\prime}\sqrt[p]{A'}| = |W'\cap\sqrt[p]{A}| = p|W'\cap A| = p|W'\cap A'|$$

By induction hypothesis $A' = P' \cap A$ is a subgroup of P.

Let H be a subgroup of index p of P, such that $N = H \cap A$ has maximal cardinality. Then N is a normal subgroup of P, since it is a subgroup by the previous argument, and it is also a normal subset of P.

Suppose that $A \not\subseteq H$, and choose $a \in A - H$. Then let $W = <a, N>$.

If $W \neq P$, then $W \cap A$ is a subgroup of P. Since W is generated by elements of A, this shows that $W \subseteq A$. Let H' be a subgroup of index p of P, containing W. Such a subgroup exists since $W \neq P$. Then $H' \cap A$ contains W, and W contains strictly N, since $a \notin H$. This contradicts the assumption on H.

Thus $W = <a, N> = P$, and then P/N is cyclic, generated by the image of a. In particular it admits a unique subgroup of index p, generated by the image of a^p. Since H/N has index p in P/N, it follows that $H = <a^p, N>$. Now H is generated by elements of A, and $H \cap A$ is a subgroup of P. Thus $H \subseteq A$. This implies in particular that $P = \sqrt[p]{A}$, since $g^p \in H$ for all $g \in P$. Hence either $A = P$, or $|P| = p|P \cap A| = p|A|$. In this case $|A| = |H|$, thus $A = H$. In both cases A is a subgroup of P.

Now if $A \subseteq H$, then $H \cap A = A$ is a subgroup of P, and the lemma follows. $\square$

8.2. Proof of proposition 8.1.1

Suppose that Ω_X is a torsion element. By the previous lemmas, the set $N = F_X$ is a subgroup of P, contained in F_X. Thus $X^N \neq \emptyset$, by 8.1.5, and there is an element $x \in X$ such that $N \subseteq P_x$, hence $N = P_x$. It follows that $N \supseteq P_y$ for all $y \in X$, and that $\Omega_X = \Omega_{P/N}$ by lemma 3.2.7.

Now if W/N is a non-trivial elementary abelian subgroup of P/N

$$|W| = |W \cap \sqrt[p]{N}| = p|W \cap N| = p|N|$$

thus $|W/N| = p$. The group P/N has p-rank at most 1, hence it is cyclic or generalized quaternion. This shows that 1) implies 2).

Conversely if N is a normal subgroup of P such that P/N is cyclic or generalized quaternion, then it it well known that $\Omega_{P/N}$ has order

- 1 if P/N is trivial or of order 2.
- 2 if P/N is cyclic of order at least 3.
- 4 if P/N is generalized quaternion.

Thus 2) implies 1), and the proof is complete. $\square$

(8.2.1) **Remark:** One can check easily that the conditions of lemma 8.1.3 hold for $X = P/N$, if P/N is cyclic or generalized quaternion: indeed in this case, the group P/N has a unique subgroup M/N of order p, if it is non trivial. Then clearly $\sqrt[p]{N} = M$.

If W is a subgroup of P not contained in N, then $W.N$ contains M, and $M = N.(W \cap M)$. Thus $|M||W \cap N| = |N||W \cap M|$, or

$$|W \cap \sqrt[p]{N}| = |W \cap M| = p|W \cap N|$$

This can be viewed as a new proof of the fact that Ω_P is a torsion element if P is quaternion. This is not a new proof for the case of cyclic groups, since it requires Dade's theorem.

8.3. Acknowledgements

I am grateful to J. Thévenaz for careful reading of a first version of this paper, for pointing out mistakes and misprints, and also for many useful suggestions.

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