The p-blocks of the Mackey algebra

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Abstract: Let p be a prime number. This paper describes the primitive idempotents and prime spectrum of the *crossed Burnside algebra* of a finite group over a p-local ring. The main application is a formula for the block idempotents of the *p-local Mackey algebra* of the group, in terms of the corresponding blocks of the group algebra.

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1 Introduction

Let p be a prime number, and $\mathcal{O}$ be a complete discrete valuation ring of characteristic 0 whith residue field of characteristic p. Let G be a finite group, and denote by $\mu_{\mathcal{O}}(G)$ the Mackey algebra of G over $\mathcal{O}$ (see [8] for definition).

Formulae for the primitive idempotents in the center of $\mu_{\mathcal{O}}(G)$ have been given by Yoshida (and slightly corrected by Oda [7]). However, those formulae are expressed in terms of ordinary irreducible characters of the centralizers of subgroups of G. The aim of this article is to state explicit formulae for the block idempotents of $\mu_{\mathcal{O}}(G)$, in terms of the blocks of the group algebra $\mathcal{O}G$.

The proof uses the natural ring homomorphism from the crossed Burnside ring $B^c_{\mathcal{O}}(G)$ to the center of the Mackey algebra, and a description of the prime spectrum and block idempotents of $B^c_{\mathcal{O}}(G)$.

The paper is organized as follows : section 2 is an exposition of definitions and basic results on the crossed Burnside ring. Section 3 describes the prime spectrum and p-blocks of this ring. Section 4 recalls the basic definitions on Mackey functors, and uses the action of the crossed Burnside ring to state explicit formulae for the block idempotents of the p-local Mackey algebra. Section 5 exposes some consequences on Mackey functors that follow from these formulae. In particular, one can show that a block b of G and the corresponding block b^μ of the Mackey algebra have the same defect groups.

2 The crossed Burnside ring

Most of the definitions and results of this section have already been discussed by Yoshida ([9]). However, they are still unpublished, or not currently available in published form. This is the reason for exposing this material here.

2.1 Definition

Let G be a finite group, and denote by G^c the set G, on which G acts by conjugation. The category $G\text{-}\mathbf{set}{\downarrow_{G^c}}$ of *crossed G-sets* is the category of G-sets over G^c : a *crossed G-set* (X, α) is a a pair consisting of a finite G-set X (i.e. a finite set with a left G-action), together with a map of G-sets α from X to G^c, and a morphism of crossed G-sets from (X, α) to (Y, β) is a morphism of G-sets φ from X to Y such that $\beta \circ \varphi = \alpha$.

There is an obvious notion of disjoint union of crossed G-sets, and the crossed Burnside group $B^c(G)$ is defined as the Grothendieck group of the category of crossed G-sets, for relation given by disjoint union decomposition : let $\mathcal{I}$ be the set of isomorphism classes of crossed G-sets, and denote by $[X, \alpha]$ the isomorphism class of the crossed G-set (X, α). Then

$$B^c(G) = \mathbb{Z}^{\mathcal{I}} / < [X \sqcup Y, \alpha \sqcup \beta] - [X, \alpha] - [Y, \beta] > \quad .$$

Let B denote the (ordinary) Burnside Mackey functor for G (see [3] or [1] Section 2.4 for definition). It follows from [1] Proposition 2.4.2 that $B^c(G)$ is isomorphic (as a $\mathbb{Z}$-module) to the evaluation $B(G^c)$ of the Mackey functor B at the G-set G^c.

If (X, α) and (Y, β) are crossed G-set, then their product is the crossed G-set $(X \times Y, \alpha.\beta)$, where $X \times Y$ is the direct product of X and Y, with diagonal G-action, and $\alpha.\beta$ is the map from $X \times Y$ to G^c defined by

$$(\alpha.\beta)(x, y) = \alpha(x)\beta(y) \quad .$$

This product on crossed G-sets clearly commutes with disjoint unions, hence it gives a product on $B^c(G)$. This turns $B^c(G)$ into a ring. The identity element of this ring is $[\bullet, u_\bullet]$, where $\bullet$ is a G-set of cardinality one, and the map $u_\bullet$ sends the unique element of $\bullet$ to the identity element of the group G.

The ring $B^c(G)$ is commutative : the map

$$(x, y) \in X \times Y \mapsto (\alpha(x)y, x) \in Y \times X$$

is an isomorphism from $(X \times Y, \alpha.\beta)$ to $(Y \times X, \beta.\alpha)$ in $G\text{-}\mathbf{set}{\downarrow_{G^c}}$, because for all $(x, y) \in X \times Y$

$$\beta\Big(\alpha(x)y\Big)\alpha(x) = \alpha(x)\beta(y)\alpha(x)^{-1}\alpha(x) = \alpha(x)\beta(y) \quad .$$

More generally, if R is a commutative ring, denote by $B^c_R(G)$ the tensor product of $B^c(G)$ with R over $\mathbb{Z}$

$$B^c_R(G) = R \otimes_{\mathbb{Z}} B^c(G) \quad .$$

It is an R-algebra. Similarly, denote by $B_R(G)$ the (ordinary) Burnside algebra of G over R.

Lemma 2.1.1: *If X is a finite G-set, denote by u_X the map from X to G sending every element to $1 \in G$. Then the correspondence $X \mapsto (X, u_X)$ extends linearly to a ring homomorphism from $B_R(G)$ to $B^c_R(G)$, which preserves identity elements.*

Proof: This amounts to checking that if X and Y are finite G-sets, then the product $(X \times Y, u_X.u_Y)$ is isomorphic to $(X \times Y, u_{X\times Y})$, which is straightforward. Moreover the trivial G-set is mapped to $(\bullet, u_\bullet)$. □

2.2 Characterization of crossed G-sets

A crossed G-set (X, α) is called *transitive* if the G-set X is. In this case, let x be any element of X, denote by H the stabilizer of x in G, and set $a = \alpha(x)$. Then a lies in the centralizer $C_G(H)$ of H in G. Define the map m_a from G/H to G^c by $m_a(gH) = {}^g a$, where ${}^g a = gag^{-1}$. Then $(G/H, m_a)$ is a crossed G-set, and the map $gH \mapsto gx$ from G/H to X is clearly an isomorphism of crossed G-sets from $(G/H, m_a)$ to (X, α).

Conversely, if H is any subgroup of G, and if $a \in C_G(H)$, then $(G/H, m_a)$ is a transitive crossed G-set. If K is a subgroup of G, and $b \in C_G(K)$, then the crossed G-sets $(G/H, m_a)$ and $(G/K, m_b)$ are isomorphic if and only if there exists an element $g \in G$ such that ${}^g H = K$ and ${}^g a = b$.

Notation 2.2.1: *I will denote by s_G the set of subgroups of G. If X is a G-set and $H \in s_G$, I denote by X^H the set of fixed points of H on X. Let $\mathcal{P}_G$ denote the set of pairs (H, a) consisting of a subgroup H of G and and element a of $C_G(H)$. The group G acts by conjugation on s_G and $\mathcal{P}_G$, and I denote by $[s_G]$ (resp. $[\mathcal{P}_G]$) a set of representative of G-orbits on s_G (resp. on $\mathcal{P}_G$). If $(H, a) \in \mathcal{P}_G$, I denote by $[H, a]_G$ the isomorphism class of the crossed G-set $(G/H, m_a)$.*

Now if (X, α) is any crossed G-set, choose a set S of representatives of the orbits of G on X. Then the map from $\bigsqcup_{s\in S} G/G_s$ to X sending gG_s to gs is clearly an isomorphism in G-$\mathbf{set}\!\downarrow_{G^c}$ from $\bigsqcup_{s\in S}(G/G_s, m_{\alpha(s)})$ to X. Thus any crossed G-set is isomorphic to a disjoint union of transitive ones.

Lemma 2.2.2: *Let (X,α) and (Y,β) be crossed G-sets. Then the following are equivalent:*

1. *The crossed G-sets (X,α) and (Y,β) are isomorphic.*
2. *For any crossed G-set (Z,γ)*
$$|\mathrm{Hom}_{G\text{-}\mathbf{set}\downarrow_{G^c}}\Big((Z,\gamma),(X,\alpha)\Big)| = |\mathrm{Hom}_{G\text{-}\mathbf{set}\downarrow_{G^c}}\Big((Z,\gamma),(Y,\beta)\Big)| \quad .$$
3. *For any $(H,a)\in[\mathcal{P}_G]$*
$$|\alpha^{-1}(a)^H| = |\beta^{-1}(a)^H| \quad .$$

Proof: Clearly 1) implies 2). Moreover 2) implies 3) since for any crossed G-set X
$$|\mathrm{Hom}_{G\text{-}\mathbf{set}\downarrow_{G^c}}\Big((G/H,m_a),(X,\alpha)\Big)| = |\alpha^{-1}(a)^H| \quad .$$
To show that 3) implies 1), I can replace (X,α) and (Y,β) by isomorphic crossed G-sets, i.e. suppose that
$$(X,\alpha) = \bigsqcup_{(K,b)\in[\mathcal{P}_G]} u_{K,b}(G/K,m_b) \qquad (Y,\beta) = \bigsqcup_{(K,b)\in[\mathcal{P}_G]} v_{K,b}(G/K,m_b)$$
where $u_{K,b}$ and $v_{K,b}$ are natural integers, and the notation $u_{K,b}(G/K,m_b)$ means a disjoint union of $u_{K,b}$ copies of $(G/K,m_b)$. Notice that if $p=(H,a)$ and $q=(K,b)$ are elements of $\mathcal{P}_G$, then
$$|\mathrm{Hom}_{G\text{-}\mathbf{set}\downarrow_{G^c}}\Big((G/H,m_a),(G/K,m_b)\Big)| = |\{g\in G/K \mid H^g\subseteq K,\ a^g=b\}| \quad .$$
Denote by $M(p,q)$ this number. Condition 3) implies that
$$\sum_{q\in[\mathcal{P}_G]} u_q M(p,q) = \sum_{q\in[\mathcal{P}_G]} v_q M(p,q)$$
for all $p\in[\mathcal{P}_G]$. It follows that the sequence $(u_q-v_q)_{q\in[\mathcal{P}_G]}$ is in the kernel of the square matrix $M(p,q)_{p,q\in[\mathcal{P}_G]}$. Now with suitable ordering of $[\mathcal{P}_G]$, this matrix is upper triangular. The diagonal coefficient $M(p,p)$ for $p=(H,a)$ is equal to
$$M(p,p) = |\{g\in N_G(H)/H \mid {}^g a = a\}| = |N_G(H)\cap C_G(a) : H|$$
This is non-zero, and M is non singular. Thus $u_{K,b}=v_{K,b}$ for any (K,b) in $[\mathcal{P}_G]$. Hence (X,α) and (Y,β) are isomorphic. □

Corollary 2.2.3: *The elements $[H,a]_G$, for $(H,a)\in[\mathcal{P}_G]$, form a basis of $B^c(G)$ over $\mathbb{Z}$.*

2.3 Brauer morphisms

Let K be a subgroup of G, and let $C_G(K)$ be the centralizer of K in G. Denote by $(X,\alpha) \mapsto (X^K,\alpha^K)$ the fixed points functor from G-$\mathbf{set}{\downarrow_{G^c}}$ to $C_G(K)$-$\mathbf{set}{\downarrow_{C_G(K)^c}}$, where X^K is viewed as a $C_G(K)$-set, and α^K is the map $X^K \to (G^c)^K = C_G(K)^c$ induced by α.

This functor induces a map Br_K, called the Brauer morphism, from $B_R^c(G)$ to $B_R^c\Big(C_G(K)\Big)$, defined by linearity from $Br_K([X,\alpha]) = [X^K,\alpha^K]$, which is clearly a ring homomorphism, which preserves identity elements.

If (X,α) is a crossed G-set, let $s_G(X,\alpha)$ denote the element of the center $\mathcal{Z}RG$ of the group algebra RG of G over R defined by

$$s_G(X,\alpha) = \sum_{x\in X} \alpha(x)$$

This clearly induces a morphism of R-algebras, still denoted by s_G, from $B_R^c(G)$ to $\mathcal{Z}RG$, which preserves identity elements.

Notation 2.3.1: *If H is a subgroup of G, then I denote by z_H the ring homomorphism $s_{C_G(H)} \circ Br_H$ from $B_R^c(G)$ to $\mathcal{Z}RC_G(H)$.*

Thus if (X,α) is a crossed G-set, then

$$z_H([X,\alpha]) = \sum_{x\in X^H} \alpha(x) = \sum_{g\in C_G(H)} |\alpha^{-1}(g)^H| g \quad .$$

Lemma 2.3.2: *If R is torsion free, then the ring homomorphism*

$$\Theta_R = \prod_{H\in[s_G]} z_H : B_R^c(G) \to \prod_{H\in[s_G]} \mathcal{Z}RC_G(H)$$

is injective.

Proof: Clearly Θ_R is a ring homomorphism. The injectivity assertion is just a reformulation of lemma 2.2.2 : if $u = \sum_{(H,a)\in[\mathcal{P}_G]} r_{H,a}[H,a]_G$ is a non-zero element the kernel of Θ_R, let K be a subgroup of G maximal such that there exists $(K,b) \in [\mathcal{P}_G]$ with $r_{K,b} \neq 0$. Now

$$z_K(u) = \sum_{\substack{a\in C_G(K)\\ a\ mod.N_G(K)}} r_{K,a} \sum_{g\in N_G(K)/K} {}^g a = \sum_{a\in C_G(K)} r_{K,a} |N_G(K)\cap C_G(a) : K| a = 0$$

Since R is torsion free, it follows that $r_{K,a} = 0$ for all $a \in C_G(K)$. This contradiction proves the lemma. □

The previous lemma can be considered from a slightly different point of view : let G^c denote the set G, on which G acts by conjugation. There is an isomorphism of G-sets

$$G^c \cong \bigsqcup_{g \in [G]} G/C_G(g)$$

where $[G]$ is a set of representatives of conjugacy classes of G. Since $B^c(G)$ is the value of the Burnside Mackey functor B at G^c, it follows that there is an isomorphism of $\mathbb{Z}$-modules

$$B^c(G) \cong \bigoplus_{g \in [G]} B\Big(C_G(g)\Big)$$

sending the crossed G-set (X, α) to the sequence $\Big(\alpha^{-1}(g)\Big)_{g \in [G]}$. The inverse isomorphism sends the element $C_G(g)/L$ of $B\Big(C_G(g)\Big)$ to $[L, g]_G$.

Now for any finite group H, it follows from Burnside's theorem (see [3] Theorem 2.3.2) that there is an injective morphism

$$\phi_H : B(H) \to \prod_{K \in [s_H]} \mathbb{Z} \qquad \mathbf{(2.3.3)}$$

defined by linearity by mapping the H-set X to the sequence $(|X^K|)_{K \in [s_H]}$. Hence there is an injective morphism of $\mathbb{Z}$-modules

$$B^c(G) \to \prod_{g \in [G]} \prod_{\in [s_{C_G(g)}]} \mathbb{Z} \cong \prod_{(H,g) \in [\mathcal{P}_G]} \mathbb{Z}$$

sending the crossed G-set (X, α) to the sequence $(|\alpha^{-1}(g)^H|)_{(H,g) \in [\mathcal{P}_G]}$. Now there is an isomorphism

$$\prod_{(H,g) \in [\mathcal{P}_G]} \mathbb{Z} \cong \prod_{H \in [s_G]} \mathcal{Z}\mathbb{Z}C_G(H) \quad .$$

This gives the injective map

$$\Theta_{\mathbb{Z}} : B^c(G) \to \prod_{H \in [s_G]} \mathcal{Z}\mathbb{Z}C_G(H)$$

and the map Θ_R of lemma 2.3.2 is obtained by tensoring this map with R, which is a flat $\mathbb{Z}$-module if R is torsion free. As a consequence :

Proposition 2.3.4: *Let K be a field of characteristic 0. Then the map*

$$\Psi_K : \prod_{H\subseteq G} z_H : B_K^c(G) \to \Big(\prod_{H\subseteq G} \mathcal{Z}KC_G(H)\Big)^G$$

is an isomorphism of K-algebras. The inverse isomorphism maps the sequence $(z_H)_{H\subseteq G}$ to

$$\frac{1}{|G|}\sum_{\substack{(L,g)\in\mathcal{P}_G\\ L\subseteq H\subseteq C_G(g)}} |L|\mu(L,H)z_H(g)[L,g]_G$$

where $\mu(L,H)$ denotes the Möbius function of the poset of subgroups of G, and $z_H(g)$ denotes the coefficient of g in z_H.

Proof: The map Ψ_K is injective by lemma 2.3.2, and its image is contained in

$$\Big(\prod_{H\subseteq G} \mathcal{Z}KC_G(H)\Big)^G \cong \prod_{H\in[s_G]} \Big(KC_G(H)\Big)^{N_G(H)} \quad .$$

Now this K-vector space has the same (finite) dimension as $B_K^c(G)$, namely the cardinality of $[\mathcal{P}_G]$. The first assertion follows.

To build the inverse map, note that for any finite group G, the Burnside algebra $B_K(G)$ is a split semi-simple commutative K-algebra. The primitive idempotents of $B_K(G)$ have been determined by Gluck ([5]). They are indexed by the (conjugacy classes of) subgroups of the group G. The idempotent e_H^G indexed by H is equal to (see [3] Theorem 3.3.2)

$$e_H^G = \frac{1}{|N_G(H)|}\sum_{L\subseteq H} |L|\mu(L,H)\; G/L \quad . \qquad \textbf{(2.3.5)}$$

This idempotent is characterized by the fact that for any $X \in B_K(G)$, one has that

$$X.e_H^G = |X^H|e_H^G \quad . \qquad \textbf{(2.3.6)}$$

It follows in particular that

$$X = \sum_{H\in[s_G]} X.e_H^G = \sum_{H\in[s_G]} |X^H|e_H^G = \frac{1}{|G|}\sum_{L\subseteq H\subseteq G} |L|\mu(L,H)|X^H|G/L \quad .$$

Now if (X,α) is a crossed G-set, the corresponding element $(z_H)_{H\subseteq G}$ is defined by

$$z_H(g) = |\alpha^{-1}(g)^H|$$

thus

$$\alpha^{-1}(g) = \frac{1}{|C_G(g)|} \sum_{L \subseteq H \subseteq C_G(g)} |L|\mu(L,H) z_H(g)\; C_G(g)/L$$

in $B\Big(C_G(g)\Big)$, and the second assertion follows. □

Remark 2.3.7: *Proposition 2.3.4 can be used to show that $B_K^c(G)$ is a semi-simple (commutative) algebra (split when K is big enough), and to state explicit formulae for its primitive idempotents. Those formulae are due to Yoshida.*

It is possible to characterize the image of the above map $\Theta_{\mathbb{Z}}$:

Proposition 2.3.8: *If $(H,g) \in \mathcal{P}_G$, and if K is a subgroup of $C_G(g)$, set*

$$n_g(H,K) = |\{x \in N_G(H) \cap C_G(g)/H \mid <H,x> =_{C_G(g)} K\}|$$

where $<H,x>$ is the subgroup of G generated by H and x, and the notation $<H,x> =_{C_G(g)} K$ means that $<H,x>$ and K are conjugate by an element of $C_G(g)$.

For $H \in [s_G]$, let $z_H = \sum_{g \in C_G(H)} z_H(g) g$ be an element of $\mathcal{Z}\mathbb{Z}C_G(H)$. Then the sequence $(z_H)_{H \in [s_G]}$ belongs to the image of $\Theta_{\mathbb{Z}}$ if and only if for any H in $[s_G]$ the following two conditions hold :

1. *The element z_H is invariant by $N_G(H)$.*
2. *For any $g \in C_G(H)$, the sum $\sum_{K \in [s_{C_G(g)}]} n_g(H,K) z_K(g)$ is a multiple of $|N_G(H) \cap C_G(g) : H|$.*

Proof: This follows from a theorem of Dress ([4], or [3] Theorem 3.2.1), characterizing the image of the map ϕ_H defined in (2.3.3): giving an element z_H in $(\mathbb{Z}C_G(H))^{N_G(H)}$, for $H \in [s_G]$ is equivalent to giving integers $z_H(g)$, for $(H,g) \in [\mathcal{P}_G]$. Now for a fixed $g \in G$, the sequence of integers $(z_H(g))_{H \in [s_{C_G(g)}]}$ is in the image of $\phi_{C_G(g)}$ if and only if condition 2) holds. □

3 The prime spectrum of $B_R^c(G)$

3.1 Prime spectrum

Lemma 3.1.1: *The Krull dimension of the ring $B_R^c(G)$ is equal to the Krull dimension of R.*

Proof: Indeed, the ring $B_R^c(G)$ is an extension of R, and it is a finitely generated (free) R-module. Hence it is integral over R. Thus $\dim B_R^c(G) = \dim R$. □

Lemma 3.1.2: *If R is torsion free, then the map Θ_R induces a surjection*

$$Spec(\Theta_R) : Spec\Big(\prod_{H\in[s_G]} \mathcal{Z}RC_G(H)\Big) \to Spec\Big(B_R^c(G)\Big) \quad .$$

Proof: Here again, the ring $C = \prod_{H\in[s_G]} \mathcal{Z}RC_G(H)$ is an extension of $B_R^c(G)$. Moreover, it is a finitely generated R-module, hence also a finitely generated $B_R^c(G)$-module. Thus C is integral over $B_R^c(G)$, and Θ_R induces a surjective maps on the spectra. □

Notation 3.1.3: *If $\mathfrak{p}$ is a prime ideal of R, denote by $k(\mathfrak{p})$ the field of fractions of $R/\mathfrak{p}$. Denote by $\varphi_\mathfrak{p}$ the canonical morphism from $\mathcal{Z}RG$ to $\mathcal{Z}k(\mathfrak{p})G$. If b is a block of $k(\mathfrak{p})G$, set*

$$I_{\mathfrak{p},b} = \{u \in \mathcal{Z}RG \mid \varphi_\mathfrak{p}(u)b \in J\Big(\mathcal{Z}k(\mathfrak{p})Gb\Big)\}$$

(where $J\Big(\mathcal{Z}k(\mathfrak{p})Gb\Big)$ denotes the Jacobson radical of the algebra $\mathcal{Z}k(\mathfrak{p})Gb$). It is an ideal of $\mathcal{Z}RG$.

Lemma 3.1.4: *Let $\mathfrak{p}$ (resp. $\mathfrak{p}'$) denote a prime ideal of R, and b (resp. b') denote a block of $k(\mathfrak{p})G$ (resp. $k(\mathfrak{p}')G$). Then :*

1. *The ideal $I_{\mathfrak{p},b}$ is a prime ideal of $\mathcal{Z}RG$. Conversely, if I is a prime ideal of $\mathcal{Z}RG$, there exist a unique prime ideal $\mathfrak{p}$ of R and a unique block b of $k(\mathfrak{p})G$ such that $I = I_{\mathfrak{p},b}$.*
2. *If $I_{\mathfrak{p},b} \subseteq I_{\mathfrak{p}',b'}$, then $\mathfrak{p} \subseteq \mathfrak{p}'$. If moreover $\mathfrak{p} = \mathfrak{p}'$, then $b = b'$.*
3. *If $\mathfrak{p} \subseteq \mathfrak{p}'$, and if the block b is given, then there exists a block b' such that $I_{\mathfrak{p},b} \subseteq I_{\mathfrak{p}',b'}$.*
4. *If $\mathfrak{p} \subseteq \mathfrak{p}'$, and if the block b' is given, then there exists a block b such that $I_{\mathfrak{p},b} \subseteq I_{\mathfrak{p}',b'}$.*

Proof: Clearly $I_{\mathfrak{p},b}$ is the kernel of the canonical morphism

$$\mathcal{Z}RG \stackrel{\varphi_\mathfrak{p}}{\to} \mathcal{Z}k(\mathfrak{p})G \stackrel{b}{\to} \mathcal{Z}k(\mathfrak{p})Gb \to \mathcal{Z}k(\mathfrak{p})Gb/J\Big(\mathcal{Z}k(\mathfrak{p})Gb\Big)$$

Since the ring on the right hand side is a field, it follows that $I_{\mathfrak{p},b}$ is a prime ideal of $\mathcal{Z}RG$.

The map $u : r \in R \mapsto r.1 \in \mathcal{Z}RG$ is an injective ring homomorphism. Thus $\mathcal{Z}RG$ is an extension ring of R. If I is a prime ideal in $\mathcal{Z}RG$, then $\mathfrak{p} = I \cap R$ is a prime ideal of R. Moreover

$$\mathcal{Z}\mathfrak{p}G = \mathfrak{p}\mathcal{Z}RG \subseteq I \quad .$$

Let π denote the projection map $\mathcal{Z}RG \to \mathcal{Z}RG/I$, and $i : \mathcal{Z}RG/I \to K$ denote the embedding of $\mathcal{Z}RG/I$ in its field of fractions K. The kernel of the map $\rho = i \circ \pi \circ u : R \to K$ is equal to $\mathfrak{p}$. Hence ρ factors through the canonical map $\lambda_\mathfrak{p} : R \to k(\mathfrak{p})$. Hence there is a map $\theta : \mathcal{Z}k(\mathfrak{p})G \to K$ and a commutative square

$$\begin{array}{ccc} \mathcal{Z}RG & \xrightarrow{\pi} & \mathcal{Z}RG/I \\ {\scriptstyle\varphi_\mathfrak{p}}\downarrow & & \downarrow{\scriptstyle i} \\ \mathcal{Z}k(\mathfrak{p})G & \xrightarrow[\theta]{} & K \end{array} \quad .$$

The ideal I is the kernel of the map $\theta \circ \varphi_\mathfrak{p}$, hence it is the inverse image under $\varphi_\mathfrak{p}$ of the prime ideal $\theta^{-1}(0)$ of $\mathcal{Z}k(\mathfrak{p})G$. This ring has dimension zero. Hence there exists a primitive idempotent b of $\mathcal{Z}k(\mathfrak{p})G$ (i.e. a block of $k(\mathfrak{p})G$) such that $\theta^{-1}(0) = J\Big(\mathcal{Z}k(\mathfrak{p})Gb\Big) + \sum_{b' \neq b} \mathcal{Z}k(\mathfrak{p})Gb'$ (where b' denotes a block of kG). Hence $I = I_{\mathfrak{p},b}$.

Finally if $\mathfrak{p}'$ is a prime ideal of R, and b' is a block of $k(\mathfrak{p}')G$ such that $I_{\mathfrak{p},b} \subseteq I_{\mathfrak{p}',b'}$, then

$$\mathfrak{p} = R \cap I_{\mathfrak{p},b} \subseteq R \cap I_{\mathfrak{p}',b'} = \mathfrak{p}' \quad .$$

Suppose moreover that $\mathfrak{p} = \mathfrak{p}'$. The idempotent b can be written

$$b = \sum_{g \in G} (r_g/s_g)g$$

where $r_g \in R/\mathfrak{p}$, and $s_g \in R/\mathfrak{p} - \{0\}$, for $g \in G$. There is an element $s \in R$ which maps to $\prod_{g \in G} s_g$ in $R/\mathfrak{p}$, and there is an element $v \in \mathcal{Z}RG$ such that

$$\varphi_\mathfrak{p}(v) = \lambda_\mathfrak{p}(s)b \quad .$$

It follows that $v - s.1 \in I_{\mathfrak{p},b} = I_{\mathfrak{p}',b'}$. Thus $\varphi_\mathfrak{p}(v - s.1)b'$ is nilpotent. But if $b' \neq b$, then

$$\varphi_\mathfrak{p}(v - s.1)b' = \lambda_\mathfrak{p}(s)bb' - \lambda_\mathfrak{p}(s)b' = -\lambda_\mathfrak{p}(s)b' \quad .$$

This cannot be nilpotent since $\lambda_{\mathfrak{p}}(s)$ is non-zero in $k(\mathfrak{p})$. Thus $b = b'$. This completes the proof of assertions 1) and 2).

Assertion 3) is nothing but the *going up* theorem, which holds between R and $\mathcal{Z}RG$ because $\mathcal{Z}RG$ is free as an R-module, hence flat over R (see [6] Theorem 9.5).

Similarly, assertion 4) is the *going down* theorem, which holds between R and $\mathcal{Z}RG$ because $\mathcal{Z}RG$ is a finitely generated R-module, hence integral over R (see [6] Theorem 9.4). □

Notation 3.1.5: *If H is a subgroup of G, if $\mathfrak{p}$ is a prime ideal of R, and if b is a block of $k(\mathfrak{p})C_G(H)$, I set*

$$I_{H,\mathfrak{p},b} = z_H^{-1}(I_{\mathfrak{p},b}) \quad .$$

It is a prime ideal of $B_R^c(G)$.

Corollary 3.1.6: *If R is torsion free, and if I is a prime ideal of $B_R^c(G)$, then there exists a subgroup H of G, a prime ideal $\mathfrak{p}$ of R, and a block b of $k(\mathfrak{p})C_G(H)$, such that $I = I_{H,\mathfrak{p},b}$.*

3.2 p-blocks

Notation 3.2.1: *From now on, the letter $\mathcal{O}$ will denote a complete discrete valuation ring of characteristic 0, with maximal ideal π, with residue field k of characteristic $p > 0$, and field of fractions K, which will be supposed big enough (i.e. K contains the $|G|^{th}$ roots of unity). I denote by $x \mapsto \bar{x}$ the reduction morphism from $\mathcal{O}$ to k, or from $\mathcal{O}G$ to kG.*

If H is a group of order dividing $|G|$, the group algebra KG is split and semi-simple. If χ is an ordinary irreducible character of H, I denote by e_χ the corresponding block of KH, and by ω_χ the morphism from $\mathcal{Z}\mathcal{O}H$ to $\mathcal{O}$ mapping u to $\chi(u)/\chi(1)$.

I will say that χ is in the block e (or belongs to the block e) of kH if e acts as identity morphism on a reduction of a simple module affording the character χ, or equivalently, if there is a block E of $\mathcal{Z}\mathcal{O}H$ such that $e_\chi.E = e_\chi$ and $\overline{E} = e$.

If P is a p-subgroup of G, I denote by br_P the Brauer morphism from $\mathcal{Z}kG$ to $\mathcal{Z}kC_G(P)$, which is the k-linear map sending $x \in G$ to itself if $x \in C_G(P)$, and to 0 otherwise.

A Brauer pair (P, e) is a pair consisting of a p-subgroup P of G, and a block e of the algebra $kC_G(P)$. If b is a block of kG, the Brauer pair (P, e) is a b-Brauer pair if $e.br_P(b) = e$.

For the remainder of this section, the ring R will be equal to $\mathcal{O}$. The primitive idempotents of the Burnside algebra $B_{\mathcal{O}}(G)$ follow from a theorem of Dress ([4] or [3] Corollary 3.3.6). They are indexed by the (conjugacy classes of) p-perfect subgroups of G. The idempotent f_H^G indexed by the p-perfect subgroup H of G is equal to the sum of the idempotents e_K^G (see equation 2.3.5) for which $O^p(K)$ is conjugate to H, each taken once (i.e. K is taken once up to conjugation in G). In particular

$$f_1^G = \sum_{P \in \underline{s}_p(G)/G} e_P^G \tag{3.2.2}$$

where $\underline{s}_p(G)$ is the set of p-subgroups of G. The corresponding block $B_{\mathcal{O}}(G)f_1^G$ of the Burnside algebra is the $\mathcal{O}$-submodule generated by the G-sets G/P, for $P \in \underline{s}_p(G)$.

Now the algebra homomorphism $\beta : B_{\mathcal{O}}(G) \to B_{\mathcal{O}}^c(G)$ of lemma 2.1.1 provides a decomposition of unity in $B_{\mathcal{O}}^c(G)$ as a sum of orthogonal idempotents $\beta(f_H^G)$, for p-perfect subgroups H of G, up to conjugation. These idempotents are no longer primitive, and in this section I will show how the idempotent $\beta(f_1^G)$ splits as a sum of primitive idempotents of $B_{\mathcal{O}}^c(G)$.

Notation 3.2.3: *I will denote by $\mathcal{A}(G)$ the $\mathcal{O}$-algebra $B_{\mathcal{O}}^c(G)\beta(f_1^G)$, and by $\overline{\mathcal{A}}(G)$ the k-algebra $k \otimes_{\mathcal{O}} \mathcal{A}(G)$. The algebra $\mathcal{A}(G)$ will be called the p-local crossed Burnside algebra (over $\mathcal{O}$).*

Proposition 3.2.4: 1. *The map*

$$\Psi_{\mathcal{O}} = \prod_{P \in \underline{s}_p(G)} z_P : \mathcal{A}(G) \to \Big(\prod_{P \in \underline{s}_p(G)} \mathcal{Z}\mathcal{O}C_G(P) \Big)^G$$

is an injective map of $\mathcal{O}$-algebras, which preserves identity elements, and induces an isomorphism of K-algebras

$$\Psi_K : K \otimes_{\mathcal{O}} \mathcal{A}(G) \cong \Big(\prod_{P \in \underline{s}_p(G)} \mathcal{Z}KC_G(P) \Big)^G \quad .$$

2. *The algebra $\mathcal{A}(G)$ is the $\mathcal{O}$-submodule of $B_{\mathcal{O}}^c(G)$ with basis the set of elements $[P,x]_G$ of $[\mathcal{P}_G]$ for which P is a p-group.*

Proof: If X is a G-set and if H is a subgroup of G

$$(z_H \circ \beta)(X) = z_H([X, u_X]) = \sum_{x \in X^H} u_X(x) = |X^H|.1 \in \mathcal{Z}RC_G(H)$$

where u_X is defined in lemma 2.1.1. It follows from equations 2.3.6 and 3.2.2 that $(z_H \circ \beta)(f_1^G) = 0$ if H is not a p-group, and that $(z_H \circ \beta)(f_1^G) = 1 \in \mathcal{Z}RC_G(H)$ otherwise. Now the map $\Psi_\mathcal{O}$ is just the restriction to $\mathcal{A}(G)$ of the map $\Theta_\mathcal{O}$ of lemma 2.3.2, hence it is injective. This proves the first part of assertion 1).

The previous argument actually shows that if P is a p-group and if x is an element of $C_G(P)$, then

$$z_H\Big([P,x]_G\beta(f_1^G)\Big) = \begin{cases} 0 & \text{if } H \text{ is not a } p\text{-group} \\ z_H([P,x]_G) & \text{otherwise} \end{cases}$$

In both cases $z_H\Big([P,x]_G\beta(f_1^G)\Big) = z_H([P,x]_G)$, thus $[P,x]_G\beta(f_1^G) = [P,x]_G$ by lemma 2.3.2. In other words $[P,x]_G \in \mathcal{A}(G)$ if P is a p-group. Conversely $\beta(f_1^G)$ is a linear combination with coefficients in $\mathcal{O}$ of pairs $[P,x]_G$ of $\mathcal{P}_G$ for which P is a p-group, and these linear combinations clearly form an ideal of $B_\mathcal{O}^c(G)$. Assertion 2) follows. The second part of assertion 1) also follows, since Ψ_K is an injective map of K-vector spaces of the same (finite) dimension. □

Remark 3.2.5: *Assertion 2) means that $\mathcal{A}(G)$ is generated over $\mathcal{O}$ by the images of the crossed G-sets (X,α) for which the stabilizer in G of any element of X is a p-group.*

Notation 3.2.6: *If P is a p-subgroup of G denote by $\bar{z}_P$ the map $\overline{\mathcal{A}}(G) \to \mathcal{Z}kC_G(G)$ such that the square*

$$\begin{array}{ccc} \mathcal{A}(G) & \xrightarrow{z_P} & \mathcal{Z}\mathcal{O}C_G(P) \\ \big\downarrow & & \big\downarrow \\ \overline{\mathcal{A}}(G) & \xrightarrow[\bar{z}_P]{} & \mathcal{Z}kC_G(P) \end{array}$$

is commutative, where the vertical arrows are the reduction maps.

Lemma 3.2.7: *For any p-subgroup P of G*

$$\bar{z}_P = br_P \circ \bar{z}_1$$

Proof: Indeed, if v is the image of the crossed G-set (X,α) in $\overline{\mathcal{A}}(G)$, then

$$\bar{z}_1(v) = \sum_{x\in X} \alpha(x) = \sum_{g\in G} |\alpha^{-1}(g)| g \in \mathcal{Z}kG$$

whereas

$$\bar{z}_P(v) = \sum_{x \in X^P} \alpha(x) = \sum_{g \in C_G(P)} |\alpha^{-1}(g)^P| g \in \mathcal{Z}kC_G(P)$$

and the lemma follows since $|\alpha^{-1}(g)| = |\alpha^{-1}(g)^P|$ in k for any p-subgroup P of $C_G(g)$: this is because k has characteritic p, and because $|\alpha^{-1}(g)|$ and $|\alpha^{-1}(g)^P|$ are congruent modulo p, since P is a p-group. □

The following theorem describes the prime spectrum of $\mathcal{A}(G)$, which is a ring of dimension 1 by lemma 3.1.1. If I is a prime ideal of $\mathcal{A}(G)$, and $q \in \{0, p\}$, I will say that I has *co-characteristic* q if the ring $\mathcal{A}(G)/I$ has characteristic q :

Theorem 3.2.8: *Denote by b a block of kG, by P (resp P') a p-subgroup of G, by χ (resp. χ') an irreducible character of $C_G(P)$ (resp. $C_G(P')$), and by e a block of $kC_G(P)$. Let I be a prime ideal of $\mathcal{A}(G)$.*

1. *The following are equivalent :*
   - (a) *The ideal I has co-characteristic 0.*
   - (b) *The ideal I is a minimal prime ideal.*
   - (c) *There exist a p-subgroup P of G and an irreducible character χ of $C_G(P)$ such that $I = I_{P,0,e_\chi}$.*
2. *The following are equivalent :*
   - (a) *The ideal I has co-characteristic p.*
   - (b) *The ideal I is a maximal prime ideal.*
   - (c) *There exist a p-subgroup P of G and a block e of $kC_G(P)$ such that $I = I_{P,\pi,e}$.*
3. *The ideal $I_{P,0,e_\chi}$ is contained in $I_{P',0,e_{\chi'}}$ if and only if the pairs (P, e_χ) and $(P', e_{\chi'})$ are conjugate in G. In this case moreover $I_{P,0,e_\chi} = I_{P',0,e_{\chi'}}$.*
4. *The ideal $I_{P,0,e_\chi}$ is contained in $I_{P,\pi,e}$ if and only if the character χ belongs to the block e. In this case, the inclusion is strict.*
5. *The ideal $I_{1,\pi,b}$ is contained in $I_{P,\pi,e}$ if and only if (P, e) is a b-Brauer pair, i.e. if $e.br_P(b) = e$. In this case moreover $I_{1,\pi,b} = I_{P,\pi,e}$.*

6. *The connected components of Spec$\Big(\mathcal{A}(G)\Big)$ are in one to one correspondence with the blocks of kG. The component $\mathcal{C}_b$ associated to the block b consists of the unique maximal prime ideal $I_{1,\pi,b}$, and of the ideals $I_{P,0,e_\chi}$, where P is a p-subgroup of G, and χ is an irreducible character of $C_G(P)$ belonging to a block e of $C_G(P)$ such that (P,e) is a b-Brauer pair.*

Proof: By corollary 3.1.6, any prime ideal I of $\mathcal{A}(G)$, which is also a prime ideal of $B_{\mathcal{O}}^c(G)$, is equal to some ideal $I_{H,\mathfrak{p},b}$, for a subgroup H of G, a prime ideal $\mathfrak{p}$ of $\mathcal{O}$ (hence $\mathfrak{p}=0$ or $\mathfrak{p}=\pi$), and a block b of $k(\mathfrak{p})C_G(H)$ (i.e. a block e_χ of $KC_G(H)$ corresponding to an irreducible character χ if $\mathfrak{p}=0$, or a block b of $kC_G(H)$ if $\mathfrak{p}=\pi$). The ideal $I_{H,\mathfrak{p},b}$ is an ideal of $\mathcal{A}(G)$ if and only if $\beta(f_1^G)\notin I_{H,\mathfrak{p},b}$, or equivalently if H is a p-group.

Now the ideal $I_{H,\mathfrak{p},b}$ has co-characteristic 0 if $\mathfrak{p}=0$, and p if $\mathfrak{p}=\pi$. In particular, the ideals $I_{H,0,e_\chi}$ and $I_{H,\pi,b}$ are distinct. By the last two assertions of lemma 3.1.4, any ideal $I_{H,\pi,b}$ contains an ideal $I_{H,0,e_\chi}$, and any ideal $I_{H,0,e_\chi}$ is contained in an ideal $I_{H,\pi,b}$. It follows that the minimal primes of $\mathcal{A}(G)$ are the ideals $I_{P,0,e_\chi}$, for p-subgroups P of G, whereas the maximal primes are the ideals $I_{P,\pi,b}$. This proves assertions 1) and 2) of the theorem.

Now

$$I_{P,0,e_\chi}=\{u\in\mathcal{A}(G)\mid \omega_\chi\circ z_P(u)=0\}$$

It follows that $I_{P,0,e_\chi}=I_{P',0,e_{\chi'}}$ if the pairs (P,e_χ) and $(P',e_{\chi'})$ are conjugate. Conversely, if $I_{P,0,e_\chi}\subseteq I_{P',0,e_{\chi'}}$, then $I_{P,0,e_\chi}=I_{P',0,e_{\chi'}}$ since both are minimal prime ideals. Let f denote the idempotent of $\Big(\prod_{Q\in\underline{s}_p(G)}\mathcal{Z}KC_G(Q)\Big)^G$ with Q-component equal to the sum of idempotents e_θ for which (Q,e_θ) is G-conjugate to (P,e_χ). Let $F=\Psi_K^{-1}(f)$ be the corresponding idempotent of $B_K^c(G)$. Then there is a non-zero integer m such that mF lies in $B_{\mathcal{O}}(G)$. Moreover $z_Q(mF)=0$ if Q is not conjugate to P in G, and $z_P(mF)$ is m times the sum of the different conjugates of e_χ under $N_G(P)$. In particular mF is not in $I_{P,0,e_\chi}$, hence not in $I_{P',0,e_{\chi'}}$. It follows that $\omega_{\chi'}\circ z_{P'}(mF)\neq 0$. In particular P' is conjugate to P, and I can suppose $P=P'$. Now there is a conjugate of e_χ under $N_G(P)$ which is not in the kernel of $\omega_{\chi'}$. Hence e_χ and $e_{\chi'}$ are conjugate under $N_G(P)$. This proves assertion 3).

The ideal $I_{P,0,e_\chi}$ is contained in the ideal $I_{P,\pi,e}$ if and only if the ideal I_{0,e_χ} of $\mathcal{Z}\mathcal{O}C_G(P)$ is contained in the ideal $I_{\pi,e}$. Now

$$I_{0,e_\chi}=\{u\in\mathcal{Z}\mathcal{O}C_G(P)\mid \omega_\chi(u)=0\}\quad .$$

On the other hand, if θ is any character in the block e, then

$$I_{\pi,e}=\{u\in\mathcal{Z}\mathcal{O}C_G(P)\mid \overline{\omega_\theta(u)}=0\}\quad .$$

It follows that $I_{0,e_\chi} \subseteq I_{\pi,e}$ if χ belongs to e. Conversely, if $I_{0,e_\chi} \subseteq I_{\pi,e}$, then let E be the block of $\mathcal{O}C_G(P)$ containing χ. Then $\omega_\chi(1-E) = 0$, hence $\overline{\omega_\theta(1-E)} = 0$ for any character θ in e. Thus $\omega_\theta(E) = 1$, and $e = \overline{E}$. Hence χ is in e, proving assertion 4).

Let $\hat{b}$ be a block of $\mathcal{O}G$ lifting b (i.e. such that $\bar{\hat{b}} = b$). Write

$$\hat{b} = \sum_{x\in G} r_x x$$

for coefficients $r_x \in \mathcal{O}$. Consider

$$\tilde{b} = \sum_{x\in[G]} r_x [C_G(x), x]_G \quad .$$

If H is any subgroup of G, then

$$z_H(\tilde{b}) = \sum_{x\in[G]} r_x \sum_{w\in(G/C_G(x))^H} {}^w x = \sum_{x\in C_G(H)} r_x x \quad .$$

The second equality comes from the fact that $w \in (G/C_G(x))^H$ if and only if ${}^w x \in C_G(H)$. Moreover $r_x = r_{{}^w x}$ since $\hat{b}$ is central in $\mathcal{O}G$.

In particular for a p-subgroup P of G

$$\overline{z_P(\tilde{b})} = br_P(b)$$

Suppose that $I_{1,\pi,p} \subseteq I_{P,\pi,e}$. For any character χ in b

$$\overline{\omega_\chi(z_1(1-\tilde{b}))} = \overline{\omega_\chi(1-\hat{b})} = 1 - \overline{\omega_\chi(b)} = 0$$

Thus $1-\tilde{b} \in I_{1,\pi,p}$. Hence $1-\tilde{b} \in I_{P,\pi,e}$, and $e.\overline{z_P(1-\tilde{b})}$ must be nilpotent. But

$$e.\overline{z_P(1-\tilde{b})} = e - e.br_P(b)$$

This is nilpotent and idempotent, hence zero.

Conversely, suppose that $e = e.br_P(b)$. Then

$$I_{1,\pi,b} = \{u \in B^c_{\mathcal{O}}(G) \mid \overline{z_1(u)}b \text{ is nilpotent}\}$$

whereas

$$I_{P,\pi,e} = \{u \in B^c_{\mathcal{O}}(G) \mid \overline{z_P(u)}e \text{ is nilpotent}\} \quad .$$

Now if $u \in I_{1,\pi,b}$

$$\overline{z_1(u)}b = \overline{z_1(u)}br_1(b) = \overline{z_1(u)}\,\overline{z_1(\tilde{b})} = \overline{z_1(u\tilde{b})} \quad .$$

$$\overline{z_P(u)}e = \overline{z_P(u)}br_P(b)e = \overline{z_P(u)}\,\overline{z_P(\tilde{b})}e = \overline{z_P(u\tilde{b})}e \quad .$$

Now lemma 3.2.7 shows that $I_{1,\pi,b} \subseteq I_{P,\pi,e}$. Hence $I_{1,\pi,b} = I_{P,\pi,e}$ since $I_{1,\pi,b}$ is a maximal ideal in $\mathcal{A}(G)$. This completes the proof of assertion 5).

Assertion 5) also shows that all the maximal ideals of $\mathcal{A}(G)$ are of the form $I_{1,\pi,b}$, for a suitable bloc b of kG. Now the minimal primes are of the form $I_{P,0,\chi}$, for a p-subgroup P of G, for an irreducible character χ of $C_G(P)$. This ideal is contained in $I_{P,\pi,e}$ for a block e of $kC_G(P)$ if and only if e contains χ. Moreover $I_{P,\pi,e}$ is equal to $I_{1,\pi,b}$ for a block b of kG if and only if (P,e) is a b-Brauer pair. This shows that each minimal prime ideal is contained in a unique maximal ideal. Assertion 6) follows, and the proof of theorem 3.2.8 is complete. □

Notation 3.2.9: *If G is a finite group, if P is a p-subgroup of G, and b is an idempotent in $\mathcal{Z}kG$, I denote by $br_P^{\mathcal{O}}(b)$ the unique idempotent in $\mathcal{ZO}C_G(P)$ lifting the idempotent $br_P(b)$ of $\mathcal{Z}kC_G(P)$. If $g \in C_G(P)$, I denote by $br_P^{\mathcal{O}}(b)(g)$ the element of $\mathcal{O}$ such that*

$$br_P^{\mathcal{O}}(b) = \sum_{g \in C_G(P)} br_P^{\mathcal{O}}(b)(g)\; g$$

Theorem 3.2.10: *If b is an idempotent of $\mathcal{Z}kG$, let*

$$b^{\mathcal{A}} = \frac{1}{|G|} \sum_{\substack{g \in G \\ Q \subseteq P \in \underline{s}_p(C_G(g))}} |Q|\mu(Q,P) br_P^{\mathcal{O}}(b)(g)\; [Q,g]_G$$

Then $b^{\mathcal{A}} \in \mathcal{A}(G)$, and as b runs through the blocks of kG, the elements $b^{\mathcal{A}}$ run through a complete set of primitive idempotents in $\mathcal{A}(G)$.

Proof: The primitive idempotents of the (commutative) ring $\mathcal{A}(G)$ are in one to one correspondence with the connected components of its spectrum, hence with the blocks of kG by theorem 3.2.8 : if b is a block of kG, then the idempotent $b^{\mathcal{A}}$ corresponding to the component $\mathcal{C}_b$ is characterised by the fact that $b^{\mathcal{A}} \notin I$ for some $I \in \mathcal{C}_b$ (or equivalently for all $I \in \mathcal{C}_b$). It follows that $b^{\mathcal{A}} \notin I_{1,\pi,b}$, but $b^{\mathcal{A}} \in I_{1,\pi,b'}$ for any block b' of kG different from b.

Thus $\omega_b\Big(\bar{z}_1(b^{\mathcal{A}})\Big) \neq 0$, but $\omega_{b'}\Big(\bar{z}_1(b^{\mathcal{A}})\Big) = 0$ for any block $b' \neq b$ of kG. Since $\bar{z}_1(b^{\mathcal{A}})$ is an idempotent of $\mathcal{Z}kG$, it follows that $\bar{z}_1(b^{\mathcal{A}}) = b$. Hence $\bar{z}_P(b^{\mathcal{A}}) = br_P(b)$ for any p-subgroup P of G, by lemma 3.2.7. Now $z_P(b^{\mathcal{A}})$ is an idempotent of $\mathcal{ZO}C_G(P)$ lifting $br_P(b)$, hence it is equal to $br_P^{\mathcal{O}}(b)$. Theorem 3.2.10 now follows from proposition 3.2.4 and from the inversion formula of proposition 2.3.4. □

Corollary 3.2.11: *Let b be an idempotent of $\mathcal{Z}kG$. If Q is a p-subgroup of G, and $g \in C_G(Q)$, then*

$$\sum_{Q \subseteq P \in \underline{s}_p(G)} \mu(Q,P) br_P^{\mathcal{O}}(g) \in |N_G(Q) \cap C_G(g) : Q| \mathcal{O}$$

Proof: This follows from the following rewriting of the formula in theorem 3.2.10

$$b^{\mathcal{A}} = \sum_{\substack{(Q,g) \in [\mathcal{P}_G] \\ Q \subseteq P \in \underline{s}_p(C_G(g))}} \frac{\mu(Q,P) br_P^{\mathcal{O}}(b)(g)}{|N_G(Q) \cap C_G(g) : Q|} \, [Q,g]_G$$

and from the fact that the elements $[Q,g]_G$, for $(Q,g) \in [\mathcal{P}_G]$ and $Q \in \underline{s}_p(G)$, form a basis of $\mathcal{A}(G)$ over $\mathcal{O}$, by proposition 3.2.4. □

4 The blocks of the Mackey algebra

4.1 Mackey functors

Let R be a commutative ring, and G be a finite group. There are several definitions of the notion of Mackey functor for G over R (see [1] Chapter 1 for a summary of these definitions). One of the most conceptual is due to Dress:

Definition 4.1.1: *Let G-**set** be the category of finite sets with a left G-action. A Mackey functor for the group G, with values in R-**Mod**, is a bivariant functor from G-**set** to R-**Mod**, i.e. a pair of functors (M^*, M_*), with M^* contravariant and M_* covariant, which coincide on objects (i.e. $M^*(X) = M_*(X) = M(X)$ for any G-set X). This bivariant functor is supposed to have the following two properties:*

- *If X and Y are G-sets, let i_X and i_Y be the respective injections from X and Y into $X \coprod Y$, then the maps $M^*(i_X) \oplus M^*(i_Y)$ and $M_*(i_X) \oplus M_*(i_Y)$ are mutual inverse R-module isomorphisms between $M(X \coprod Y)$ and $M(X) \oplus M(Y)$.*

- *If*

$$\begin{array}{ccc} T & \xrightarrow{\gamma} & Y \\ \delta \big\downarrow & & \big\downarrow \alpha \\ Z & \xrightarrow[\beta]{} & X \end{array}$$

is a cartesian (or pull-back) square of G-sets, then $M^*(\beta).M_*(\alpha) = M_*(\delta).M^*(\gamma)$*.*

A morphism θ *from the Mackey functor* M *to the Mackey functor* N *is a natural transformation of bivariant functors, consisting of a morphism* $\theta_X : M(X) \to N(X)$ *for any G-set* X*, such that for any morphism of G-sets* $f : X \to Y$*, the squares*

$$\begin{array}{ccc} M(X) & \xrightarrow{\theta_X} & N(X) \\ {\scriptstyle M_*(f)}\downarrow & & \downarrow{\scriptstyle N_*(f)} \\ M(Y) & \xrightarrow[\theta_Y]{} & N(Y) \end{array} \qquad \begin{array}{ccc} M(X) & \xrightarrow{\theta_X} & N(X) \\ {\scriptstyle M^*(f)}\uparrow & & \uparrow{\scriptstyle N^*(f)} \\ M(Y) & \xrightarrow[\theta_Y]{} & N(Y) \end{array}$$

are commutative.

I will denote by $Mack_R(G)$ the category of Mackey functors for G over R.

4.2 The Mackey algebra

The Mackey algebra $\mu_R(G)$ of R over G was defined by Thévenaz and Webb ([8]) : it is the R-algebra generated by the elements t_K^H, r_K^H, and $c_{x,H}$, where H and K are subgroups of G such that $K \subseteq H$, and $x \in G$, with the following relations:

$$t_K^H t_L^K = t_L^H \ \ \forall \ \ L \subseteq K \subseteq H$$
$$r_L^K r_K^H = r_L^H \ \ \forall \ \ L \subseteq K \subseteq H$$
$$c_{y,{}^xH} c_{x,H} = c_{yx,H} \ \ \forall \ \ x, y, H$$
$$t_H^H = r_H^H = c_{h,H} \ \ \forall \ \ h, H \text{ such that } h \in H$$
$$c_{x,H} t_K^H = t_{{}^xK}^{{}^xH} c_{x,K} \ \ \forall \ \ x, K, H \text{ such that } K \subseteq H$$
$$c_{x,K} r_K^H = r_{{}^xK}^{{}^xH} c_{x,H} \ \ \forall \ \ x, K, H \text{ such that } K \subseteq H$$
$$\sum_H t_H^H = \sum_H r_H^H = 1$$
$$r_K^H t_L^H = \sum_{x \in K\backslash H/L} t_{K \cap {}^xL}^K c_{x,K^x \cap L} r_{K^x \cap L}^L \ \ \forall \ \ K \subseteq H \supseteq L$$

any other product of r_H^K, t_H^K and $c_{g,H}$ being zero.

It can be shown from Proposition 3.4 of [8] that the map $x \in G \mapsto \sum_{H \subseteq G} c_{x,H}$ extends to an injective R-algebra homomorphism from RG to $\mu_R(G)$, and it is handy to identify G with its image in $\mu_R(G)$ via this map.

Thévenaz and Webb show that the category of $\mu_R(G)$-modules is equivalent to the category of Mackey functors for G over R. This equivalence is build as follows : if $K \subseteq H$ are subgroups of G, denote by $p_K^H : G/K \to G/H$ the map of G-sets sending xK to xH, for $x \in G$. If $g \in G$, denote by $\gamma_g : G/H \to G/{}^gH$ the map sending xH to $xg.{}^gH$, for $x \in G$. Then if M is a Mackey functor, the R-module $\oplus_{L\subseteq G} M(L)$ can be endowed with a $\mu_R(G)$-module structure : the generator t_K^H maps $M(L)$ to 0 if $L \neq K$, and $M(K)$ to $M(H)$ via the map $M_*(p_K^H)$. Similarly, the generator r_K^H maps $M(L)$ to 0 if $L \neq H$, and $M(H)$ to $M(K)$ via the map $M^*(p_K^H)$. Finally the generator $c_{g,H}$ maps $M(L)$ to 0 if $L \neq H$, and $M(H)$ to $M({}^gH)$ via the map $M_*(\gamma_g)$.

4.3 Action of crossed G-sets

Any crossed G-set gives an endofunctor of the category of Mackey functors for G over R, and this will lead to a ring homomorphism from $B_R^c(G)$ to the center $\mathcal{Z}\mu_R(G)$ of the Mackey algebra. This action of crossed G-sets on Mackey functors was already observed by Yoshida.

Let (X, α) be a crossed G-set. If M is any Mackey functor for G over R, and if Y is a finite G-set, let $\zeta(X,\alpha)_{M,Y}$ denote the endomorphism of $M(Y)$ defined by

$$\zeta(X,\alpha)_{M,Y} = M_* \begin{pmatrix} {\scriptstyle x,y} \\ \downarrow \\ {\scriptstyle \alpha(x)y} \end{pmatrix} M^* \begin{pmatrix} {\scriptstyle x,y} \\ \downarrow \\ {\scriptstyle y} \end{pmatrix}$$

where $\begin{pmatrix} {\scriptstyle x,y} \\ \downarrow \\ {\scriptstyle \alpha(x)y} \end{pmatrix}$ denotes the map from $X \times Y$ to Y sending (x, y) to $\alpha(x)y$, and $\begin{pmatrix} {\scriptstyle x,y} \\ \downarrow \\ {\scriptstyle y} \end{pmatrix}$ is the projection map from $X \times Y$ to Y. This definition extends the one given by Thévenaz and Webb ([8] Section 9) for the action of $B(G)$ on Mackey functors.

Proposition 4.3.1: *Let (X, α) be a crossed G-set. The the maps $\zeta(X,\alpha)_{M,Y}$ define a natural transformation $\zeta(X,\alpha)$ of the identity functor of the category $Mack_R(G)$. Moreover, if (Y, β) is another crossed G-set, then*

$$\zeta(X,\alpha) + \zeta(Y,\beta) = \zeta(X \sqcup Y, \alpha \sqcup \beta)$$

$$\zeta(X,\alpha) \circ \zeta(Y,\beta) = \zeta(X \times Y, \alpha.\beta)$$

Proof: This amounts to a series of verifications : first the maps $\zeta(X,\alpha)_{M,Y}$ define an endomorphism $\zeta(X,\alpha)_M$ of the Mackey functor M. It means that for any morphism of G-sets $f : Y \to Z$, one has that

$$\zeta(X,\alpha)_{M,Z} \circ M_*(f) = M_*(f) \circ \zeta(X,\alpha)_{M,Y} \qquad (\mathbf{4.3.2})$$

$$\zeta(X,\alpha)_{M,Y} \circ M^*(f) = M^*(f) \circ \zeta(X,\alpha)_{M,Z} \tag{4.3.3}$$

Equation 4.3.2 reads

$$M_*\left(\begin{smallmatrix} x,z \\ \downarrow \\ \alpha(x)z \end{smallmatrix}\right) M^*\left(\begin{smallmatrix} x,z \\ \downarrow \\ z \end{smallmatrix}\right) M_*(f) = M_*(f) M_*\left(\begin{smallmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{smallmatrix}\right) M^*\left(\begin{smallmatrix} x,y \\ \downarrow \\ y \end{smallmatrix}\right) \tag{4.3.4}$$

The square

$$\begin{array}{ccc} X \times Y & \xrightarrow{\left(\begin{smallmatrix} x,y \\ \downarrow \\ y \end{smallmatrix}\right)} & Y \\ {\scriptstyle \left(\begin{smallmatrix} x,y \\ \downarrow \\ x,f(y) \end{smallmatrix}\right)} \big\downarrow & & \big\downarrow {\scriptstyle f} \\ X \times Z & \xrightarrow[\left(\begin{smallmatrix} x,z \\ \downarrow \\ z \end{smallmatrix}\right)]{} & Z \end{array}$$

is a cartesian square of G-sets. Hence

$$M^*\left(\begin{smallmatrix} x,z \\ \downarrow \\ z \end{smallmatrix}\right) M_*(f) = M_*\left(\begin{smallmatrix} x,y \\ \downarrow \\ x,f(y) \end{smallmatrix}\right) M^*\left(\begin{smallmatrix} x,y \\ \downarrow \\ y \end{smallmatrix}\right)$$

so equation 4.3.4 is equivalent to

$$M_*\left(\begin{smallmatrix} x,y \\ \downarrow \\ \alpha(x)f(y) \end{smallmatrix}\right) M^*\left(\begin{smallmatrix} x,y \\ \downarrow \\ y \end{smallmatrix}\right) = M_*\left(\begin{smallmatrix} x,y \\ \downarrow \\ f(\alpha(x)y) \end{smallmatrix}\right) M^*\left(\begin{smallmatrix} x,y \\ \downarrow \\ y \end{smallmatrix}\right)$$

which holds since f is a morphism of G-sets.

Similarly equation 4.3.3 reads

$$M_*\left(\begin{smallmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{smallmatrix}\right) M^*\left(\begin{smallmatrix} x,y \\ \downarrow \\ y \end{smallmatrix}\right) M^*(f) = M^*(f) M_*\left(\begin{smallmatrix} x,z \\ \downarrow \\ \alpha(x)z \end{smallmatrix}\right) M^*\left(\begin{smallmatrix} x,z \\ \downarrow \\ z \end{smallmatrix}\right) \tag{4.3.5}$$

Similarly the square

$$\begin{array}{ccc} X \times Y & \xrightarrow{\left(\begin{smallmatrix} x,y \\ \downarrow \\ x,f(y) \end{smallmatrix}\right)} & X \times Z \\ {\scriptstyle \left(\begin{smallmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{smallmatrix}\right)} \big\downarrow & & \big\downarrow {\scriptstyle \left(\begin{smallmatrix} x,z \\ \downarrow \\ \alpha(x)z \end{smallmatrix}\right)} \\ Y & \xrightarrow[f]{} & Z \end{array}$$

is cartesian, and both sides of equation 4.3.5 are equal to

$$M_*\left(\begin{smallmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{smallmatrix}\right) M^*\left(\begin{smallmatrix} x,y \\ \downarrow \\ f(y) \end{smallmatrix}\right) \quad .$$

Hence $\zeta(X,\alpha)_M$ is an endomorphism of the Mackey functor M, for any M. If $\phi : M \to N$ is a morphism of Mackey functors, given by maps $\phi_Y : M(Y) \to N(Y)$ for any G-set Y, then

$$\begin{aligned}
\zeta(X,\alpha)_{N,Y} \circ \phi_Y &= N_*\left(\begin{smallmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{smallmatrix}\right) \circ N^*\left(\begin{smallmatrix} x,y \\ \downarrow \\ y \end{smallmatrix}\right) \circ \phi_Y \\
&= N_*\left(\begin{smallmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{smallmatrix}\right) \circ \phi_{X\times Y} \circ M^*\left(\begin{smallmatrix} x,y \\ \downarrow \\ y \end{smallmatrix}\right) \\
&\ \text{(since } \phi \text{ is a morphism of Mackey functors)} \\
&= \phi_Y \circ M_*\left(\begin{smallmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{smallmatrix}\right) \circ M^*\left(\begin{smallmatrix} x,y \\ \downarrow \\ y \end{smallmatrix}\right) \\
&= \phi_Y \circ \zeta(X,\alpha)_{M,Y} \quad .
\end{aligned}$$

It follows that the maps $\zeta(X,\alpha)_M$ define an endomorphism of the identity functor of $Mack_R(G)$.

Finally, if (Y,β) is another crossed G-set, then for any Mackey functor M and any G-set Z, the assertion

$$\zeta(X,\alpha)_{M,Z} + \zeta(Y,\beta)_{M,Z} = \zeta(X \sqcup Y, \alpha \sqcup \beta)_{M,Z}$$

follows easily from the first condition in definition 4.1.1. Concerning composition, one has that

$$\zeta(X,\alpha)_{M,Z} \circ \zeta(Y,\beta)_{M,Z} = M_*\left(\begin{smallmatrix} x,z \\ \downarrow \\ \alpha(x)z \end{smallmatrix}\right) M^*\left(\begin{smallmatrix} x,z \\ \downarrow \\ z \end{smallmatrix}\right) M_*\left(\begin{smallmatrix} y,z \\ \downarrow \\ \beta(y)z \end{smallmatrix}\right) M^*\left(\begin{smallmatrix} y,z \\ \downarrow \\ z \end{smallmatrix}\right) \quad .$$

Now the square

$$\begin{array}{ccc}
X \times Y \times Z & \xrightarrow{\left(\begin{smallmatrix} x,y,z \\ \downarrow \\ y,z \end{smallmatrix}\right)} & Y \times Z \\
{\scriptstyle \left(\begin{smallmatrix} x,y,z \\ \downarrow \\ x,\beta(y)z \end{smallmatrix}\right)} \downarrow & & \downarrow {\scriptstyle \left(\begin{smallmatrix} y,z \\ \downarrow \\ \beta(y)z \end{smallmatrix}\right)} \\
X \times Z & \xrightarrow[\left(\begin{smallmatrix} x,z \\ \downarrow \\ z \end{smallmatrix}\right)]{} & Z
\end{array}$$

is cartesian. It follows that

$$\zeta(X,\alpha)_{M,Z} \circ \zeta(Y,\beta)_{M,Z} = M_*\left(\begin{smallmatrix} x,y,z \\ \downarrow \\ \alpha(x)\beta(y)z \end{smallmatrix}\right) M^*\left(\begin{smallmatrix} x,y,z \\ \downarrow \\ z \end{smallmatrix}\right) = \zeta(X \times Y, \alpha.\beta)_{M,Z}$$

as was to be shown. □

4.4 The center of the Mackey algebra

In any category of modules over a ring A, the natural transformations from the identity functor to itself identify with the center of A : any element z of the center of A defines an endomorphism of any A-module M (by multiplication by z on M), and this gives a natural transformation of the identity functor of the category of A-modules.

Notation 4.4.1: *The ring homomorphism from $B_R^c(G)$ to $\mathcal{Z}\mu_R(G)$ following from proposition 4.3.1 will be still denoted by ζ.*

It is given as follows :

Proposition 4.4.2: *Let $(L,x)\in\mathcal{P}_G$. Then*

$$\zeta([L,x]_G)=\sum_{H\subseteq G}\sum_{w\in L\backslash G/H} t^H_{L^w\cap H}\; w^{-1}xw\; r^H_{L^w\cap H}\quad .$$

Proof: Let H be a subgroup of G, and M be a Mackey functor for G over R. The element $[L,x]_G$ corresponds to the crossed G-set $(G/L,m_x)$. The action of $\zeta(G/L,m_x)$ on $M(G/H)$ is defined as

$$M_*\begin{pmatrix} uL,vH\\ \downarrow\\ {}^ux.vH\end{pmatrix} M^*\begin{pmatrix} uL,vH\\ \downarrow\\ vH\end{pmatrix}: M(G/H)\to M(G/H)\quad .$$

This maps factors through $M(G/L\times G/H)$, which is isomorphic to

$$\bigoplus_{w\in L\backslash G/H} M(G/L^w\cap H)$$

since the map

$$g(L^w\cap H)\in\bigsqcup_{w\in L\backslash G/H} G/L^w\cap H\mapsto (gw^{-1}L,gH)\in G/L\times G/H$$

is an isomorphism of G-sets.

Up to this isomorphism, the map

$$M^*\begin{pmatrix} uL,vH\\ \downarrow\\ vH\end{pmatrix}: M(G/H)\to\bigoplus_{w\in L\backslash G/H} M(G/L^w\cap H)$$

is just the map $\bigoplus_{w\in L\backslash G/H} r^H_{L^w\cap H}$, and on the component $M(G/L^w\cap H)$, the map $M_*\begin{pmatrix} uL,vH\\ \downarrow\\ {}^ux.vH\end{pmatrix}$ is equal to $M_*(\phi_w)$, where $\phi_w\Big(g.(L^w\cap H)\Big)=g.w^{-1}xwH$. Hence $M_*(\phi_w)=t^H_{L^w\cap H}\,w^{-1}xw$, and the action of $\zeta([L,x]_G)$ on $M(G/H)$ is equal to

$$\sum_{w\in L\backslash G/H} t^H_{L^w\cap H}\; w^{-1}xw\; r^H_{L^w\cap H}\quad .$$

The proposition follows, taking for M the free Mackey functor $\mu_R(G)$. □

Remark 4.4.3: *Proposition 4.4.2 shows that the composition $\zeta\circ\beta$ is equal to the morphism from the Burnside ring $B_R(G)$ to the center of the Mackey algebra considered in Section 9 of [8].*

4.5 The p-blocks of $\mu_{\mathcal{O}}(G)$

Notation 4.5.1: *Let b be an idempotent of $\mathcal{Z}kG$. I denote by b^μ the central idempotent of $\mu_{\mathcal{O}}(G)$ defined by*

$$b^\mu = \zeta(b^{\mathcal{A}}) \quad .$$

I denote by $\mu_{\mathcal{O}}^1(G)$ the algebra

$$(\zeta \circ \beta)(f_1^G)\mu_{\mathcal{O}}(G) \quad .$$

This algebra will be called the p-local Mackey algebra.

This notation is consistent with the notation of [2]. It is motivated by Sections 9 and 10 of [8], in which Thévenaz and Webb show that the category $Mack_{\mathcal{O}}(G)$ splits as a direct sum of subcategories $Mack_{\mathcal{O}}(G,J)$, indexed by p-perfect subgroups J of G (up to G-conjugation) : the category $Mack_{\mathcal{O}}(G,J)$ consists of $\mu_{\mathcal{O}}(G)$-modules M such that $(\zeta\circ\beta)(f_J^G)M = M$. In particular, the category $Mack_{\mathcal{O}}(G,1)$ is the category of $\mu_{\mathcal{O}}^1(G)$-modules. Moreover, the category $Mack_{\mathcal{O}}(G,J)$ is equivalent to $Mack_{\mathcal{O}}\left(N_G(J)/J,1\right)$.

It follows that studying the p-blocks of $\mu_{\mathcal{O}}(G)$ is equivalent to studying the p-blocks of the algebras $\mu_{\mathcal{O}}^1\left(N_G(H)/H\right)$, for p-perfect subgroups H of G.

Theorem 4.5.2: *Let b be an idempotent of $\mathcal{Z}kG$. Then*

$$b^\mu = \sum_{H\subseteq G}\frac{1}{|H|}\sum_{\substack{Q\subseteq P\in\underline{s}_p(H)\\ x\in C_G(P)}} |Q|\mu(Q,P)br_P^{\mathcal{O}}(b)(x)\; t_Q^H x r_Q^H \quad .$$

The idempotents b^μ, as b runs through the blocks of kG, are a complete set of primitive idempotents of the center of the p-local Mackey algebra $\mu_{\mathcal{O}}^1(G)$.

Proof: By proposition 4.4.2 and theorem 3.2.10

$$b^\mu = \frac{1}{|G|}\sum_{\substack{g\in G\\ Q\subseteq P\in\underline{s}_p(C_G(g))}} |Q|\mu(Q,P)br_P^{\mathcal{O}}(b)(g)\sum_{H\subseteq G}\sum_{w\in Q\backslash G/H} t_{Q^w\cap H}^H g^w r_{Q^w\cap H}^H \quad .$$

Changing the order of summations gives

$$\begin{aligned} b^\mu &= \frac{1}{|G|}\sum_{H\subseteq G}\sum_{\substack{g\in G\\ Q\subseteq P\in\underline{s}_p(C_G(g))\\ w\in Q\backslash G/H}} |Q|\mu(Q,P)br_P^{\mathcal{O}}(b)(g)t_{Q^w\cap H}^H g^w r_{Q^w\cap H}^H \\ &= \frac{1}{|G|}\sum_{H\subseteq G}\sum_{\substack{g,w\in G\\ Q\subseteq P\in\underline{s}_p(C_G(g))}} \frac{|Q^w\cap H|}{|Q||H|}|Q|\mu(Q,P)br_P^{\mathcal{O}}(b)(g)t_{Q^w\cap H}^H g^w r_{Q^w\cap H}^H \quad . \end{aligned}$$

Summing on Q^w, P^w and g^w instead of Q, P, w leads to

$$\begin{aligned} b^\mu &= \frac{1}{|G|}\sum_{H\subseteq G}\frac{1}{|H|}\sum_{\substack{g,w\in G\\ Q\subseteq P\in \underline{s}_p(C_G(g))}} |Q\cap H|\mu(Q,P)br_P^{\mathcal{O}}(b)(g)t_{Q\cap H}^H g r_{Q\cap H}^H \\ &= \sum_{H\subseteq G}\frac{1}{|H|}\sum_{\substack{g\in G\\ Q\subseteq P\in \underline{s}_p(C_G(g))}} |Q\cap H|\mu(Q,P)br_P^{\mathcal{O}}(b)(g)t_{Q\cap H}^H g r_{Q\cap H}^H \quad . \end{aligned}$$

Fix a subgroup H of G, an element g of G, and a p-subgroup P of $C_G(g)$. By classical combinatorial formulae, if R is any subgroup of $H\cap P$

$$\sum_{\substack{Q\subseteq P\\ Q\cap H=R}} \mu(Q,P) = \begin{cases} 0 & \text{if } H\cap P\neq P \\ \mu(R,P) & \text{otherwise} \end{cases} \quad .$$

It follows that

$$b^\mu = \sum_{H\subseteq G}\frac{1}{|H|}\sum_{\substack{R\subseteq P\in \underline{s}_p(H)\\ g\in C_G(P)}} |R|\mu(R,P)br_P^{\mathcal{O}}(b)(g)t_R^H g r_R^H$$

which states the formula in theorem 4.5.2.

To complete the proof of this theorem, it remains to show that if b is a block of kG, i.e. a primitive idempotent of $\mathcal{Z}kG$, then b^μ is a primitive idempotent of $\mathcal{Z}\mu_{\mathcal{O}}(G)$. Suppose that b^μ splits as the sum of two orthogonal idempotents c and d of $\mathcal{Z}\mu_{\mathcal{O}}(G)$. Then in particular

$$t_1^1 b^\mu = t_1^1 b^\mu t_1^1 = t_1^1 c t_1^1 + t_1^1 d t_1^1$$

is a decomposition of $t_1^1 b^\mu t_1^1$ as a sum of orthogonal central idempotents in the algebra $t_1^1\mu_{\mathcal{O}}(G)t_1^1$. This algebra is isomorphic to $\mathcal{O}G$ (via $x\in G\mapsto t_1^1 x t_1^1$), and moreover

$$t_1^1 b^\mu t_1^1 = t_1^1 b t_1^1 \quad .$$

Indeed if $R\subseteq H$ are subgroups of G, and if $g\in G$, then $t_1^1.t_R^H g r_R^H.t_1^1 = 0$ in $\mu_{\mathcal{O}}(G)$, unless H is the trivial subgroup of G.

If b is primitive, it follows that one of $t_1^1 c t_1^1$ or $t_1^1 d t_1^1$ is zero, say $t_1^1 c t_1^1$. Now $\mu_{\mathcal{O}}(G)c$ is a projective Mackey functor in $Mack_{\mathcal{O}}(G,1)$ (i.e. a projective $\mu_{\mathcal{O}}^1(G)$-module), whose evaluation at the trivial subgroup is equal to

$$t_1^1\mu_{\mathcal{O}}(G)c = t_1^1 c t_1^1 \mu_{\mathcal{O}}(G) = 0 \quad .$$

It follows that $\mu_{\mathcal{O}}(G)c = 0$, by Corollary 12.2 of [8]. Hence $c = 0$, and the proof is complete. □

5 Consequences

5.1 Another formula

The following gives another expression for b^μ, which may be easier to remember :

Proposition 5.1.1: *If H is a subgroup of G, denote by $X \mapsto [X]$ the $\mathcal{O}$-linear morphism from $B_{\mathcal{O}}(G)$ to $\mu_{\mathcal{O}}(G)$ mapping H/L to $t_L^H r_L^H$. If b is an idempotent of $\mathcal{Z}kG$, then*

$$b^\mu = \sum_{H\subseteq G} \frac{1}{|H|} \sum_{P\in \underline{s}_p(G)} |P| t_P^H \left([e_P^P] br_P^{\mathcal{O}}(b)\right) r_P^H \quad .$$

Proof: Indeed in $B_K(P)$

$$e_P^P = \frac{1}{|P|} \sum_{Q\subseteq P} |Q| \mu(Q,P)\; P/Q \quad .$$

Thus

$$[e_P^P] = \frac{1}{|P|} \sum_{Q\subseteq P} |Q| \mu(Q,P)\; t_Q^P r_Q^P \quad .$$

Now by theorem 4.5.2

$$\begin{aligned}
b^\mu &= \sum_{H\subseteq G} \frac{1}{|H|} \sum_{\substack{Q\subseteq P\in \underline{s}_p(H)\\ x\in C_G(P)}} |Q|\mu(Q,P) br_P^{\mathcal{O}}(b)(x)\; t_Q^H x r_Q^H \\
&= \sum_{H\subseteq G} \frac{1}{|H|} \sum_{P\in \underline{s}_p(H)} t_P^H \left(\sum_{\substack{Q\subseteq P\\ x\in C_G(P)}} |Q|\mu(Q,P) t_Q^P br_P^{\mathcal{O}}(b)(x) x r_Q^P \right) r_P^H \\
&= \sum_{H\subseteq G} \frac{1}{|H|} \sum_{P\in \underline{s}_p(H)} t_P^H \left(\sum_{\substack{Q\subseteq P\\ x\in C_G(P)}} |Q|\mu(Q,P) t_Q^P r_Q^P br_P^{\mathcal{O}}(b)(x) x \right) r_P^H \\
&\quad \text{(since } C_G(P)\subseteq C_G(Q) \text{ for } Q\subseteq P) \\
&= \sum_{H\subseteq G} \frac{1}{|H|} \sum_{P\in \underline{s}_p(H)} t_P^H \left((\sum_{Q\subseteq P} |Q|\mu(Q,P) t_Q^P r_Q^P)(\sum_{x\in C_G(P)} br_P^{\mathcal{O}}(b)(x) x) \right) r_P^H
\end{aligned}$$

as was to be shown. □

5.2 Residues

Notation 5.2.1: *Let M be a Mackey functor for G over R, and H be a subgroup of G. Then Brauer residue of M at H is defined as*

$$\overline{M}(H) = M(H) / \sum_{L \subset H} t_L^H M(L)$$

and the projection $M(H) \to \overline{M}(H)$ is denoted by br_H. By duality, the Brauer co-residue of M at H is defined as

$$\underline{M}(H) = \bigcap_{L \subset H} \mathrm{Ker}\ r_L^H \subset M(H)$$

Proposition 5.2.2: *Let M be a Mackey functor for G over $\mathcal{O}$, in $Mack_{\mathcal{O}}(G,1)$, and b be a block of kG.*

1. *If H is a subgroup of G, then $\overline{M}(H) = 0$, unless H is a p-group. Similarly $\underline{M}(H) = 0$, unless H is a p-group.*
2. *If L is a p-subgroup of G, and if $m \in M(L)$, then*
$$br_L(b^\mu.m) = br_L^{\mathcal{O}}(b) br_L(m)$$
Similarly, if $m \in \underline{M}(L)$, then
$$b^\mu.m = br_L^{\mathcal{O}}(b) m$$
3. *The following are equivalent :*
   - *(a) The block b^μ acts as the identity of M.*
   - *(b) For any p-subgroup P of G, the idempotent $br_P^{\mathcal{O}}(b)$ acts as the identity of $\overline{M}(P)$.*
   - *(c) For any p-subgroup P of G, the idempotent $br_P^{\mathcal{O}}(b)$ acts as the identity of $\underline{M}(P)$.*

Proof: The first assertion follows from the fact that $(\zeta \circ \beta)(f_1^G)$ acts as the identity of M if M is in $Mack_{\mathcal{O}}(G,1)$. Moreover, the action of this idempotent on $M(H)$ is a linear combination with coefficients in $\mathcal{O}$ of elements $t_P^H r_P^H$, for p-subgroups P of H.

For assertion 2, observe that b^μ acts on $M(L)$ via

$$b_L^\mu = \frac{1}{|L|} \sum_{\substack{Q \subseteq P \subseteq L \\ x \in C_G(P)}} |Q| \mu(Q,P) br_P^{\mathcal{O}}(b)(x) t_Q^L x r_Q^L \quad .$$

Thus if $m \in M(L)$

$$b_L^\mu m = \frac{1}{|L|} \sum_{\substack{Q \subseteq P \subseteq L \\ x \in C_G(P)}} |Q|\mu(Q,P) br_P^{\mathcal{O}}(b)(x) t_Q^L x r_Q^L m \quad .$$

In this summation, all the terms are traces from proper subgroups of L, except those for which $Q = P = L$. This gives :

$$br_L(b_L^\mu m) = br_L(\sum_{x \in C_G(L)} br_L^{\mathcal{O}}(b)(x)x.m) = br_L(br_L^{\mathcal{O}}(b)m) = br_L^{\mathcal{O}}(b) br_L(m) \quad .$$

Similarly, if $m \in \underline{M}(L)$, then all proper restrictions of m are zero, hence

$$b_L^\mu.m = \sum_{x \in C_G(L)} br_L^{\mathcal{O}}(b)(x)x.m = br_L^{\mathcal{O}}(b)m$$

as was to be shown.

For assertion 3), suppose first that b^μ acts as the identity of M. Then by assertion 2), the idempotent $br_L^{\mathcal{O}}(b)$ acts as the identity of $\overline{M}(L)$ and $\underline{M}(L)$, for any p-subgroup L of G. Thus assertion (a) implies (b) and (c).

Conversely, suppose that (b) holds. Then in particular b^μ acts as the identity of $\overline{M}(1) = M(1)$. By induction on the order of the subgroup H of G, this shows that b^μ acts as the identity of $M(H)$: let $m \in M(H)$. If H is not a p-group, then by assertion 1) $br_H(m) = 0$, and m can be written

$$m = \sum_{L \subset H} t_L^H m_L$$

for proper subgroups L of H and elements $m_L \in M(L)$. Now by induction

$$b^\mu.m = b_H^\mu \sum_{L \subset H} t_L^H m_L = \sum_{L \subset H} t_L^H (b_L^\mu m_L) = \sum_{L \subset H} t_L^H m_L = m \quad .$$

And if H is a p-subgroup of G, then by assertion (b)

$$m - b^\mu.m \in \text{Ker } br_H \quad .$$

Hence

$$m - b^\mu.m = \sum_{L \subset H} t_L^H m_L$$

for proper subgroups L of H and elements $m_L \in M(L)$. Now

$$b^\mu.m - (b^\mu)^2.m = m - b^\mu.m$$

by the same argument as above. Hence $b^\mu.m = m$, and (b) implies (a).

Similarly if (c) holds, then in particular b^μ acts as the identity of $\underline{M}(1) = M(1)$. By induction on the order of the subgroup H of G, this shows that b^μ acts as the identity of $M(H)$: let $m \in M(H)$. If H is not a p-group, then m is a linear combination of traces from proper subgroups of H, hence $b^\mu.m = m$ by induction. And if H is a p-subgroup of G, then by induction for any proper subgroup L of H

$$r_L^H(b^\mu.m) = r_L^H(b_H^\mu.m) = b_L^\mu.r_L^H m = r_L^H m$$

This shows that $m - b^\mu.m \in \underline{M}(L)$. Now (c) implies that

$$m - b^\mu.m = b^\mu(m - b^\mu.m) = b^\mu.m - b^\mu.m = 0$$

thus $m = b^\mu.m$, as was to be shown. □

Corollary 5.2.3: *Let M be a Mackey functor in $Mack_{\mathcal{O}}(G,1)$. If M is indecomposable, and if there is a p-subgroup P such that $\overline{M}(P) \neq 0$ and $C_G(P)$ acts trivially on $\overline{M}(P)$ (resp. $\underline{M}(P) \neq 0$ and $C_G(P)$ acts trivially on $\underline{M}(P)$), then M is in the principal block of $Mack_{\mathcal{O}}(G,1)$.*

Proof: Since M is indecomposable, there is a block b of kG such that b^μ acts as the identity of M. Hence $br_P^{\mathcal{O}}(b)$ acts as the identity of $\overline{M}(P)$, for any p-subgroup P of G. Moreover, since $C_G(P)$ acts trivially on $\overline{M}(P)$, the action of $br_P^{\mathcal{O}}(b)$ on $\overline{M}(P)$ is multiplication by the sum $\sum_{x \in C_G(P)} br_P^{\mathcal{O}}(b)(x)$, which is zero if b is not the principal block of kG : indeed if $e = \sum_{x \in C_G(P)} e_x x$ is any block of $\mathcal{O}C_G(P)$, then the sum $\sum_{x \in C_G(P)} e_x$ is equal to 1 or 0, according to the fact that e is the principal block of $\mathcal{O}C_G(P)$ or not. And by Brauer's Third Main Theorem, the principal block of $\mathcal{O}C_G(P)$ appears in the decomposition of $br_P^{\mathcal{O}}(b)$ if and only if b is the principal block of $\mathcal{O}G$. The argument is the same with $\underline{M}(P)$. □

Remark 5.2.4: *This corollary shows in particular that if M is an indecomposable Mackey functor in $Mack_{\mathcal{O}}(G,1)$, and if $C_G(P)$ acts trivially on $M(P)$ for any p-subgroup P of G, then M is in the principal block of $Mack_{\mathcal{O}}(G,1)$.*

5.3 The defect of a block of $\mu_{\mathcal{O}}(G)$

In [1] Chapter 12, it is shown that there exists a natural Green functor for G whose evaluation at G is isomorphic to the center of the Mackey algebra. This functor is denoted by ζ_B, and its value at the G-set X is the set

$$\zeta_B(X) = \mathrm{Hom}_{Funct}(\mathcal{I}, \mathcal{I}_X)$$

of natural transformations from the identity functor $\mathcal{I}$ on the category of Mackey functors to the endofunctor $\mathcal{I}_X$ on this category, mapping the Mackey functor M to its Dress construction M_X, defined on the G-set Y by

$$M_X(Y) = M(Y \times X) \quad .$$

This functor ζ_B has also been considered by Oda (the functor $\mathcal{T}$ of [7]) from a different point of view. The equivalence of those two points of view follows from Proposition 12.2.8 of [1] : the value of ζ_B at a subgroup H of G is the set of sequences (z_L), indexed by the subgroups L of H, such that

$$z_L \in t_L^L \mu_{\mathcal{O}}(G) t_L^L \qquad t_Q^L z_L = z_Q t_Q^L \qquad z_Q r_Q^L = r_Q^L z_L \qquad x z_L = z_{^xL} x$$

for any subgroups $Q \subseteq L$ of H and any $x \in H$. The isomorphism $\zeta_B(G) \cong \mathcal{Z}\mu_{\mathcal{O}}(G)$ is obtained by mapping the sequence $(z_L)_{L \subseteq G}$ to the element $\sum_{L \subseteq G} z_L$.

Also recall from Proposition 12.2.8 of [1] that if Q is a subgroup of G, and $(z_L)_{L \subseteq Q}$ is an element of $\zeta_B(Q)$, then for any subgroup H of G, the component of $t_Q^G(z)$ at the subgroup H is equal to

$$t_Q^G(z)_H = \sum_{w \in Q \backslash G / H} t_{Q^w \cap H}^H w^{-1} z_{Q \cap {^wH}} w r_{Q^w \cap H}^H \quad .$$

If $g \in C_G(Q)$, one can define an element $z(Q,g)$ of $\zeta_B(Q)$ by setting, for any subgroup L of Q

$$z(Q,g)_L = t_L^L g r_L^L \quad .$$

Let b be a block of kG, with defect group D. With the previous notation, the expression of b^μ given in proposition 4.5.2 can be written as

$$b^\mu = \frac{1}{|G|} \sum_{\substack{Q \subseteq P \in \underline{s}_p(G) \\ g \in C_G(P)}} |Q| \mu(Q,P) br_P^{\mathcal{O}}(b)(g) t_Q^G z(Q,g) \quad .$$

The summation can be restricted to p-subgroups P of G for which $br_P^{\mathcal{O}}(b) \neq 0$. Such subgroups are all contained in D up to G-conjugation. It follows that

$$b^\mu \in t_D^G \zeta_B(D) \quad .$$

Definition 5.3.1: *A defect group of the block $b^\mu \in \mathcal{Z}\mu_{\mathcal{O}}(G) = \zeta_B(G)$ is a subgroup E of G, minimal subject to the condition*

$$b^\mu \in t_E^G \zeta_B(E) \quad .$$

The following argument is then classical : if D is any subgroup of G such that $b^\mu \in t_D^G \zeta_B(D)$, then

$$b^\mu = (b^\mu)^2 \in (t_E^G \zeta_B(E))(t_D^G \zeta_B(D)) \subseteq \sum_{x \in E\backslash G/D} t_{E\cap {}^xD}^G \zeta_B(E \cap {}^xD)$$

and by Rosenberg's lemma and primitivity of b^μ, there exists $x \in G$ such that $E \subseteq {}^xD$. In particular E is unique up to G-conjugation.

Proposition 5.3.2: *Let b be a block of kG, with defect group D. Then D is also a defect group of b^μ.*

Proof: Let E be a defect of b^μ. The above argument shows that E is contained in D up to G-conjugation. The reverse inclusion has been proved by Oda ([7] Theorem 6). It also follows from the following fact : if $b^\mu = t_E^G(c)$, for $c \in \zeta_B(E)$, then

$$(b^\mu)_1 = t_1^1 Tr_E^G(c_1) r_1^1 \in t_1^1 Tr_E^G(\mathcal{O}G)^E r_1^1$$

where Tr_E^G is the relative trace map. Since moreover $(b^\mu)_1 = t_1^1 br_1^{\mathcal{O}}(b) r_1^1$, it follows that the block $b^{\mathcal{O}} = br_1^{\mathcal{O}}(b)$ lifting b to $\mathcal{O}G$ belongs to $Tr_E^G(\mathcal{O}G)^E$. Hence D is contained in E up to G-conjugation. □

References

[1] S. Bouc. *Green-functors and G-sets*, volume 1671 of *Lecture Notes in Mathematics*. Springer, October 1997.

[2] S. Bouc. Résolutions de foncteurs de Mackey. In *Group representations: Cohomology, Group Actions and Topology*, volume 63, pages 31–83. AMS Proceedings of Symposia in Pure Mathematics, 1998.

[3] S. Bouc. Burnside rings. In *Handbook of Algebra*, volume 2, chapter 6D, pages 739–804. Elsevier, 2000.

[4] A. Dress. A characterization of solvable groups. *Math. Zeit.*, 110:213–217, 1969.

[5] D. Gluck. Idempotent formula for the Burnside ring with applications to the p-subgroup simplicial complex. *Illinois J. Math.*, 25:63–67, 1981.

[6] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1986.

[7] F. Oda. On defect groups of the Mackey algebras for finite groups. In *Group representations: Cohomology, Group Actions and Topology*, volume 63. AMS Proceedings of Symposia in Pure Mathematics, 1998.

[8] J. Thévenaz and P. Webb. The structure of Mackey functors. *Trans. Amer. Math. Soc.*, 347(6):1865–1961, June 1995.

[9] T. Yoshida. Talk at the AMS Conference on Group Representations and Topology. Seattle, 1996.