# The $p$-blocks of the Mackey algebra

Serge Bouc

*Institut de Mathématiques de Jussieu*
*Université Paris 7-Denis Diderot, 75251, Paris Cedex 05, France*
*email: bouc@math.jussieu.fr*

## 1 Introduction

Let $p$ be a prime number, and $\mathcal{O}$ be a complete discrete valuation ring of characteristic 0 whith residue field of characteristic $p$. Let $G$ be a finite group, and denote by $\mu_{\mathcal{O}}(G)$ the Mackey algebra of $G$ over $\mathcal{O}$ (see [8] for definition).

Formulae for the primitive idempotents in the center of $\mu_{\mathcal{O}}(G)$ have been given by Yoshida (and slightly corrected by Oda [7]). However, those formulae are expressed in terms of ordinary irreducible characters of the centralizers of subgroups of $G$. The aim of this article is to state explicit formulae for the block idempotents of $\mu_{\mathcal{O}}(G)$, in terms of the blocks of the group algebra $\mathcal{O}G$.

The proof uses the natural ring homomorphism from the crossed Burnside ring $B^c_{\mathcal{O}}(G)$ to the center of the Mackey algebra, and a description of the prime spectrum and block idempotents of $B^c_{\mathcal{O}}(G)$.

The paper is organized as follows : section 2 is an exposition of definitions and basic results on the crossed Burnside ring. Section 3 describes the prime spectrum and $p$-blocks of this ring. Section 4 recalls the basic definitions on Mackey functors, and uses the action of the crossed Burnside ring to state explicit formulae for the block idempotents of the $p$-local Mackey algebra. Section 5 exposes some consequences on Mackey functors that follow from these formulae. In particular, one can show that a block $b$ of $G$ and the corresponding block $b^\mu$ of the Mackey algebra have the same defect groups.

## 2 The crossed Burnside ring

Most of the definitions and results of this section have already been discussed by Yoshida ([9]). However, they are still unpublished, or not currently available in published form. This is the reason for exposing this material here.

## 2.1 Definition

Let $G$ be a finite group, and denote by $G^c$ the set $G$, on which $G$ acts by conjugation. The category $G$-$\mathbf{set}\!\downarrow_{G^c}$ of *crossed $G$-sets* is the category of $G$-sets over $G^c$ : a *crossed $G$-set* $(X,\alpha)$ is a a pair consisting of a finite $G$-set $X$ (i.e. a finite set with a left $G$-action), together with a map of $G$-sets $\alpha$ from $X$ to $G^c$, and a morphism of crossed $G$-sets from $(X,\alpha)$ to $(Y,\beta)$ is a morphism of $G$-sets $\varphi$ from $X$ to $Y$ such that $\beta\circ\varphi=\alpha$.

There is an obvious notion of disjoint union of crossed $G$-sets, and the crossed Burnside group $B^c(G)$ is defined as the Grothendieck group of the category of crossed $G$-sets, for relation given by disjoint union decomposition : let $\mathcal{I}$ be the set of isomorphism classes of crossed $G$-sets, and denote by $[X,\alpha]$ the isomorphism class of the crossed $G$-set $(X,\alpha)$. Then

$$B^c(G)=\mathbb{Z}^{\mathcal{I}}/<[X\sqcup Y,\alpha\sqcup\beta]-[X,\alpha]-[Y,\beta]> \quad .$$

Let $B$ denote the (ordinary) Burnside Mackey functor for $G$ (see [3] or [1] Section 2.4 for definition). It follows from [1] Proposition 2.4.2 that $B^c(G)$ is isomorphic (as a $\mathbb{Z}$-module) to the evaluation $B(G^c)$ of the Mackey functor $B$ at the $G$-set $G^c$.

If $(X,\alpha)$ and $(Y,\beta)$ are crossed $G$-set, then their product is the crossed $G$-set $(X\times Y,\alpha.\beta)$, where $X\times Y$ is the direct product of $X$ and $Y$, with diagonal $G$-action, and $\alpha.\beta$ is the map from $X\times Y$ to $G^c$ defined by

$$(\alpha.\beta)(x,y)=\alpha(x)\beta(y) \quad .$$

This product on crossed $G$-sets clearly commutes with disjoint unions, hence it gives a product on $B^c(G)$. This turns $B^c(G)$ into a ring. The identity element of this ring is $[\bullet,u_\bullet]$, where $\bullet$ is a $G$-set of cardinality one, and the map $u_\bullet$ sends the unique element of $\bullet$ to the identity element of the group $G$.

The ring $B^c(G)$ is commutative : the map

$$(x,y)\in X\times Y\mapsto(\alpha(x)y,x)\in Y\times X$$

is an isomorphism from $(X\times Y,\alpha.\beta)$ to $(Y\times X,\beta.\alpha)$ in $G$-$\mathbf{set}\!\downarrow_{G^c}$, because for all $(x,y)\in X\times Y$

$$\beta\Big(\alpha(x)y\Big)\alpha(x)=\alpha(x)\beta(y)\alpha(x)^{-1}\alpha(x)=\alpha(x)\beta(y) \quad .$$

More generally, if $R$ is a commutative ring, denote by $B^c_R(G)$ the tensor product of $B^c(G)$ with $R$ over $\mathbb{Z}$

$$B^c_R(G)=R\otimes_{\mathbb{Z}}B^c(G) \quad .$$

It is an $R$-algebra. Similarly, denote by $B_R(G)$ the (ordinary) Burnside algebra of $G$ over $R$.

**Lemma 2.1.1:** *If $X$ is a finite $G$-set, denote by $u_X$ the map from $X$ to $G$ sending every element to $1 \in G$. Then the correspondence $X \mapsto (X, u_X)$ extends linearly to a ring homomorphism from $B_R(G)$ to $B_R^c(G)$, which preserves identity elements.*

**Proof:** This amounts to checking that if $X$ and $Y$ are finite $G$-sets, then the product $(X \times Y, u_X.u_Y)$ is isomorphic to $(X \times Y, u_{X\times Y})$, which is straightforward. Moreover the trivial $G$-set is mapped to $(\bullet, u_\bullet)$. $\square$

## 2.2 Characterization of crossed $G$-sets

A crossed $G$-set $(X, \alpha)$ is called *transitive* if the $G$-set $X$ is. In this case, let $x$ be any element of $X$, denote by $H$ the stabilizer of $x$ in $G$, and set $a = \alpha(x)$. Then $a$ lies in the centralizer $C_G(H)$ of $H$ in $G$. Define the map $m_a$ from $G/H$ to $G^c$ by $m_a(gH) = {}^g a$, where ${}^g a = gag^{-1}$. Then $(G/H, m_a)$ is a crossed $G$-set, and the map $gH \mapsto gx$ from $G/H$ to $X$ is clearly an isomorphism of crossed $G$-sets from $(G/H, m_a)$ to $(X, \alpha)$.

Conversely, if $H$ is any subgroup of $G$, and if $a \in C_G(H)$, then $(G/H, m_a)$ is a transitive crossed $G$-set. If $K$ is a subgroup of $G$, and $b \in C_G(K)$, then the crossed $G$-sets $(G/H, m_a)$ and $(G/K, m_b)$ are isomorphic if and only if there exists an element $g \in G$ such that ${}^g H = K$ and ${}^g a = b$.

**Notation 2.2.1:** *I will denote by $s_G$ the set of subgroups of $G$. If $X$ is a $G$-set and $H \in s_G$, I denote by $X^H$ the set of fixed points of $H$ on $X$. Let $\mathcal{P}_G$ denote the set of pairs $(H, a)$ consisting of a subgroup $H$ of $G$ and and element $a$ of $C_G(H)$. The group $G$ acts by conjugation on $s_G$ and $\mathcal{P}_G$, and I denote by $[s_G]$ (resp. $[\mathcal{P}_G]$) a set of representative of $G$-orbits on $s_G$ (resp. on $\mathcal{P}_G$). If $(H, a) \in \mathcal{P}_G$, I denote by $[H, a]_G$ the isomorphism class of the crossed $G$-set $(G/H, m_a)$.*

Now if $(X, \alpha)$ is any crossed $G$-set, choose a set $S$ of representatives of the orbits of $G$ on $X$. Then the map from $\bigsqcup_{s\in S} G/G_s$ to $X$ sending $gG_s$ to $gs$ is clearly an isomorphism in $G\text{-}\mathbf{set}{\downarrow_{G^c}}$ from $\bigsqcup_{s\in S}(G/G_s, m_{\alpha(s)})$ to $X$. Thus any crossed $G$-set is isomorphic to a disjoint union of transitive ones.

**Lemma 2.2.2:** *Let $(X, \alpha)$ and $(Y, \beta)$ be crossed $G$-sets. Then the following are equivalent:*

1. *The crossed $G$-sets $(X, \alpha)$ and $(Y, \beta)$ are isomorphic.*
2. *For any crossed $G$-set $(Z, \gamma)$*

$$|\mathrm{Hom}_{G\text{-}\mathbf{set}{\downarrow_{G^c}}}\Big((Z,\gamma),(X,\alpha)\Big)| = |\mathrm{Hom}_{G\text{-}\mathbf{set}{\downarrow_{G^c}}}\Big((Z,\gamma),(Y,\beta)\Big)| \quad .$$

*3. For any* $(H,a)\in[\mathcal{P}_G]$

$$|\alpha^{-1}(a)^H| = |\beta^{-1}(a)^H| \quad .$$

**Proof:** Clearly 1) implies 2). Moreover 2) implies 3) since for any crossed $G$-set $X$

$$|\mathrm{Hom}_{G\text{-}\mathbf{set}\downarrow_{G^c}}\Big((G/H,m_a),(X,\alpha)\Big)| = |\alpha^{-1}(a)^H| \quad .$$

To show that 3) implies 1), I can replace $(X,\alpha)$ and $(Y,\beta)$ by isomorphic crossed $G$-sets, i.e. suppose that

$$(X,\alpha) = \bigsqcup_{(K,b)\in[\mathcal{P}_G]} u_{K,b}(G/K,m_b) \qquad (Y,\beta) = \bigsqcup_{(K,b)\in[\mathcal{P}_G]} v_{K,b}(G/K,m_b)$$

where $u_{K,b}$ and $v_{K,b}$ are natural integers, and the notation $u_{K,b}(G/K,m_b)$ means a disjoint union of $u_{K,b}$ copies of $(G/K,m_b)$. Notice that if $p=(H,a)$ and $q=(K,b)$ are elements of $\mathcal{P}_G$, then

$$|\mathrm{Hom}_{G\text{-}\mathbf{set}\downarrow_{G^c}}\Big((G/H,m_a),(G/K,m_b)\Big)| = |\{g\in G/K \mid H^g\subseteq K,\ a^g=b\}| \quad .$$

Denote by $M(p,q)$ this number. Condition 3) implies that

$$\sum_{q\in[\mathcal{P}_G]} u_q M(p,q) = \sum_{q\in[\mathcal{P}_G]} v_q M(p,q)$$

for all $p\in[\mathcal{P}_G]$. It follows that the sequence $(u_q-v_q)_{q\in[\mathcal{P}_G]}$ is in the kernel of the square matrix $M(p,q)_{p,q\in[\mathcal{P}_G]}$. Now with suitable ordering of $[\mathcal{P}_G]$, this matrix is upper triangular. The diagonal coefficient $M(p,p)$ for $p=(H,a)$ is equal to

$$M(p,p) = |\{g\in N_G(H)/H \mid {}^g a = a\}| = |N_G(H)\cap C_G(a):H|$$

This is non-zero, and $M$ is non singular. Thus $u_{K,b}=v_{K,b}$ for any $(K,b)$ in $[\mathcal{P}_G]$. Hence $(X,\alpha)$ and $(Y,\beta)$ are isomorphic. □

**Corollary 2.2.3:** *The elements* $[H,a]_G$*, for* $(H,a)\in[\mathcal{P}_G]$*, form a basis of* $B^c(G)$ *over* $\mathbb{Z}$*.*

## 2.3 Brauer morphisms

Let $K$ be a subgroup of $G$, and let $C_G(K)$ be the centralizer of $K$ in $G$. Denote by $(X,\alpha)\mapsto(X^K,\alpha^K)$ the fixed points functor from $G$-$\mathbf{set}\downarrow_{G^c}$ to

$C_G(K)$-**set**$\downarrow_{C_G(K)^c}$, where $X^K$ is viewed as a $C_G(K)$-set, and $\alpha^K$ is the map $X^K \to (G^c)^K = C_G(K)^c$ induced by $\alpha$.

This functor induces a map $Br_K$, called the Brauer morphism, from $B_R^c(G)$ to $B_R^c\Big(C_G(K)\Big)$, defined by linearity from $Br_K([X,\alpha]) = [X^K, \alpha^K]$, which is clearly a ring homomorphism, which preserves identity elements.

If $(X,\alpha)$ is a crossed $G$-set, let $s_G(X,\alpha)$ denote the element of the center $\mathcal{Z}RG$ of the group algebra $RG$ of $G$ over $R$ defined by

$$s_G(X,\alpha) = \sum_{x\in X} \alpha(x)$$

This clearly induces a morphism of $R$-algebras, still denoted by $s_G$, from $B_R^c(G)$ to $\mathcal{Z}RG$, which preserves identity elements.

**Notation 2.3.1:** *If $H$ is a subgroup of $G$, then I denote by $z_H$ the ring homomorphism $s_{C_G(H)} \circ Br_H$ from $B_R^c(G)$ to $\mathcal{Z}RC_G(H)$.*

Thus if $(X,\alpha)$ is a crossed $G$-set, then

$$z_H([X,\alpha]) = \sum_{x\in X^H} \alpha(x) = \sum_{g\in C_G(H)} |\alpha^{-1}(g)^H| g \quad .$$

**Lemma 2.3.2:** *If the characteristic of $R$ is zero, then the ring homomorphism*

$$\Theta_R = \prod_{H\in[s_G]} z_H : B_R^c(G) \to \prod_{H\in[s_G]} \mathcal{Z}RC_G(H)$$

*is injective.*

**Proof:** Clearly $\Theta_R$ is a ring homomorphism. The injectivity assertion is just a reformulation of lemma 2.2.2 : if $u = \sum_{(H,a)\in[\mathcal{P}_G]} r_{H,a}[H,a]_G$ is a non-zero element the kernel of $\Theta_R$, let $K$ be a subgroup of $G$ maximal such that there exists $(K,b) \in [\mathcal{P}_G]$ with $r_{K,b} \neq 0$. Now

$$z_K(u) = \sum_{\substack{a\in C_G(K)\\ a\ mod. N_G(K)}} r_{K,a} \sum_{g\in N_G(K)/K} {}^g a = \sum_{a\in C_G(K)} r_{K,a} |N_G(K) \cap C_G(a) : K| a = 0$$

Since $R$ has characteristic zero, it follows that $r_{K,a} = 0$ for all $a \in C_G(K)$. This contradiction proves the lemma. □

The previous lemma can be considered from a slightly different point of view : let $G^c$ denote the set $G$, on which $G$ acts by conjugation. There is an isomorphism of $G$-sets

$$G^c \cong \bigsqcup_{g\in[G]} G/C_G(g)$$

where $[G]$ is a set of representatives of conjugacy classes of $G$. Since $B^c(G)$ is the value of the Burnside Mackey functor $B$ at $G^c$, it follows that there is an isomorphism of $\mathbb{Z}$-modules

$$B^c(G) \cong \bigoplus_{g\in[G]} B\Big(C_G(g)\Big)$$

sending the crossed $G$-set $(X,\alpha)$ to the sequence $\Big(\alpha^{-1}(g)\Big)_{g\in[G]}$. The inverse isomorphism sends the element $C_G(g)/L$ of $B\Big(C_G(g)\Big)$ to $[L,g]_G$.

Now for any finite group $H$, it follows from Burnside's theorem (see [3] Theorem 2.3.2) that there is an injective morphism

$$\phi_H : B(H) \to \prod_{K\in[s_H]} \mathbb{Z} \tag{2.3.3}$$

defined by linearity by mapping the $H$-set $X$ to the sequence $(|X^K|)_{K\in[s_H]}$. Hence there is an injective morphism of $\mathbb{Z}$-modules

$$B^c(G) \to \prod_{g\in[G]} \prod_{\in[s_{C_G(g)}]} \mathbb{Z} \cong \prod_{(H,g)\in[\mathcal{P}_G]} \mathbb{Z}$$

sending the crossed $G$-set $(X,\alpha)$ to the sequence $(|\alpha^{-1}(g)^H|)_{(H,g)\in[\mathcal{P}_G]}$. Now there is an isomorphism

$$\prod_{(H,g)\in[\mathcal{P}_G]} \mathbb{Z} \cong \prod_{H\in[s_G]} \mathcal{Z}\mathbb{Z}C_G(H) \quad .$$

This gives the injective map

$$\Theta_\mathbb{Z} : B^c(G) \to \prod_{H\in[s_G]} \mathcal{Z}\mathbb{Z}C_G(H)$$

and the map $\Theta_R$ of lemma 2.3.2 is obtained by tensoring this map with $R$, which is a flat $\mathbb{Z}$-module if $R$ has characteristic 0. As a consequence :

**Proposition 2.3.4:** *Let $K$ be a field of characteristic 0. Then the map*

$$\Psi_K : \prod_{H\subseteq G} z_H : B^c_K(G) \to \Big(\prod_{H\subseteq G} \mathcal{Z}KC_G(H)\Big)^G$$

*is an isomorphism of $K$-algebras. The inverse isomorphism maps the sequence $(z_H)_{H\subseteq G}$ to*

$$\frac{1}{|G|} \sum_{\substack{(L,g)\in\mathcal{P}_G \\ L\subseteq H\subseteq C_G(g)}} |L|\mu(L,H)z_H(g)[L,g]_G$$

*where $\mu(L,H)$ denotes the Möbius function of the poset of subgroups of $G$, and $z_H(g)$ denotes the coefficient of $g$ in $z_H$.*

**Proof:** The map $\Psi_K$ is injective by lemma 2.3.2, and its image is contained in

$$\Big(\prod_{H\subseteq G} \mathcal{Z}KC_G(H)\Big)^G \cong \prod_{H\in[s_G]} \Big(KC_G(H)\Big)^{N_G(H)} \quad .$$

Now this $K$-vector space has the same (finite) dimension as $B_K^c(G)$, namely the cardinality of $[\mathcal{P}_G]$. The first assertion follows.

To build the inverse map, note that for any finite group $G$, the Burnside algebra $B_K(G)$ is a split semi-simple commutative $K$-algebra. The primitive idempotents of $B_K(G)$ have been determined by Gluck ([5]). They are indexed by the (conjugacy classes of) subgroups of the group $G$. The idempotent $e_H^G$ indexed by $H$ is equal to (see [3] Theorem 3.3.2)

$$e_H^G = \frac{1}{|N_G(H)|}\sum_{L\subseteq H} |L|\mu(L,H)\; G/L \quad . \qquad \mathbf{(2.3.5)}$$

This idempotent is characterized by the fact that for any $X\in B_K(G)$, one has that

$$X.e_H^G = |X^H|e_H^G \quad . \qquad \mathbf{(2.3.6)}$$

It follows in particular that

$$X = \sum_{H\in[s_G]} X.e_H^G = \sum_{H\in[s_G]} |X^H|e_H^G = \frac{1}{|G|}\sum_{L\subseteq H\subseteq G} |L|\mu(L,H)|X^H|G/L \quad .$$

Now if $(X,\alpha)$ is a crossed $G$-set, the corresponding element $(z_H)_{H\subseteq G}$ is defined by

$$z_H(g) = |\alpha^{-1}(g)^H|$$

thus

$$\alpha^{-1}(g) = \frac{1}{|C_G(g)|}\sum_{L\subseteq H\subseteq C_G(g)} |L|\mu(L,H)z_H(g)\; C_G(g)/L$$

in $B\Big(C_G(g)\Big)$, and the second assertion follows. □

**Remark 2.3.7:** *Proposition 2.3.4 can be used to show that $B_K^c(G)$ is a split semi-simple (commutative) algebra, and to state explicit formulae for its primitive idempotents. Those formulae are due to Yoshida.*

It is possible to characterize the image of the above map $\Theta_{\mathbb{Z}}$ :

**Proposition 2.3.8:** *If $(H, g) \in \mathcal{P}_G$, and if $K$ is a subgroup of $C_G(g)$, set*

$$n_g(H, K) = |\{x \in N_G(H) \cap C_G(g)/H \mid <H, x> =_{C_G(g)} K\}|$$

*where $<H, x>$ is the subgroup of $G$ generated by $H$ and $x$, and the notation $<H, x> =_{C_G(g)} K$ means that $<H, x>$ and $K$ are conjugate by an element of $C_G(g)$.*

*For $H \in [s_G]$, let $z_H = \sum_{g \in C_G(H)} z_H(g)g$ be an element of $\mathcal{Z}\mathbb{Z}C_G(H)$. Then the sequence $(z_H)_{H \in [s_G]}$ belongs to the image of $\Theta_{\mathbb{Z}}$ if and only if for any $H$ in $[s_G]$ the following two conditions hold :*

1. *The element $z_H$ is invariant by $N_G(H)$.*
2. *For any $g \in C_G(H)$, the sum $\sum_{K \in [s_{C_G(g)}]} n_g(H, K) z_K(g)$ is a multiple of $|N_G(H) \cap C_G(g) : H|$.*

**Proof:** This follows from a theorem of Dress ([4], or [3] Theorem 3.2.1), characterizing the image of the map $\phi_H$ defined in (2.3.3): giving an element $z_H$ in $(\mathbb{Z}C_G(H))^{N_G(H)}$, for $H \in [s_G]$ is equivalent to giving integers $z_H(g)$, for $(H, g) \in [\mathcal{P}_G]$. Now for a fixed $g \in G$, the sequence of integers $(z_H(g))_{H \in [s_{C_G(g)}]}$ is in the image of $\phi_{C_G(g)}$ if and only if condition 2) holds. □

# 3 The prime spectrum of $B_R^c(G)$

## 3.1 Prime spectrum

**Lemma 3.1.1:** *The Krull dimension of the ring $B_R^c(G)$ is equal to the Krull dimension of $R$.*

**Proof:** Indeed, the ring $B_R^c(G)$ is an extension of $R$, and it is a finitely generated (free) $R$-module. Hence it is integral over $R$. Thus $\dim B_R^c(G) = \dim R$. □

**Lemma 3.1.2:** *If $R$ has characteristic zero, then the map $\Theta_R$ induces a surjection*

$$Spec(\Theta_R) : Spec\Big(\prod_{H \in [s_G]} \mathcal{Z}RC_G(H)\Big) \to Spec\Big(B_R^c(G)\Big) \quad .$$

**Proof:** Here again, the ring $C = \prod_{H\in[s_G]} \mathcal{Z}RC_G(H)$ is an extension of $B_R^c(G)$. Moreover, it is a finitely generated $R$-module, hence also a finitely generated $B_R^c(G)$-module. Thus $C$ is integral over $B_R^c(G)$, and $\Theta_R$ induces a surjective maps on the spectra. □

**Notation 3.1.3:** *If $\mathfrak{p}$ is a prime ideal of $R$, denote by $k(\mathfrak{p})$ the field of fractions of $R/\mathfrak{p}$. Denote by $\varphi_\mathfrak{p}$ the canonical morphism from $\mathcal{Z}RG$ to $\mathcal{Z}k(\mathfrak{p})G$. If $b$ is a block of $k(\mathfrak{p})G$, set*

$$I_{\mathfrak{p},b} = \{u \in \mathcal{Z}RG \mid \varphi_\mathfrak{p}(u)b \in J\Big(\mathcal{Z}k(\mathfrak{p})Gb\Big)\}$$

*(where $J\Big(\mathcal{Z}k(\mathfrak{p})Gb\Big)$ denotes the Jacobson radical of the algebra $\mathcal{Z}k(\mathfrak{p})Gb$). It is an ideal of $\mathcal{Z}RG$.*

**Lemma 3.1.4:** *Let $\mathfrak{p}$ (resp. $\mathfrak{p}'$) denote a prime ideal of $R$, and $b$ (resp. $b'$) denote a block of $k(\mathfrak{p})G$ (resp. $k(\mathfrak{p}')G$). Then :*

1. *The ideal $I_{\mathfrak{p},b}$ is a prime ideal of $\mathcal{Z}RG$. Conversely, if $I$ is a prime ideal of $\mathcal{Z}RG$, there exist a unique prime ideal $\mathfrak{p}$ of $R$ and a unique block $b$ of $k(\mathfrak{p})G$ such that $I = I_{\mathfrak{p},b}$.*
2. *If $I_{\mathfrak{p},b} \subseteq I_{\mathfrak{p}',b'}$, then $\mathfrak{p} \subseteq \mathfrak{p}'$. If moreover $\mathfrak{p} = \mathfrak{p}'$, then $b = b'$.*
3. *If $\mathfrak{p} \subseteq \mathfrak{p}'$, and if the block $b$ is given, then there exists a block $b'$ such that $I_{\mathfrak{p},b} \subseteq I_{\mathfrak{p}',b'}$.*
4. *If $\mathfrak{p} \subseteq \mathfrak{p}'$, and if the block $b'$ is given, then there exists a block $b$ such that $I_{\mathfrak{p},b} \subseteq I_{\mathfrak{p}',b'}$.*

**Proof:** Clearly $I_{\mathfrak{p},b}$ is the kernel of the canonical morphism

$$\mathcal{Z}RG \stackrel{\varphi_\mathfrak{p}}{\to} \mathcal{Z}k(\mathfrak{p})G \stackrel{b}{\to} \mathcal{Z}k(\mathfrak{p})Gb \to \mathcal{Z}k(\mathfrak{p})Gb/J\Big(\mathcal{Z}k(\mathfrak{p})Gb\Big)$$

Since the ring on the right hand side is a field, it follows that $I_{\mathfrak{p},b}$ is a prime ideal of $\mathcal{Z}RG$.

The map $u : r \in R \mapsto r.1 \in \mathcal{Z}RG$ is an injective ring homomorphism. Thus $\mathcal{Z}RG$ is an extension ring of $R$. If $I$ is a prime ideal in $\mathcal{Z}RG$, then $\mathfrak{p} = I \cap R$ is a prime ideal of $R$. Moreover

$$\mathcal{Z}\mathfrak{p}G = \mathfrak{p}\mathcal{Z}RG \subseteq I \quad .$$

Let $\pi$ denote the projection map $\mathcal{Z}RG \to \mathcal{Z}RG/I$, and $i : \mathcal{Z}RG/I \to K$ denote the embedding of $\mathcal{Z}RG/I$ in its field of fractions $K$. The kernel of the map $\rho = i \circ \pi \circ u : R \to K$ is equal to $\mathfrak{p}$. Hence $\rho$ factors through the canonical map $\lambda_{\mathfrak{p}} : R \to k(\mathfrak{p})$. Hence there is a map $\theta : \mathcal{Z}k(\mathfrak{p})G \to K$ and a commutative square

$$\begin{array}{ccc} \mathcal{Z}RG & \xrightarrow{\pi} & \mathcal{Z}RG/I \\ {\scriptstyle\varphi_{\mathfrak{p}}}\downarrow & & \downarrow{\scriptstyle i} \\ \mathcal{Z}k(\mathfrak{p})G & \xrightarrow[\theta]{} & K \end{array} \quad .$$

The ideal $I$ is the kernel of the map $\theta \circ \varphi_{\mathfrak{p}}$, hence it is the inverse image under $\varphi_{\mathfrak{p}}$ of the prime ideal $\theta^{-1}(0)$ of $\mathcal{Z}k(\mathfrak{p})G$. This ring has dimension zero. Hence there exists a primitive idempotent $b$ of $\mathcal{Z}k(\mathfrak{p})G$ (i.e. a block of $k(\mathfrak{p})G$) such that $\theta^{-1}(0) = J\Big(\mathcal{Z}k(\mathfrak{p})Gb\Big) + \sum_{b' \neq b} \mathcal{Z}k(\mathfrak{p})Gb'$ (where $b'$ denotes a block of $kG$). Hence $I = I_{\mathfrak{p},b}$.

Finally if $\mathfrak{p}'$ is a prime ideal of $R$, and $b'$ is a block of $k(\mathfrak{p}')G$ such that $I_{\mathfrak{p},b} \subseteq I_{\mathfrak{p}',b'}$, then

$$\mathfrak{p} = R \cap I_{\mathfrak{p},b} \subseteq R \cap I_{\mathfrak{p}',b'} = \mathfrak{p}' \quad .$$

Suppose moreover that $\mathfrak{p} = \mathfrak{p}'$. The idempotent $b$ can be written

$$b = \sum_{g \in G} (r_g/s_g)g$$

where $r_g \in R/\mathfrak{p}$, and $s_g \in R/\mathfrak{p} - \{0\}$, for $g \in G$. There is an element $s \in R$ which maps to $\prod_{g \in G} s_g$ in $R/\mathfrak{p}$, and there is an element $v \in \mathcal{Z}RG$ such that

$$\varphi_{\mathfrak{p}}(v) = \lambda_{\mathfrak{p}}(s)b \quad .$$

It follows that $v - s.1 \in I_{\mathfrak{p},b} = I_{\mathfrak{p}',b'}$. Thus $\varphi_{\mathfrak{p}}(v - s.1)b'$ is nilpotent. But if $b' \neq b$, then

$$\varphi_{\mathfrak{p}}(v - s.1)b' = \lambda_{\mathfrak{p}}(s)bb' - \lambda_{\mathfrak{p}}(s)b' = -\lambda_{\mathfrak{p}}(s)b' \quad .$$

This cannot be nilpotent since $\lambda_{\mathfrak{p}}(s)$ is non-zero in $k(\mathfrak{p})$. Thus $b = b'$. This completes the proof of assertions 1) and 2).

Assertion 3) is nothing but the *going up* theorem, which holds between $R$ and $\mathcal{Z}RG$ because $\mathcal{Z}RG$ is free as an $R$-module, hence flat over $R$ (see [6] Theorem 9.5).

Similarly, assertion 4) is the *going down* theorem, which holds between $R$ and $\mathcal{Z}RG$ because $\mathcal{Z}RG$ is a finitely generated $R$-module, hence integral over $R$ (see [6] Theorem 9.4). □

**Notation 3.1.5:** *If $H$ is a subgroup of $G$, if $\mathfrak{p}$ is a prime ideal of $R$, and if $b$ is a block of $k(\mathfrak{p})C_G(H)$, I set*

$$I_{H,\mathfrak{p},b} = z_H^{-1}(I_{\mathfrak{p},b}) \quad .$$

*It is a prime ideal of $B_R^c(G)$.*

**Corollary 3.1.6:** *If $R$ has characteristic zero, and if $I$ is a prime ideal of $B_R^c(G)$, then there exists a subgroup $H$ of $G$, a prime ideal $\mathfrak{p}$ of $R$, and a block $b$ of $k(\mathfrak{p})C_G(H)$, such that $I = I_{H,\mathfrak{p},b}$.*

## 3.2 $p$-blocks

**Notation 3.2.1:** *From now on, the letter $\mathcal{O}$ will denote a complete discrete valuation ring of characteristic 0, with maximal ideal $\pi$, with residue field $k$ of characteristic $p > 0$, and field of fractions $K$, which will be supposed big enough (i.e. $K$ contains the $|G|^{th}$ roots of unity). I denote by $x \mapsto \bar{x}$ the reduction morphism from $\mathcal{O}$ to $k$, or from $\mathcal{O}G$ to $kG$.*

*If $H$ is a group of order dividing $|G|$, the group algebra $KG$ is split and semi-simple. If $\chi$ is an ordinary irreducible character of $H$, I denote by $e_\chi$ the corresponding block of $KH$, and by $\omega_\chi$ the morphism from $\mathcal{Z}\mathcal{O}H$ to $\mathcal{O}$ mapping $u$ to $\chi(u)/\chi(1)$.*

*I will say that $\chi$ is in the block $e$ (or belongs to the block $e$) of $kH$ if $e$ acts as identity morphism on a reduction of a simple module affording the character $\chi$, or equivalently, if there is a block $E$ of $\mathcal{Z}\mathcal{O}H$ such that $e_\chi.E = e_\chi$ and $\overline{E} = e$.*

*If $P$ is a $p$-subgroup of $G$, I denote by $br_P$ the Brauer morphism from $\mathcal{Z}kG$ to $\mathcal{Z}kC_G(P)$, which is the $k$-linear map sending $x \in G$ to itself if $x \in C_G(P)$, and to 0 otherwise.*

*A Brauer pair $(P, e)$ is a pair consisting of a $p$-subgroup $P$ of $G$, and a block $e$ of the algebra $kC_G(P)$. If $b$ is a block of $kG$, the Brauer pair $(P, e)$ is a $b$-Brauer pair if $e.br_P(b) = e$.*

For the remainder of this section, the ring $R$ will be equal to $\mathcal{O}$. The primitive idempotents of the Burnside algebra $B_\mathcal{O}(G)$ follow from a theorem of Dress ([4] or [3] Corollary 3.3.6). They are indexed by the (conjugacy classes of) $p$-perfect subgroups of $G$. The idempotent $f_H^G$ indexed by the $p$-perfect subgroup $H$ of $G$ is equal to the sum of the idempotents $e_K^G$ (see equation 2.3.5) for which $O^p(K)$ is conjugate to $H$, each taken once (i.e. $K$ is taken once up to conjugation in $G$). In particular

$$f_1^G = \sum_{P \in \underline{s}_p(G)/G} e_P^G \qquad \textbf{(3.2.2)}$$

where $\underline{s}_p(G)$ is the set of $p$-subgroups of $G$. The corresponding block $B_{\mathcal{O}}(G)f_1^G$ of the Burnside algebra is the $\mathcal{O}$-submodule generated by the $G$-sets $G/P$, for $P \in \underline{s}_p(G)$.

Now the algebra homomorphism $\beta : B_{\mathcal{O}}(G) \to B_{\mathcal{O}}^c(G)$ of lemma 2.1.1 provides a decomposition of unity in $B_{\mathcal{O}}^c(G)$ as a sum of orthogonal idempotents $\beta(f_H^G)$, for $p$-perfect subgroups $H$ of $G$, up to conjugation. These idempotents are no longer primitive, and in this section I will show how the idempotent $\beta(f_1^G)$ splits as a sum of primitive idempotents of $B_{\mathcal{O}}^c(G)$.

**Notation 3.2.3:** *I will denote by $\mathcal{A}(G)$ the $\mathcal{O}$-algebra $B_{\mathcal{O}}^c(G)\beta(f_1^G)$, and by $\overline{\mathcal{A}}(G)$ the $k$-algebra $k \otimes_{\mathcal{O}} \mathcal{A}(G)$. The algebra $\mathcal{A}(G)$ will be called the $p$-local crossed Burnside algebra (over $\mathcal{O}$).*

**Proposition 3.2.4:** *1. The map*

$$\Psi_{\mathcal{O}} = \prod_{P \in \underline{s}_p(G)} z_P : \mathcal{A}(G) \to \left( \prod_{P \in \underline{s}_p(G)} \mathcal{Z}\mathcal{O}C_G(P) \right)^G$$

*is an injective map of $\mathcal{O}$-algebras, which preserves identity elements, and induces an isomorphism of $K$-algebras*

$$\Psi_K : K \otimes_{\mathcal{O}} \mathcal{A}(G) \cong \left( \prod_{P \in \underline{s}_p(G)} \mathcal{Z}KC_G(P) \right)^G .$$

*2. The algebra $\mathcal{A}(G)$ is the $\mathcal{O}$-submodule of $B_{\mathcal{O}}^c(G)$ with basis the set of elements $[P, x]_G$ of $[\mathcal{P}_G]$ for which $P$ is a $p$-group.*

**Proof:** If $X$ is a $G$-set and if $H$ is a subgroup of $G$

$$(z_H \circ \beta)(X) = z_H([X, u_X]) = \sum_{x \in X^H} u_X(x) = |X^H|.1 \in \mathcal{Z}RC_G(H)$$

where $u_X$ is defined in lemma 2.1.1. It follows from equations 2.3.6 and 3.2.2 that $(z_H \circ \beta)(f_1^G) = 0$ if $H$ is not a $p$-group, and that $(z_H \circ \beta)(f_1^G) = 1 \in \mathcal{Z}RC_G(H)$ otherwise. Now the map $\Psi_{\mathcal{O}}$ is just the restriction to $\mathcal{A}(G)$ of the map $\Theta_{\mathcal{O}}$ of lemma 2.3.2, hence it is injective. This proves the first part of assertion 1).

The previous argument actually shows that if $P$ is a $p$-group and if $x$ is an element of $C_G(P)$, then

$$z_H\Big([P,x]_G\beta(f_1^G)\Big) = \begin{cases} 0 & \text{if } H \text{ is not a } p\text{-group} \\ z_H([P,x]_G) & \text{otherwise} \end{cases}$$

In both cases $z_H\Big([P,x]_G\beta(f_1^G)\Big) = z_H([P,x]_G)$, thus $[P,x]_G\beta(f_1^G) = [P,x]_G$ by lemma 2.3.2. In other words $[P,x]_G \in \mathcal{A}(G)$ if $P$ is a $p$-group. Conversely $\beta(f_1^G)$ is a linear combination with coefficients in $\mathcal{O}$ of pairs $[P,x]_G$ of $\mathcal{P}_G$ for which $P$ is a $p$-group, and these linear combinations clearly form an ideal of $B_{\mathcal{O}}^c(G)$. Assertion 2) follows. The second part of assertion 1) also follows, since $\Psi_K$ is an injective map of $K$-vector spaces of the same (finite) dimension. □

**Remark 3.2.5:** *Assertion 2) means that $\mathcal{A}(G)$ is generated over $\mathcal{O}$ by the images of the crossed $G$-sets $(X,\alpha)$ for which the stabilizer in $G$ of any element of $X$ is a $p$-group.*

**Notation 3.2.6:** *If $P$ is a $p$-subgroup of $G$ denote by $\bar{z}_P$ the map $\overline{\mathcal{A}}(G) \to \mathcal{Z}kC_G(G)$ such that the square*

$$\begin{array}{ccc} \mathcal{A}(G) & \xrightarrow{z_P} & \mathcal{Z}\mathcal{O}C_G(P) \\ \downarrow & & \downarrow \\ \overline{\mathcal{A}}(G) & \xrightarrow[\bar{z}_P]{} & \mathcal{Z}kC_G(P) \end{array}$$

*is commutative, where the vertical arrows are the reduction maps.*

**Lemma 3.2.7:** *For any $p$-subgroup $P$ of $G$*

$$\bar{z}_P = br_P \circ \bar{z}_1$$

**Proof:** Indeed, if $v$ is the image of the crossed $G$-set $(X,\alpha)$ in $\overline{\mathcal{A}}(G)$, then

$$\bar{z}_1(v) = \sum_{x\in X}\alpha(x) = \sum_{g\in G}|\alpha^{-1}(g)|g \in \mathcal{Z}kG$$

whereas

$$\bar{z}_P(v) = \sum_{x\in X^P}\alpha(x) = \sum_{g\in C_G(P)}|\alpha^{-1}(g)^P|g \in \mathcal{Z}kC_G(P)$$

and the lemma follows since $|\alpha^{-1}(g)| = |\alpha^{-1}(g)^P|$ in $k$ for any $p$-subgroup $P$ of $C_G(g)$ : this is because $k$ has characteritic $p$, and because $|\alpha^{-1}(g)|$ and $|\alpha^{-1}(g)^P|$ are congruent modulo $p$, since $P$ is a $p$-group. □

The following theorem describes the prime spectrum of $\mathcal{A}(G)$, which is a ring of dimension 1 by lemma 3.1.1. If $I$ is a prime ideal of $\mathcal{A}(G)$, and $q \in \{0, p\}$, I will say that $I$ has *co-characteristic* $q$ if the ring $\mathcal{A}(G)/I$ has characteristic $q$ :

**Theorem 3.2.8:** *Denote by $b$ a block of $kG$, by $P$ (resp $P'$) a $p$-subgroup of $G$, by $\chi$ (resp. $\chi'$) an irreducible character of $C_G(P)$ (resp. $C_G(P')$), and by $e$ a block of $kC_G(P)$. Let $I$ be a prime ideal of $\mathcal{A}(G)$.*

1. *The following are equivalent :*
   - (a) *The ideal $I$ has co-characteristic 0.*
   - (b) *The ideal $I$ is a minimal prime ideal.*
   - (c) *There exist a $p$-subgroup $P$ of $G$ and an irreducible character $\chi$ of $C_G(P)$ such that $I = I_{P,0,e_\chi}$.*
2. *The following are equivalent :*
   - (a) *The ideal $I$ has co-characteristic $p$.*
   - (b) *The ideal $I$ is a maximal prime ideal.*
   - (c) *There exist a $p$-subgroup $P$ of $G$ and a block $e$ of $kC_G(P)$ such that $I = I_{P,\pi,e}$.*
3. *The ideal $I_{P,0,e_\chi}$ is contained in $I_{P',0,e_{\chi'}}$ if and only if the pairs $(P, e_\chi)$ and $(P', e_{\chi'})$ are conjugate in $G$. In this case moreover $I_{P,0,e_\chi} = I_{P',0,e_{\chi'}}$.*
4. *The ideal $I_{P,0,e_\chi}$ is contained in $I_{P,\pi,e}$ if and only if the character $\chi$ belongs to the block $e$. In this case, the inclusion is strict.*
5. *The ideal $I_{1,\pi,b}$ is contained in $I_{P,\pi,e}$ if and only if $(P, e)$ is a $b$-Brauer pair, i.e. if $e.br_P(b) = e$. In this case moreover $I_{1,\pi,b} = I_{P,\pi,e}$.*
6. *The connected components of $Spec\Big(\mathcal{A}(G)\Big)$ are in one to one correspondence with the blocks of $kG$. The component $\mathcal{C}_b$ associated to the block $b$ consists of the unique maximal prime ideal $I_{1,\pi,b}$, and of the ideals $I_{P,0,e_\chi}$, where $P$ is a $p$-subgroup of $G$, and $\chi$ is an irreducible character of $C_G(P)$ belonging to a block $e$ of $C_G(P)$ such that $(P, e)$ is a $b$-Brauer pair.*

**Proof:** By corollary 3.1.6, any prime ideal $I$ of $\mathcal{A}(G)$, which is also a prime ideal of $B_{\mathcal{O}}^c(G)$, is equal to some ideal $I_{H,\mathfrak{p},b}$, for a subgroup $H$ of $G$, a prime ideal $\mathfrak{p}$ of $\mathcal{O}$ (hence $\mathfrak{p}=0$ or $\mathfrak{p}=\pi$), and a block $b$ of $k(\mathfrak{p})C_G(H)$ (i.e. a block $e_\chi$ of $KC_G(H)$ corresponding to an irreducible character $\chi$ if $\mathfrak{p}=0$, or a block $b$ of $kC_G(H)$ if $\mathfrak{p}=\pi$). The ideal $I_{H,\mathfrak{p},b}$ is an ideal of $\mathcal{A}(G)$ if and only if $\beta(f_1^G)\notin I_{H,\mathfrak{p},b}$, or equivalently if $H$ is a $p$-group.

Now the ideal $I_{H,\mathfrak{p},b}$ has co-characteristic 0 if $\mathfrak{p}=0$, and $p$ if $\mathfrak{p}=\pi$. In particular, the ideals $I_{H,0,e_\chi}$ and $I_{H,\pi,b}$ are distinct. By the last two assertions of lemma 3.1.4, any ideal $I_{H,\pi,b}$ contains an ideal $I_{H,0,e_\chi}$, and any ideal $I_{H,0,e_\chi}$ is contained in an ideal $I_{H,\pi,b}$. It follows that the minimal primes of $\mathcal{A}(G)$ are the ideals $I_{P,0,e_\chi}$, for $p$-subgroups $P$ of $G$, whereas the maximal primes are the ideals $I_{P,\pi,b}$. This proves assertions 1) and 2) of the theorem.

Now

$$I_{P,0,e_\chi}=\{u\in\mathcal{A}(G)\mid \omega_\chi\circ z_P(u)=0\}$$

It follows that $I_{P,0,e_\chi}=I_{P',0,e_{\chi'}}$ if the pairs $(P,e_\chi)$ and $(P',e_{\chi'})$ are conjugate. Conversely, if $I_{P,0,e_\chi}\subseteq I_{P',0,e_{\chi'}}$, then $I_{P,0,e_\chi}=I_{P',0,e_{\chi'}}$ since both are minimal prime ideals. Let $f$ denote the idempotent of $\left(\prod_{Q\in\underline{s}_p(G)}\mathcal{Z}KC_G(Q)\right)^G$ with $Q$-component equal to the sum of idempotents $e_\theta$ for which $(Q,e_\theta)$ is $G$-conjugate to $(P,e_\chi)$. Let $F=\Psi_K^{-1}(f)$ be the corresponding idempotent of $B_K^c(G)$. Then there is a non-zero integer $m$ such that $mF$ lies in $B_{\mathcal{O}}(G)$. Moreover $z_Q(mF)=0$ if $Q$ is not conjugate to $P$ in $G$, and $z_P(mF)$ is $m$ times the sum of the different conjugates of $e_\chi$ under $N_G(P)$. In particular $mF$ is not in $I_{P,0,e_\chi}$, hence not in $I_{P',0,e_{\chi'}}$. It follows that $\omega_{\chi'}\circ z_{P'}(mF)\neq 0$. In particular $P'$ is conjugate to $P$, and I can suppose $P=P'$. Now there is a conjugate of $e_\chi$ under $N_G(P)$ which is not in the kernel of $\omega_{\chi'}$. Hence $e_\chi$ and $e_{\chi'}$ are conjugate under $N_G(P)$. This proves assertion 3).

The ideal $I_{P,0,e_\chi}$ is contained in the ideal $I_{P,\pi,e}$ if and only if the ideal $I_{0,e_\chi}$ of $\mathcal{Z}\mathcal{O}C_G(P)$ is contained in the ideal $I_{\pi,e}$. Now

$$I_{0,e_\chi}=\{u\in\mathcal{Z}\mathcal{O}C_G(P)\mid \omega_\chi(u)=0\}\quad .$$

On the other hand, if $\theta$ is any character in the block $e$, then

$$I_{\pi,e}=\{u\in\mathcal{Z}\mathcal{O}C_G(P)\mid \overline{\omega_\theta(u)}=0\}\quad .$$

It follows that $I_{0,e_\chi}\subseteq I_{\pi,e}$ if $\chi$ belongs to $e$. Conversely, if $I_{0,e_\chi}\subseteq I_{\pi,e}$, then let $E$ be the block of $\mathcal{O}C_G(P)$ containing $\chi$. Then $\omega_\chi(1-E)=0$, hence $\overline{\omega_\theta(1-E)}=0$ for any character $\theta$ in $e$. Thus $\omega_\theta(E)=1$, and $e=\overline{E}$. Hence $\chi$ is in $e$, proving assertion 4).

Let $\hat{b}$ be a block of $\mathcal{O}G$ lifting $b$ (i.e. such that $\bar{\hat{b}} = b$). Write

$$\hat{b} = \sum_{x \in G} r_x x$$

for coefficients $r_x \in \mathcal{O}$. Consider

$$\tilde{b} = \sum_{x \in [G]} r_x [C_G(x), x]_G \quad .$$

If $H$ is any subgroup of $G$, then

$$z_H(\tilde{b}) = \sum_{x \in [G]} r_x \sum_{w \in (G/C_G(x))^H} {}^w x = \sum_{x \in C_G(H)} r_x x \quad .$$

The second equality comes from the fact that $w \in (G/C_G(x))^H$ if and only if ${}^w x \in C_G(H)$. Moreover $r_x = r_{{}^w x}$ since $\hat{b}$ is central in $\mathcal{O}G$.

In particular for a $p$-subgroup $P$ of $G$

$$\overline{z_P(\tilde{b})} = br_P(b)$$

Suppose that $I_{1,\pi,p} \subseteq I_{P,\pi,e}$. For any character $\chi$ in $b$

$$\overline{\omega_\chi(z_1(1-\tilde{b}))} = \overline{\omega_\chi(1-\hat{b})} = 1 - \overline{\omega_\chi(b)} = 0$$

Thus $1 - \tilde{b} \in I_{1,\pi,p}$. Hence $1 - \tilde{b} \in I_{P,\pi,e}$, and $e.\overline{z_P(1-\tilde{b})}$ must be nilpotent. But

$$e.\overline{z_P(1-\tilde{b})} = e - e.br_P(b)$$

This is nilpotent and idempotent, hence zero.

Conversely, suppose that $e = e.br_P(b)$. Then

$$I_{1,\pi,b} = \{u \in B^c_{\mathcal{O}}(G) \mid \overline{z_1(u)}b \text{ is nilpotent}\}$$

whereas

$$I_{P,\pi,e} = \{u \in B^c_{\mathcal{O}}(G) \mid \overline{z_P(u)}e \text{ is nilpotent}\} \quad .$$

Now if $u \in I_{1,\pi,b}$

$$\overline{z_1(u)}b = \overline{z_1(u)}br_1(b) = \overline{z_1(u)}\,\overline{z_1(\tilde{b})} = \overline{z_1(u\tilde{b})} \quad .$$

$$\overline{z_P(u)}e = \overline{z_P(u)}br_P(b)e = \overline{z_P(u)}\,\overline{z_P(\tilde{b})}e = \overline{z_P(u\tilde{b})}e \quad .$$

Now lemma 3.2.7 shows that $I_{1,\pi,b} \subseteq I_{P,\pi,e}$. Hence $I_{1,\pi,b} = I_{P,\pi,e}$ since $I_{1,\pi,b}$ is a maximal ideal in $\mathcal{A}(G)$. This completes the proof of assertion 5).

Assertion 5) also shows that all the maximal ideals of $\mathcal{A}(G)$ are of the form $I_{1,\pi,b}$, for a suitable bloc $b$ of $kG$. Now the minimal primes are of the form $I_{P,0,\chi}$, for a $p$-subgroup $P$ of $G$, for an irreducible character $\chi$ of $C_G(P)$. This ideal is contained in $I_{P,\pi,e}$ for a block $e$ of $kC_G(P)$ if and only if $e$ contains $\chi$. Moreover $I_{P,\pi,e}$ is equal to $I_{1,\pi,b}$ for a block $b$ of $kG$ if and only if $(P,e)$ is a $b$-Brauer pair. This shows that each minimal prime ideal is contained in a unique maximal ideal. Assertion 6) follows, and the proof of theorem 3.2.8 is complete. □

**Notation 3.2.9:** *If $G$ is a finite group, if $P$ is a $p$-subgroup of $G$, and $b$ is an idempotent in $\mathcal{Z}kG$, I denote by $br_P^{\mathcal{O}}(b)$ the unique idempotent in $\mathcal{ZO}C_G(P)$ lifting the idempotent $br_P(b)$ of $\mathcal{Z}kC_G(P)$. If $g\in C_G(P)$, I denote by $br_P^{\mathcal{O}}(b)(g)$ the element of $\mathcal{O}$ such that*

$$br_P^{\mathcal{O}}(b)=\sum_{g\in C_G(g)} br_P^{\mathcal{O}}(b)(g)\;g$$

**Theorem 3.2.10:** *If $b$ is an idempotent of $\mathcal{Z}kG$, let*

$$b^{\mathcal{A}}=\frac{1}{|G|}\sum_{\substack{g\in G\\ Q\subseteq P\in \underline{s}_p(C_G(g))}}|Q|\mu(Q,P)br_P^{\mathcal{O}}(b)(g)\;[Q,g]_G$$

*Then $b^{\mathcal{A}}\in\mathcal{A}(G)$, and as $b$ runs through the blocks of $kG$, the elements $b^{\mathcal{A}}$ run through a complete set of primitive idempotents in $\mathcal{A}(G)$.*

**Proof:** The primitive idempotents of the (commutative) ring $\mathcal{A}(G)$ are in one to one correspondence with the connected components of its spectrum, hence with the blocks of $kG$ by theorem 3.2.8 : if $b$ is a block of $kG$, then the idempotent $b^{\mathcal{A}}$ corresponding to the component $\mathcal{C}_b$ is characterised by the fact that $b^{\mathcal{A}}\notin I$ for some $I\in\mathcal{C}_b$ (or equivalently for all $I\in\mathcal{C}_b$). It follows that $b^{\mathcal{A}}\notin I_{1,\pi,b}$, but $b^{\mathcal{A}}\in I_{1,\pi,b'}$ for any block $b'$ of $kG$ different from $b$.

Thus $\omega_b\Big(\bar{z}_1(b^{\mathcal{A}})\Big)\neq 0$, but $\omega_{b'}\Big(\bar{z}_1(b^{\mathcal{A}})\Big)=0$ for any block $b'\neq b$ of $kG$. Since $\bar{z}_1(b^{\mathcal{A}})$ is an idempotent of $\mathcal{Z}kG$, it follows that $\bar{z}_1(b^{\mathcal{A}})=b$. Hence $\bar{z}_P(b^{\mathcal{A}})=br_P(b)$ for any $p$-subgroup $P$ of $G$, by lemma 3.2.7. Now $z_P(b^{\mathcal{A}})$ is an idempotent of $\mathcal{ZO}C_G(P)$ lifting $br_P(b)$, hence it is equal to $br_P^{\mathcal{O}}(b)$. Theorem 3.2.10 now follows from proposition 3.2.4 and from the inversion formula of proposition 2.3.4. □

**Corollary 3.2.11:** *Let $b$ be an idempotent of $\mathcal{Z}kG$. If $Q$ is a $p$-subgroup of $G$, and $g\in C_G(Q)$, then*

$$\sum_{Q\subseteq P\in\underline{s}_p(G)}\mu(Q,P)br_P^{\mathcal{O}}(g)\in|N_G(Q)\cap C_G(g):Q|\mathcal{O}$$

**Proof:** This follows from the following rewriting of the formula in theorem 3.2.10

$$b^{\mathcal{A}} = \sum_{\substack{(Q,g)\in[\mathcal{P}_G] \\ Q\subseteq P\in \underline{s}_p(C_G(g))}} \frac{\mu(Q,P)br_P^{\mathcal{O}}(b)(g)}{|N_G(Q)\cap C_G(g):Q|}\,[Q,g]_G$$

and from the fact that the elements $[Q,g]_G$, for $(Q,g)\in[\mathcal{P}_G]$ and $Q\in \underline{s}_p(G)$, form a basis of $\mathcal{A}(G)$ over $\mathcal{O}$, by proposition 3.2.4. □

# 4 The blocks of the Mackey algebra

## 4.1 Mackey functors

Let $R$ be a commutative ring, and $G$ be a finite group. There are several definitions of the notion of Mackey functor for $G$ over $R$ (see [1] Chapter 1 for a summary of these definitions). One of the most conceptual is due to Dress:

**Definition 4.1.1:** *Let $G$-**set** be the category of finite sets with a left $G$-action. A Mackey functor for the group $G$, with values in $R$-**Mod**, is a bivariant functor from $G$-**set** to $R$-**Mod**, i.e. a pair of functors $(M^*, M_*)$, with $M^*$ contravariant and $M_*$ covariant, which coincide on objects (i.e. $M^*(X) = M_*(X) = M(X)$ for any $G$-set $X$). This bivariant functor is supposed to have the following two properties:*

- *If $X$ and $Y$ are $G$-sets, let $i_X$ and $i_Y$ be the respective injections from $X$ and $Y$ into $X\coprod Y$, then the maps $M^*(i_X)\oplus M^*(i_Y)$ and $M_*(i_X)\oplus M_*(i_Y)$ are mutual inverse $R$-module isomorphisms between $M(X\coprod Y)$ and $M(X)\oplus M(Y)$.*

- *If*

$$\begin{array}{ccc} T & \xrightarrow{\gamma} & Y \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ Z & \xrightarrow[\beta]{} & X \end{array}$$

  *is a cartesian (or pull-back) square of $G$-sets, then $M^*(\beta).M_*(\alpha) = M_*(\delta).M^*(\gamma)$.*

*A morphism $\theta$ from the Mackey functor $M$ to the Mackey functor $N$ is a natural transformation of bivariant functors, consisting of a morphism $\theta_X$ :*

*$M(X) \to N(X)$ for any $G$-set $X$, such that for any morphism of $G$-sets $f : X \to Y$, the squares*

$$\begin{array}{ccc} M(X) & \xrightarrow{\theta_X} & N(X) \\ {\scriptstyle M_*(f)}\downarrow & & \downarrow{\scriptstyle N_*(f)} \\ M(Y) & \xrightarrow[\theta_Y]{} & N(Y) \end{array} \qquad \begin{array}{ccc} M(X) & \xrightarrow{\theta_X} & N(X) \\ {\scriptstyle M^*(f)}\uparrow & & \uparrow{\scriptstyle N^*(f)} \\ M(Y) & \xrightarrow[\theta_Y]{} & N(Y) \end{array}$$

*are commutative.*

I will denote by $Mack_R(G)$ the category of Mackey functors for $G$ over $R$.

## 4.2 The Mackey algebra

The Mackey algebra $\mu_R(G)$ of $R$ over $G$ was defined by Thévenaz and Webb ([8]) : it is the $R$-algebra generated by the elements $t_K^H$, $r_K^H$, and $c_{x,H}$, where $H$ and $K$ are subgroups of $G$ such that $K \subseteq H$, and $x \in G$, with the following relations:

$$t_K^H t_L^K = t_L^H \;\; \forall \;\; L \subseteq K \subseteq H$$

$$r_L^K r_K^H = r_L^H \;\; \forall \;\; L \subseteq K \subseteq H$$

$$c_{y,{}^xH} c_{x,H} = c_{yx,H} \;\; \forall \;\; x, y, H$$

$$t_H^H = r_H^H = c_{h,H} \;\; \forall \;\; h, H \text{ such that } h \in H$$

$$c_{x,H} t_K^H = t_{{}^xK}^{{}^xH} c_{x,K} \;\; \forall \;\; x, K, H \text{ such that } K \subseteq H$$

$$c_{x,K} r_K^H = r_{{}^xK}^{{}^xH} c_{x,H} \;\; \forall \;\; x, K, H \text{ such that } K \subseteq H$$

$$\sum_H t_H^H = \sum_H r_H^H = 1$$

$$r_K^H t_L^H = \sum_{x \in K \backslash H / L} t_{K \cap {}^xL}^K c_{x, K^x \cap L} r_{K^x \cap L}^L \;\; \forall \;\; K \subseteq H \supseteq L$$

any other product of $r_H^K$, $t_H^K$ and $c_{g,H}$ being zero.

It can be shown from Proposition 3.4 of [8] that the map $x \in G \mapsto \sum_{H \subseteq G} c_{x,H}$ extends to an injective $R$-algebra homomorphism from $RG$ to $\mu_R(G)$, and it is handy to identify $G$ with its image in $\mu_R(G)$ via this map.

Thévenaz and Webb show that the category of $\mu_R(G)$-modules is equivalent to the category of Mackey functors for $G$ over $R$. This equivalence is build as follows : if $K \subseteq H$ are subgroups of $G$, denote by $p_K^H : G/K \to G/H$ the map of $G$-sets sending $xK$ to $xH$, for $x \in G$. If $g \in G$, denote by

$\gamma_g : G/H \to G/{}^gH$ the map sending $xH$ to $xg.{}^gH$, for $x \in G$. Then if $M$ is a Mackey functor, the $R$-module $\oplus_{L\subseteq G} M(L)$ can be endowed with a $\mu_R(G)$-module structure : the generator $t_K^H$ maps $M(L)$ to 0 if $L \neq K$, and $M(K)$ to $M(H)$ via the map $M_*(p_K^H)$. Similarly, the generator $r_K^H$ maps $M(L)$ to 0 if $L \neq H$, and $M(H)$ to $M(K)$ via the map $M^*(p_K^H)$. Finally the generator $c_{g,H}$ maps $M(L)$ to 0 if $L \neq H$, and $M(H)$ to $M({}^gH)$ via the map $M_*(\gamma_g)$.

## 4.3 Action of crossed $G$-sets

Any crossed $G$-set gives an endofunctor of the category of Mackey functors for $G$ over $R$, and this will lead to a ring homomorphism from $B_R^c(G)$ to the center $\mathcal{Z}\mu_R(G)$ of the Mackey algebra. This action of crossed $G$-sets on Mackey functors was already observed by Yoshida.

Let $(X,\alpha)$ be a crossed $G$-set. If $M$ is any Mackey functor for $G$ over $R$, and if $Y$ is a finite $G$-set, let $\zeta(X,\alpha)_{M,Y}$ denote the endomorphism of $M(Y)$ defined by

$$\zeta(X,\alpha)_{M,Y} = M_*\begin{pmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{pmatrix} M^*\begin{pmatrix} x,y \\ \downarrow \\ y \end{pmatrix}$$

where $\begin{pmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{pmatrix}$ denotes the map from $X \times Y$ to $Y$ sending $(x,y)$ to $\alpha(x)y$, and $\begin{pmatrix} x,y \\ \downarrow \\ y \end{pmatrix}$ is the projection map from $X \times Y$ to $Y$. This definition extends the one given by Thévenaz and Webb ([8] Section 9) for the action of $B(G)$ on Mackey functors.

**Proposition 4.3.1:** *Let $(X,\alpha)$ be a crossed $G$-set. The the maps $\zeta(X,\alpha)_{M,Y}$ define a natural transformation $\zeta(X,\alpha)$ of the identity functor of the category $Mack_R(G)$. Moreover, if $(Y,\beta)$ is another crossed $G$-set, then*

$$\zeta(X,\alpha) + \zeta(Y,\beta) = \zeta(X \sqcup Y, \alpha \sqcup \beta)$$

$$\zeta(X,\alpha) \circ \zeta(Y,\beta) = \zeta(X \times Y, \alpha.\beta)$$

**Proof:** This amounts to a series of verifications : first the maps $\zeta(X,\alpha)_{M,Y}$ define an endomorphism $\zeta(X,\alpha)_M$ of the Mackey functor $M$. It means that for any morphism of $G$-sets $f : Y \to Z$, one has that

$$\zeta(X,\alpha)_{M,Z} \circ M_*(f) = M_*(f) \circ \zeta(X,\alpha)_{M,Y} \qquad \textbf{(4.3.2)}$$

$$\zeta(X,\alpha)_{M,Y} \circ M^*(f) = M^*(f) \circ \zeta(X,\alpha)_{M,Z} \qquad \textbf{(4.3.3)}$$

Equation 4.3.2 reads

$$M_*\begin{pmatrix} x,z \\ \downarrow \\ \alpha(x)z \end{pmatrix} M^*\begin{pmatrix} x,z \\ \downarrow \\ z \end{pmatrix} M_*(f) = M_*(f) M_*\begin{pmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{pmatrix} M^*\begin{pmatrix} x,y \\ \downarrow \\ y \end{pmatrix} \qquad \textbf{(4.3.4)}$$

The square

$$\begin{array}{ccc} X\times Y & \xrightarrow{\begin{pmatrix} x,y \\ \downarrow \\ y \end{pmatrix}} & Y \\ {\scriptstyle\begin{pmatrix} x,y \\ \downarrow \\ x,f(y) \end{pmatrix}}\Big\downarrow & & \Big\downarrow f \\ X\times Z & \xrightarrow[\begin{pmatrix} x,z \\ \downarrow \\ z \end{pmatrix}]{} & Z \end{array}$$

is a cartesian square of $G$-sets. Hence

$$M^*\begin{pmatrix} x,z \\ \downarrow \\ z \end{pmatrix} M_*(f) = M_*\begin{pmatrix} x,y \\ \downarrow \\ x,f(y) \end{pmatrix} M^*\begin{pmatrix} x,y \\ \downarrow \\ y \end{pmatrix}$$

so equation 4.3.4 is equivalent to

$$M_*\begin{pmatrix} x,y \\ \downarrow \\ \alpha(x)f(y) \end{pmatrix} M^*\begin{pmatrix} x,y \\ \downarrow \\ y \end{pmatrix} = M_*\begin{pmatrix} x,y \\ \downarrow \\ f(\alpha(x)y) \end{pmatrix} M^*\begin{pmatrix} x,y \\ \downarrow \\ y \end{pmatrix}$$

which holds since $f$ is a morphism of $G$-sets.

Similarly equation 4.3.3 reads

$$M_*\begin{pmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{pmatrix} M^*\begin{pmatrix} x,y \\ \downarrow \\ y \end{pmatrix} M^*(f) = M^*(f) M_*\begin{pmatrix} x,z \\ \downarrow \\ \alpha(x)z \end{pmatrix} M^*\begin{pmatrix} x,z \\ \downarrow \\ z \end{pmatrix} \qquad (\mathbf{4.3.5})$$

Similarly the square

$$\begin{array}{ccc} X\times Y & \xrightarrow{\begin{pmatrix} x,y \\ \downarrow \\ x,f(y) \end{pmatrix}} & X\times Z \\ {\scriptstyle\begin{pmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{pmatrix}}\Big\downarrow & & \Big\downarrow{\scriptstyle\begin{pmatrix} x,z \\ \downarrow \\ \alpha(x)z \end{pmatrix}} \\ Y & \xrightarrow[f]{} & Z \end{array}$$

is cartesian, and both sides of equation 4.3.5 are equal to

$$M_*\begin{pmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{pmatrix} M^*\begin{pmatrix} x,y \\ \downarrow \\ f(y) \end{pmatrix} \quad .$$

Hence $\zeta(X,\alpha)_M$ is an endomorphism of the Mackey functor $M$, for any $M$. If $\phi : M \to N$ is a morphism of Mackey functors, given by maps $\phi_Y : M(Y) \to N(Y)$ for any $G$-set $Y$, then

$$\begin{aligned} \zeta(X,\alpha)_{N,Y}\circ\phi_Y &= N_*\begin{pmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{pmatrix} \circ N^*\begin{pmatrix} x,y \\ \downarrow \\ y \end{pmatrix} \circ \phi_Y \\ &= N_*\begin{pmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{pmatrix} \circ \phi_{X\times Y} \circ M^*\begin{pmatrix} x,y \\ \downarrow \\ y \end{pmatrix} \\ &\quad \text{(since } \phi \text{ is a morphism of Mackey functors)} \\ &= \phi_Y \circ M_*\begin{pmatrix} x,y \\ \downarrow \\ \alpha(x)y \end{pmatrix} \circ M^*\begin{pmatrix} x,y \\ \downarrow \\ y \end{pmatrix} \\ &= \phi_Y\circ\zeta(X,\alpha)_{M,Y} \quad . \end{aligned}$$

It follows that the maps $\zeta(X,\alpha)_M$ define an endomorphism of the identity functor of $Mack_R(G)$.

Finally, if $(Y,\beta)$ is another crossed $G$-set, then for any Mackey functor $M$ and any $G$-set $Z$, the assertion

$$\zeta(X,\alpha)_{M,Z}\circ\zeta(Y,\beta)_{M,Z}=\zeta(X\sqcup Y,\alpha\sqcup\beta)_{M,Z}$$

follows easily from the first condition in definition 4.1.1. Concerning composition, one has that

$$\zeta(X,\alpha)_{M,Z}\circ\zeta(Y,\beta)_{M,Z}=M_*\begin{pmatrix}x,z\\ \downarrow\\ \alpha(x)z\end{pmatrix}M^*\begin{pmatrix}x,z\\ \downarrow\\ z\end{pmatrix}M_*\begin{pmatrix}y,z\\ \downarrow\\ \beta(y)z\end{pmatrix}M^*\begin{pmatrix}y,z\\ \downarrow\\ z\end{pmatrix}\quad .$$

Now the square

$$\begin{array}{ccc}
X\times Y\times Z & \xrightarrow{\begin{pmatrix}x,y,z\\ \downarrow\\ y,z\end{pmatrix}} & Y\times Z\\
\begin{pmatrix}x,y,z\\ \downarrow\\ x,\beta(y)z\end{pmatrix}\Big\downarrow & & \Big\downarrow\begin{pmatrix}y,z\\ \downarrow\\ \beta(y)z\end{pmatrix}\\
X\times Z & \xrightarrow[\begin{pmatrix}x,z\\ \downarrow\\ z\end{pmatrix}]{} & Z
\end{array}$$

is cartesian. It follows that

$$\zeta(X,\alpha)_{M,Z}\circ\zeta(Y,\beta)_{M,Z}=M_*\begin{pmatrix}x,y,z\\ \downarrow\\ \alpha(x)\beta(y)z\end{pmatrix}M^*\begin{pmatrix}x,y,z\\ \downarrow\\ z\end{pmatrix}=\zeta(X\times Y,\alpha.\beta)_{M,Z}$$

as was to be shown. □

## 4.4 The center of the Mackey algebra

In any category of modules over a ring $A$, the natural transformations from the identity functor to itself identify with the center of $A$ : any element $z$ of the center of $A$ defines an endomorphism of any $A$-module $M$ (by multiplication by $z$ on $M$), and this gives a natural transformation of the identity functor of the category of $A$-modules.

**Notation 4.4.1:** *The ring homomorphism from $B_R^c(G)$ to $\mathcal{Z}\mu_R(G)$ following from proposition 4.3.1 will be still denoted by $\zeta$.*

It is given as follows :

**Proposition 4.4.2:** *Let $(L,x)\in\mathcal{P}_G$. Then*

$$\zeta([L,x]_G)=\sum_{H\subseteq G}\sum_{w\in L\backslash G/H}t^H_{L^w\cap H}\,w^{-1}xw\,r^H_{L^w\cap H}\quad .$$

**Proof:** Let $H$ be a subgroup of $G$, and $M$ be a Mackey functor for $G$ over $R$. The element $[L,x]_G$ corresponds to the crossed $G$-set $(G/L,m_x)$. The action of $\zeta(G/L,m_x)$ on $M(G/H)$ is defined as

$$M_*\begin{pmatrix} uL,vH \\ \downarrow \\ {}^u x.vH \end{pmatrix} M^*\begin{pmatrix} uL,vH \\ \downarrow \\ vH \end{pmatrix} : M(G/H) \to M(G/H) \quad .$$

This maps factors through $M(G/L \times G/H)$, which is isomorphic to

$$\bigoplus_{w\in L\backslash G/H} M(G/L^w \cap H)$$

since the map

$$g(L^w \cap H) \in \bigsqcup_{w\in L\backslash G/H} G/L^w \cap H \mapsto (gw^{-1}L, gH) \in G/L \times G/H$$

is an isomorphism of $G$-sets.

Up to this isomorphism, the map

$$M^*\begin{pmatrix} uL,vH \\ \downarrow \\ vH \end{pmatrix} : M(G/H) \to \bigoplus_{w\in L\backslash G/H} M(G/L^w \cap H)$$

is just the map $\bigoplus_{w\in L\backslash G/H} r^H_{L^w\cap H}$, and on the component $M(G/L^w \cap H)$, the map $M_*\begin{pmatrix} uL,vH \\ \downarrow \\ {}^u x.vH \end{pmatrix}$ is equal to $M_*(\phi_w)$, where $\phi_w\Big(g.(L^w \cap H)\Big) = g.w^{-1}xwH$. Hence $M_*(\phi_w) = t^H_{L^w\cap H}\, w^{-1}xw$, and the action of $\zeta([L,x]_G)$ on $M(G/H)$ is equal to

$$\sum_{w\in L\backslash G/H} t^H_{L^w\cap H}\, w^{-1}xw\, r^H_{L^w\cap H} \quad .$$

The proposition follows, taking for $M$ the free Mackey functor $\mu_R(G)$. □

**Remark 4.4.3:** *Proposition 4.4.2 shows that the composition $\zeta \circ \beta$ is equal to the morphism from the Burnside ring $B_R(G)$ to the center of the Mackey algebra considered in Section 9 of [8].*

## 4.5 The $p$-blocks of $\mu_{\mathcal{O}}(G)$

**Notation 4.5.1:** *Let $b$ be an idempotent of $\mathcal{Z}kG$. I denote by $b^\mu$ the central idempotent of $\mu_{\mathcal{O}}(G)$ defined by*

$$b^\mu = \zeta(b^{\mathcal{A}}) \quad .$$

*I denote by $\mu^1_{\mathcal{O}}(G)$ the algebra*

$$(\zeta \circ \beta)(f^G_1)\mu_{\mathcal{O}}(G) \quad .$$

*This algebra will be called the $p$-local Mackey algebra.*

This notation is consistent with the notation of [2]. It is motivated by Sections 9 and 10 of [8], in which Thévenaz and Webb show that the category $Mack_{\mathcal{O}}(G)$ splits as a direct sum of subcategories $Mack_{\mathcal{O}}(G,J)$, indexed by $p$-perfect subgroups $J$ of $G$ (up to $G$-conjugation) : the category $Mack_{\mathcal{O}}(G,J)$ consists of $\mu_{\mathcal{O}}(G)$-modules $M$ such that $(\zeta\circ\beta)(f_J^G)M=M$. In particular, the category $Mack_{\mathcal{O}}(G,1)$ is the category of $\mu_{\mathcal{O}}^1(G)$-modules. Moreover, the category $Mack_{\mathcal{O}}(G,J)$ is equivalent to $Mack_{\mathcal{O}}\left(N_G(J)/J,1\right)$.

It follows that studying the $p$-blocks of $\mu_{\mathcal{O}}(G)$ is equivalent to studying the $p$-blocks of the algebras $\mu_{\mathcal{O}}^1\left(N_G(H)/H\right)$, for $p$-perfect subgroups $H$ of $G$.

**Theorem 4.5.2:** *Let $b$ be an idempotent of $\mathcal{Z}kG$. Then*

$$b^\mu=\sum_{H\subseteq G}\frac{1}{|H|}\sum_{\substack{Q\subseteq P\in\underline{s}_p(H)\\ x\in C_G(P)}}|Q|\mu(Q,P)br_P^{\mathcal{O}}(b)(x)\; t_Q^Hxr_Q^H\quad .$$

*The idempotents $b^\mu$, as $b$ runs through the blocks of $kG$, are a complete set of primitive idempotents of the center of the $p$-local Mackey algebra $\mu_{\mathcal{O}}^1(G)$.*

**Proof:** By proposition 4.4.2 and theorem 3.2.10

$$b^\mu=\frac{1}{|G|}\sum_{\substack{g\in G\\ Q\subseteq P\in\underline{s}_p(C_G(g))}}|Q|\mu(Q,P)br_P^{\mathcal{O}}(b)(g)\sum_{H\subseteq G}\sum_{w\in Q\backslash G/H}t_{Q^w\cap H}^H g^w r_{Q^w\cap H}^H\quad .$$

Changing the order of summations gives

$$\begin{aligned}
b^\mu &= \frac{1}{|G|}\sum_{H\subseteq G}\sum_{\substack{g\in G\\ Q\subseteq P\in\underline{s}_p(C_G(g))\\ w\in Q\backslash G/H}}|Q|\mu(Q,P)br_P^{\mathcal{O}}(b)(g)t_{Q^w\cap H}^H g^w r_{Q^w\cap H}^H\\
&= \frac{1}{|G|}\sum_{H\subseteq G}\sum_{\substack{g,w\in G\\ Q\subseteq P\in\underline{s}_p(C_G(g))}}\frac{|Q^w\cap H|}{|Q||H|}|Q|\mu(Q,P)br_P^{\mathcal{O}}(b)(g)t_{Q^w\cap H}^H g^w r_{Q^w\cap H}^H\quad .
\end{aligned}$$

Summing on $Q^w$, $P^w$ and $g^w$ instead of $Q$, $P$, $w$ leads to

$$\begin{aligned}
b^\mu &= \frac{1}{|G|}\sum_{H\subseteq G}\frac{1}{|H|}\sum_{\substack{g,w\in G\\ Q\subseteq P\in\underline{s}_p(C_G(g))}}|Q\cap H|\mu(Q,P)br_P^{\mathcal{O}}(b)(g)t_{Q\cap H}^H g r_{Q\cap H}^H\\
&= \sum_{H\subseteq G}\frac{1}{|H|}\sum_{\substack{g\in G\\ Q\subseteq P\in\underline{s}_p(C_G(g))}}|Q\cap H|\mu(Q,P)br_P^{\mathcal{O}}(b)(g)t_{Q\cap H}^H g r_{Q\cap H}^H\quad .
\end{aligned}$$

Fix a subgroup $H$ of $G$, an element $g$ of $G$, and a $p$-subgroup $P$ of $C_G(g)$. By classical combinatorial formulae, if $R$ is any subgroup of $H \cap P$

$$\sum_{\substack{Q \subseteq P \\ Q \cap H = R}} \mu(Q,P) = \begin{cases} 0 & \text{if } H \cap P \neq P \\ \mu(R,P) & \text{otherwise} \end{cases} \quad .$$

It follows that

$$b^\mu = \sum_{H \subseteq G} \frac{1}{|H|} \sum_{\substack{R \subseteq P \in \mathcal{S}_p(H) \\ g \in C_G(P)}} |R|\mu(R,P) br_P^{\mathcal{O}}(b)(g) t_R^H g r_R^H$$

which states the formula in theorem 4.5.2.

To complete the proof of this theorem, it remains to show that if $b$ is a block of $kG$, i.e. a primitive idempotent of $\mathcal{Z}kG$, then $b^\mu$ is a primitive idempotent of $\mathcal{Z}\mu_{\mathcal{O}}(G)$. Suppose that $b^\mu$ splits as the sum of two orthogonal idempotents $c$ and $d$ of $\mathcal{Z}\mu_{\mathcal{O}}(G)$. Then in particular

$$t_1^1 b^\mu = t_1^1 b^\mu t_1^1 = t_1^1 c t_1^1 + t_1^1 d t_1^1$$

is a decomposition of $t_1^1 b^\mu t_1^1$ as a sum of orthogonal central idempotents in the algebra $t_1^1 \mu_{\mathcal{O}}(G) t_1^1$. This algebra is isomorphic to $\mathcal{O}G$ (via $x \in G \mapsto t_1^1 x t_1^1$), and moreover

$$t_1^1 b^\mu t_1^1 = t_1^1 b t_1^1 \quad .$$

Indeed if $R \subseteq H$ are subgroups of $G$, and if $g \in G$, then $t_1^1 . t_R^H g r_R^H . t_1^1 = 0$ in $\mu_{\mathcal{O}}(G)$, unless $H$ is the trivial subgroup of $G$.

If $b$ is primitive, it follows that one of $t_1^1 c t_1^1$ or $t_1^1 d t_1^1$ is zero, say $t_1^1 c t_1^1$. Now $\mu_{\mathcal{O}}(G)c$ is a projective Mackey functor in $Mack_{\mathcal{O}}(G,1)$ (i.e. a projective $\mu_{\mathcal{O}}^1(G)$-module), whose evaluation at the trivial subgroup is equal to

$$t_1^1 \mu_{\mathcal{O}}(G) c = t_1^1 c t_1^1 \mu_{\mathcal{O}}(G) = 0 \quad .$$

It follows that $\mu_{\mathcal{O}}(G)c = 0$, by Corollary 12.2 of [8]. Hence $c = 0$, and the proof is complete. □

# 5 Consequences

## 5.1 Another formula

The following gives another expression for $b^\mu$, which may be easier to remember :

**Proposition 5.1.1:** *If $H$ is a subgroup of $G$, denote by $X \mapsto [X]$ the $\mathcal{O}$-linear morphism from $B_{\mathcal{O}}(G)$ to $\mu_{\mathcal{O}}(G)$ mapping $H/L$ to $t_L^H r_L^H$. If $b$ is an idempotent of $\mathcal{Z}kG$, then*

$$b^{\mu} = \sum_{H \subseteq G} \frac{1}{|H|} \sum_{P \in \underline{s}_p(G)} |P| t_P^H \left( [e_P^P] br_P^{\mathcal{O}}(b) \right) r_P^H \quad .$$

**Proof:** Indeed in $B_K(P)$

$$e_P^P = \frac{1}{|P|} \sum_{Q \subseteq P} |Q| \mu(Q, P) \; P/Q \quad .$$

Thus

$$[e_P^P] = \frac{1}{|P|} \sum_{Q \subseteq P} |Q| \mu(Q, P) \; t_Q^P r_Q^P \quad .$$

Now by theorem 4.5.2

$$\begin{aligned} b^{\mu} &= \sum_{H \subseteq G} \frac{1}{|H|} \sum_{\substack{Q \subseteq P \in \underline{s}_p(H) \\ x \in C_G(P)}} |Q| \mu(Q, P) br_P^{\mathcal{O}}(b)(x) \; t_Q^H x r_Q^H \\ &= \sum_{H \subseteq G} \frac{1}{|H|} \sum_{P \in \underline{s}_p(H)} t_P^H \left( \sum_{\substack{Q \subseteq P \\ x \in C_G(P)}} |Q| \mu(Q, P) t_Q^P br_P^{\mathcal{O}}(b)(x) x r_Q^P \right) r_P^H \\ &= \sum_{H \subseteq G} \frac{1}{|H|} \sum_{P \in \underline{s}_p(H)} t_P^H \left( \sum_{\substack{Q \subseteq P \\ x \in C_G(P)}} |Q| \mu(Q, P) t_Q^P r_Q^P br_P^{\mathcal{O}}(b)(x) x \right) r_P^H \\ &\quad \text{(since } C_G(P) \subseteq C_G(Q) \text{ for } Q \subseteq P\text{)} \\ &= \sum_{H \subseteq G} \frac{1}{|H|} \sum_{P \in \underline{s}_p(H)} t_P^H \left( (\sum_{Q \subseteq P} |Q| \mu(Q, P) t_Q^P r_Q^P)( \sum_{x \in C_G(P)} br_P^{\mathcal{O}}(b)(x) x) \right) r_P^H \end{aligned}$$

as was to be shown. □

## 5.2 Residues

**Notation 5.2.1:** *Let $M$ be a Mackey functor for $G$ over $R$, and $H$ be a subgroup of $G$. Then Brauer residue of $M$ at $H$ is defined as*

$$\overline{M}(H) = M(H) / \sum_{L \subset H} t_L^H M(L)$$

*and the projection $M(H) \to \overline{M}(H)$ is denoted by $br_H$. By duality, the Brauer co-residue of $M$ at $H$ is defined as*

$$\underline{M}(H) = \bigcap_{L \subseteq H} \mathrm{Ker}\, r_L^H \subset M(H)$$

**Proposition 5.2.2:** *Let $M$ be a Mackey functor for $G$ over $\mathcal{O}$, in $Mack_{\mathcal{O}}(G,1)$, and $b$ be a block of $kG$.*

1. *If $H$ is a subgroup of $G$, then $\overline{M}(H) = 0$, unless $H$ is a $p$-group. Similarly $\underline{M}(H) = 0$, unless $H$ is a $p$-group.*

2. *If $L$ is a $p$-subgroup of $G$, and if $m \in M(L)$, then*

$$br_L(b^\mu.m) = br_L^{\mathcal{O}}(b)br_L(m)$$

*Similarly, if $m \in \underline{M}(L)$, then*

$$b^\mu.m = br_L^{\mathcal{O}}(b)m$$

3. *The following are equivalent :*
   - *(a) The block $b^\mu$ acts as the identity of $M$.*
   - *(b) For any $p$-subgroup $P$ of $G$, the idempotent $br_P^{\mathcal{O}}(b)$ acts as the identity of $\overline{M}(P)$.*
   - *(c) For any $p$-subgroup $P$ of $G$, the idempotent $br_P^{\mathcal{O}}(b)$ acts as the identity of $\underline{M}(P)$.*

**Proof:** The first assertion follows from the fact that $(\zeta \circ \beta)(f_1^G)$ acts as the identity of $M$ if $M$ is in $Mack_{\mathcal{O}}(G,1)$. Moreover, the action of this idempotent on $M(H)$ is a linear combination with coefficients in $\mathcal{O}$ of elements $t_P^H r_P^H$, for $p$-subgroups $P$ of $H$.

For assertion 2, observe that $b^\mu$ acts on $M(L)$ via

$$b_L^\mu = \frac{1}{|L|} \sum_{\substack{Q \subseteq P \subseteq L \\ x \in C_G(P)}} |Q|\mu(Q,P) br_P^{\mathcal{O}}(b)(x) t_Q^L x r_Q^L \quad .$$

Thus if $m \in M(L)$

$$b_L^\mu m = \frac{1}{|L|} \sum_{\substack{Q \subseteq P \subseteq L \\ x \in C_G(P)}} |Q|\mu(Q,P) br_P^{\mathcal{O}}(b)(x) t_Q^L x r_Q^L m \quad .$$

In this summation, all the terms are traces from proper subgroups of $L$, except those for which $Q = P = L$. This gives :

$$br_L(b^\mu_L m) = br_L(\sum_{x\in C_G(L)} br^{\mathcal{O}}_L(b)(x)x.m) = br_L(br^{\mathcal{O}}_L(b)m) = br^{\mathcal{O}}_L(b)br_L(m) \quad .$$

Similarly, if $m \in \underline{M}(L)$, then all proper restrictions of $m$ are zero, hence

$$b^\mu_L.m = \sum_{x\in C_G(L)} br^{\mathcal{O}}_L(b)(x)x.m = br^{\mathcal{O}}_L(b)m$$

as was to be shown.

For assertion 3), suppose first that $b^\mu$ acts as the identity of $M$. Then by assertion 2), the idempotent $br^{\mathcal{O}}_L(b)$ acts as the identity of $\overline{M}(L)$ and $\underline{M}(L)$, for any $p$-subgroup $L$ of $G$. Thus assertion (a) implies (b) and (c).

Conversely, suppose that (b) holds. Then in particular $b^\mu$ acts as the identity of $\overline{M}(1) = M(1)$. By induction on the order of the subgroup $H$ of $G$, this shows that $b^\mu$ acts as the identity of $M(H)$ : let $m \in M(H)$. If $H$ is not a $p$-group, then by assertion 1) $br_H(m) = 0$, and $m$ can be written

$$m = \sum_{L\subset H} t^H_L m_L$$

for proper subgroups $L$ of $H$ and elements $m_L \in M(L)$. Now by induction

$$b^\mu.m = b^\mu_H \sum_{L\subset H} t^H_L m_L = \sum_{L\subset H} t^H_L (b^\mu_L m_L) = \sum_{L\subset H} t^H_L m_L = m \quad .$$

And if $H$ is a $p$-subgroup of $G$, then by assertion (b)

$$m - b^\mu.m \in \mathrm{Ker}\ br_H \quad .$$

Hence

$$m - b^\mu.m = \sum_{L\subset H} t^H_L m_L$$

for proper subgroups $L$ of $H$ and elements $m_L \in M(L)$. Now

$$b^\mu.m - (b^\mu)^2.m = m - b^\mu.m$$

by the same argument as above. Hence $b^\mu.m = m$, and (b) implies (a).

Similarly if (c) holds, then in particular $b^\mu$ acts as the identity of $\underline{M}(1) = M(1)$. By induction on the order of the subgroup $H$ of $G$, this shows that $b^\mu$ acts as the identity of $M(H)$ : let $m \in M(H)$. If $H$ is not a $p$-group, then $m$ is a linear combination of traces from proper subgroups of $H$, hence $b^\mu.m = m$ by induction. And if $H$ is a $p$-subgroup of $G$, then by induction for any proper subgroup $L$ of $H$

$$r_L^H(b^\mu.m) = r_L^H(b_H^\mu.m) = b_L^\mu.r_L^H m = r_L^H m$$

This shows that $m - b^\mu.m \in \underline{M}(L)$. Now (c) implies that

$$m - b^\mu.m = b^\mu(m - b^\mu.m) = b^\mu.m - b^\mu.m = 0$$

thus $m = b^\mu.m$, as was to be shown. □

**Corollary 5.2.3:** *Let $M$ be a Mackey functor in $Mack_{\mathcal{O}}(G,1)$. If $M$ is indecomposable, and if there is a $p$-subgroup $P$ such that $\overline{M}(P) \neq 0$ and $C_G(P)$ acts trivially on $\overline{M}(P)$ (resp. $\underline{M}(P) \neq 0$ and $C_G(P)$ acts trivially on $\underline{M}(P)$), then $M$ is in the principal block of $Mack_{\mathcal{O}}(G,1)$.*

**Proof:** Since $M$ is indecomposable, there is a block $b$ of $kG$ such that $b^\mu$ acts as the identity of $M$. Hence $br_P^{\mathcal{O}}(b)$ acts as the identity of $\overline{M}(P)$, for any $p$-subgroup $P$ of $G$. Moreover, since $C_G(P)$ acts trivially on $\overline{M}(P)$, the action of $br_P^{\mathcal{O}}(b)$ on $\overline{M}(P)$ is multiplication by the sum $\sum_{x\in C_G(P)} br_P^{\mathcal{O}}(b)(x)$, which is zero if $b$ is not the principal block of $kG$ : indeed if $e = \sum_{x\in C_G(P)} e_x x$ is any block of $\mathcal{O}C_G(P)$, then the sum $\sum_{x\in C_G(P)} e_x$ is equal to 1 or 0, according to the fact that $e$ is the principal block of $\mathcal{O}C_G(P)$ or not. And by Brauer main theorem, the principal block of $\mathcal{O}C_G(P)$ appears in the decomposition of $br_P^{\mathcal{O}}(b)$ if and only if $b$ is the principal block of $\mathcal{O}G$. The argument is the same with $\underline{M}(P)$. □

**Remark 5.2.4:** *This corollary shows in particular that if $M$ is an indecomposable Mackey functor in $Mack_{\mathcal{O}}(G,1)$, and if $C_G(P)$ acts trivially on $M(P)$ for any $p$-subgroup $P$ of $G$, then $M$ is in the principal block of $Mack_{\mathcal{O}}(G,1)$.*

## 5.3 The defect of a block of $\mu_{\mathcal{O}}(G)$

In [1] Chapter 12, it is shown that there exists a natural Green functor for $G$ whose evaluation at $G$ is isomorphic to the center of the Mackey algebra. This functor is denoted by $\zeta_B$, and its value at the $G$-set $X$ is the set

$$\zeta_B(X) = \mathrm{Hom}_{Funct}(\mathcal{I}, \mathcal{I}_X)$$

of natural transformations from the identity functor $\mathcal{I}$ on the category of Mackey functors to the endofunctor $\mathcal{I}_X$ on this category, mapping the Mackey functor $M$ to its Dress construction $M_X$, defined on the $G$-set $Y$ by

$$M_X(Y) = M(Y \times X) \quad .$$

This functor $\zeta_B$ has also been considered by Oda (the functor $\mathcal{T}$ of [7]) from a different point of view. The equivalence of those two points of view follows from Proposition 12.2.8 of [1] : the value of $\zeta_B$ at a subgroup $H$ of $G$ is the set of sequences $(z_L)$, indexed by the subgroups $L$ of $H$, such that

$$z_L \in t_L^L \mu_{\mathcal{O}}(G) t_L^L \qquad t_Q^L z_L = z_Q t_Q^L \qquad z_Q r_Q^L = r_Q^L z_L \qquad x z_L = z_{{}^xL} x$$

for any subgroups $Q \subseteq L$ of $H$ and any $x \in H$. The isomorphism $\zeta_B(G) \cong \mathcal{Z}\mu_{\mathcal{O}}(G)$ is obtained by mapping the sequence $(z_L)_{L\subseteq G}$ to the element $\sum_{L\subseteq G} z_L$.

Also recall from Proposition 12.2.8 of [1] that if $Q$ is a subgroup of $G$, and $(z_L)_{L\subseteq Q}$ is an element of $\zeta_B(Q)$, then for any subgroup $H$ of $G$, the component of $t_Q^G(z)$ at the subgroup $H$ is equal to

$$t_Q^G(z)_H = \sum_{w\in Q\backslash G/H} t^H_{Q^w\cap H} w^{-1} z_{Q\cap {}^wH} w r^H_{Q^w\cap H} \quad .$$

If $g \in C_G(Q)$, one can define an element $z(Q,g)$ of $\zeta_B(Q)$ by setting, for any subgroup $L$ of $Q$

$$z(Q,g)_L = t_L^L g r_L^L \quad .$$

Let $b$ be a block of $kG$, with defect group $D$. With the previous notation, the expression of $b^\mu$ given in proposition 4.5.2 can be written as

$$b^\mu = \frac{1}{|G|} \sum_{\substack{Q\subseteq P\in \underline{s}_p(G)\\ g\in C_G(P)}} |Q|\mu(Q,P) br_P^{\mathcal{O}}(b)(g) t_Q^G z(Q,g) \quad .$$

The summation can be restricted to $p$-subgroups $P$ of $G$ for which $br_P^{\mathcal{O}}(b) \neq 0$. Such subgroups are all contained in $D$ up to $G$-conjugation. It follows that

$$b^\mu \in t_D^G \zeta_B(D) \quad .$$

**Definition 5.3.1:** *A defect group of the block $b^\mu \in \mathcal{Z}\mu_{\mathcal{O}}(G) = \zeta_B(G)$ is a subgroup $E$ of $G$, minimal subject to the condition*

$$b^\mu \in t_E^G \zeta_B(E) \quad .$$

The following argument is then classical : if $D$ is any subgroup of $G$ such that $b^\mu \in t_D^G \zeta_B(D)$, then

$$b^\mu = (b^\mu)^2 \in (t_E^G \zeta_B(E))(t_D^G \zeta_B(D)) \subseteq \sum_{x\in E\backslash G/D} t^G_{E\cap {}^xD} \zeta_B(E\cap {}^xD)$$

and by Rosenberg's lemma and primitivity of $b^\mu$, there exists $x \in G$ such that $E \subseteq {}^xD$. In particular $E$ is unique up to $G$-conjugation.

**Proposition 5.3.2:** *Let $b$ be a block of $kG$, with defect group $D$. Then $D$ is also a defect group of $b^\mu$.*

**Proof:** Let $E$ be a defect of $b^\mu$. The above argument shows that $E$ is contained in $D$ up to $G$-conjugation. The reverse inclusion has been proved by Oda ([7] Theorem 6). It also follows from the following fact : if $b^\mu = t_E^G(c)$, for $c \in \zeta_B(E)$, then

$$(b^\mu)_1 = t_1^1 Tr_E^G(c_1) r_1^1 \in t_1^1 Tr_E^G(\mathcal{O}G)^E r_1^1$$

where $Tr_E^G$ is the relative trace map. Since moreover $(b^\mu)_1 = t_1^1 br_1^{\mathcal{O}}(b) r_1^1$, it follows that the block $b^{\mathcal{O}} = br_1^{\mathcal{O}}(b)$ lifting $b$ to $\mathcal{O}G$ belongs to $Tr_E^G(\mathcal{O}G)^E$. Hence $D$ is contained in $E$ up to $G$-conjugation. □

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