

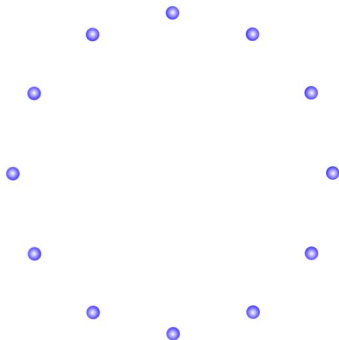
Around Möbius inversion

Serge Bouc

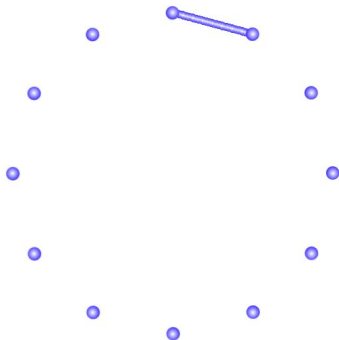
CNRS-LAMFA
Université de Picardie

UNAM, August 5, 2011

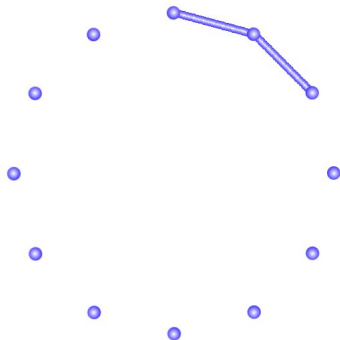
Regular 12-gons



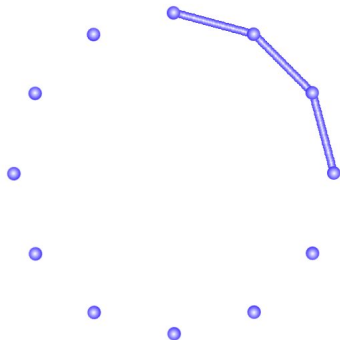
Regular 12-gons



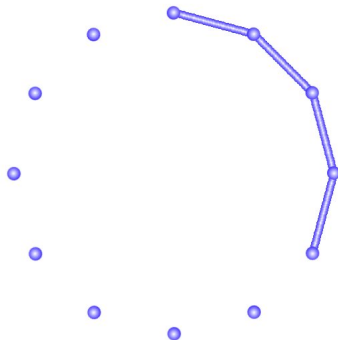
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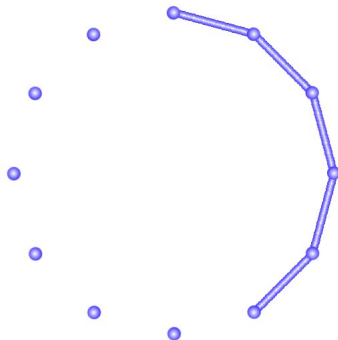
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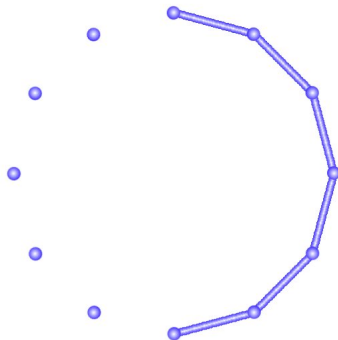
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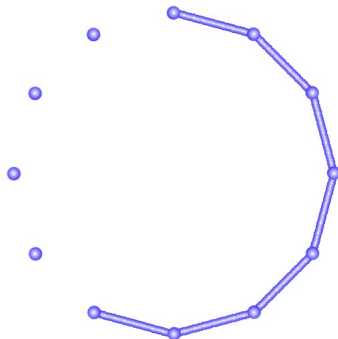
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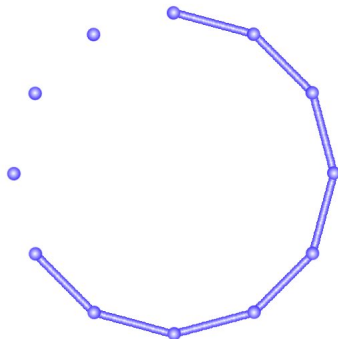
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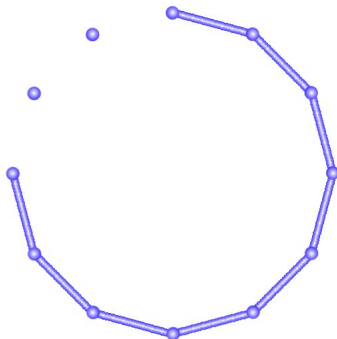
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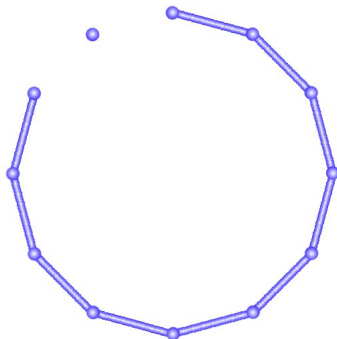
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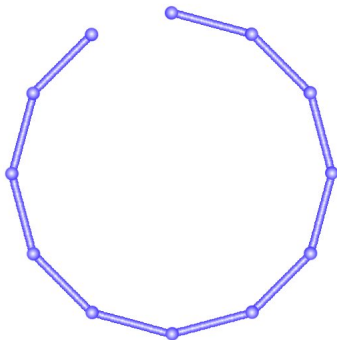
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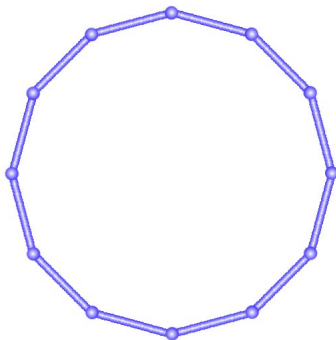
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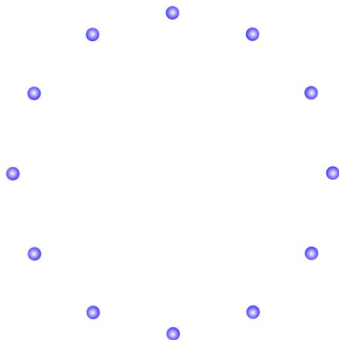
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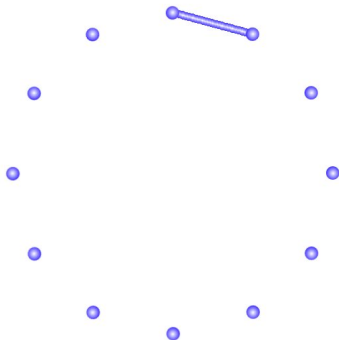
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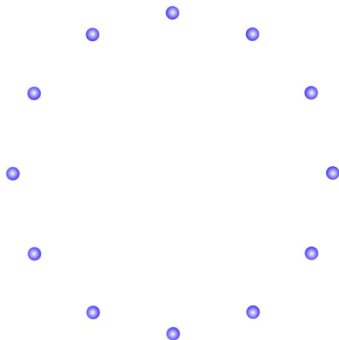
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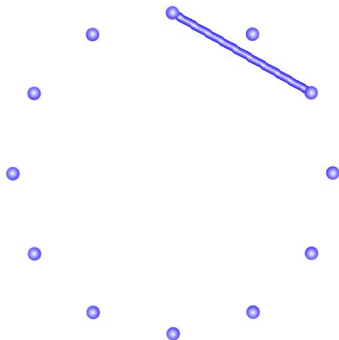
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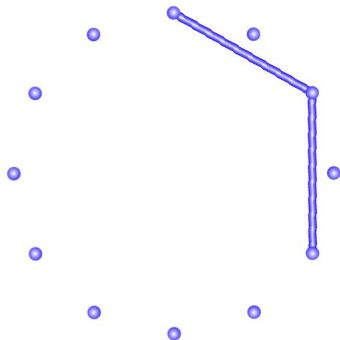
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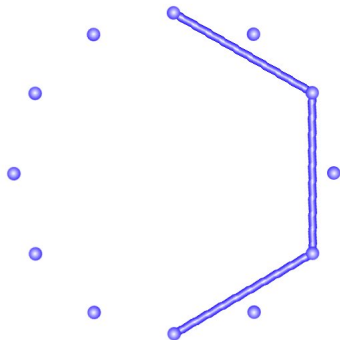
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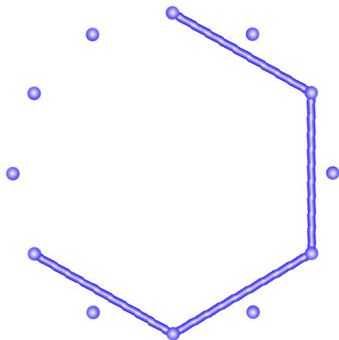
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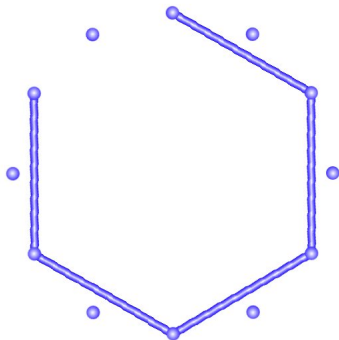
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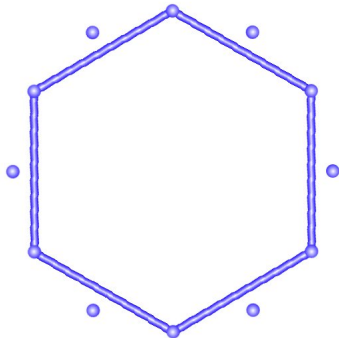
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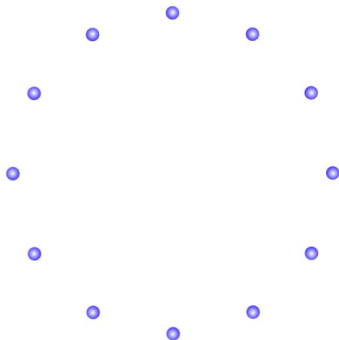
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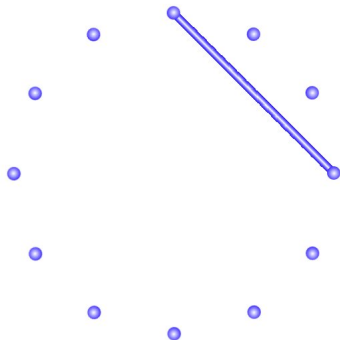
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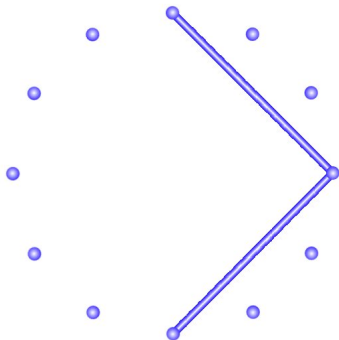
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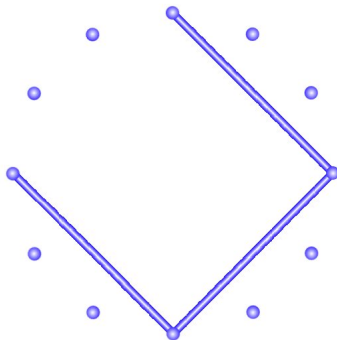
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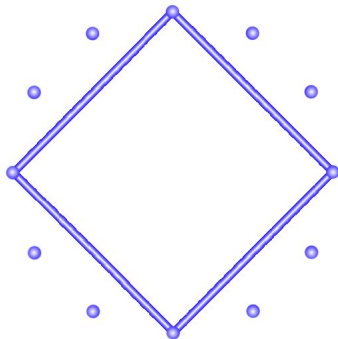
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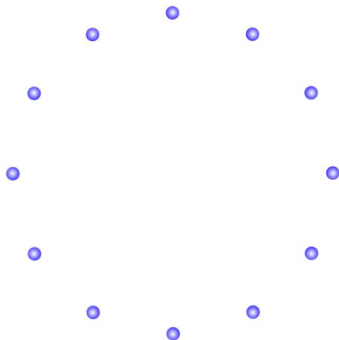
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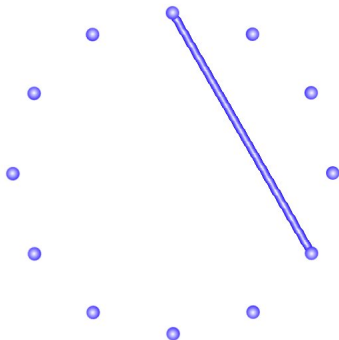
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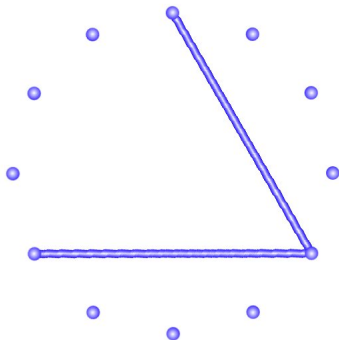
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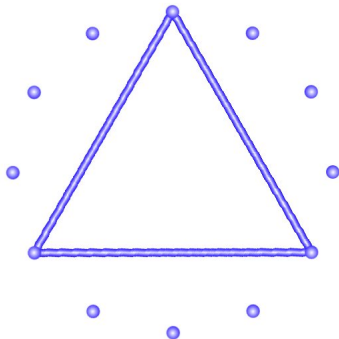
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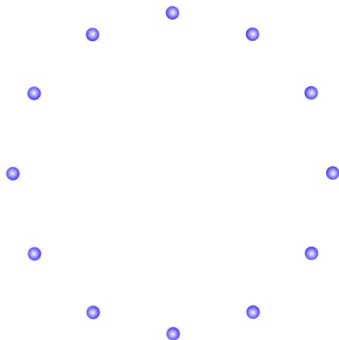
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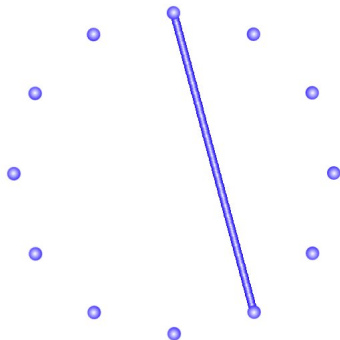
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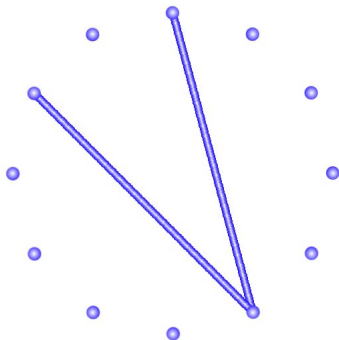
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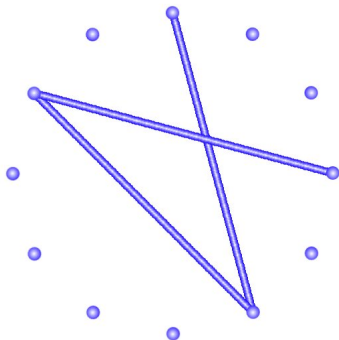
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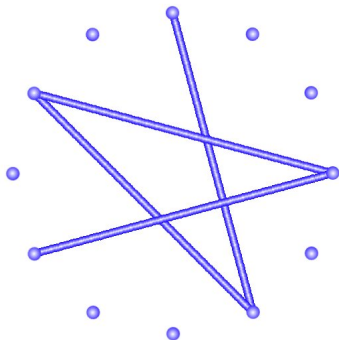
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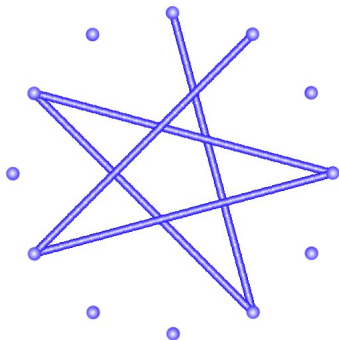
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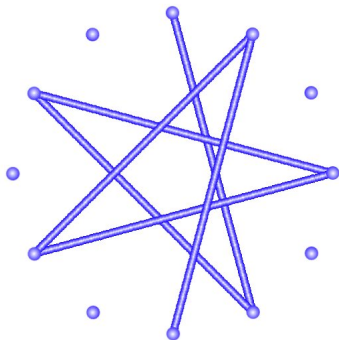
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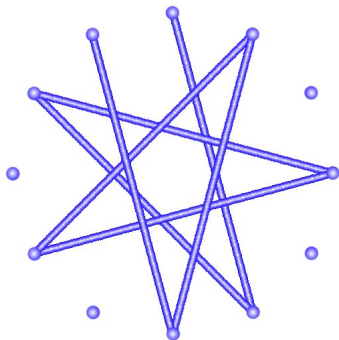
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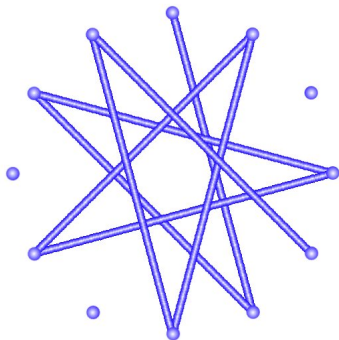
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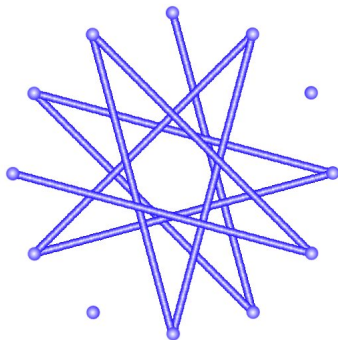
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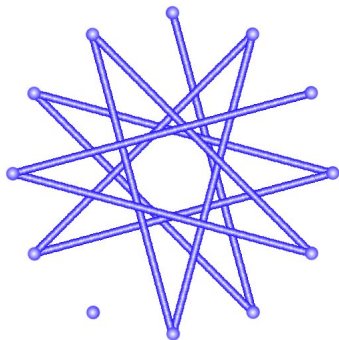
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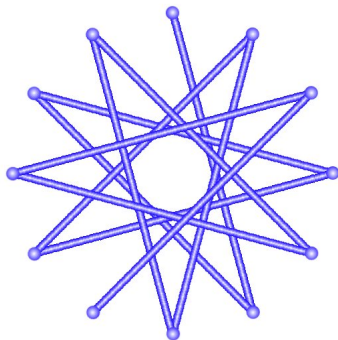
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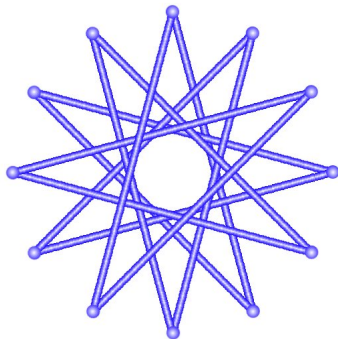
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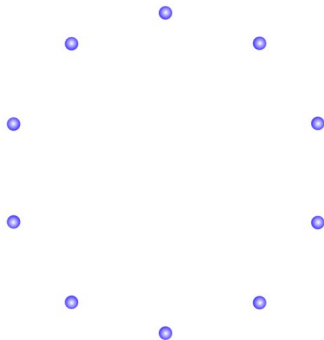


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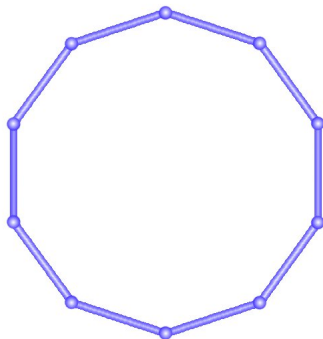


Regular 12-gons

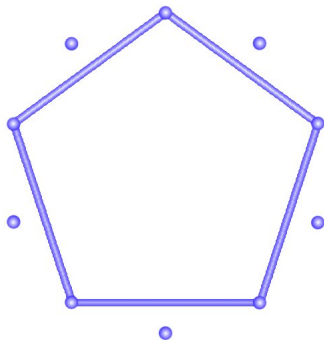
Regular 10-gons



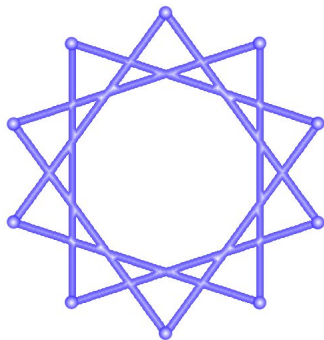
Regular 10-gons



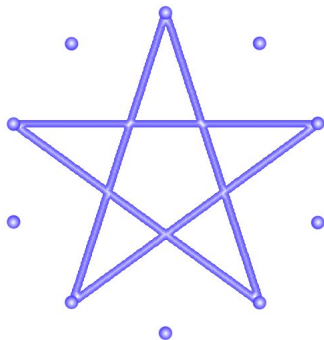
Regular 10-gons



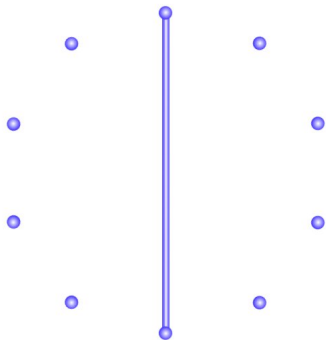
Regular 10-gons



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Regular n -gons

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Irreducible polynomials

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where $\psi_p(n)$ is the **answer**.

So far...

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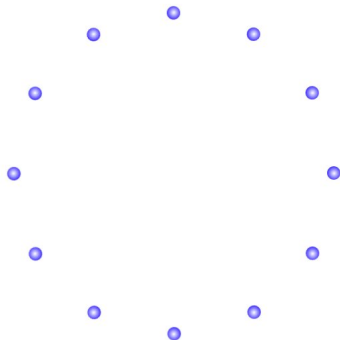
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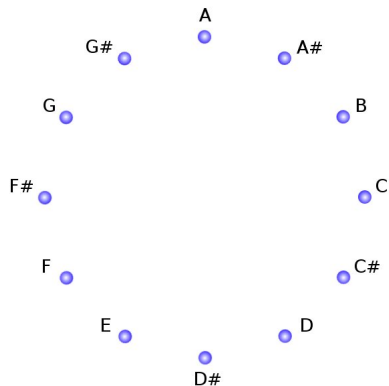
Musical interlude

Silent musical interlude

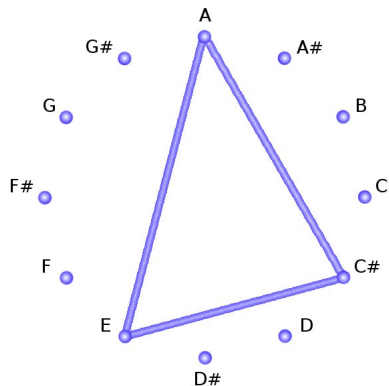
Silent musical interlude



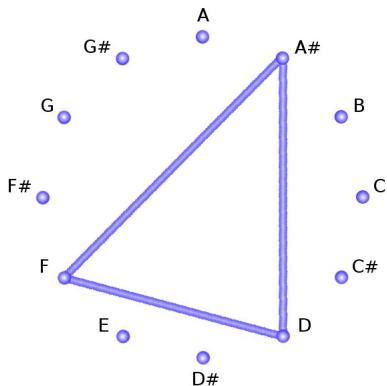
Silent musical interlude



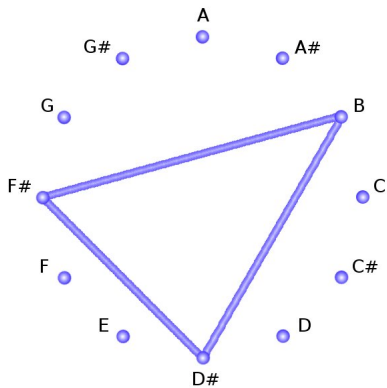
Silent musical interlude



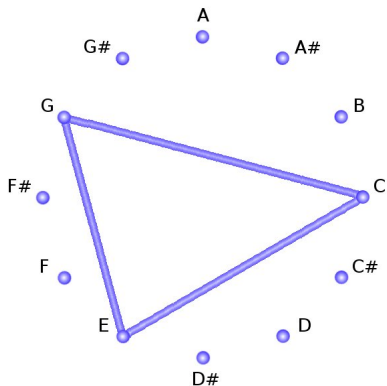
Silent musical interlude



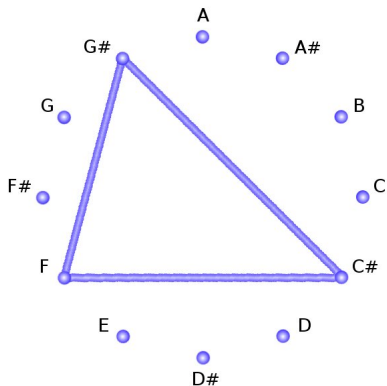
Silent musical interlude



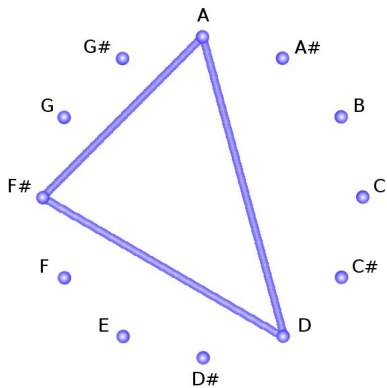
Silent musical interlude



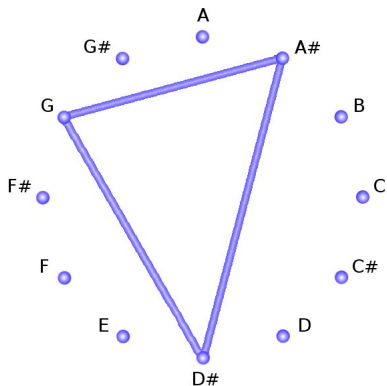
Silent musical interlude



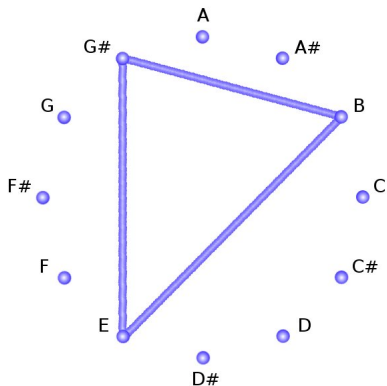
Silent musical interlude



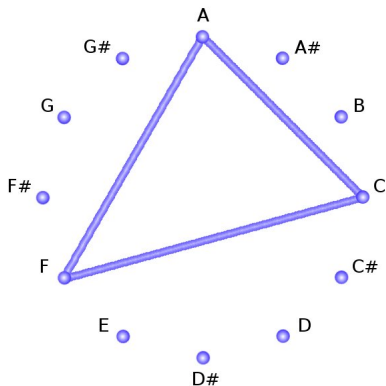
Silent musical interlude



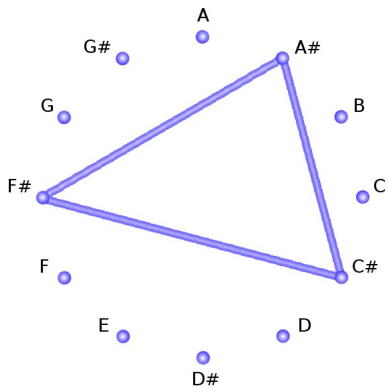
Silent musical interlude



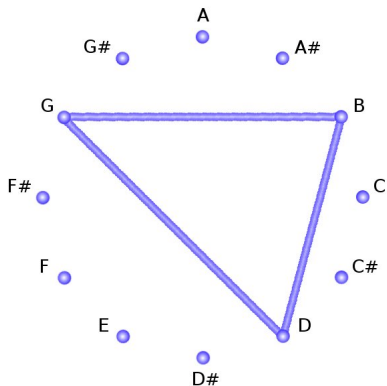
Silent musical interlude



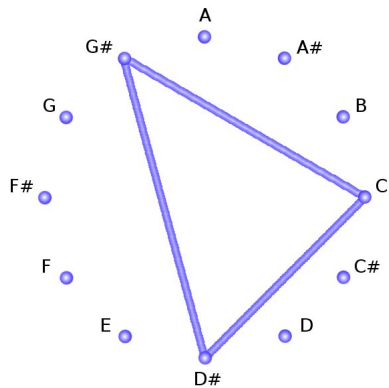
Silent musical interlude



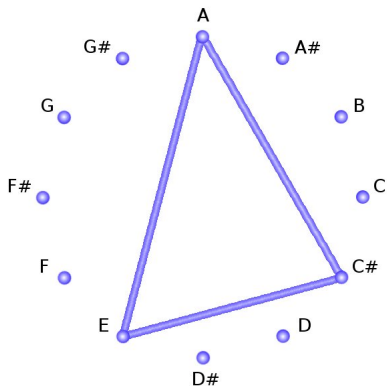
Silent musical interlude



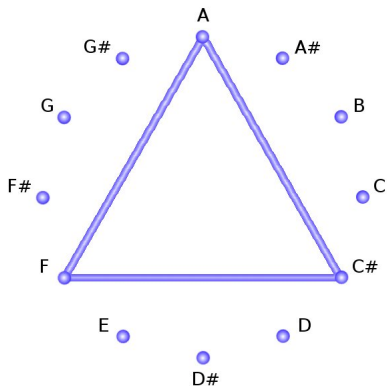
Silent musical interlude



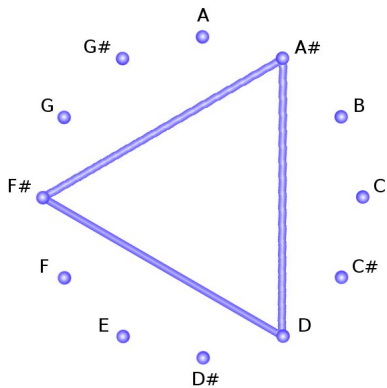
Silent musical interlude



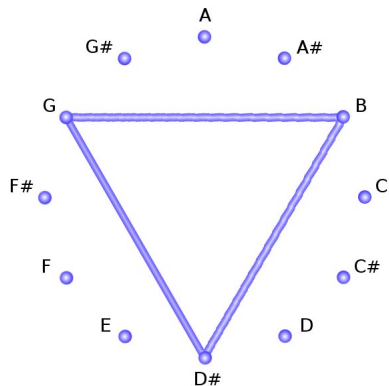
Silent musical interlude



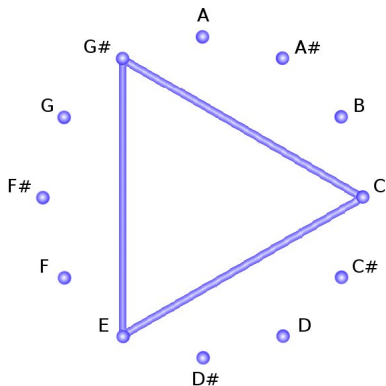
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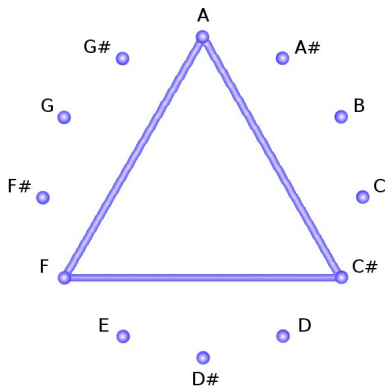
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where $\gamma(n)$ is the **answer**.

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$$\mu(n) = \begin{cases} 0 & \text{if } \exists l > 1, l^2 \mid n \\ (-1)^r & \text{where } r = |\{p \mid p \text{ prime, } p \mid n\}| , \text{ otherwise.} \end{cases}$$

August Ferdinand Möbius (1790-1868)



A. F. Möbius

Applications

- Euler totient function

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$$\gamma(n) = \sum_{d|n} \mu(n/d)2^d$$

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Generalization

$$\phi(n) =$$

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Gian-Carlo Rota (1932-1999)

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