Mackey functors

Throughout this presentation of Mackey functors, the letter R denotes a commutative and associative ring with unit, and G denotes a finite group.

1. Three equivalent definitions

There are (at least) three equivalent definitions of Mackey functors for G over R. Each of them uses a different object associated to G.

1.1. Using the poset of subgroups of G.

This definition of Mackey functors uses the poset of subgroups of G. It goes back to Green ([2])

Definition 1.1.1 : *A Mackey functor M for G over R consists of the following data :*

- *An R-module $M(H)$, for any subgroup H of G.*
- *Maps of R-modules*

  $$t_H^K : M(H) \to M(K) \quad r_H^K : M(K) \to M(H) \quad c_{x,H} : M(H) \to M({}^xH) \quad ,$$

  whenever $H \subseteq K$ are subgroups of G, and $x \in G$.

These maps are subject to the following conditions :

- *(transitivity) If $H \subseteq K \subseteq L$ are subgroups of G, then*

  $$t_K^L t_H^K = t_H^L \qquad r_H^K r_K^L = r_H^L \quad .$$

  If x and y are elements of G, and if H is a subgroup of G, then

  $$c_{y,{}^xH} c_{x,H} = c_{yx,H} \quad .$$

- *(compatibility) If $H \subseteq K$ are subgroups of G, and if $x \in G$, then*

  $$c_{x,K} t_H^K = t_{{}^xH}^{{}^xK} c_{x,H} \qquad c_{x,H} r_H^K = r_{{}^xH}^{{}^xK} c_{x,K} \quad .$$

- *(triviality) If H is a subgroup of G and x is an element of H, then*

  $$t_H^H = r_H^H = c_{x,H} = Id_{M(H)} \quad .$$

- *(Mackey axiom) If $H \subseteq K \supseteq L$ are subgroups of G, then*

$$r^K_H t^K_L = \sum_{x \in [H\backslash K/L]} t^H_{H\cap {}^xL} c_{x,H^x\cap L} r^L_{H^x\cap L} \quad ,$$

where $[H\backslash K/L]$ is a set of representatives of the double cosets modulo H and L in K.

If M and M' are Mackey functors for G over R, a morphism of Mackey functors $f : M \to M'$ is a collection of morphisms $f_H : M(H) \to M'(H)$, where H is a subgroup of G, such that for any subgroups $H \subseteq K$ of G and any $x \in G$

$$f_K t^K_H = t^K_H f_H \qquad f_H r^K_H = r^K_H f_K \qquad f_{{}^xH} c_{x,H} = c_{x,H} f_H \quad .$$

If $f : M \to M'$ and $f' : M' \to M$" are morphisms of Mackey functors, the composition $f'f$ is the morphism of Mackey functors defined by $(f'f)_H = f'_H f_H$, for $H \subseteq G$. The Mackey functors for G over R, with these morphisms and this composition of morphism, form a category, denoted by $\mathsf{Mack}_R(G)$.

It follows in particular from these definitions that if H is a subgroup of G and M is a Mackey functor for G over R, then $M(H)$ is an $R\overline{N}_G(H)$-module, where $\overline{N}_G(H)$ denotes the group $N_G(H)/H$.

1.2. Using the category of G-sets. The second definition of Mackey functors for G over R is due to Dress ([1]). It uses the category G-set of finite G-sets : its objects are the finite sets with a left G-action, its morphisms are the G-equivariant maps, and composition of morphisms is composition of maps.

Definition 1.2.1 : *A Mackey functor $\tilde{M}$ for G over R is a bivariant functor from G-set to the category RG-Mod of left RG-modules, i.e. a pair of functors $(\tilde{M}_*, \tilde{M}^*)$ from G-set to RG-Mod, with $\tilde{M}_*$ covariant, and $\tilde{M}^*$ contravariant, which coincide on objects (i.e. $\tilde{M}_*(X) = \tilde{M}^*(X)$ for any finite G-set X, and this common value is the denoted by $\tilde{M}(X)$). This bivariant functor is subject to the two following additional conditions :*

- *(additivity) If X and Y are finite G-sets, and if i_X and i_Y are the respective inclusion maps from X and Y to their disjoint union $X \sqcup Y$, then the maps*

$$\tilde{M}(X) \oplus \tilde{M}(Y) \xrightarrow{(\tilde{M}_*(i_X), \tilde{M}_*(i_Y))} \tilde{M}(X \sqcup Y) \xrightarrow{\begin{pmatrix} \tilde{M}^*(i_X) \\ \tilde{M}^*(i_Y) \end{pmatrix}} \tilde{M}(X) \oplus \tilde{M}(Y)$$

are mutual inverse isomorphisms.

- *(cartesian squares) If*

$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ Z & \xrightarrow[d]{} & T \end{array}$$

  is a cartesian (i.e. pull-back) square of finite G-sets, then

$$\tilde{M}_*(b)\tilde{M}^*(a) = \tilde{M}^*(d)\tilde{M}_*(c) \quad .$$

A morphism of Mackey functors is a natural transformation of bivariant functors, and composition of morphisms is composition of natural transformations. The category of Mackey functors for this definition will be denoted by $\widetilde{\mathsf{Mack}}_R(G)$.

1.3. Using the Mackey algebra. The following definition is due to Thévenaz and Webb ([4]) :

Definition 1.3.1 : *The Mackey algebra* $\mu_R(G)$ *of* G *over* R *is the associative* R*-algebra defined by generators* $\hat{t}_H^K$, $\hat{r}_H^K$, *and* $\hat{c}_{x,H}$, *for subgroups* $H \subseteq K$ *of* G *and* $x \in G$, *subject to the following relations :*

- *(transitivity) If* $H \subseteq K \subseteq L$ *are subgroups of* G, *then*

$$\hat{t}_K^L \hat{t}_H^K = \hat{t}_H^L \qquad \hat{r}_H^K \hat{r}_K^L = \hat{r}_H^L \quad .$$

  If x *and* y *are elements of* G, *and if* H *is a subgroup of* G, *then*

$$\hat{c}_{y,{}^xH}\hat{c}_{x,H} = \hat{c}_{yx,H} \quad .$$

- *(compatibility) If* $H \subseteq K$ *are subgroups of* G, *and if* $x \in G$, *then*

$$\hat{c}_{x,K}\hat{t}_H^K = \hat{t}_{{}^xH}^{{}^xK}\hat{c}_{x,H} \qquad \hat{c}_{x,H}\hat{r}_H^K = \hat{r}_{{}^xH}^{{}^xK}\hat{c}_{x,K} \quad .$$

- *(triviality) If* H *is a subgroup of* G *and* x *is an element of* H, *then*

$$\hat{t}_H^H = \hat{r}_H^H = \hat{c}_{x,H} \quad .$$

- *(Mackey axiom) If* $H \subseteq K \supseteq L$ *are subgroups of* G, *then*

$$\hat{r}_H^K \hat{t}_L^K = \sum_{x \in [H\backslash K/L]} \hat{t}_{H\cap {}^xL}^H \hat{c}_{x,H^x\cap L} \hat{r}_{H^x\cap L}^L \quad ,$$

  where $[H\backslash K/L]$ *is a set of representatives of the double cosets modulo* H *and* L *in* K.

- *(vanishing) All products of generators, different from those appearing in the previous four relations, are zero.*
- *(identity) The sum of the elements $\hat{t}_H^H$, over all subgroups H of G, is equal to the identity element of $\mu_R(G)$.*

Definition 1.3.2 : *A Mackey functors $\hat{M}$ for G over R is a $\mu_R(G)$-module, and a morphism of Mackey functors is a morphism of $\mu_R(G)$-modules.*

Remark 1.3.3 : The Mackey algebra $\mu_R(G)$ has a natural anti-automorphism σ_G, defined by R-linearity from

$$t_H^K \mapsto r_H^K \qquad r_H^K \mapsto t_H^K \qquad c_{x,H} \mapsto c_{x^{-1},H} \quad ,$$

for $H \subseteq K \subseteq G$ and $x \in G$. In particular, any left $\mu_R(G)$-module can be viewed as a right $\mu_R(G)$-module via this anti-automorphism.

1.4. Equivalence of the definitions.

• $[1 \to 2]$ If M is a Mackey functor in the first sense, and X is a finite G-set, define

$$\tilde{M}(X) = \big(\bigoplus_{x\in X} M(G_x)\big)^G \quad ,$$

where the group G permutes the components of the direct sum via its action on X, and via the maps $c_{g,G_x} : M(G_x) \to M(G_{gx})$, for $g \in G$ and $x \in X$.

If $f : X \to Y$ is a map of G-sets, and if $u = (u_x)_{x\in X}$ is an element of $\tilde{M}(X)$, define the element $M_*(f)(u)$ of $\tilde{M}(Y)$ by

$$\forall y \in Y,\ M_*(f)(u)_y = \sum_{x\in G_y\backslash f^{-1}(y)} t_{G_x}^{G_y} u_x \quad .$$

Conversely, if $v = (v_y)_{y\in Y}$ is an element of $\tilde{M}(Y)$, define the element $M^*(f)(v)$ of $\tilde{M}(X)$ by

$$\forall x \in X,\ (M^*(f)(v)_x = r_{G_x}^{G_{f(x)}} v_{f(x)} \quad .$$

Then $\tilde{M}$ is a Mackey functor in the second sense, and the correspondence $M \mapsto \tilde{M}$ is an equivalence of category from $\mathsf{Mack}_R(G)$ to $\widetilde{\mathsf{Mack}}_R(G)$.

• $[2 \to 1]$ If $\tilde{M}$ is a Mackey functor in the second sense, and if H is a subgroup of G, set

$$M(H) = \tilde{M}(G/H) \quad .$$

If $H \subseteq K$ are subgroups of G, let $p_H^K : G/H \to G/K$ denote the natural projection map. Define then

$$t_H^K = \tilde{M}_*(p_H^K) \qquad r_H^K = \tilde{M}^*(p_H^K) \quad .$$

If $x \in G$, and if H is a subgroup of G, let $\gamma_{x,H}$ denote the map of G-sets from G/H to $G/{}^xH$ defined by $\gamma_{x,H}(gH) = gx^{-1x}H$. Then set

$$c_{x,H} = \tilde{M}_*(\gamma_{x,H}) \quad .$$

Then M is a Mackey functor in the first sense, and the correspondence $\tilde{M} \mapsto M$ extends to an equivalence of categories from $\widetilde{\mathsf{Mack}}_R(G)$ to $\mathsf{Mack}_R(G)$, which is a quasi-inverse to the equivalence $1 \to 2$.

• $[1 \to 3]$ If M is a Mackey functor in the first sense, set

$$\hat{M} = \mathop{\oplus}_{H \subseteq G} M(H) \quad .$$

Let the generator $\hat{t}_H^K$ of $\mu_R(G)$ act on $\hat{M}$ by sending the component $M(L)$, for $L \neq H$, to 0, and the component $M(H)$ to $M(K)$ via the map t_H^K. Similarly, let $\hat{r}_H^K$ send $M(L)$ to 0 if $L \neq K$, and to $M(H)$ via the map r_H^K otherwise. Finally, if $x \in G$, let $\hat{c}_{x,H}$ send the component $M(L)$ to 0 if $L \neq H$, and to $M({}^xH)$ via the map $c_{x,H}$ otherwise.

Then $\hat{M}$ is a $\mu_R(G)$-module, and the correspondence $M \mapsto \hat{M}$ extends to an equivalence of categories from $\mathsf{Mack}_R(G)$ to $\mu_R(G)$-Mod.

• $[3 \to 1]$ Let $\hat{M}$ be a $\mu_R(G)$-module. If H is a subgroup of G, define

$$M(H) = \hat{t}_H^H \hat{M} \quad .$$

The relations on the generators of $\mu_R(G)$ imply that if $H \subseteq K$ and $x \in G$, then

$$\hat{t}_H^K M(H) \subseteq M(K) \qquad \hat{r}_H^K M(K) \subseteq M(H) \qquad \hat{c}_{x,H} M(H) \subseteq M({}^xH) \quad ,$$

and this defines respectively a map $t_H^K : M(H) \to M(K)$, a map $r_H^K : M(K) \to M(H)$, and a map $c_{x,H} : M(H) \to M({}^xH)$.

Now M is a Mackey functor in the first sense, and the correspondence $\hat{M} \mapsto M$ extends to an equivalence of categories from $\mu_R(G)$-Mod to $\mathsf{Mack}_R(G)$, which is a quasi-inverse to the equivalence $1 \to 3$.

Notation 1.4.1 : *In view of these equivalence, the three categories* $\mathsf{Mack}_R(G)$, $\widetilde{\mathsf{Mack}}_R(G)$ *and* $\mu_R(G)$-Mod *will sometimes be identified, and a Mackey functor will be considered as an object of any of them. This will lead in particular to equalities such as* $M(H) = M(G/H)$ *for a Mackey functor* M *and a subgroup* H *of* G.

2. Examples

2.1. Representation groups. The various representation groups attached to subgroups of G can generally be viewed as Mackey functors :

• Let k be a field. If H is a subgroup of G, denote by $R_k(G)$ the Grothendieck group of the category of finitely generated kG-modules, for relations given by short exact sequences.

If $H \subseteq K$ are subgroups of G, the restriction of modules induces a map $r_H^K = \mathrm{Res}_H^K : R_k(K) \to R_k(H)$, whereas induction of modules induces a map $t_H^K = \mathrm{Ind}_H^K : R_k(H) \to R_k(K)$. If $x \in G$, there is an obvious conjugation map $c_{x,H} : R_k(H) \to R_k({}^xH)$. With this notation R_k becomes a Mackey functor for G over $\mathbb{Z}$.

One can check easily that if X is a finite G-set, then $R_k(X)$ is the Grothendieck group of G-equivariant k-vector bundles over X.

Similarly, one can consider the Grothendieck group $P_k(G)$ of the category of finitely generated *projective* kG-modules, for relations given by direct sum decomposition. It is also a Mackey functor for the previous operations of restriction, induction, and conjugation.

• If H is a subgroup of G, denote by $B(H)$ the Burnside group of H, i.e. the Grothendieck group of the category of finite H-sets, for relations given by disjoint unions decomposition. Then if $H \subseteq K$ are subgroups of G, the restriction of K-sets gives a map $r_H^K = \mathrm{Res}_H^K : B(K) \to B(H)$. Similarly, induction of H-sets (defined by $\mathrm{Ind}_H^K X = K \times_H X$) induces a map $B(H) \to B(K)$. If $x \in G$, there is again an obvious conjugation map $c_{x,H} : B(H) \to B({}^xH)$, and B is a Mackey functor for G over $\mathbb{Z}$.

One can show that for any finite G-set X, the group $B(X)$ is the Grothendieck group of the category of finite G-sets over X, for relations given by disjoint union decomposition.

2.2. (Co)homology functors. Let M be an RG-module, and l be a non-negative integer. If K is a subgroup of G, define $M(K)$ as the cohomology group $H^l(K, \mathrm{Res}_K^G M)$ (or $H_l(K, \mathrm{Res}_K^G M)$). Then the operations of transfer, restriction, and conjugation, define a structure of Mackey functor on M. If X is a finite G-set, then $\tilde{M}(X) \cong H^l(G, RX \otimes_R M)$ (or $H_l(G, RX \otimes_R M)$). Similarly, the Tate (co)homology groups have a natural structure of Mackey functor.

These Mackey functors have the following additional property : the composition $t_H^K r_H^K$ is equal to the multiplication by $|K : H|$, for any subgroups $H \subseteq K$ of G. Mackey functors with this property are called *cohomological*.

3. Fixed points, fixed quotients, and simple Mackey functors

Since the category of Mackey functors for G over R is equivalent to the category of modules over an R-algebra, it is an abelian category. One can try to classify and describe its simple and projective objects.

3.1. Fixed points and fixed quotients. Let V be an RG-module. The zero-th cohomology functor $H^0(-,V)$ is denoted by FP_V, and it is called the *fixed points functor* associated to V. Similarly, the zero-th homology functor $H_0(-,V)$ is denoted by FQ_V, and it is called the *fixed quotient functor* associated to V. If H is a subgroup of G

$$FP_V(H) = V^H \qquad FQ_V(H) = V_H \quad .$$

If $H \subseteq K$ are subgroups of G, then the maps t_H^K and r_H^K for FP_V are the relative trace maps and restriction maps, i.e.

$$\forall v \in V^H,\ t_H^K(v) = \sum_{x \in K/H} xv \qquad \forall w \in V^K,\ r_H^K(v) = v \quad .$$

If $g \in G$, then the conjugation map $c_{g,H}$ is given by

$$\forall v \in V^H,\ c_{g,H}(v) = gv \quad .$$

Similarly, the maps t_H^K and r_H^K for FQ_V are given by

$$\forall v \in V_H,\ t_H^K(v) = v \qquad \forall w \in V_K,\ r_H^K(w) = \sum_{x \in H\backslash K} xw \quad ,$$

and the map $c_{g,H}$ is given by $c_{g,H}(v) = gv$, for $v \in V_H$.

If X is a finite G-set, one can show that

$$FP_V(X) = \mathrm{Hom}_{RG}(RX, V) \qquad FQ_V(X) = RX \otimes_{RG} V \quad ,$$

where RX is the permutation module associated to X, i.e. the free R-module with basis X.

If $f : X \to Y$ is a map of finite G-sets, let $Rf : RX \to RY$ and ${}^tRf : RY \to RX$ be the R-linear maps defined by

$$\forall x \in X,\ (Rf)(x) = f(x) \qquad \forall y \in Y,\ ({}^tRf)(y) = \sum_{x \in f^{-1}(y)} x \quad .$$

Then the map $(FP_V)_*(f)$ is given by composition with ${}^tRf$, and the map $(FP_V)^*(f)$ is given by composition with Rf. Similarly, the map $(FQ_V)_*(f)$ is the map $Rf \otimes_{RG} Id_V$, and $(FQ_V)^*(f) = {}^tRf \otimes_{RG} Id_V$.

The main property of these constructions is the following :

Proposition 3.1.1 : *The correspondence $V \mapsto FP_V$ (resp. $V \mapsto FQ_V$) is a functor from RG-Mod to $\mathsf{Mack}_R(G)$, which is right adjoint (resp. left adjoint) to the evaluation functor $M \mapsto M(1)$.*

Notation 3.1.2 : *Let H be a subgroup of G, and let V be an $R\overline{N}_G(H)$-module. The composite bivariant functor*

$$X \mapsto X^H \mapsto FP_V(X^H) = \mathrm{Hom}_{R\overline{N}_G(H)}\Big(R(X^H), V\Big)$$

from G-set to R-Mod is denoted by $FP_{H,V}$. If X is a finite G-set, set

$$S_{H,V}(X) = Tr_1^{\overline{N}_G(H)}\Big(\mathrm{Hom}_R\Big(R(X^H), V\Big)\Big) \subset FP_{H,V}(X) \quad ,$$

where $Tr_1^{\overline{N}_G(H)}$ is the relative trace map.

One can can show that $FP_{H,V}$ is a Mackey functor, and that $S_{H,V}$ is a sub-Mackey functor of $FP_{H,V}$.

Definition 3.1.3 : *Let M be a Mackey functor for G over R. A minimal subgroup for M is a subgroup H of G, such that $M(H) \neq 0$, but $M(K) = 0$ for every proper subgroup K of H.*

Proposition 3.1.4 : *(Thévenaz-Webb [3])*

1. *Let H be a subgroup of G, and V be a simple $R\overline{N}_G(H)$-module. Then $S_{H,V}$ is a simple Mackey functor for G over R. Moreover H is a minimal subgroup for $S_{H,V}$, and $S_{H,V}(H) \cong V$.*
2. *Let S be a simple Mackey functor for G over R. Let H be a minimal subgroup for S, and set $V = S(H)$. Then V is a simple $R\overline{N}_G(H)$-module, and S is isomorphic to $S_{H,V}$.*
3. *If K is a subgroup of G and W is a simple $R\overline{N}_G(K)$-module, the simple functors $S_{H,V}$ and $S_{K,W}$ are isomorphic if and only if the pairs (H, V) and (K, W) are conjugate under G.*

4. The Dress construction and projective Mackey functors

4.1. The Dress construction. Let M be a Mackey functor for G over R, and let X be a fixed finite G-set. The functor $\pi_X : Y \mapsto Y \times X$ is an endofunctor of the category G-set, which commutes to disjoint unions, and preserves cartesian squares. If M is a Mackey functor for G over R, it follows that the functor $M \circ \pi_X$ is also a Mackey functor for G over R, which is denoted by M_X, and called the *Dress construction* associated to M and X.

The construction $M \mapsto M_X$ is an endofunctor of the category $\mathsf{Mack}_R(G)$. One can show that this functor is self-adjoint.

4.2. The Burnside functor. Let RB be the Burnside functor over R, defined by composing the Burnside functor B from G-set to $\mathbb{Z}$-Mod with the functor $A \mapsto R \otimes_{\mathbb{Z}} A$ from $\mathbb{Z}$-Mod to R-Mod. The main properties of RB are summarized in the following proposition :

Proposition 4.2.1 : *1. Let M be a Mackey functor for G over R. Then*

$$\mathrm{Hom}_{\mathsf{Mack}_R(G)}(RB, M) \cong M(G) \quad .$$

2. *More generally, if X is a finite G-set, then*

$$\mathrm{Hom}_{\mathsf{Mack}_R(G)}(RB_X, M) \cong M(X) \quad .$$

3. *The functor B_X is a projective Mackey functor, and any Mackey functor is isomorphic to a quotient of a direct sum of such Mackey functors.*

4. *Let $\Omega = \bigsqcup_{H \subseteq G} G/H$. Then RB_Ω is a progenerator in $\mathsf{Mack}_R(G)$, and there is an isomorphism of R-algebras*

$$\mathrm{End}_{\mathsf{Mack}_R(G)}(RB_\Omega) \cong \mu_R(G) \quad .$$

5. *There is an isomorphism $\mathrm{End}_{\mathsf{Mack}_R(G)}(RB_\Omega) \cong RB(\Omega \times \Omega)$, and the corresponding algebra structure on $RB(\Omega \times \Omega)$ is obtained by linearity from the pull-back product $(X, Y) \mapsto X \times_\Omega Y$ in the following diagram*

$$\begin{tikzcd}
 & & X \times_\Omega Y \arrow[dl] \arrow[dr] & & \\
 & X \arrow[dl] \arrow[dr] & & Y \arrow[dl] \arrow[dr] & \\
\Omega & & \Omega & & \Omega
\end{tikzcd} \quad .$$

With this identification $\mu_R(G) \cong RB(\Omega \times \Omega)$, the anti-automorphism σ_G of $\mu_R(G)$ sends the G-set $\Big(X, (f, g)\Big)$ over $\Omega \times \Omega$ to $\Big(X, (g, f)\Big)$.

References

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