Idempotents of double Burnside algebras, L-enriched bisets, and decomposition of p-biset functors

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Abstract: Let R be a (unital) commutative ring, and G be a finite group with order invertible in R. We introduce new idempotents $\epsilon_{T,S}^G$ in the double Burnside algebra $RB(G,G)$ of G over R, indexed by conjugacy classes of minimal sections (T,S) of G (i.e. sections such that $S \leq \Phi(T)$). These idempotents are orthogonal, and their sum is equal to the identity. It follows that for any biset functor F over R, the evaluation $F(G)$ splits as a direct sum of specific R-modules indexed by minimal sections of G, up to conjugation.

The restriction of these constructions to the biset category of p-groups, where p is a prime number invertible in R, leads to a decomposition of the category of p-biset functors over R as a direct product of categories $\mathcal{F}_L$ indexed by *atoric* p-groups L up to isomorphism.

We next introduce the notions of L*-enriched biset* and L*-enriched biset functor* for an arbitrary finite group L, and show that for an atoric p-group L, the category $\mathcal{F}_L$ is equivalent to the category of L-enriched biset functors defined over elementary abelian p-groups.

Finally, the notion of *vertex* of an indecomposable p-biset functor is introduced (when $p \in R^\times$), and when R is a field of characteristic different from p, the objects of the category $\mathcal{F}_L$ are characterized in terms of vertices of their composition factors.

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1. Introduction

Let R denote throughout a commutative ring (with identity element). For a finite group G, we consider the double Burnside algebra $RB(G,G)$ of G over R. In the case where the order of G is invertible in R, we introduce idempotents $\epsilon_{T,S}^G$ in $RB(G,G)$, indexed by the set $\mathcal{M}(G)$ of minimal sections of G, i.e. the set of pairs (T,S) of subgroups of G with $S \trianglelefteq T$ and $S \leq \Phi(T)$, where $\Phi(T)$ is the Frattini subgroup of T (such sections have been considered in Section 5 of [9]). The idempotent $\epsilon_{T,S}^G$ only depends on the conjugacy class of (T,S) in G. Moreover, the idempotents $\epsilon_{T,S}^G$, where (T,S) runs through a set $[\mathcal{M}(G)]$ of representatives of orbits of G acting on $\mathcal{M}(G)$ by conjugation, are orthogonal, and their sum is equal to the identity element of $RB(G,G)$.

The idempotent $\epsilon_{G,\mathbf{1}}^G$ plays a special role in our construction, and it is denoted by $\varphi_{\mathbf{1}}^G$. In particular, when F is a biset functor over R (and the order of G is invertible in R), we set $\delta_\Phi F(G) = \varphi_{\mathbf{1}}^G F(G)$. We show that $\delta_\Phi F(G)$ consists of those elements $u \in F(G)$ such that $\mathrm{Res}_H^G u = 0$ whenever H is a proper subgroup of G, and $\mathrm{Def}_{G/N}^G u = 0$ whenever N is a non-trivial normal subgroup of G contained in $\Phi(G)$. This yields moreover a decomposition

$$F(G) \cong \big(\mathop{\oplus}_{(T,S)\in\mathcal{M}(G)} \delta_\Phi F(T/S)\big)^G \cong \mathop{\oplus}_{(T,S)\in[\mathcal{M}(G)]} \delta_\Phi F(T/S)^{N_G(T,S)/T} \;\; .$$

In view of the fact that the Frattini subgroup is well behaved for p-groups, it is natural to restrict these constructions to the biset category $R\mathcal{C}_p$ of p-groups with coefficients in R, where p is a prime invertible in R, and to consider p-biset functors over R. Then we get orthogonal idempotents b_L in the *center* of $R\mathcal{C}_p$, indexed by *atoric* p-groups, i.e. finite p-groups which cannot be split as a direct product $C_p \times Q$, for some p-group Q. We show next that every finite p-group P admits a unique largest atoric quotient $P^@$, well defined up to isomorphism, and that there exists an elementary abelian p-subgroup E of P (non unique in general) such that $P \cong E \times P^@$. For a given atoric p-group L, we introduce a category $R\mathcal{C}_p^{\sharp L}$, defined as a quotient of the subcategory of $R\mathcal{C}_p$ consisting of p-groups P such that $P^@ \cong L$. This leads to a decomposition of the category $\mathcal{F}_{p,R}$ of p-biset functors over R as a direct product

$$\mathcal{F}_{p,R} \cong \prod_{L\in[\mathcal{A}t_p]} \mathsf{Fun}_R\big(R\mathcal{C}_p^{\sharp L}, R\text{-}\mathsf{Mod}\big)$$

of categories of representations of $R\mathcal{C}_p^{\sharp L}$ over R, where L runs through a set $[\mathcal{A}t_p]$ of isomorphism classes of atoric p-groups. Similar questions on idempotents in double Burnside algebras and decomposition of biset functors categories have been considered by L. Barker ([1]), R. Boltje and S. Danz ([2], [3]), R. Boltje and B. Külshammer ([4]), and P. Webb ([16]).

In particular, via the above decomposition, to any indecomposable p-biset functor F is associated a unique atoric p-group, called the *vertex* of F. We show that this vertex is isomorphic to $Q^@$, for any p-group Q such that $F(Q) \neq \{0\}$ but F vanishes on any proper subquotient of Q.

Going back to arbitrary finite groups, we next introduce the notions of *L-enriched biset* and *L-enriched biset functor*, and show that when L is an atoric p-group, the abelian category $\mathsf{Fun}_R\big(R\mathcal{C}_p^{\sharp L}, R\text{-}\mathsf{Mod}\big)$ is equivalent to the category of L-enriched biset functors from elementary abelian p-groups to R-modules.

The paper is organized as follows: Section 2 is a review of definitions and basic results on Burnside rings and biset functors. Section 3 is concerned with the algebra $\mathcal{E}(G)$ obtained by "cutting" the double Burnside algebra $RB(G,G)$ of a finite group G by the idempotent $\widetilde{e_G^G}$ corresponding to the "top" idempotent e_G^G of the Burnside algebra $RB(G)$. Orthogonal idempotents φ_N^G of $\mathcal{E}(G)$ are introduced, indexed by normal subgroups N of G contained in $\Phi(G)$. It is shown moreover that if G is nilpotent, then $\varphi_{\mathbf{1}}^G$ is central in $\mathcal{E}(G)$. In Section 4, the idempotents $\epsilon_{T,S}^G$ of $RB(G,G)$ are introduced, leading in Section 5 to the corresponding direct sum decomposition of the evaluation at G of any biset functor over R. In Section 6, atoric p-groups are introduced, and their main properties are stated. In Section 7, the biset category of p-groups over R is considered, leading to a splitting of the category $\mathcal{F}_{p,R}$ of p-biset functors over R as a direct product of abelian categories $\mathcal{F}_L = \mathsf{Fun}_R\big(R\mathcal{C}_p^{\sharp L}, R\text{-}\mathsf{Mod}\big)$ indexed by atoric p-groups L up to isomorphism. In Section 8, for an arbitrary finite group L, the notions of L-enriched biset and L-enriched biset functor are introduced, and it is shown that when L is an atoric p-group, the category $\mathcal{F}_L$ is equivalent to the category of L-enriched biset functors on elementary abelian p-groups. Finally, in Section 9, for a given atoric p-group L, and when p is invertible in R, the structure of the category $\mathcal{F}_L$ is considered, and the notion of vertex of an indecomposable p-biset functor over R is introduced. In particular, when R is a field of characteristic different from p, it is shown that the objects of $\mathcal{F}_L$ are those p-biset functors all composition factors of which have vertex L.

Contents

2. Review of Burnside rings and biset functors

This section recalls some basic definitions and notation on bisets, Burnside rings, and biset functors. Details can be found in [7].

2.1. Let G be a finite group, let s_G denote the set of subgroups of G, let $\overline{s_G}$ denote the set of conjugacy classes of subgroups of G, and let $[s_G]$ denote a set of representatives of $\overline{s_G}$.

Let $B(G)$ denote the Burnside ring of G, i.e. the Grothendieck ring of the category of finite G-sets. It is a commutative ring, with an identity element, equal to the class of a G-set of cardinality 1. The additive group $B(G)$ is a free abelian group on the set $\{[G/H] \mid H \in [s_G]\}$ of isomorphism classes of transitive G-sets.

2.2. • When G and H are finite groups, and L is a subgroup of $G \times H$, set

$$\begin{aligned} p_1(L) &= \{g \in G \mid \exists h \in H,\ (g,h) \in L\} \ , \\ p_2(L) &= \{h \in H \mid \exists g \in G,\ (g,h) \in L\} \ , \\ k_1(L) &= \{g \in G \mid (g,1) \in L\} \ , \\ k_2(L) &= \{h \in H \mid (1,h) \in L\} \ . \end{aligned}$$

Recall that $k_i(L) \trianglelefteq p_i(L)$, for $i \in \{1,2\}$, that $\big(k_1(L) \times k_2(L)\big) \trianglelefteq L$, and that there are canonical isomorphisms ([7], Proposition 2.3.21)

$$p_1(L)/k_1(L) \cong L/\big(k_1(L) \times k_2(L)\big) \cong p_2(L)/k_2(L) \ .$$

Set moreover $q(L) = L/\big(k_1(L) \times k_2(L)\big)$.

• When Z is a subgroup of G, set

$$\Delta(Z) = \{(z,z) \mid z \in Z\} \leq (G \times G) \ .$$

When N is a normal subgroup of a subgroup H of G, set

$$\Delta_N(H) = \{(a,b) \in G \times G \mid a,b \in H,\ ab^{-1} \in N\} \ .$$

It is a subgroup of $G \times G$.

• When G, H, and K are groups, when $L \leq (G \times H)$ and $M \leq (H \times K)$, set

$$L * M = \{(g,k) \in (G \times K) \mid \exists h \in H,\ (g,h) \in L \text{ and } (h,k) \in M\} \ .$$

It is a subgroup of $(G \times K)$.

2.3. When G and H are finite groups, a (G,H)-*biset* U is a set endowed with a left action of G and a right action of H which commute. In other words U is a $G \times H^{op}$-set, where H^{op} is the opposite group of H. The *opposite biset* U^{op} is the (H, G)-biset equal to U as a set, with actions defined for $h \in H$, $u \in U$ and $g \in G$ by $h \cdot u \cdot g$ (in U^{op}) $= g^{-1}uh^{-1}$ (in U).

The Burnside group $B(G, H)$ is the Grothendieck group of the category of finite (G, H)-bisets. It is a free abelian group on the set of isomorphism classes $[(G \times H)/L]$, for $L \in [s_{G\times H}]$, where the (G, H)-biset structure on $(G \times H)/L$ is given by

$$\forall a, g \in G,\ \forall b, h \in H,\ a \cdot (g, h)L \cdot b = (ag, b^{-1}h)L \ \ .$$

When G, H, and K are finite groups, there is a unique bilinear product

$$\times_H : B(G, H) \times B(H, K) \to B(G, K)$$

induced by the usual product $(U, V) \mapsto U \times_H V = (U \times V)/H$ of bisets, where the right action of H on $U \times V$ is defined for $u \in U$, $v \in V$ and $h \in H$ by $(u, v)\cdot h = (uh, h^{-1}v)$. As the group H is generally clear from the context, this product will often simply be denoted $(\alpha, \beta) \mapsto \alpha\beta$.

This leads to the following definitions:

2.4. Definition: *The* biset category of finite groups $\mathcal{C}$ *is defined as follows:*

- *The objects of $\mathcal{C}$ are the finite groups.*
- *When G and H are finite groups,*

$$\mathrm{Hom}_{\mathcal{C}}(G, H) = B(H, G) \ \ .$$

- *When G, H, and K are finite groups, the composition*

$$\circ : \mathrm{Hom}_{\mathcal{C}}(H, K) \times \mathrm{Hom}_{\mathcal{C}}(G, H) \to \mathrm{Hom}_{\mathcal{C}}(G, K)$$

  is the product

$$\times_H : B(K, H) \times B(H, G) \to B(K, G) \ \ .$$

- *The identity morphism of the group G is the class of the set G, viewed as a (G, G)-biset by left and right multiplication.*

A biset functor *is an additive functor from $\mathcal{C}$ to the category of abelian groups.*

When R is a commutative (unital) ring, the category $R\mathcal{C}$ is defined similarly by extending coefficients to R, i.e. by setting

$$\mathrm{Hom}_{R\mathcal{C}}(G,H) = R \otimes_{\mathbb{Z}} B(H,G) \enspace ,$$

which will be simply denoted by $RB(H,G)$. A *biset functor over* R is an R-linear functor from $R\mathcal{C}$ to the category R-Mod of R-modules. The category of biset functors over R (where morphisms are natural transformations of functors) is denoted by $\mathcal{F}_R$.

For simplicity, the composition of morphisms $\alpha \in RB(H,G)$ and $\beta \in RB(K,H)$ in the category $R\mathcal{C}$ will generally be simply denoted by $\beta\,\alpha$ instead of $\beta \times_H \alpha$.

The correspondence sending a (G,H)-biset U to its opposite U^{op} extends to an isomorphism of R-modules $RB(G,H) \to RB(H,G)$. These isomorphisms give an equivalence of R-linear categories from $R\mathcal{C}$ to its opposite category, which is the identity on objects.

2.5. Let G and H be finite groups, and F be a biset functor (with values in R-Mod). For any finite (H,G)-biset U, the isomorphism class $[U]$ of U belongs to $B(H,G)$, and it yields an R-linear map $F([U]) : F(G) \to F(H)$, simply denoted by $F(U)$, or even $f \in F(G) \mapsto U(f) \in F(H)$. This is a very convenient abuse of notation. In particular:

- When H is a subgroup of G, denote by Ind_H^G the set G, viewed as a (G,H)-biset for left and right multiplication, and by Res_H^G the same set, viewed as an (H,G)-biset. This gives a map $\mathrm{Ind}_H^G : F(H) \to F(G)$, called induction, and a map $\mathrm{Res}_H^G : F(G) \to F(H)$, called restriction. We observe that $(\mathrm{Ind}_H^G)^{op}$ and Res_H^G are isomorphic (H,G)-bisets (and similarly $(\mathrm{Res}_H^G)^{op} \cong \mathrm{Ind}_H^G$ as (G,H)-bisets).

- When N is a normal subgroup of G, let $\mathrm{Inf}_{G/N}^G$ denote the set G/N, viewed as a $(G,G/N)$-biset for the left action of G, and right action of G/N by multiplication. Also let $\mathrm{Def}_{G/N}^G$ denote the set G/N, viewed as a $(G/N,G)$-biset. This gives a map $\mathrm{Inf}_{G/N}^G : F(G/N) \to F(G)$, called inflation, and a map $\mathrm{Def}_{G/N}^G : F(G) \to F(G/N)$, called deflation. We observe that $(\mathrm{Inf}_{G/N}^G)^{op}$ and $\mathrm{Def}_{G/N}^G$ are isomorphic $(G/N,G)$-bisets (and similarly $(\mathrm{Def}_{G/N}^G)^{op} \cong \mathrm{Inf}_{G/N}^G$ as $(G,G/N)$-bisets).

- Finally, when $f : G \to G'$ is a group isomorphism, denote by $\mathrm{Iso}(f)$ the set G', viewed as a (G',G)-biset for left multiplication in G', and right action of G given by multiplication by the image under f. This gives a map $\mathrm{Iso}(f) : F(G) \to F(G')$, called transport by isomorphism. Clearly $\big(\mathrm{Iso}(f)\big)^{op} \cong \mathrm{Iso}(f^{-1})$ as (G,G')-bisets.

The above bisets $\mathrm{Ind}_H^G, \mathrm{Res}_H^G, \mathrm{Inf}_{G/N}^G, \mathrm{Def}_{G/N}^G$ and $\mathrm{Iso}(f)$ are called *elementary bisets*, as they generate the biset category, in the following sense: when G and H are finite groups, any (G, H)-biset is a disjoint union of transitive ones. It follows that any element of $B(G, H)$ is a linear combination of morphisms of the form $[(G \times H)/L]$, where $L \in s_{G\times H}$. Moreover, any such morphism factors as

$$(\mathbf{2.6}) \quad [(G \times H)/L] = \mathrm{Ind}_{p_1(L)}^G \,\mathrm{Inf}_{p_1(L)/k_1(L)}^{p_1(L)} \,\mathrm{Iso}(f)\, \mathrm{Def}_{p_2(L)/k_2(L)}^{p_2(L)} \,\mathrm{Res}_{p_2(L)}^H \ ,$$

where $f : p_2(L)/k_2(L) \to p_1(L)/k_1(L)$ is the canonical group isomorphism.

It follows that elementary bisets satisfy a (rather long) list of relations: the composition of two of them, when it makes sense, can always be expressed as a sum of compositions of the form Ind Inf Iso Def Res (in that order), given explicitly by (2.6). These compatibility relations are listed in Section 1.1.3 of [7]. We will use them freely.

For finite groups G, H, K, for $L \leq (G \times H)$ and $M \leq (H \times K)$, one has that

$$(\mathbf{2.7}) \quad [(G \times H)/L] \times_H [(H \times K)/M] = \sum_{h \in p_2(L)\backslash H/p_1(M)} [(G \times K)/(L * {}^{(h,1)}M)$$

in $B(G, K)$.

2.8. Definition: *When G is a finite group, a* section (T, S) *of G is a pair of subgroups of G such that $S \trianglelefteq T$.*

A group H is called a subquotient *of G (notation $H \sqsubseteq G$) if there exists a section (T, S) of G such that $T/S \cong H$.*

When (T, S) is a section of G, we denote by $\mathrm{Indinf}_{T/S}^G$ the set G/S, viewed as a $(G, T/S)$-biset for the natural actions given by multiplication of G and T/S. One checks easily that $\mathrm{Indinf}_{T/S}^G$ is isomorphic to the composition $\mathrm{Ind}_T^G\mathrm{Inf}_{T/S}^T$ as $(G, T/S)$-biset. Similarly, we denote by $\mathrm{Defres}_{T/S}^G$ the set $S\backslash G$, viewed as a $(T/S, G)$-biset. It is isomorphic to the composition $\mathrm{Def}_{T/S}^T\mathrm{Res}_T^G$. We observe that $(\mathrm{Indinf}_{T/S}^G)^{op} \cong \mathrm{Defres}_{T/S}^G$ as $(T/S, G)$-bisets, and that $(\mathrm{Defres}_{T/S}^G)^{op} \cong \mathrm{Indinf}_{T/S}^G$ as $(G, T/S)$-bisets.

With this notation, (2.6) gives in particular

$$(\mathbf{2.9}) \quad [(G \times G)/\Delta_S(T)] = \mathrm{Indinf}_{T/S}^G \,\mathrm{Defres}_{T/S}^G \ .$$

Two special cases are worth noticing, as they will be used intensively in the sequel:

$$(\mathbf{2.10}) \quad \text{for } N \trianglelefteq G,\ \ [(G \times G)/\Delta_N(G)] = \mathrm{Inf}_{G/N}^G \,\mathrm{Def}_{G/N}^G \ .$$

$$(\mathbf{2.11}) \qquad \text{for } H \le G,\ \ [(G\times G)/\Delta(H)] = \mathrm{Ind}_H^G \,\mathrm{Res}_H^G\ \ .$$

2.12. When G is a finite group, the group $B(G,G)$ is the ring of endomorphisms of G in the category $\mathcal{C}$. This ring is called the double Burnside ring of G. It is a non-commutative ring (if G is non trivial), with identity element equal to the class of the set G, viewed as a (G,G)-biset for left and right multiplication.

There is a unitary ring homomorphism $\alpha \mapsto \widetilde{\alpha}$ from $B(G)$ to $B(G,G)$, induced by the functor $X \mapsto \widetilde{X}$ from G-sets to (G,G)-bisets, where $\widetilde{X} = G\times X$, with (G,G)-biset structure given by

$$\forall a,b,g\in G,\ \forall x\in X,\ a(g,x)b = (agb, b^{-1}x)\ \ .$$

This construction has in particular the following properties (Corollary 2.5.12 of [7]):

2.13. Lemma: *Let G be a finite group.*

1. *If H is a subgroup of G, and X is a finite G-set, then there is an isomorphism of (G,H)-bisets*

$$\widetilde{X}\times_G \mathrm{Ind}_H^G \cong \mathrm{Ind}_H^G \times_H \widetilde{\mathrm{Res}_H^G X}\ \ ,$$

and an isomorphism of (H,G)-bisets

$$\mathrm{Res}_H^G \times_G \widetilde{X} \cong \widetilde{\mathrm{Res}_H^G X}\times_H \mathrm{Res}_H^G\ \ ,$$

where $\mathrm{Res}_H^G X$ denotes the set X, viewed as an H-set by restriction.

2. *If H is a subgroup of G, and Y is a finite H-set, then there is an isomorphism of (G,G)-bisets*

$$\mathrm{Ind}_H^G \times_H \widetilde{Y}\times_H \mathrm{Res}_H^G \cong \widetilde{\mathrm{Ind}_H^G Y}\ \ ,$$

where $\mathrm{Ind}_H^G Y = G\times_H Y$ is the G-set induced from Y.

3. *If N is a normal subgroup of G, and X is a finite G/N-set, then there is an isomorphism of $(G/N,G)$-bisets*

$$\widetilde{X}\times_{G/N} \mathrm{Def}_{G/N}^G \cong \mathrm{Def}_{G/N}^G \times_G \widetilde{\mathrm{Inf}_{G/N}^G X}\ \ ,$$

where $\mathrm{Inf}_{G/N}^{G}X$ *denotes the set* X*, viewed as a* G*-set by inflation.*

4. *If* N *is a normal subgroup of* G*, and* X *is a finite* G*-set, then there is an isomorphism of* $(G/N, G/N)$*-bisets*

$$\mathrm{Def}_{G/N}^{G} \times_G \widetilde{X} \times_G \mathrm{Inf}_{G/N}^{G} \cong \widetilde{\mathrm{Def}_{G/N}^{G}X} \;\;,$$

where $\mathrm{Def}_{G/N}^{G}X$ *is the set* $N\backslash X$ *of* N*-orbits on* X*, viewed as a* G/N*-set.*

2.14. Remark: One checks easily from the definition that if $Y = H/H$, then $\widetilde{H}$ is isomorphic to the identity (H,H)-biset. By Assertion 2 of Lemma 2.13, it follows more generally that if $H \leq G$, then $\widetilde{G/H}$ is isomorphic to the composition $\mathrm{Ind}_H^G \mathrm{Res}_H^G$ as a (G,G)-biset. By (2.11), it is also isomorphic to $(G \times G)/\Delta(H)$. By linearity, it also follows that $(\widetilde{X})^{op} \cong \widetilde{X}$ as (G,G)-biset, for any G-set X.

2.15. Lemma: *If* $f : G \to H$ *is a group isomorphism, and* X *is a finite* G*-set, then there is an isomorphism of* (H,G)*-bisets*

$$\mathrm{Iso}(f) \times_G \widetilde{X} \cong \widetilde{{}^{f}X} \times_H \mathrm{Iso}(f) \;\;,$$

where ${}^{f}X$ *is the set* X*, on which* H *acts by* $h.x = f^{-1}(h)x$*, for* $h \in H$ *and* $x \in X$*.*

Proof : This follows by linearity from the case $X = G/K$, for $K \leq G$. In this case indeed

$$\mathrm{Iso}(f) \times_G \widetilde{X} \cong \mathrm{Iso}(f)\mathrm{Ind}_K^G\mathrm{Res}_K^G \cong \mathrm{Ind}_{f(K)}^H\mathrm{Res}_{f(K)}^H\mathrm{Iso}(f) \cong \widetilde{H/f(K)} \times_H \mathrm{Iso}(f) \;\;,$$

and there is an obvious isomorphism of H-sets ${}^{f}(G/K) \cong H/f(K)$. □

2.16. Let $RB(G)$ denote the R-algebra $R \otimes_{\mathbb{Z}} B(G)$. Assume moreover that the order of G is invertible in R. Then for $H \leq G$, let $e_H^G \in RB(G)$ be defined by

$$\textbf{(2.17)} \qquad e_H^G = \frac{1}{|N_G(H)|} \sum_{K \leq H} |K|\mu(K,H)\,[G/K] \;\;,$$

where μ is the Möbius function of the poset of subgroups of G. The elements e_H^G, for $H \in [s_G]$, are orthogonal idempotents of $RB(G)$, and their sum is equal to the identity element of $RB(G)$. It follows that the elements $\widetilde{e_H^G}$, for $H \in [s_G]$, are orthogonal idempotents of the R-algebra $RB(G,G) =$

$R \otimes_{\mathbb{Z}} B(G,G)$, and the sum of these idempotents is equal to the identity element of $RB(G,G)$. The idempotents $\widetilde{e_G^G}$ play a special role, due to the following lemma:

2.18. Lemma: *Let R be a commutative ring, and G be a finite group with order invertible in R.*

1. *Let H be a proper subgroup of G. Then*

$$\mathrm{Res}_H^G \, \widetilde{e_G^G} = 0 \ \ \textit{and} \ \ \widetilde{e_G^G} \, \mathrm{Ind}_H^G = 0 \ \ .$$

2. *When $N \trianglelefteq G$, let $m_{G,N} \in R$ be defined by*

$$m_{G,N} = \frac{1}{|G|} \sum_{\substack{X \in s_G \\ XN = G}} |X| \mu(X,G) \ \ .$$

Then

$$\mathrm{Def}_{G/N}^G \, \widetilde{e_G^G} \, \mathrm{Inf}_{G/N}^G = m_{G,N} \widetilde{e_{G/N}^{G/N}} \ \ .$$

3. *Let $N \trianglelefteq G$, and suppose that N is contained in the Frattini subgroup $\Phi(G)$ of G. Then*

$$\widetilde{e_{G/N}^{G/N}} \, \mathrm{Def}_{G/N}^G = \mathrm{Def}_{G/N}^G \, \widetilde{e_G^G} \ \ \textit{and} \ \ \mathrm{Inf}_{G/N}^G \, \widetilde{e_{G/N}^{G/N}} = \widetilde{e_G^G} \, \mathrm{Inf}_{G/N}^G \ \ .$$

Proof : Assertion 1 follows from Lemma 2.13 and Assertion 1 of Theorem 5.2.4. of [7].

Assertion 2 follows from Lemma 2.13 and Assertion 4 of Theorem 5.2.4. of [7].

Finally the first part of Assertion 3 follows from Lemma 2.13 and Assertion 3 of Theorem 5.2.4. of [7]: indeed $\mathrm{Inf}_{G/N}^G e_{G/N}^{G/N}$ is equal to the sum of the different idempotents e_X^G of $RB(G)$ indexed by subgroups X such that $XN = G$. If $N \leq \Phi(G)$, then $XN = G$ implies $X\Phi(G) = G$, hence $X = G$. The second part of Assertion 3 follows by taking opposite bisets, since $\widetilde{e_G^G}$ and $\widetilde{e_{G/N}^{G/N}}$ are equal to their opposite bisets, and since $(\mathrm{Def}_{G/N}^G)^{op} \cong \mathrm{Inf}_{G/N}^G$. □

2.19. Remark: For the same reason, if $N \leq \Phi(G)$, then $m_{G,N} = 1$.

2.20. Remark: It follows from Assertion 1 and Equation 2.6 that if G and H are finite groups and if $L \leq (G \times H)$, then $\widetilde{e_G^G} \, [(G \times H)/L] = 0$ if $p_1(L) \neq G$, and $[(G\times H)/L]\widetilde{e_H^H}=0$ if $p_2(L)\neq H$.

3. Idempotents in $\mathcal{E}(G)$

3.1. Notation: *When G is a finite group with order invertible in R, denote by $\mathcal{E}(G)$ the R-algebra defined by*

$$\mathcal{E}(G)=\widetilde{e_G^G}\,RB(G,G)\,\widetilde{e_G^G}\ \ .$$

Set

$$\Sigma(G,G)=\{L\in s_{G\times G}\mid p_1(L)=p_2(L)=G\}\ \ ,$$

and for $L\in s_{G\times G}$, set

$$Y_L=\widetilde{e_G^G}\,[(G\times G)/L]\,\widetilde{e_G^G}\in\mathcal{E}(G)\ \ .$$

The R-algebra $\mathcal{E}(G)$ has been considered in [5], Section 9, in the case R is a field of characteristic 0. The extension of the results proved there to the case where R is a commutative ring in which the order of G is invertible is straightforward. In particular:

3.2. Proposition: *Let G be a finite group with order invertible in R.*

1. *If $L\in s_{G\times G}-\Sigma(G,G)$, then $Y_L=0$.*
2. *The elements Y_L, for L in a set of representatives of $(G\times G)$-conjugacy classes on $\Sigma(G,G)$, form an R-basis of $\mathcal{E}(G)$.*
3. *For $L,M\in\Sigma(G,G)$*

$$Y_LY_M=\frac{m_{G,k_2(L)\cap k_1(M)}}{|G|}\sum_{\substack{Z\leq G\\ Zk_2(L)=Zk_1(M)=G\\ Z\geq k_2(L)\cap k_1(M)}}|Z|\mu(Z,G)\,Y_{L*\Delta(Z)*M}$$

in $\mathcal{E}(G)$.

3.3. Corollary: *Let $L,M\in\Sigma(G,G)$. If one of the groups $k_2(L)$ or $k_1(M)$*

is contained in $\Phi(G)$*, then*

$$Y_L\,Y_M = Y_{L*M} \quad .$$

Proof : Indeed if $k_2(L) \leq \Phi(G)$, then $Zk_2(L) = G$ implies $Z\Phi(G) = G$, hence $Z = G$. Similarly, if $k_1(M) \leq \Phi(G)$, then $Zk_1(M) = G$ implies $Z = G$. In each case, Proposition 3.2 then gives

$$Y_L\,Y_M = m_{G,k_2(L)\cap k_1(M)} Y_{L*M} \quad ,$$

and moreover $m_{G,k_2(L)\cap k_1(M)} = 1$ since $k_2(L) \cap k_1(M) \leq \Phi(G)$, by Remark 2.19. □

3.4. Notation: *For a normal subgroup* N *of* G *such that* $N \leq \Phi(G)$*, set*

$$\varphi_N^G = \sum_{\substack{M \trianglelefteq G \\ N \leq M \leq \Phi(G)}} \mu_{\trianglelefteq G}(N, M)\, Y_{\Delta_M(G)} \quad ,$$

where $\mu_{\trianglelefteq G}$ *is the Möbius function of the poset of normal subgroups of* G*.*

3.5. Proposition: *Let* $N \trianglelefteq G$ *with* $N \leq \Phi(G)$*. Then*

$$\varphi_N^G = \mathrm{Inf}_{G/N}^G\, \varphi_{\mathbf{1}}^{G/N}\, \mathrm{Def}_{G/N}^G \quad .$$

Proof : Indeed if $N \leq M \trianglelefteq G$, then $\mu_{\trianglelefteq G}(N, M) = \mu_{\trianglelefteq G/N}(\mathbf{1}, M/N)$. Since moreover $N \leq \Phi(G)$, setting $\overline{G} = G/N$ and $\overline{M} = M/N$, we have by Lemma 2.18

$$\begin{aligned}
\mathrm{Inf}_{G/N}^G\, Y_{\Delta_{\overline{G}}(\overline{M})}\, \mathrm{Def}_{G/N}^G &= \mathrm{Inf}_{G/N}^G\, \widetilde{e_{\overline{G}}^{\overline{G}}}\, \big[(\overline{G} \times \overline{G})/\Delta_{\overline{G}}(\overline{M})\big]\, \widetilde{e_{\overline{G}}^{\overline{G}}}\, \mathrm{Def}_{G/N}^G \\
&= \widetilde{e_G^G}\, \mathrm{Inf}_{G/N}^G\, \big[(\overline{G} \times \overline{G})/\Delta_{\overline{G}}(\overline{M})\big]\, \mathrm{Def}_{G/N}^G\, \widetilde{e_G^G} \\
&= \widetilde{e_G^G}\, \big[(G \times G)/\Delta_M(G)\big]\, \widetilde{e_G^G} \\
&= Y_{\Delta_M(G)} \quad ,
\end{aligned}$$

since $\mathrm{Inf}_{G/N}^G\, \big[(\overline{G} \times \overline{G}))/\Delta_{\overline{G}}(\overline{M})\big]\, \mathrm{Def}_{G/N}^G = (G \times G)/\Delta_M(G)$, by 2.10 and transitivity of inflation. Moreover summing over normal subgroups $\overline{M}$ of $\overline{G}$ contained in $\Phi(\overline{G})$ amounts to summing over normal subgroups M of G with $N \leq M \leq \Phi(G)$. □

3.6. Proposition:

1. *Let $N \trianglelefteq G$, with $N \leq \Phi(G)$. Then*

$$\begin{aligned}\varphi_N^G &= \widetilde{e_G^G}\Big(\sum_{\substack{M \trianglelefteq G\\ N\leq M\leq \Phi(G)}} \mu_{\trianglelefteq G}(N,M)\big[(G\times G)/\Delta_M(G)\big]\Big)\\ &= \Big(\sum_{\substack{M \trianglelefteq G\\ N\leq M\leq \Phi(G)}} \mu_{\trianglelefteq G}(N,M)\big[(G\times G)/\Delta_M(G)\big]\Big)\,\widetilde{e_G^G}\ \ .\end{aligned}$$

2. *In particular*

$$\varphi_{\mathbf{1}}^G = \frac{1}{|G|}\sum_{\substack{X\leq G, M\trianglelefteq G\\ M\leq\Phi(G)\leq X\leq G}} |X|\mu(X,G)\mu_{\trianglelefteq G}(\mathbf{1},M)\,\mathrm{Indinf}_{X/M}^G\,\mathrm{Defres}_{X/M}^G\ \ .$$

3. *Let $N \trianglelefteq G$ with $N \leq \Phi(G)$, and $f : G \to H$ be a group isomorphism. Then*

$$\mathrm{Iso}(f)\,\varphi_N^G = \varphi_{f(N)}^H\,\mathrm{Iso}(f)\ \ .$$

Proof : For Assertion 1, by definition

$$\varphi_N^G = \sum_{\substack{M \trianglelefteq G\\ N\leq M\leq \Phi(G)}} \mu_{\trianglelefteq G}(N,M)\widetilde{e_G^G}[(G\times G)/\Delta_M(G)]\sum_{X\leq G}\tfrac{|X|}{|G|}\mu(X,G)[(G\times G)/\Delta(X)].$$

Moreover $[(G\times G)/\Delta_M(G)]\,[(G\times G)/\Delta(X)] = [(G\times G)/\big(\Delta_M(G)*\Delta(X)\big)]$, by (2.7), and $\Delta_M(G)*\Delta(X) = \{(mx,x)\mid x\in X, m\in M\}$. The first projection of this group is equal to MX, hence it is equal to G if and only if $X=G$, since $M\leq\Phi(G)$. The first equality of Assertion 1 follows, by Remark 2.20, since moreover $\Delta_M(G)*\Delta(G) = \Delta_M(G)$. The second one follows by taking opposite bisets, since $\widetilde{e_G^G}$ and $[(G\times G)/\Delta_M(G)]$ are equal to their opposite, by (2.10) and Remark 2.14.

Assertion 2 follows in the special case where $N=1$, expanding $\widetilde{e_G^G}$ as

$$\widetilde{e_G^G} = \frac{1}{|G|}\sum_{X\leq G}|X|\mu(X,G)\,\big[(G\times G)/\Delta(X)\big]\ \ ,$$

observing that $\mu(X,G)=0$ unless $X\geq\Phi(G)$, and that if $X\geq\Phi(G)\geq M$, then

$$\big[(G\times G)/\Delta(X)\big]\,\big[(G\times G)/\Delta_M(G)\big] = \big[(G\times G)/\Delta_M(X)\big]\ \ ,$$

which is equal to $\mathrm{Indinf}_{X/M}^G \, \mathrm{Defres}_{X/M}^G$ by (2.9).

Now for Assertion 3

$$\mathrm{Iso}(f)\, \varphi_N^G \, \mathrm{Iso}(f^{-1}) = \mathrm{Iso}(f)\, \widetilde{e_G^G}\, \mathrm{Iso}(f^{-1})\ \mathrm{Iso}(f)\, \Sigma\, \mathrm{Iso}(f)^{-1} \quad ,$$

where $\Sigma = \sum_{\substack{M \trianglelefteq G \\ N \leq M \leq \Phi(G)}} \mu_{\trianglelefteq G}(N, M)\, \big[(G\times G)/\Delta_M(G)\big]$. Moreover

$$\mathrm{Iso}(f)\, \widetilde{e_G^G}\, \mathrm{Iso}(f^{-1}) = \widetilde{e_H^H}$$

by Lemma 2.15, since obviously ${}^f(e_G^G) = e_H^H$. Finally

$$\begin{aligned}
\mathrm{Iso}(f)\, \Sigma\, \mathrm{Iso}(f)^{-1} &= \sum_{\substack{M \trianglelefteq G \\ N \leq M \leq \Phi(G)}} \mu_{\trianglelefteq G}(N, M)\, \mathrm{Iso}(f)\, \mathrm{Inf}_{G/M}^G \mathrm{Def}_{G/M}^G\, \mathrm{Iso}(f^{-1}) \\
&= \sum_{\substack{M \trianglelefteq G \\ N \leq M \leq \Phi(G)}} \mu_{\trianglelefteq G}(N, M)\, \mathrm{Inf}_{H/f(M)}^H\, \mathrm{Def}_{H/f(M)}^H \\
&= \sum_{\substack{M' \trianglelefteq H \\ f(N) \leq M' \leq \Phi(H)}} \mu_{\trianglelefteq H}\big(f(N), M'\big)\, \mathrm{Inf}_{H/M'}^H\, \mathrm{Def}_{H/M'}^H
\end{aligned}$$

where $M' = f(M)$ in the last summation. It follows that

$$\mathrm{Iso}(f)\varphi_N^G \mathrm{Iso}(f^{-1}) = \widetilde{e_H^H} \sum_{\substack{M' \trianglelefteq H \\ f(N) \leq M' \leq \Phi(H)}} \mu_{\trianglelefteq H}\big(f(N), M'\big)\, \mathrm{Inf}_{H/M'}^H \mathrm{Def}_{H/M'}^H = \varphi_{f(N)}^H \quad ,$$

as was to be shown. □

3.7. Corollary:

1. *Let* $H < G$. *Then* $\mathrm{Res}_H^G\, \varphi_N^G = 0$ *and* $\varphi_N^G\, \mathrm{Ind}_H^G = 0$.
2. *Let* $M \trianglelefteq G$. *If* $M \cap \Phi(G) \not\leq N$, *then* $\mathrm{Def}_{G/M}^G\, \varphi_N^G = 0$ *and* $\varphi_N^G\, \mathrm{Inf}_{G/M}^G = 0$.

Proof : The first part of Assertion 1 follows from Lemma 2.18, since

$$\mathrm{Res}_H^G\, \varphi_N^G = \mathrm{Res}_H^G\, \widetilde{e_G^G}\, \varphi_N^G = 0 \quad .$$

The second part follows by taking opposite bisets.

For Assertion 2, let $P = M \cap \Phi(G)$. Since $\mathrm{Def}_{G/M}^G = \mathrm{Def}_{G/M}^{G/P}\, \mathrm{Def}_{G/P}^G$, it suffices to consider the case $M = P$, i.e. the case where $M \leq \Phi(G)$. Then, since $\big[(G\times G)/\Delta_M(G)\big]=\mathrm{Inf}_{G/M}^G\,\mathrm{Def}_{G/M}^G$ for any $M\trianglelefteq G$, by 2.10, and since $\mathrm{Def}_{G/M}^G\,\mathrm{Inf}_{G/Q}^G=\mathrm{Inf}_{G/MQ}^{G/M}\,\mathrm{Def}_{G/MQ}^{G/Q}$ for any $M,Q\trianglelefteq G$, we have

$$\begin{aligned}
\mathrm{Def}_{G/M}^G\,\varphi_N^G &= \mathrm{Def}_{G/M}^G \sum_{\substack{Q\trianglelefteq G\\ N\leq Q\leq \Phi(G)}} \mu_{\trianglelefteq G}(N,Q)\,\mathrm{Inf}_{G/Q}^G\,\mathrm{Def}_{G/Q}^G\,\widetilde{e_G^G}\\
&= \sum_{\substack{Q\trianglelefteq G\\ N\leq Q\leq \Phi(G)}} \mu_{\trianglelefteq G}(N,Q)\,\mathrm{Inf}_{G/MQ}^{G/M}\,\mathrm{Def}_{G/MQ}^G\,\widetilde{e_G^G}\\
&= \sum_{\substack{P\trianglelefteq G\\ NM\leq P\leq \Phi(G)}} \Big(\sum_{\substack{Q\trianglelefteq G\\ N\leq Q\leq \Phi(G)\\ QM=P}} \mu_{\trianglelefteq G}(N,Q)\Big)\,\mathrm{Inf}_{G/P}^{G/M}\,\mathrm{Def}_{G/P}^G\,\widetilde{e_G^G}\ .
\end{aligned}$$

Now for a given $P\trianglelefteq G$ with $P\leq\Phi(G)$, the sum $\sum\limits_{\substack{Q\trianglelefteq G\\ N\leq Q\leq \Phi(G)\\ QM=P}} \mu_{\trianglelefteq G}(N,Q)$ is equal to zero unless $NM=N$, that is $M\leq N$, by classical properties of the Möbius function ([15] Corollary 3.9.3). This proves the first part of Assertion 2, and the second part follows by taking opposite bisets. □

3.8. Theorem: *Let G be a finite group with order invertible in R.*

1. *The elements φ_N^G, for $N\trianglelefteq G$ with $N\leq\Phi(G)$, form a set of orthogonal idempotents in the algebra $\mathcal{E}(G)$, and their sum is equal to the identity element $\widetilde{e_G^G}$ of $\mathcal{E}(G)$.*
2. *Let $N\trianglelefteq G$ with $N\leq\Phi(G)$, and let H be a finite group.*
   (a) *If $L\leq(G\times H)$, then $\varphi_N^G\,[(G\times H)/L]=0$ unless $p_1(L)=G$ and $k_1(L)\cap\Phi(G)\leq N$.*
   (b) *If $L'\leq(H\times G)$, then $[(H\times G)/L']\,\varphi_N^G=0$ unless $p_2(L')=G$ and $k_2(L')\cap\Phi(G)\leq N$.*

Proof : For $N\trianglelefteq G$, set $u_N^G=Y_{\Delta_N(G)}$. Since $\Delta_N(G)*\Delta_M(G)=\Delta_{NM}(G)$ for any normal subgroups N and M of G, it follows from Corollary 3.3 that if either N or M is contained in $\Phi(G)$, then $u_N^G\,u_M^G=u_{NM}^G$.

Now Assertion 1 follows from the following general procedure for building orthogonal idempotents (see [13] Theorem 10.1 for details): we have a finite lattice P (here P is the lattice of normal subgroups of G contained in $\Phi(G)$), and a set of elements g_x of a ring A, for $x\in P$ (here $A=\mathcal{E}(G)$ and $g_N=u_N^G$), with the property that $g_xg_y=g_{x\vee y}$ for any $x,y\in P$, and $g_0=1$, where 0 is the smallest element of P (here this element is the trivial subgroup of G, and $u_{\mathbf{1}}^G = Y_{\Delta_{\mathbf{1}}(G)} = \widetilde{e_G^G}$). Then the elements f_x defined for $x \in P$ by

$$f_x = \sum_{\substack{y \in P \\ x \leq y}} \mu(x, y) g_y \quad ,$$

where μ is the Möbius function of P, are orthogonal idempotents of A, and their sum is equal to the identity element of A. This is exactly Assertion 1 (since $f_x = \varphi_N^G$ here, for $x = N \in P$).

Let $L \leq (G \times H)$. Assertion (a) follows from (2.6) and Corollary 3.7, since

$$\varphi_N^G \operatorname{Ind}_{p_1(L)}^G \operatorname{Inf}_{p_1(L)/k_1(L)}^{p_1(L)} = 0$$

unless $p_1(L) = G$ and $k_1(L) \cap \Phi(G) \leq N$. The proof of Assertion (b) is similar. Alternatively, one can take opposite bisets in (a). □

3.9. Proposition: *Let G be a finite group with order invertible in R.*

1. *Let $L \in \Sigma(G, G)$. Then*

$$\varphi_{\mathbf{1}}^G Y_L = \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N)\, Y_{(N \times \mathbf{1})L} \quad .$$

   This is non zero if and only if $k_1(L) \cap \Phi(G) = \mathbf{1}$. Similarly

$$Y_L\, \varphi_{\mathbf{1}}^G = \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N)\, Y_{L(\mathbf{1} \times N)} \quad ,$$

   and $Y_L\, \varphi_1^G \neq 0$ if and only if $k_2(L) \cap \Phi(G) = \mathbf{1}$.

2. *The elements $\varphi_{\mathbf{1}}^G Y_L$ (resp. $Y_L\, \varphi_{\mathbf{1}}^G$), when L runs through a set of representatives of conjugacy classes of elements of $\Sigma(G, G)$ such that $k_1(L) \cap \Phi(G) = \mathbf{1}$ (resp. $k_2(L) \cap \Phi(G) = \mathbf{1}$), form an R-basis of the right ideal $\varphi_{\mathbf{1}}^G \mathcal{E}(G)$ (resp. the left ideal $\mathcal{E}(G)\, \varphi_{\mathbf{1}}^G$) of $\mathcal{E}(G)$.*

Proof : Let $L \in \Sigma(G, G)$. By Proposition 3.6, we have

$$\begin{aligned}
\varphi_{\mathbf{1}}^G Y_L &= \widetilde{e_G^G} \Big(\sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) \big[(G \times G)/\Delta_N(G)\big] \Big) \big[(G \times G)/L\big]\, \widetilde{e_G^G} \\
&= \widetilde{e_G^G} \Big(\sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) \big[(G \times G)/(\Delta_N(G) * L)\big] \Big)\, \widetilde{e_G^G}
\end{aligned}$$

$$= \widetilde{e_G^G}\Big(\sum_{\substack{N \trianglelefteq G\\ N\leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1},N)\,\big[(G\times G)/(N\times \mathbf{1})L\big]\Big)\,\widetilde{e_G^G}\;\;.$$

$$= \sum_{\substack{N \trianglelefteq G\\ N\leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1},N)\,Y_{(N\times \mathbf{1})L}\;\;.$$

Set $M = k_1(L)\cap\Phi(G)$. Then $M \trianglelefteq G$, and $(N\times\mathbf{1})L = (NM\times\mathbf{1})L$ for any normal subgroup N of G contained in $\Phi(G)$. Thus

$$\textbf{(3.10)}\qquad \varphi_{\mathbf{1}}^G\,Y_L = \sum_{\substack{P\trianglelefteq G\\ M\leq P\leq\Phi(G)}}\Big(\sum_{\substack{N\trianglelefteq G\\ NM=P}}\mu_{\trianglelefteq G}(\mathbf{1},N)\Big)\,Y_{(P\times\mathbf{1})L}\;\;.$$

If $M\neq\mathbf{1}$, then $\Big(\sum_{\substack{N\trianglelefteq G\\ NM=P}}\mu_{\trianglelefteq G}(\mathbf{1},N)\Big)=0$ for any $P\trianglelefteq G$ with $M\leq P\leq\Phi(G)$, again by [15], Corollary 3.9.3. Hence $\varphi_{\mathbf{1}}^G\,Y_L=0$ in this case. And if $M=\mathbf{1}$, Equation (3.10) reads

$$\varphi_{\mathbf{1}}^G\,Y_L = \sum_{\substack{P\trianglelefteq G\\ P\leq\Phi(G)}}\mu_{\trianglelefteq G}(\mathbf{1},P)\,Y_{(P\times\mathbf{1})L}\;\;.$$

The element $Y_{(P\times\mathbf{1})L}$ is equal to Y_L if and only if $(P\times\mathbf{1})L$ is conjugate to L. This implies that $k_1\big((P\times\mathbf{1})L\big)$ is conjugate to (hence equal to) $k_1(L)$. Thus $P\leq k_1\big((P\times\mathbf{1})L\big)\leq k_1(L)$, so $P\leq k_1(L)\cap\Phi(G)=\mathbf{1}$, hence $P=\mathbf{1}$. So the coefficient of Y_L in $\varphi_{\mathbf{1}}^G\,Y_L$ is equal to 1, hence $\varphi_{\mathbf{1}}^G\,Y_L\neq 0$. The remaining statements of Assertion 1 follow by taking opposite bisets.

Assertion 2 follows from Proposition 3.2, and from the fact that the coefficient of Y_L in $\varphi_{\mathbf{1}}^G\,Y_L$ is equal to 1 when $k_1(L)\cap\Phi(G)=\mathbf{1}$. □

3.11. Corollary: *Let G be a finite group of order invertible in R. If every minimal (non-trivial) normal subgroup of G is contained in $\Phi(G)$, then $\varphi_{\mathbf{1}}^G$ is central in $\mathcal{E}(G)$, and the algebra $\varphi_{\mathbf{1}}^G\,\mathcal{E}(G)$ is isomorphic to $R\mathrm{Out}(G)$.*

Proof : Indeed if $L\in\Sigma(L,L)$ and $\varphi_{\mathbf{1}}^G\,Y_L\neq 0$, then $k_1(L)\cap\Phi(G)=\mathbf{1}$. It follows that $k_1(L)$ contains no minimal normal subgroup of G, and then $k_1(L)=\mathbf{1}$. Equivalently $q(L)\cong p_1(L)/k_1(L)\cong G\cong p_2(L)/k_2(L)$, i.e. $k_2(L)=\mathbf{1}$ also, or equivalently $k_2(L)\cap\Phi(G)=\mathbf{1}$. Hence $\varphi_{\mathbf{1}}^G Y_L\neq 0$ if and only if $Y_L\varphi_{\mathbf{1}}^G\neq 0$, and in this case, there exists an automorphism θ of G such that

$$L=\Delta_\theta(G)=\{\big(\theta(x),x\big)\mid x\in G\}\;\;.$$

In this case for any normal subgroup N of G contained in $\Phi(G)$

$$\begin{aligned}(N\times \mathbf{1})L &= \{(a,b)\in G\times G \mid a\theta(b)^{-1}\in N\}\\ &= \{(a,b)\in G\times G \mid a^{-1}\theta(b)\in N\}\\ &= L\big(\mathbf{1}\times \theta^{-1}(N)\big)\ .\end{aligned}$$

Now $N\mapsto \theta^{-1}(N)$ is a permutation of the set of normal subgroups of G contained in $\Phi(G)$. Moreover $\mu_{\trianglelefteq G}(\mathbf{1},N)=\mu_{\trianglelefteq G}\big(\mathbf{1},\theta^{-1}(N)\big)$.

Summing over all $N\leq\Phi(G)$, it follows that $\varphi_{\mathbf{1}}^G\,Y_L=Y_L\,\varphi_{\mathbf{1}}^G$, so $\varphi_{\mathbf{1}}^G$ is central in $\mathcal{E}(G)$. Moreover the map $\theta\in \mathrm{Aut}(G)\mapsto \varphi_{\mathbf{1}}^G\,Y_{\Delta_\theta(G)}$ clearly induces an algebra isomorphism $R\mathrm{Out}(G)\to\varphi_{\mathbf{1}}^G\,\mathcal{E}(G)$ (observe indeed that if θ is an inner automorphism of G, then $\Delta_\theta(G)$ is conjugate to $\Delta(G)$ in $G\times G$, so $Y_{\Delta_\theta(G)}=Y_{\Delta(G)}=\widetilde{e_G^G}$). □

3.12. Theorem: *Let G be a finite group with order invertible in R. If G is nilpotent, then $\varphi_{\mathbf{1}}^G$ is a central idempotent of $\mathcal{E}(G)$.*

Proof : Let $L\in\Sigma(G,G)$. Setting $Q=q(L)$, there are two surjective group homomorphisms $s,t:G\to Q$ such that $L=\{(x,y)\in G\times G\mid s(x)=t(y)\}$. Then $k_1(L)=\mathrm{Ker}\,s$ and $k_2(L)=\mathrm{Ker}\,t$. Now by Proposition 3.9

$$\varphi_{\mathbf{1}}^G Y_L=\sum_{\substack{N\trianglelefteq G\\ N\leq\Phi(G)}}\mu_{\trianglelefteq G}(\mathbf{1},N)\,Y_{(N\times\mathbf{1})L}\ \ ,$$

and this is non zero if and only if $\mathrm{Ker}\,s\cap\Phi(G)=\mathbf{1}$. Now $s\big(\Phi(G)\big)$ is equal to $\Phi(Q)$ since G is nilpotent: indeed $G=\prod_p G_p$ (resp. $Q=\prod_p Q_p$) is the direct product of its p-Sylow subgroups G_p (resp. Q_p), and s induces a surjective group homomorphism $G_p\to Q_p$, for any prime p. Moreover $\Phi(G)=\prod_p\Phi(G_p)$ (resp. $\Phi(Q)=\prod_p\Phi(Q_p)$). Finally $\Phi(G_p)$ is the subgroup of G_p generated by commutators and p-powers of elements of G_p, hence it maps by s *onto* the subgroup of Q_p generated by commutators and p-powers of elements of Q_p, that is $\Phi(Q_p)$. Similarly $t\big(\Phi(G)\big)=\Phi(Q)$.

If $\mathrm{Ker}\,s\cap\Phi(G)=\mathbf{1}$, it follows that s induces an isomorphism from $\Phi(G)$ to $\Phi(Q)$. Then the surjective homomorphism $\Phi(G)\to\Phi(Q)$ induced by t is also an isomorphism, and in particular $\mathrm{Ker}\,t\cap\Phi(G)=\mathbf{1}$.

Let $D=L\cap\big(\Phi(G)\times\Phi(G)\big)$. Then $k_1(D)\subseteq k_1(L)\cap\Phi(G)=\mathrm{Ker}\,s\cap\Phi(G)$, hence $k_1(D)=\mathbf{1}$. Similarly $k_2(L)\subseteq k_2(L)\cap\Phi(G)=\mathrm{Ker}\,t\cap\Phi(G)=\mathbf{1}$, hence $k_2(D)=\mathbf{1}$. Since $s\big(\Phi(G)\big)=\Phi(Q)=t\big(\Phi(G)\big)$, we have moreover $p_1(D)=\Phi(G)=p_2(D)$. It follows that there is an automorphism α of $\Phi(G)$ such that $D=\{\big(x,\alpha(x)\big)\mid x\in\Phi(G)\}$.

Moreover for any $y \in G$, there exists $z \in G$ such that $(y,z) \in L$. It follows that $\big(x^y, \alpha(x)^z\big) \in D$ for any $x \in \Phi(G)$, that is $\alpha(x^y) = \alpha(x)^z$. In particular if N is a normal subgroup of G contained in $\Phi(G)$, then so is $\alpha(N)$. Hence α induces an automorphism of the poset of normal subgroups of G contained in $\Phi(G)$. In particular $\mu_{\trianglelefteq G}(\mathbf{1}, N) = \mu_{\trianglelefteq G}\big(\mathbf{1}, \alpha(N)\big)$.

Moreover for $n \in N$ and $(y,z) \in L$, we have

$$(n,1)(y,z) = (y,z)(n^y,1) = (y,z)\big(n^y, \alpha(n^y)\big)\big(1, \alpha(n^y)^{-1}\big) \ .$$

Since $\big(n^y, \alpha(n^y)\big) \in D \leq L$, we have $(N \times \mathbf{1})L = L\big(\mathbf{1} \times \alpha(N)\big)$. It follows that

$$\begin{aligned} \varphi_{\mathbf{1}}^G Y_L &= \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N)\, Y_{(N\times\mathbf{1})L} = \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N)\, Y_{L(\mathbf{1}\times\alpha(N))} \\ &= \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}\big(\mathbf{1}, \alpha(N)\big)\, Y_{L(\mathbf{1}\times\alpha(N))} = \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}\big(\mathbf{1}, N\big)\, Y_{L(\mathbf{1}\times N)} \\ &= Y_L \varphi_{\mathbf{1}}^G \ , \end{aligned}$$

as was to be shown. □

3.13. Remark: When G is not nilpotent, it is not true in general that $\varphi_{\mathbf{1}}^G$ is central in $\mathcal{E}(G)$. This is because $t\big(\Phi(G)\big)$ need not be equal to $\Phi(Q)$ for a surjective group homomorphism $t : G \to Q$. For example, there is a surjection t from the group $G = C_4 \times (C_5 \rtimes C_4)$ to $Q = C_4$ with kernel $C_4 \times C_5$ *containing* $\Phi(G) = C_2 \times \mathbf{1}$, and another surjection $s : G \to Q$ with kernel $\mathbf{1} \times (C_5 \rtimes C_4)$ intersecting $\Phi(G)$ trivially. In this case, the group $L = \{(x,y) \in G \times G \mid s(x) = t(y)\}$ is in $\Sigma(G,G)$, and $k_1(L) \cap \Phi(G) = \mathbf{1}$, but $k_2(L) \cap \Phi(G) = \Phi(G) \neq \mathbf{1}$. By Proposition 3.9, we have $\varphi_{\mathbf{1}}^G\, Y_L \neq 0$ and $Y_L\, \varphi_{\mathbf{1}}^G = 0$, so $\varphi_{\mathbf{1}}^G$ is not central in $\mathcal{E}(G)$.

4. Idempotents in $RB(G,G)$

Recall from Definition 2.8 that a *section* (T,S) of a finite group G is a pair of subgroups of G such that $S \trianglelefteq T$. For such a section (T,S) of G, recall that $\mathrm{Indinf}_{T/S}^G \in B(G, T/S)$ denotes (the isomorphism class of) the $(G, T/S)$-biset G/S, and that $\mathrm{Defres}_{T/S}^G \in B(T/S, G)$ denote (the isomorphism class of) the $(T/S, G)$-biset $S\backslash G$.

The group G acts by conjugation on the set of its sections: if $g \in G$ and (T,S) is a section of G, then ${}^g(T,S) = ({}^gT, {}^gS)$ is another section of G.

4.1. Notation: *Let G be a finite group, and let (T,S) be a section of G.*

1. *Let R be a commutative ring in which the order of G is invertible. Let $u_{T,S}^G \in RB(G,T/S)$ be defined by*

$$u_{T,S}^G = \mathrm{Indinf}_{T/S}^G \varphi_{\mathbf{1}}^{T/S} \quad ,$$

and let $v_{T,S}^G \in RB(T/S,G)$ be defined by

$$v_{T,S}^G = \varphi_{\mathbf{1}}^{T/S} \mathrm{Defres}_{T/S}^G \quad .$$

4.2. Remark: Observe that $v_{T,S}^G = (u_{T,S}^G)^{op}$: indeed $(G/S)^{op} \cong S\backslash G$, and $(\varphi_{\mathbf{1}}^{T/S})^{op} = \varphi_{\mathbf{1}}^{T/S}$.

4.3. Definition: *A section (T,S) of a finite group G is called* minimal *(cf. [9]) if $S \leq \Phi(T)$. Let $\mathcal{M}(G)$ denote the set of minimal sections of G.*

4.4. Remark: The terminology comes from the following observation: if (T,S) is any section of G, and H is a subgroup of T minimal subject to $HS = T$, then the section $(H, H\cap S)$ is such that $H/(H\cap S) \cong T/S$, and it is moreover minimal in the sense of Definition 4.3 (for if $K \leq H$ is such that $K(H\cap S) = H$, then $KS = HS = T$, thus $K = H$, showing that $H\cap S \leq \Phi(H)$). In other words a section (T,S) is minimal if and only if the only subgroup H of T such that $H/(H\cap S) \cong T/S$ is T itself.

4.5. Theorem: *Let G be a finite group with order invertible in R.*

1. *If (T,S) and (T',S') are minimal sections of G, then*

$$v_{T',S'}^G \, u_{T,S}^G = 0$$

unless (T,S) and (T',S') are conjugate in G.

2. *If (T,S) is a minimal section of G, then*

$$v_{T,S}^G \, u_{T,S}^G = \varphi_{\mathbf{1}}^{T/S}\Big(\sum_{g\in N_G(T,S)/T} \mathrm{Iso}(c_g)\Big) = \Big(\sum_{g\in N_G(T,S)/T} \mathrm{Iso}(c_g)\Big)\varphi_{\mathbf{1}}^{T/S} \quad ,$$

where $N_G(T,S) = N_G(T)\cap N_G(S)$, and c_g is the automorphism of T/S induced by conjugation by g.

Proof : Indeed $(S'\backslash G)\times_G (G/S)\cong S'\backslash G/S$ as a $(T'/S',T/S)$-biset. Hence

$$v_{T',S'}^G u_{T,S}^G=\varphi_{\mathbf{1}}^{T'/S'}\Big(\sum_{g\in T'\backslash G/T} S'\backslash T'gT/S\Big)\varphi_{\mathbf{1}}^{T/S}\;\;.$$

For any $g\in G$, the $(T'/S',T/S)$-biset $S'\backslash T'gT/S$ is transitive, isomorphic to $\big((T'/S')\times(T/S)\big)/L_g$, where

$$L_g=\{(t'S',tS)\in(T'/S')\times(T/S)\mid t'gt^{-1}\in S'gS\}\;\;.$$

Then $t'S'\in p_1(L_g)$ if and only if $t'\in S'\cdot gTg^{-1}\cap T'$. Hence

$$p_1(L_g)=({}^gT\cap T')S'/S'\;\;.$$

Similarly $p_2(L_g)=(T'^g\cap T)S/S$. In particular $p_1(L_g)=T'/S'$ if and only if $({}^gT\cap T')S'=T'$, i.e. ${}^gT\cap T'=T'$, since $S'\leq\Phi(T')$. Thus $p_1(L_g)=T'/S'$ if and only if $T'\leq{}^gT$. Similarly $p_2(L_g)=T/S$ if and only if $T\leq T'^g$. By Theorem 3.8, it follows that $\varphi_{\mathbf{1}}^{T'/S'}(S'\backslash T'gT/S)\varphi_{\mathbf{1}}^{T/S}=0$ unless $T'={}^gT$.

Assume now that $T'={}^gT$. Then $t'S'\in k_1(L_G)$ if and only if t' lies in $S'\cdot gSg^{-1}\cap T'$. Hence

$$k_1(L_g)=({}^gS\cap T')S'/S'\;\;,$$

and similarly $k_2(L_g)=(S'^g\cap T)S/S$. But since $S\leq\Phi(T)$ and $S\trianglelefteq T$, it follows that ${}^gS\trianglelefteq{}^gT=T'$ and ${}^gS\leq{}^g\Phi(T)=\Phi(T')$. Hence ${}^gS\cdot S'/S'$ is contained in $k_1(L_g)\cap\Phi(T')/S'$. Moreover $\Phi(T')/S'=\Phi(T'/S')$, as

$$\Phi(T'/S')=\bigcap_{S'\leq M'<T'}(M'/S')=\bigcap_{M'<T'}(M'/S')=(\bigcap_{M'<T'}M')/S'=\Phi(T')/S'\;\;,$$

where M' runs through maximal subgroups of T', which all contain S' since $S'\leq\Phi(T')$.

It follows that if $k_1(L_g)\cap\Phi(T'/S')=\mathbf{1}$, then ${}^gS\cdot S'=S'$, that is ${}^gS\leq S'$. Similarly if $k_2(L_g)\cap\Phi(T/S)=\mathbf{1}$, then $S'^g\leq S$. By Theorem 3.8, it follows that $\varphi_{\mathbf{1}}^{T'/S'}(S'\backslash T'gT/S)\varphi_{\mathbf{1}}^{T/S}=0$ unless $T'={}^gT$ and $S'={}^gS$. This proves Assertion 1.

For Assertion 2, the same computation shows that

$$v_{T,S}^G\,u_{T,S}^G=\sum_{g\in N_G(T,S)/T}\varphi_{\mathbf{1}}^{T/S}\,(S\backslash TgT/S)\,\varphi_{\mathbf{1}}^{T/S}\;\;.$$

But $S\backslash TgT/S=gT/S$ if $g\in N_G(T,S)$, and this $(T/S,T/S)$-biset is isomorphic to $\mathrm{Iso}(c_g)$. Assertion 2 follows, since moreover $\varphi_{\mathbf{1}}^{T/S}$ commutes with any biset of the form Iso(θ), where θ is an automorphism of T/S, by Proposition 3.6. □

4.6. Notation: *For a minimal section (T,S) of the group G, set*

$$\epsilon_{T,S}^G = \frac{1}{|N_G(T,S):T|} u_{T,S}^G v_{T,S}^G = \frac{1}{|N_G(T,S):T|} \mathrm{Indinf}_{T/S}^G \varphi_{\mathbf{1}}^{T/S} \mathrm{Defres}_{T/S}^G \in RB(G,G) \ .$$

Note that $\epsilon_{T,S}^G = \epsilon_{{}^gT,{}^gS}^G$ for any $g \in G$, and that $\epsilon_{G,N}^G = \varphi_N^G$ when $N \trianglelefteq G$ and $N \leq \Phi(G)$, by Proposition 3.5.

4.7. Proposition: *Let (T,S) be a minimal section of G. Then*

$$\epsilon_{T,S}^G = \frac{1}{|N_G(T,S)|} \sum_{\substack{X \leq T, M \trianglelefteq T \\ S \leq M \leq \Phi(T) \leq X \leq T}} |X| \mu(X,T) \mu_{\trianglelefteq T}(S,M) \, \mathrm{Indinf}_{X/M}^G \, \mathrm{Defres}_{X/M}^G \ .$$

Proof : This is a straightforward consequence of the above definition of $\epsilon_{T,S}^G$, and of Assertion 2 of Proposition 3.6, using the transitivity of Defres and Indinf involved. □

4.8. Theorem: *Let G be a finite group with order invertible in R, let $[\mathcal{M}(G)]$ be a set of representatives of conjugacy classes of minimal sections of G. Then the elements $\epsilon_{T,S}^G$, for $(T,S) \in [\mathcal{M}(G)]$, are orthogonal idempotents of $RB(G,G)$, and their sum is equal to the identity element of $RB(G,G)$.*

Proof : Let (T,S) and (T',S') be distinct elements of $[\mathcal{M}(G)]$. Then

$$\epsilon_{T',S'}^G \, \epsilon_{T,S}^G = \frac{1}{|N_G(T',S'):T'|} \frac{1}{|N_G(T,S):T|} \, u_{T',S'}^G \, v_{T',S'}^G \, u_{T,S}^G \, v_{T,S}^G = 0 \ \ ,$$

since $v_{T',S'}^G u_{T,S}^G = 0$ by Theorem 4.5. Moreover:

$$\begin{aligned} \Sigma = \sum_{(T,S)\in[\mathcal{M}(G)]} \epsilon_{T,S}^G &= \sum_{(T,S)\in[\mathcal{M}(G)]} \frac{1}{|N_G(T,S):T|} \, u_{T,S}^G \, v_{T,S}^G \\ &= \sum_{(T,S)\in\mathcal{M}(G)} \frac{1}{|G:T|} \, u_{T,S}^G \, v_{T,S}^G \\ &= \sum_{(T,S)\in\mathcal{M}(G)} \frac{1}{|G:T|} \, \mathrm{Indinf}_{T/S}^G \, \varphi_{\mathbf{1}}^{T/S} \, \mathrm{Defres}_{T/S}^G \end{aligned}$$

Now for a given $T \leq G$

$$\sum_{\substack{S \trianglelefteq T \\ S \leq \Phi(T)}} \mathrm{Indinf}_{T/S}^G \, \varphi_{\mathbf{1}}^{T/S} \, \mathrm{Defres}_{T/S}^G = \mathrm{Ind}_T^G \big(\sum_{\substack{S \trianglelefteq T \\ S \leq \Phi(T)}} \mathrm{Inf}_{T/S}^T \, \varphi_{\mathbf{1}}^{T/S} \, \mathrm{Def}_{T/S}^T \big) \mathrm{Res}_T^G$$

$$= \mathrm{Ind}_T^G \big(\sum_{\substack{S \trianglelefteq T \\ S \leq \Phi(T)}} \varphi_S^T \big) \mathrm{Res}_T^G = \mathrm{Ind}_T^G \, \widetilde{e_T^T} \, \mathrm{Res}_T^G$$

by Proposition 3.5 and Theorem 3.8.

Moreover $\mathrm{Ind}_T^G \, \widetilde{e_T^T} \, \mathrm{Res}_T^G = |N_G(T) : T| \, \widetilde{e_T^G}$, by (2.17) and Lemma 2.13. Thus

$$\sum_{(T,S) \in [\mathcal{M}(G)]} \epsilon_{T,S}^G = \sum_{T \leq G} \frac{|N_G(T):T|}{|G:T|} \, \widetilde{e_T^G} = \sum_{T \in [s_G]} \widetilde{e_T^G} = \widetilde{G/G} = \big[(G \times G)/\Delta(G)\big] \ .$$

So the sum Σ is equal to the identity of $RB(G,G)$. Since $\epsilon_{T,S}^G \epsilon_{T',S'}^G = 0$ if (T,S) and (T',S') are distinct elements of $[\mathcal{M}(G)]$, it follows that for any $(T,S) \in [\mathcal{M}(G)]$

$$\epsilon_{T,S}^G = \epsilon_{T,S}^G \Sigma = (\epsilon_{T,S}^G)^2 \ ,$$

which completes the proof of the theorem. □

5. Application to biset functors

5.1. Notation: *Let F be a biset functor over R. When G is a finite group with order invertible in R, we set*

$$\delta_\Phi F(G) = \varphi_{\mathbf{1}}^G F(G)$$

5.2. Proposition: *Let F be a biset functor over R. Then for any finite group G with order invertible in R, the R-submodule $\delta_\Phi F(G)$ of $F(G)$ is the set of elements $u \in F(G)$ such that*

$$\begin{cases} \mathrm{Res}_H^G u = 0 & \forall H < G \\ \mathrm{Def}_{G/N}^G u = 0 & \forall N \trianglelefteq G, \ N \cap \Phi(G) \neq \mathbf{1} \end{cases} .$$

Proof : If $u \in \delta_\Phi F(G) = \varphi_{\mathbf{1}}^G F(G)$, then $\mathrm{Res}_H^G u = 0$ for any proper subgroup H of G, and $\mathrm{Def}_{G/N}^G u = 0$ for any $N \trianglelefteq G$ such that $N \cap \Phi(G) \neq \mathbf{1}$, by Corollary 3.7.

Conversely, if $u \in F(G)$ fulfills the two conditions of the proposition, then $\widetilde{e_G^G} u = u$, because $\widetilde{e_G^G}$ is equal to the identity element $\big[(G\times G)/\Delta(G)\big]$ of $RB(G,G)$, plus a linear combination of elements of the form $\big[(G\times G)/\Delta(H)\big] = \mathrm{Ind}_H^G \mathrm{Res}_H^G$, for proper subgroups H of G. Similarly, it follows again from Corollary 3.7 that $\mathrm{Inf}_{G/N}^G \mathrm{Def}_{G/N}^G u = 0$ for any non-trivial normal subgroup of G contained in $\Phi(G)$, thus $\varphi_{\mathbf{1}}^G u = u$. □

5.3. Remark: Since $\mathrm{Def}_{G/N}^G = \mathrm{Def}_{G/N}^{G/M} \mathrm{Def}_{G/M}^G$, where $M = N \cap \Phi(G)$, saying that $\mathrm{Def}_{G/N}^G u = 0$ for any $N \trianglelefteq G$ with $N \cap \Phi(G) \neq \mathbf{1}$ is equivalent to saying that $\mathrm{Def}_{G/N}^G u = 0$ for any non trivial normal subgroup N of G contained in $\Phi(G)$.

5.4. Theorem: *Let F be a biset functor over R. Then for any finite group G with order invertible in R, the maps*

$$\begin{array}{ccc} F(G) & \rightleftarrows & \bigoplus_{(T,S)\in[\mathcal{M}(G)]} \big(\delta_\Phi F(T/S)\big)^{N_G(T,S)/T} \\ w & \stackrel{V}{\longmapsto} & \bigoplus_{(T,S)} \frac{1}{|N_G(T,S):T|}\, v_{T,S}^G\, w \\ \sum_{(T,S)} u_{T,S}^G\, w_{T,S} & \stackrel{U}{\longleftarrow\!\shortmid} & \bigoplus_{(T,S)} w_{T,S} \end{array}$$

are well defined isomorphisms of R-modules, inverse to one other.

Proof : We have first to check that if $w \in F(G)$, then the element $v_{T,S}^G\, w$ of $\varphi_{\mathbf{1}}^{T/S} F(T/S) = \delta_\Phi F(T/S)$ is invariant under the action of $N_G(T,S)/T$. But for any $g \in N_G(T/S)$

$$\mathrm{Iso}(c_g)\, v_{T,S}^G = v_{{}^gT,{}^gS}^G\, \mathrm{Iso}(c_g) = v_{T,S}^G\, \mathrm{Iso}(c_g) \quad ,$$

where $\mathrm{Iso}(c_g) : F(G) \to F(G)$ on the right hand side is conjugation by g, that is an inner automorphism, hence the identity map, for $g \in G$.

Now for $w \in F(G)$

$$\begin{aligned} UV(w) &= \sum_{(T,S)\in[\mathcal{M}(G)]} \tfrac{1}{|N_G(T,S):T|}\, u_{T,S}^G\, v_{T,S}^G\, w \\ &= \sum_{(T,S)\in[\mathcal{M}(G)]} \epsilon_{T,S}^G\, w = w \quad , \end{aligned}$$

so UV is the identity map of $F(G)$.

Conversely, if $w_{T,S} \in \big(\delta_\Phi F(T/S)\big)^{N_G(T,S)/T}$, for $(T,S) \in [\mathcal{M}(G)]$, then by Theorem 4.5

$$\begin{aligned} VU\big(\bigoplus_{(T,S)\in[\mathcal{M}(G)]} w_{T,S}\big) &= \bigoplus_{(T,S)\in[\mathcal{M}(G)]} \sum_{(T',S')\in[\mathcal{M}(G)]} \tfrac{1}{|N_G(T,S):T|}\, v_{T,S}^G\, u_{T',S'}^G\, w_{T',S'} \\ &= \bigoplus_{(T,S)\in[\mathcal{M}(G)]} \tfrac{1}{|N_G(T,S):T|}\, v_{T,S}^G\, u_{T,S}^G\, w_{T,S} \\ &= \bigoplus_{(T,S)\in[\mathcal{M}(G)]} \tfrac{1}{|N_G(T,S):T|} \sum_{g\in N_G(T,S)/T} \mathrm{Iso}(c_g)\, w_{T,S} \\ &= \bigoplus_{(T,S)\in[\mathcal{M}(G)]} w_{T,S} \;\;, \end{aligned}$$

so VU is also equal to the identity map. □

6. Atoric p-groups

For the remainder of the paper, we denote by p a (fixed) prime number.

6.1. Notation and Definition:

- *If P is a finite p-group, let $\Omega_1 P$ denote the subgroup of P generated by the elements of order p.*
- *A finite p-group P is called* atoric *if it does not admit any decomposition $P = E \times Q$, where E is a non-trivial elementary abelian p-group. Let $\mathcal{A}t_p$ denote the class of atoric p-groups, and let $[\mathcal{A}t_p]$ denote a set of representatives of isomorphism classes in $\mathcal{A}t_p$.*

The terminology "atoric" is inspired by [14], where elementary abelian p-groups are called *p-tori*. Atoric p-groups have been considered (without naming them) in [6], Example 5.8.

6.2. Lemma: *Let P be a finite p-group, and N be a normal subgroup of P. The following conditions are equivalent:*

1. $N \cap \Phi(P) = \mathbf{1}$
2. *N is elementary abelian and central in P, and admits a complement in P.*
3. *N is elementary abelian and there exists a subgroup Q of P such that $P = N \times Q$.*

Proof : $\boxed{1 \Rightarrow 3}$ Let $N \trianglelefteq P$ with $N \cap \Phi(P) = \mathbf{1}$. Then N maps injectively in the elementary abelian p-group $P/\Phi(P)$, so N is elementary abelian. Let $Q/\Phi(P)$ be a complement of $N\Phi(P)/\Phi(P)$ in $P/\Phi(P)$. Then $Q \geq \Phi(P) \geq [P,P]$, so Q is normal in P. Moreover $Q \cdot N = P$ and $Q \cap N\Phi(P) = (Q \cap N)\Phi(P) = \Phi(P)$, thus $Q \cap N \leq \Phi(P) \cap N = \mathbf{1}$. Now N and Q are normal subgroups of P which intersect trivially, hence they centralize each other. It follows that $P = N \times Q$.

$\boxed{3 \Rightarrow 2}$ This is clear.

$\boxed{2 \Rightarrow 1}$ If $P = N \cdot Q$ for some subgroup Q of P, and if N is central in P, then $P = N \times Q$. Thus $\Phi(P) = \mathbf{1} \times \Phi(Q)$, as N is elementary abelian. Then $N \cap \Phi(P) \leq N \cap Q = \mathbf{1}$. □

6.3. Lemma: *Let P be a finite p-group. The following conditions are equivalent:*

1. *P is atoric.*
2. *If $N \trianglelefteq P$ and $N \cap \Phi(P) = \mathbf{1}$, then $N = \mathbf{1}$.*
3. *$\Omega_1 Z(P) \leq \Phi(P)$.*

Proof : $\boxed{1 \Rightarrow 2}$ Suppose that P is atoric. Let $N \trianglelefteq P$ with $N \cap \Phi(P) = \mathbf{1}$. Then by Lemma 6.2, the group N is elementary abelian and there exists a subgroup Q of P such that $P = N \times Q$. Hence $N = \mathbf{1}$.

$\boxed{2 \Rightarrow 3}$ Suppose now that Assertion 2 holds. If x is a central element of order p of P, then the subgroup N of P generated by x is normal in P, and non trivial. Then $N \cap \Phi(P) \neq \mathbf{1}$, hence $N \leq \Phi(P)$ since N has order p, thus $x \in \Phi(P)$.

$\boxed{3 \Rightarrow 1}$ Finally, if Assertion 3 holds, and if $P = E \times Q$ for some subgroups E and Q of P with E elementary abelian, then $\Phi(P) = \mathbf{1} \times \Phi(Q)$. Moreover $E \leq \Omega_1 Z(P) \leq \Phi(P) \leq Q$, so $E = E \cap Q = \mathbf{1}$, and P is atoric. □

6.4. Proposition: *Let P be a finite p-group, and N be a maximal normal subgroup of P such that $N \cap \Phi(P) = \mathbf{1}$. Then:*

1. *The group N is elementary abelian and there exists a subgroup T of P such that $P = N \times T$.*
2. *The group $P/N \cong T$ is atoric.*
3. *If Q is an atoric p-group and $s : P \twoheadrightarrow Q$ is a surjective group homo-*

morphism, then $s(T) = Q$. In particular Q is isomorphic to a quotient of T.

Proof : (1) This follows from Lemma 6.2.

(2) By (1), there exists $T \leq P$ such that $P = N \times T$. In particular $P/N \cong T$. Now if $T = E \times S$, for some subgroups E and S of T with E elementary abelian, then $P = (N \times E) \times S$, and $N \times E$ is an elementary abelian normal subgroup of P which intersects trivially $\Phi(P) = \Phi(S)$. By maximality of N, it follows that $E = \mathbf{1}$, so $T \cong P/N$ is atoric.

(3) Let $s : P \twoheadrightarrow Q$ be a surjective group homomorphism, where Q is atoric. By (1), the group N is elementary abelian, and there exists a subgroup T of P such that $P = N \times T$. Moreover $s\big(\Phi(P)\big) = \Phi(Q)$ as P is a p-group, as already shown in the proof of Theorem 3.12, and $s\big(Z(P)\big) \leq Z(Q)$ as s is surjective. It follows that $s(N) \leq \Omega_1 Z(Q)$, so $s(N) \leq \Phi(Q)$ since Q is atoric, by Lemma 6.3. Now $s(P) = Q = s(N)s(T)$, thus $Q = \Phi(Q)s(T)$, and $s(T) = Q$, as was to be shown. □

6.5. Notation: *When P is a finite p-group, and N is a maximal normal subgroup of P such that $N \cap \Phi(P) = \mathbf{1}$, we set $P^{@} = P/N$.*

By Proposition 6.4, the group $P^{@}$ does not depend on the choice of N, up to isomorphism: it is the greatest atoric quotient of P, in the sense that any atoric quotient of P is isomorphic to a quotient of $P^{@}$. In particular $P^{@}$ is trivial if and only if P is elementary abelian.

6.6. Proposition: *Let $s : P \twoheadrightarrow Q$ be a surjective group homomorphism. Then $P^{@} \cong Q^{@}$ if and only if $\mathrm{Ker}(s) \cap \Phi(P) = \mathbf{1}$.*

Proof : Let $N = \mathrm{Ker}(s)$. If $N \cap \Phi(P) = \mathbf{1}$, then by Lemma 6.2, the group N is elementary abelian, and there exists a subgroup T of P such that $P = N \times T$. Moreover $T \cong Q$. So by Proposition 6.4, there exists an elementary abelian subgroup E of T, and a subgroup S of T with $S \cong T^{@} \cong Q^{@}$ such that $T = E \times S$. Then $P = N \times E \times S$, so $P^{@} \cong S$, since S is atoric and $N \times E$ is elementary abelian. Hence $P^{@} \cong Q^{@}$.

Conversely if $P^{@} \cong Q^{@}$, to prove that $\mathrm{Ker}(s) \cap \Phi(P) = \mathbf{1}$, it suffices to prove that $\mathrm{Ker}(\pi \circ s) \cap \Phi(P) = \mathbf{1}$, where π is a surjective group homomorphism $Q \to Q^{@}$. Now there is an elementary abelian subgroup E of P and an atoric subgroup $T \cong P^{@}$ of P such that $P = E \times T$. By Proposition 6.4, we have $(\pi \circ s)(T) = Q^{@} \cong T$, so $\pi \circ s$ induces an isomorphism from T to $Q^{@}$. In particular $\mathrm{Ker}(\pi \circ s) \cap T = \mathbf{1}$, so $\mathrm{Ker}(\pi \circ s) \cap \Phi(P) = \mathbf{1}$ since $\Phi(P) \leq T$. □

6.7. Proposition: *Let P be a finite p-group, let N be a normal subgroup of P such that $P/N \cong P^{@}$, and let Q be a subgroup of P. The following are equivalent:*

1. $Q^{@} \cong P^{@}$.
2. $QN = P$.
3. *There exists a central elementary abelian subgroup E of P such that $P = EQ$.*
4. *There exists an elementary abelian subgroup E of P such that $P = E \times Q$.*

Proof : $\boxed{1 \Rightarrow 2}$ Suppose $Q^{@} \cong P^{@}$. We have $N \cap \Phi(P) = \mathbf{1}$, by Proposition 6.6. Moreover $\Phi(Q) \leq \Phi(P)$, as P is a p-group. Setting $M = N \cap Q$, we have $M \cap \Phi(Q) = \mathbf{1}$, so $(Q/M)^{@} \cong Q^{@} \cong P^{@}$. But $\overline{Q} = Q/M \cong QN/N$ is a subgroup of $P/N \cong P^{@}$, and moreover there exists an elementary abelian subgroup E of $\overline{Q}$ such that $\overline{Q} \cong E \times \overline{Q}^{@} \cong E \times P^{@}$. Hence $E = \mathbf{1}$ and $\overline{Q} \cong QN/N \cong P/N$, so $QN = P$, as was to be shown.

$\boxed{2 \Rightarrow 3}$ We have $N \cap \Phi(P) = \mathbf{1}$, by Proposition 6.6. Hence N is elementary abelian, and central in P, and 2 implies 3.

$\boxed{3 \Rightarrow 4}$ Let E be an elementary abelian central subgroup of P such that $P = EQ$. Let F be a complement of $E \cap Q$ in E. Then F is elementary abelian and central in P. Moreover $QF = QE = P$, and $Q \cap F = \mathbf{1}$. Hence $P = F \times Q$.

$\boxed{4 \Rightarrow 1}$ If $P = E \times Q$ and E is elementary abelian, then $\Phi(P) = \mathbf{1} \times \Phi(Q)$. Thus $E \cap \Phi(P) = \mathbf{1}$, so $(P/E)^{@} \cong P^{@}$ by Proposition 6.6, and $Q^{@} \cong P^{@}$. □

6.8. Proposition: *Let P be a finite p-group, and Q be a subquotient of P. Then $Q^{@}$ is a subquotient of $P^{@}$.*

Proof : Let (V, U) be a section of P such that $V/U \cong Q$. Then $Q^{@}$ is isomorphic to a quotient of $V^{@}$, by Proposition 6.4. Hence it suffices to prove that $V^{@}$ is a subquotient of $P^{@}$.

Let E be a maximal normal subgroup of P such that $E \cap \Phi(P) = \mathbf{1}$, and $T \cong P^{@}$ be a subgroup of P such that $P = E \times T$. Let $W = EV$. Then $W^{@} \cong V^{@}$ by Proposition 6.7. Moreover $E \leq W \leq E \times T$, so $W = E \times S$, where $S = W \cap T$. Then $V^{@} \cong W^{@} \cong S^{@}$, and $S^{@}$ is a quotient of S, hence a subquotient of $T \cong P^{@}$. This completes the proof. □

7. Splitting the biset category of p-groups, when $p \in R^\times$

7.1. Notation and Definition: *Let $\mathcal{C}_p$ (resp. $R\mathcal{C}_p$) denote the full subcategory of the biset category $\mathcal{C}$ (resp. $R\mathcal{C}$) consisting of finite p-groups. A* p-biset functor *over R is an R-linear functor from $R\mathcal{C}_p$ to the category of R-modules. Let $\mathcal{F}_{p,R}$ denote the category of p-biset functors over R.*

In the statements below, we indicate by $[\ p \in R^\times]$ the assumption that p is invertible in R.

7.2. Theorem: $[\ p \in R^\times]$ *Let P and Q be finite p-groups, let (T,S) be a minimal section of P, and (V,U) be a minimal section of Q. Then*

$$\epsilon_{V,U}^Q\, RB(Q,P)\, \epsilon_{T,S}^P \neq \{0\} \implies (V/U)^@ \cong (T/S)^@ \ \ .$$

Proof : If $\epsilon_{V,U}^Q\, RB(Q,P)\, \epsilon_{T,S}^P \neq \{0\}$, there exists $a \in RB(Q,P)$ such that

$$\epsilon_{V,U}^Q\, a\, \epsilon_{T,S}^P = \mathrm{Indinf}_{V/U}^Q\, \varphi_{\mathbf{1}}^{V/U}\, \mathrm{Defres}_{V/U}^Q\, a\, \mathrm{Indinf}_{T/S}^P\, \varphi_1^{T/S}\, \mathrm{Defres}_{T/S}^P \neq 0 \ \ ,$$

and in particular the element $b = \mathrm{Defres}_{V/U}^Q\, a\, \mathrm{Indinf}_{T/S}^P$ of $RB(V/U, T/S)$ is such that $\varphi_{\mathbf{1}}^{V/U}\, b\, \varphi_{\mathbf{1}}^{T/S} \neq 0$. It follows that there is a subgroup L of the product $(V/U) \times (T/S)$ such that

$$\varphi_{\mathbf{1}}^{V/U}\big[\big((V/U) \times (T/S)\big)/L\big]\varphi_{\mathbf{1}}^{T/S} \neq 0 \ \ .$$

Then Theorem 3.8 implies that $p_1(L) = V/U$, $k_1(L) \cap \Phi(V/U) = \mathbf{1}$, $p_2(L) = T/S$, and $k_2(L) \cap \Phi(T/S) = \mathbf{1}$. By Proposition 6.6, it follows that

$$(V/U)^@ \cong \big(p_1(L)/k_1(L)\big)^@ \cong \big(p_2(L)/k_2(L)\big)^@ \cong (T/S)^@ \ \ ,$$

as was to be shown. □

7.3. Notation: $[\ p \in R^\times]$ *Let L be an atoric p-group. If P is a finite p-group, we denote by b_L^P the element of $RB(P,P)$ defined by*

$$b_L^P = \sum_{\substack{(T,S) \in [\mathcal{M}(P)] \\ (T/S)^@ \cong L}} \epsilon_{T,S}^P \ \ .$$

Recall that the *center* $Z(\mathcal{D})$ of an essentially small category $\mathcal{D}$ is by definition the set of natural transformations from the identity functor $\mathrm{Id}_{\mathcal{D}}$ to itself. Thus an element θ of $Z(\mathcal{D})$ assigns to each object D of $\mathcal{D}$ an endomorphism θ_D of D, in such a way that for any morphism $f : D \to D'$ in $\mathcal{D}$, the diagram

$$\begin{array}{ccc} D & \xrightarrow{\theta_D} & D \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ D' & \xrightarrow{\theta_{D'}} & D' \end{array}$$

is commutative. The center $Z(\mathcal{D})$ is in general a monoid for the composition of natural transformations. If $\mathcal{D}$ is R-linear (for some commutative ring R), then $Z(\mathcal{D})$ becomes an R-algebra in a natural way (the R-module structure is given by R-linear combination of natural transformations).

7.4. Theorem: $[\, p \in R^\times]$

1. *Let L be an atoric p-group, and P be a finite p-group. Then $b_L^P \neq 0$ if and only if $L \sqsubseteq P^@$.*
2. *Let L and M be atoric p-groups, and let P and Q be finite p-groups. If $b_M^Q\, RB(Q,P)\, b_L^P \neq \{0\}$, then $M \cong L$.*
3. *Let L be an atoric p-group, and let P and Q be finite p-groups. Then for any $a \in RB(Q,P)$*
$$b_L^Q\, a = a\, b_L^P \quad .$$
4. *The family of elements $b_L^P \in RB(P,P)$, for finite p-groups P, is an idempotent endomorphism b_L of the identity functor of the category $R\mathcal{C}_p$ (i.e. an idempotent of the center of $R\mathcal{C}_p$). The idempotents b_L, for $L \in [\mathcal{A}t_p]$, are orthogonal, and their sum is equal to the identity element of the center of $R\mathcal{C}_p$.*
5. *For a given finite p-group P, the elements b_L^P, for $L \in [\mathcal{A}t_p]$ such that $L \sqsubseteq P^@$, are non zero orthogonal central idempotents of $RB(P,P)$, and their sum is equal to the identity of $RB(P,P)$.*
6. *For given finite p-groups P and Q, and a given atoric p-group L, let $\mathcal{S}$ be a set of representatives of conjugacy classes of subgroups Y of $Q \times P$ such that $q(Y)^@ \cong L$. Then the elements $b_L^Q\,[(Q\times P)/Y] = [(Q\times P)/Y]\, b_L^P$, for $Y \in \mathcal{S}$, form an R-basis of $b_L^Q RB(Q,P)$.*

Proof : (1) The idempotent b_L^P is non zero if and only if there exists a minimal section (T,S) of P such that $(T/S)^@ \cong L$. Then $L \sqsubseteq P^@$, by Proposition 6.8. Conversely, if $L \sqsubseteq P^@$, then $L \sqsubseteq P$, and by Remark 4.4, there exists a minimal section (T,S) of P such that $T/S \cong L$. Then $(T/S)^@ \cong L^@ \cong L$, so $\epsilon^P_{T,S}$ appears in the sum defining b^P_L, thus $b^P_L \neq 0$.

(2) If $b^Q_M\, RB(Q,P)\, b^P_L \neq \{0\}$, then there exist a minimal section (V,U) of Q with $(V/U)^@ \cong M$ and a minimal section (T,S) of P with $(T/S)^@ \cong L$ such that $\epsilon^Q_{V,U}\, RB(Q,P)\, \epsilon^P_{T,S} \neq 0$. Then $(V/U)^@ \cong (T/S)^@$ by Theorem 7.2, that is $M \cong L$.

(3) By Theorem 4.8, the identity element of $RB(P,P)$ is equal to the sum of the idempotents $\epsilon^P_{T,S}$, for $(T,S) \in [\mathcal{M}(P)]$. Grouping those idempotents $\epsilon^P_{T,S}$ for which $(T/S)^@$ is isomorphic to a given $L \in [\mathcal{A}t_p]$ shows that the identity element of $RB(P,P)$ is equal to the sum of the idempotents b^P_L, for $L \in [\mathcal{A}t_p]$ (and there are finitely many non zero b^P_L, by (1)). It follows that

$$\begin{aligned} b^Q_M\, a &= b^Q_M\, a \sum_{L\in[\mathcal{A}t_p]} b^P_L = \sum_{L\in[\mathcal{A}t_p]} b^Q_M\, a\, b^P_L \\ &= b^Q_M\, a\, b^P_M \;\; \text{[by (2)]} \\ &= \sum_{L\in[\mathcal{A}t_p]} b^Q_L\, a\, b^P_M \;\; \text{[by (2)]} \\ &= a\, b^P_M \;\;, \end{aligned}$$

since $\sum\limits_{L\in[\mathcal{A}t_p]} b^Q_L$ is the identity element of $RB(Q,Q)$.

(4) It follows that the family b^P_L, where P runs over finite p-groups, is an element b_L of the center of $R\mathcal{C}_p$. Clearly $b_L^2 = b_L$, and if L and M are non isomorphic atoric p-groups, then $b_L b_M = 0$, by (2). Moreover the infinite sum $\sum\limits_{L\in[\mathcal{A}t_p]} b_L$ is actually locally finite, i.e. for each finite p-group P, the sum $\sum\limits_{L\in[\mathcal{A}t_p]} b^P_L$ has only finitely many non zero terms. The sum $\sum\limits_{L\in[\mathcal{A}t_p]} b_L$ is clearly equal to the identity endomorphism of the identity functor of $R\mathcal{C}_p$.

(5) This is a straightforward consequence of (1) and (3).

(6) Let Y be any subgroup of $Q \times P$. By 2.6, we can factorize $[(Q\times P)/Y]$ as $[(Q\times P)/Y] = ab$, where $a \in RB\big(Q, q(Y)\big)$ and $b \in RB\big(q(Y), P\big)$. If $b^Q_L\,[(Q\times P)/Y]$ is non zero, then $b^Q_L a$, equal to $a b^{q(Y)}_L$ by Assertion 3, is also non zero, hence $b^{q(Y)}_L \neq 0$, so $L \sqsubseteq q(Y)$ by Assertion 1. Thus $L \cong L^@ \sqsubseteq q(Y)^@$.

But on the other hand b^Q_L is the sum of the distinct idempotents $\epsilon^Q_{T,S}$ corresponding to minimal sections (T,S) of Q such that $(T/S)^@ \cong L$. By Proposition 4.7, together with (2.9), it follows that b^Q_L is a linear combination of terms of the form $\big[(Q\times Q)/\Delta_M(X)\big]$, where (X,M) is a section of Q such that $S \le M \le \Phi(T) \le X \le T$ for one of these minimal sections (T,S) of Q.

Now the composition $b_L^Q\,[(Q\times P)/Y]$ is a linear combination of terms of the form $[(Q\times Q)/\Delta_M(X)]\,[(Q\times P)/Y]$, that is by (2.7), a linear combination of terms $[(Q\times P)/\big(\Delta_M(X)*^{(x,1)}Y\big)]$, for some $x\in Q$. By Lemma 2.3.22 of [7], the group $q\big(\Delta_M(X)*^{(x,1)}Y\big)$ is a subquotient of $q\big(\Delta_M(X)\big)\cong X/M$, hence it is a subquotient of T/S. It follows that $b_L^Q\,[(Q\times P)/Y]$ is a linear combination of terms of the form $[(Q\times P)/Z]$, where $q(Z)\sqsubseteq T/S$ for some minimal section (T,S) of Q with $(T/S)^@\cong L$. In particular $q(Z)^@\sqsubseteq (T/S)^@\cong L$.

But then, composing with b_L^Q, we get that $b_L^Q\,[(Q\times P)/Y]$ is a linear combination of terms of the form $b_L^Q\,[(Q\times P)/Z]$, where $q(Z)^@\sqsubseteq L$. On the other hand, we have seen that $b_L^Q\,[(Q\times P)/Z]=0$ unless $L\sqsubseteq q(Z)^@$. It follows that the elements $b_L^Q\,[(Q\times P)/Z]$, for $q(Z)^@\cong L$, generate $b_L^Q\,RB(Q,P)$.

Allowing L to run through all atoric p-groups, we see that the elements $b^Q_{q(Z)^@}[(Q\times P)/Z]$, when Z runs through subgroups of $Q\times P$ up to conjugation, generate $RB(Q,P)$. In other words the linear endomorphism β of $RB(Q,P)$ sending $[(Q\times P)/Z]$ to $b^Q_{q(Z)^@}[(Q\times P)/Z]$ is surjective. As $RB(Q,P)$ is a free R-module, the linear map β must be split surjective, and there is a linear endomorphism γ of $RB(Q,P)$ such that $\beta\gamma=\mathrm{Id}$. This can be viewed as a product of square matrices with coefficients in R. Taking determinants (which makes sense since R is commutative), we get that β and γ are both isomorphisms, and in particular the elements $b^Q_{q(Z)^@}[(Q\times P)/Z]$, for Z in a set of representatives of conjugacy classes of subgroups of $Q\times P$, are linearly independent. In particular, for a fixed atoric p-group L, the elements $b_L^Q\,[(Q\times P)/Z]$, for $Z\in\mathcal{S}$, are linearly independent. This completes the proof. □

7.5. Corollary: [$p\in R^\times$]

1. *Let L be an atoric p-group. For a p-biset functor F, the family of maps $F(b_L^P):F(P)\to F(P)$, for finite p-groups P, is an endomorphism of F, denoted by $F(b_L)$.*
2. *If $\theta:F\to G$ is a natural transformation of p-biset functors, the diagram*

$$\begin{array}{ccc} F & \xrightarrow{F(b_L)} & F \\ {\scriptstyle\theta}\big\downarrow & & \big\downarrow{\scriptstyle\theta} \\ G & \xrightarrow[G(b_L)]{} & G \end{array}$$

is commutative. Hence the family of maps $F(b_L^P):F(P)\to F(P)$, for p-groups P and p-biset functors F, is an idempotent of the center of *the category $\mathcal{F}_{p,R}$, denoted by $\widehat{b}_L$.*

3. *The idempotents $\widehat{b}_L$, for $L \in [\mathcal{A}t_p]$, are orthogonal idempotents of the center of $\mathcal{F}_{p,R}$, and their sum is the identity.*
4. *If F is a p-biset functor over R, let $\widehat{b}_L F$ denote the image of the endomorphism $F(b_L)$ of F. Then $F = \bigoplus_{L\in[\mathcal{A}t_p]} \widehat{b}_L F$.*
5. *Let $\widehat{b}_L\mathcal{F}_{p,R}$ denote the full subcategory of $\mathcal{F}_{p,R}$ consisting of functors F such that $F = \widehat{b}_L F$. Then $\widehat{b}_L\mathcal{F}_{p,R}$ is an abelian subcategory of $\mathcal{F}_{p,R}$. Moreover the functor*

$$F \in \mathcal{F}_{p,R} \mapsto (\widehat{b}_L F)_{L\in[\mathcal{A}t_p]} \in \prod_{L\in[\mathcal{A}t_p]} \widehat{b}_L\mathcal{F}_{p,R} \tag{7.6}$$

is an equivalence of categories.

Proof : All assertions are straightforward consequences of Theorem 7.4. □

In order to study the categories appearing in the above decomposition (7.6) of $\mathcal{F}_{p,R}$, it will be convenient to consider first the product of those categories $\widehat{b}_H\mathcal{F}_{p,R}$ obtained when H runs through atoric subquotients of a given atoric p-group L. This motivates the following notation:

7.7. Notation: *For an atoric p-group L, let $R\mathcal{C}_p^L$ denote the full subcategory of $R\mathcal{C}_p$ consisting of the class $\mathcal{Y}_L$ of finite p-groups P such that $P^@ \sqsubseteq L$. When $p \in R^\times$, let moreover*

$$b_L^+ = \sum_{\substack{H\in[\mathcal{A}t_p]\\ H\sqsubseteq L}} b_H$$

be the sum of the idempotents b_H corresponding to atoric subquotients of L, up to isomorphism. When P is any finite p-group, we get a corresponding central idempotent of $RB(P,P)$, defined by

$$b_L^{+P} = \sum_{\substack{H\in[\mathcal{A}t_p]\\ H\sqsubseteq L}} b_H^P \ .$$

Similarly, we denote by

$$\widehat{b}_L^+ = \sum_{\substack{H\in[\mathcal{A}t_p]\\ H\sqsubseteq L}} \widehat{b}_H$$

the central idempotent of $\mathcal{F}_{p,R}$ corresponding to b_L^+. For any finite p-group P and any p-biset functor F, we get a linear map

$$F(b_L^{+P}) = \sum_{\substack{H\in[\mathcal{A}t_p]\\ H\sqsubseteq L}} F(b_H^P) : F(P) \to F(P) \ \ .$$

The class $\mathcal{Y}_L$ is closed under taking subquotients, by Proposition 6.8. It follows that we can apply the results of Section 6 (Appendix) of [12]: if F is a p-biset functor over R, we can restrict F to an R-linear functor from $R\mathcal{C}_p^L$ to R-Mod. This yields a forgetful functor $\mathcal{O}_{\mathcal{Y}_L} : \mathcal{F}_{p,R} \to \mathsf{Fun}_R\big(R\mathcal{C}_p^L, R\text{-}\mathsf{Mod}\big)$. The right adjoint $\mathcal{R}_{\mathcal{Y}_L}$ of this functor is described in full detail in Section 6 of [12], as follows: if G is an R-linear functor from $R\mathcal{C}_p^L$ to R-Mod, and P is a finite p-group, set

$$\textbf{(7.8)}\qquad \mathcal{R}_{\mathcal{Y}_L}(G)(P) = \varprojlim_{(X,M)\in\Sigma_L(P)} G(X/M)$$

the inverse limit of modules $G(X/M)$ on the set $\Sigma_L(P)$ of sections (X,M) of P such that $(X/M)^@ \sqsubseteq L$, i.e. the set of sequences $(l_{X,M})_{(X,M)\in\Sigma_L(P)}$ with the following properties:

(7.9)
1. if $(X,M)\in\Sigma_L(P)$, then $l_{X,M}\in G(X/M)$.
2. if $(X,M),(Y,N)\in\Sigma_L(P)$ and $M\leq N\leq Y\leq X$, then
$$\mathrm{Defres}_{Y/N}^{X/M}\, l_{X,M} = l_{Y,N} \ \ .$$
3. if $x\in P$ and $(X,M)\in\Sigma_L(P)$, then ${}^x l_{X,M} = l_{{}^xX,{}^xM}$.

7.10. Remark: Observe that in Condition 2, there is no need to assume that $(Y,N)\in\Sigma_L(P)$: indeed if $M\leq N\leq Y\leq X$ and if $(X,M)\in\Sigma_L(P)$, then Y/N is a subquotient of X/M, so $(Y/N)^@$ is a subquotient of L, by Proposition 6.8, that is $(Y,N)\in\Sigma_L(P)$.

Recall now that for finite groups P and Q, and for a finite (Q,P)-biset U, for a subgroup T of Q and an element u of U, the subgroup T^u of P is defined by $T^u = \{x\in P \mid \exists t\in T \ \ tu = ux\}$. By Lemma 6.4 of [12], if (T,S) is a section of Q, then (T^u,S^u) is a section of P, and T^u/S^u is a subquotient of T/S.

With this notation, when P and Q are finite p-groups, when U is a finite (Q,P)-biset, and $l=(l_{X,M})_{(X,M)\in\Sigma_L(P)}$ is an element of $\mathcal{R}_{\mathcal{Y}_L}(G)(P)$, we denote by Ul the sequence indexed by $\Sigma_L(Q)$ defined by

$$(Ul)_{Y,N} = \sum_{u\in[Y\backslash U/P]_L} (N\backslash Yu)(l_{Y^u,N^u})$$

where $[Y\backslash U/P]$ is a set of representatives of $(Y\times P)$-orbits on U, and $N\backslash Yu$ is viewed as a $(Y/N, Y^u/N^u)$-biset. It is shown in Section 6 of [12] that $Ul \in \mathcal{R}_{\mathcal{Y}_L}(G)(Q)$, and that $\mathcal{R}_{\mathcal{Y}_L}(G)$ becomes a p-biset functor in this way. Moreover[1]:

7.11. Theorem: [[12] Theorem 6.15] *The assignment $G\mapsto \mathcal{R}_{\mathcal{Y}_L}(G)$ is an R-linear functor $\mathcal{R}_{\mathcal{Y}_L}$ from $\mathsf{Fun}_R\big(R\mathcal{C}_p^L, R\text{-}\mathsf{Mod}\big)$ to $\mathcal{F}_{p,R}$, which is right adjoint to the forgetful functor $\mathcal{O}_{\mathcal{Y}_L}$. Moreover the composition $\mathcal{O}_{\mathcal{Y}_L}\circ\mathcal{R}_{\mathcal{Y}_L}$ is isomorphic to the identity functor of $\mathsf{Fun}_R\big(R\mathcal{C}_p^L, R\text{-}\mathsf{Mod}\big)$.*

7.12. Theorem: $[\ p\in R^\times]$ *For an atoric p-group L, let $\widehat{b}_L^+\mathcal{F}_{p,R}$ be the full subcategory of $\mathcal{F}_{p,R}$ consisting of functors F such that $\widehat{b}_L^+F = F$. Then the forgetful functor $\mathcal{O}_{\mathcal{Y}_L}$ and its right adjoint $\mathcal{R}_{\mathcal{Y}_L}$ restrict to quasi-inverse equivalences of categories*

$$\widehat{b}_L^+\mathcal{F}_{p,R} \underset{\mathcal{R}_{\mathcal{Y}_L}}{\overset{\mathcal{O}_{\mathcal{Y}_L}}{\rightleftarrows}} \mathsf{Fun}_R\big(R\mathcal{C}_p^L, R\text{-}\mathsf{Mod}\big)\ .$$

Proof : First step: The first thing to check is that the image of the functor $\mathcal{R}_{\mathcal{Y}_L}$ is contained in $\widehat{b}_L^+\mathcal{F}_{p,R}$. We first prove that if H is an atoric p-group, if $F\in\mathcal{F}_{p,R}$, and if $\mathcal{O}_{\mathcal{Y}_L}(\widehat{b}_HF)\neq 0$, then $H\sqsubseteq L$: indeed in that case, there exists $P\in\mathcal{Y}_L$ such that $b_H^PF(P)\neq 0$. In particular $b_H^P\neq 0$, hence $H\sqsubseteq P^@$, by Theorem 7.4. Since $P^@\sqsubseteq L$ as $P\in\mathcal{Y}_L$, it follows that $H\sqsubseteq L$, as claimed.

In particular

$$\mathcal{O}_{\mathcal{Y}_L}(F) = \mathcal{O}_{\mathcal{Y}_L}\big(\sum_{\substack{H\in[\mathcal{A}t_p]\\ H\sqsubseteq L}} \widehat{b}_HF\big) = \mathcal{O}_{\mathcal{Y}_L}\big(\widehat{b}_L^+F\big)\ .$$

Set $\mathcal{G}_p^L = \mathsf{Fun}_R\big(R\mathcal{C}_p^L, R\text{-}\mathsf{Mod}\big)$, and let $G\in\mathcal{G}_p^L$. Let H be an atoric p-group such that $H\not\sqsubseteq L$, and let $F\in\mathcal{F}_{p,R}$. Then $\mathcal{O}_{\mathcal{Y}_L}(\widehat{b}_HF)=\{0\}$ by the above

[1]In Theorem 6.15 of [12], only the case $R=\mathbb{Z}$ is considered, but the proofs extend trivially to the case of an arbitrary commutative ring R

claim. Moreover

$$\begin{aligned}\mathrm{Hom}_{\mathcal{F}_{p,R}}\big(F,\widehat{b}_H\mathcal{R}_{\mathcal{Y}_L}(G)\big) &= \mathrm{Hom}_{\mathcal{F}_{p,R}}\big(\widehat{b}_HF,\widehat{b}_H\mathcal{R}_{\mathcal{Y}_L}(G)\big)\\ &= \mathrm{Hom}_{\mathcal{F}_{p,R}}\big(\widehat{b}_HF,\mathcal{R}_{\mathcal{Y}_L}(G)\big)\\ &\cong \mathrm{Hom}_{\mathcal{G}_p^L}\big(\mathcal{O}_{\mathcal{Y}_L}\big(\widehat{b}_HF\big),G\big)=\{0\}\ .\end{aligned}$$

So the functor $F\mapsto \mathrm{Hom}_{\mathcal{F}_{p,R}}\big(F,\widehat{b}_H\mathcal{R}_{\mathcal{Y}_L}(G)\big)$ is the zero functor, and it follows from Yoneda's lemma that $\widehat{b}_H\mathcal{R}_{\mathcal{Y}_L}(G)=0$ if $H\not\sqsubseteq L$. In other words $\mathcal{R}_{\mathcal{Y}_L}(G)=\widehat{b}_L^+\mathcal{R}_{\mathcal{Y}_L}(G)$, as was to be shown.

Second step: The first step shows that we have adjoint functors

$$\widehat{b}_L^+\mathcal{F}_{p,R}\ \underset{\mathcal{R}_{\mathcal{Y}_L}}{\overset{\mathcal{O}_{\mathcal{Y}_L}}{\rightleftarrows}}\ \mathsf{Fun}_R\big(R\mathcal{C}_p^L,R\text{-}\mathsf{Mod}\big)=\mathcal{G}_p^L\ .$$

Moreover, the composition $\mathcal{O}_{\mathcal{Y}_L}\circ\mathcal{R}_{\mathcal{Y}_L}$ is isomorphic to the identity functor, by Theorem 7.11. All we have to show is that the unit of the adjunction is also an isomorphism, in other words, that for any $F\in\widehat{b}_L^+\mathcal{F}_{p,R}$ and any finite p-group P, the natural map

$$\textbf{(7.13)}\qquad \eta_P: F(P)\to\mathcal{R}_{\mathcal{Y}_L}\mathcal{O}_{\mathcal{Y}_L}(F)(P)=\lim_{\substack{\longleftarrow\\(X,M)\in\Sigma_L(P)}}F(X/M)$$

sending $u\in F(P)$ to the sequence $\big(\mathrm{Defres}^P_{X/M}u\big)_{(X,M)\in\Sigma_L(P)}$, is an isomorphism.

The map η_P is injective: indeed, if $u\in F(P)$, then $u=\sum\limits_{\substack{H\in[\mathcal{A}t_p]\\H\sqsubseteq L}}b_H^Pu$, as $F=\widehat{b}_L^+F$. If $\mathrm{Defres}^P_{X/M}u=0$ for any section (X,M) of P with $(X/M)^@\sqsubseteq L$, then $F(\epsilon^P_{T,S})(u)=0$ for any section (T,S) of P such that $(T/S)^@\sqsubseteq L$, by Proposition 4.7 and Proposition 6.8. In particular $b_H^Pu=0$ for any atoric subquotient H of L, hence $u=0$.

To prove that η_P is also surjective, we generalize the construction of Theorem A.2 of [11] (which is the case $L=\mathbf{1}$), and we define, for an element $v=(v_{X,M})_{(X,M)\in\Sigma_L(P)}$ in $\mathcal{R}_{\mathcal{Y}_L}\mathcal{O}_{\mathcal{Y}_L}(F)(P)$, an element $u=\iota_P(v)$ of $F(P)$ by

$$u=\frac{1}{|P|}\sum_{\substack{(T,S)\in\mathcal{M}(P)\\(T/S)^@\sqsubseteq L}}\ \sum_{\substack{X\le T,\,M\trianglelefteq T\\S\le M\le\Phi(T)\le X\le T}}|X|\mu(X,T)\mu_{\trianglelefteq T}(S,M)\mathrm{Indinf}^P_{X/M}v_{X,M}\ .$$

This yields an R-linear map $\iota_P:\mathcal{R}_{\mathcal{Y}_L}\mathcal{O}_{\mathcal{Y}_L}(F)(P)\to F(P)$.

For $(Y,N)\in\Sigma_L(P)$, set $u_{Y,N}=\mathrm{Defres}^P_{Y/N}u$. Then:

$$u_{Y,N}=\sum_{\substack{(T,S)\in\mathcal{M}(P)\\(T/S)^@\sqsubseteq L}}\;\sum_{\substack{X\leq T,\,M\trianglelefteq T\\S\leq M\leq\Phi(T)\leq X\leq T}}\frac{|X|}{|P|}\mu(X,T)\mu_{\trianglelefteq T}(S,M)\mathrm{Defres}^P_{Y/N}\mathrm{Indinf}^P_{X/M}v_{X,M}.$$

Moreover, by Proposition A.1 of [11]

$$\mathrm{Defres}^P_{Y/N}\,\mathrm{Indinf}^P_{X/M}v_{X,M}=\sum_{g\in[Y\backslash P/X]}\mathrm{Indinf}^{Y/N}_{J_g/J'_g}\,\mathrm{Iso}(\phi_g)\,\mathrm{Defres}^{{}^gX/{}^gM_g}_{I_g/I'_g}v_{X,N}\quad,$$

where $J_g=N(Y\cap{}^gX)$, $J'_g=N(Y\cap{}^gM)$, $I_g={}^gM(Y\cap{}^gX)$, $I'_g={}^gM(N\cap{}^gX)$, and ϕ_g is the isomorphism $I_g/I'_g\to J_g/J'_g$ sending xI'_g to xJ'_g, for $x\in Y\cap{}^gX$. Moreover $\mathrm{Defres}^{{}^gX/{}^gM_g}_{I_g/I'_g}v_{X,N}=v_{I_g,I'_g}$ by Conditions 2 and 3 in the definition (7.9) of the inverse limit on $\Sigma_L(P)$, since moreover $(I_g,I'_g)\in\Sigma_L(P)$ by Remark 7.10. Hence

$$\begin{aligned}\mathrm{Defres}^P_{Y/N}\,\mathrm{Indinf}^P_{X/M}v_{X,M}&=\sum_{g\in[Y\backslash P/X]}\mathrm{Indinf}^{Y/N}_{J_g/J'_g}\,\mathrm{Iso}(\phi_g)v_{I_g,I'_g}\\&=\sum_{g\in P}\frac{|Y\cap{}^gX|}{|Y||X|}\,\mathrm{Indinf}^{Y/N}_{J_g/J'_g}\,\mathrm{Iso}(\phi_g)v_{I_g,I'_g}\quad.\end{aligned}$$

Thus

$$u_{Y,N}=\sum_{\substack{(T,S)\in\mathcal{M}(P)\\(T/S)^@\sqsubseteq L\\X\leq T,\,M\trianglelefteq T\\S\leq M\leq\Phi(T)\leq X\leq T\\g\in P}}\frac{|Y\cap{}^gX|}{|P||Y|}\mu(X,T)\mu_{\trianglelefteq T}(S,M)\,\mathrm{Indinf}^{Y/N}_{J_g/J'_g}\,\mathrm{Iso}(\phi_g)v_{I_g,I'_g}.$$

Now $\mu(X,T)=\mu({}^gX,{}^gT)$ and $\mu_{\trianglelefteq T}(S,M)=\mu_{\trianglelefteq {}^gT}({}^gS,{}^gM)$, so summing over $({}^gT,{}^gS,{}^gX,{}^gM)$ instead of (T,S,X,M) we get

$$u_{Y,N}=\sum_{\substack{(T,S)\in\mathcal{M}(P)\\(T/S)^@\sqsubseteq L\\X\leq T,\,M\trianglelefteq T\\S\leq M\leq\Phi(T)\leq X\leq T}}\frac{|Y\cap X|}{|Y|}\mu(X,T)\mu_{\trianglelefteq T}(S,M)\,\mathrm{Indinf}^{Y/N}_{J_1/J'_1}\,\mathrm{Iso}(\phi_1)v_{I_1,I'_1}.$$

Setting $W=Y\cap X$, we have $J_1=NW$, $J'_1=N(W\cap M)$, $I_1=MW$, $I'_1=M(N\cap W)$, and these four groups only depend on W, once M and N are given. Hence, for given T,S and M, we can group together the terms of the above summation for which $Y\cap X$ is a given subgroup W of $Y\cap T$. This gives

$$u_{Y,N}=\sum_{\substack{(T,S)\in\mathcal{M}(P)\\(T/S)^@\sqsubseteq L\\M\trianglelefteq T\\S\leq M\leq\Phi(T)\\W\leq Y\cap T}}\Big(\sum_{\substack{\Phi(T)\leq X\leq T\\X\cap Y=W}}\mu(X,T)\Big)\frac{|W|}{|Y|}\mu_{\trianglelefteq T}(S,M)\,\mathrm{Indinf}_{J_1/J_1'}^{Y/N}\,\mathrm{Iso}(\phi_1)v_{I_1,I_1'}.$$

Moreover $\sum\limits_{\substack{\Phi(T)\leq X\leq T\\X\cap Y=W}}\mu(X,T)=\sum\limits_{\substack{X\leq T\\X\cap(Y\cap T)=W}}\mu(X,T)$, since $\mu(X,T)=0$ unless $X\geq\Phi(T)$, and the latter summation vanishes unless $Y\cap T=T$, by classical combinatorial lemmas ([15] Corollary 3.9.3). This gives:

$$u_{Y,N}=\sum_{\substack{(T,S)\in\mathcal{M}(P)\\(T/S)^@\sqsubseteq L\\M\trianglelefteq T\\S\leq M\leq\Phi(T)\leq W\leq T\leq Y}}\frac{|W|}{|Y|}\mu(W,T)\mu_{\trianglelefteq T}(S,M)\,\mathrm{Indinf}_{J_1/J_1'}^{Y/N}\,\mathrm{Iso}(\phi_1)v_{I_1,I_1'}\ .$$

Moreover in this summation $J_1=NW$, $J_1'=N(W\cap M)=NM$, $I_1=MW=W$, $I_1'=M(N\cap W)=MN\cap W$. All these groups remain unchanged if we replace M by $M\big(N\cap\Phi(T)\big)$, so for given T,S and W, we can group together those terms for which $M\big(N\cap\Phi(T)\big)$ is a given normal subgroup U of T with $U\leq\Phi(T)$. The sum $\sum\limits_{\substack{S\leq M\trianglelefteq T\\M(N\cap\Phi(T))=U}}\mu_{\trianglelefteq T}(S,M)$ is equal to 0 (by the same above-mentioned classical combinatorial lemmas, applied to the normal subgroup $S\big(N\cap\Phi(T)\big)$ of T) unless $S\big(N\cap\Phi(T)\big)=S$, i.e. $N\cap\Phi(T)\leq S$. Hence

$$u_{Y,N}=\sum_{\substack{(T,S)\in\mathcal{M}(P)\\(T/S)^@\sqsubseteq L\\U\trianglelefteq T\\N\cap\Phi(T)\leq S\leq U\leq\Phi(T)\leq W\leq T\leq Y}}\frac{|W|}{|Y|}\mu(W,T)\mu_{\trianglelefteq T}(S,U)\,\mathrm{Indinf}_{J_1/J_1'}^{Y/N}\,\mathrm{Iso}(\phi_1)v_{I_1,I_1'},$$

where $J_1=NW$, $J_1'=NU$, $I_1=W$, $I_1'=UN\cap W$.

Now if $N\cap\Phi(T)\leq S\leq\Phi(T)\leq T\leq Y$, then $(TN/N)^@\sqsubseteq(Y/N)^@$. Moreover the normal subgroup $(N\cap T)/\big(N\cap\Phi(T)\big)$ of $T/\big(N\cap\Phi(T)\big)$ intersects trivially the Frattini subgroup

$$\Phi\Big(T/\big(N\cap\Phi(T)\big)\Big)=\Phi(T)/\big(N\cap\Phi(T)\big)\ ,$$

so $\Big(T/\big(N\cap\Phi(T)\big)\Big)^@\cong(T/(N\cap T)^@\cong(TN/N)^@$ by Proposition 6.6, applied to the quotient map $T/\big(N\cap\Phi(T)\big)\to T/(N\cap T)$.

Then $(T/S)^@ \sqsubseteq \Big(T/\big(N\cap\Phi(T)\big)\Big)^@ \sqsubseteq (TN/N)^@ \sqsubseteq (Y/N)^@$. Since $(Y/N)^@ \sqsubseteq L$ by assumption, it follows that

$$u_{Y,N} = \sum_{\substack{S \trianglelefteq T \leq Y\\ U \trianglelefteq T\\ N\cap\Phi(T)\leq S\leq U\leq \Phi(T)\leq W\leq T\leq Y}} \frac{|W|}{|Y|}\mu(W,T)\mu_{\trianglelefteq T}(S,U)\,\mathrm{Indinf}_{J_1/J_1'}^{Y/N}\,\mathrm{Iso}(\phi_1)v_{I_1,I_1'}.$$

Now the sum $\sum\limits_{\substack{S\trianglelefteq T\\ N\cap\Phi(T)\leq S\leq U}} \mu_{\trianglelefteq T}(S,U)$ is equal to zero unless $U = N\cap\Phi(T)$. Hence

$$u_{Y,N} = \sum_{\Phi(T)\leq W\leq T\leq Y} \frac{|W|}{|Y|}\mu(W,T)\,\mathrm{Indinf}_{J_1/J_1'}^{Y/N}\,\mathrm{Iso}(\phi_1)v_{I_1,I_1'}.$$

For a given subgroup W of Y, the sum $\sum\limits_{\Phi(T)\leq W\leq T\leq Y} \mu(W,T)$ is equal to $\sum\limits_{W\leq T\leq Y} \mu(W,T)$ since $\mu(W,T)=0$ unless $W\geq\Phi(T)$, and the latter is equal to zero if $W\neq Y$, and to 1 if $W=Y$. Thus

$$u_{Y,N} = \frac{|Y|}{|Y|}\mathrm{Indinf}_{J_1/J_1'}^{Y/N}\,\mathrm{Iso}(\phi_1)v_{I_1,I_1'},$$

where $J_1 = NY = Y$, $J_1' = NU = N$, $I_1 = Y$, $I_1' = UN\cap Y = N$. Hence $I_1 = J_1 = Y$ and $I_1' = J_1' = N$, so ϕ_1 is equal to the identity. It follows that $u_{Y,N} = v_{Y,N}$ for any $(Y,N)\in\Sigma_L(P)$, so $\eta_P(u) = v$. This proves that the map η_P is surjective, hence an isomorphism, with inverse ι_P. This completes the proof of Theorem 7.12. □

7.14. Definition: *Let L be an atoric p-group, and let $R\mathcal{C}_p^{\sharp L}$ be the following category:*

- *The objects of $R\mathcal{C}_p^{\sharp L}$ are the finite p-groups P such that $P^@\cong L$.*
- *If P and Q are finite p-groups such that $P^@\cong Q^@\cong L$, then*

$$\mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}(P,Q) = RB(Q,P)/\sum_{L\not\sqsubseteq S} RB(Q,S)B(S,P)$$

  is the quotient of $RB(Q,P)$ by the R-submodule generated by all morphisms from P to Q in $R\mathcal{C}_p$ which factor through a p-group S which does not admit L as a subquotient.

- *The composition of morphisms in $R\mathcal{C}_p^{\sharp L}$ is induced by the composition of morphisms in $R\mathcal{C}_p$.*

7.15. Remark: Morphisms in $R\mathcal{C}_p$ which factor through a p-group S such that $L \not\sqsubseteq S$ clearly generate a two-sided ideal, so the composition in $R\mathcal{C}_p^{\sharp L}$ is well defined. Moreover the category $R\mathcal{C}_p^{\sharp L}$ is R-linear. Let $\mathsf{Fun}_R\big(R\mathcal{C}_p^{\sharp L}, R\text{-}\mathsf{Mod}\big)$ denote the category of R-linear functors from $R\mathcal{C}_p^{\sharp L}$ to the category R-Mod of R-modules.

7.16. Lemma: *Let p be a prime, and L be an atoric p-group. Let P and Q be finite p-groups.*

1. *If $P^@ \cong L$ or $Q^@ \cong L$, and if $M \leq (Q \times P)$, then $q(M)^@ \sqsubseteq L$. Moreover $q(M)^@ \cong L$ if and only if $L \sqsubseteq q(M)$.*
2. *If $P^@ \cong Q^@ \cong L$, then*

$$\mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}(P,Q) = RB(Q,P)/\sum_{S^@ \sqsubset L} RB(Q,S)B(S,P)$$

   is also the quotient of $RB(Q,P)$ by the R-submodule generated by all morphisms from P to Q in $R\mathcal{C}_p$ which factor through a p-group S such that $S^@$ is a proper subquotient of L.
3. *If $P^@ \cong Q^@ \cong L$, then $\mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}(P,Q)$ has an R-basis consisting of the (images of the) transitive (Q,P)-bisets $(Q \times P)/M$, where M is a subgroup of $(Q \times P)$ such that $q(M)^@ \cong L$ (up to conjugation).*

Proof : (1) Indeed $q(M)$ is a subquotient of P, and a subquotient of Q. Hence $q(M)^@$ is a subquotient of $P^@$ and a subquotient of $Q^@$, so $q(M)^@ \sqsubseteq L^@ \cong L$. Now suppose that $q(M)^@ \cong L$. Then L is a quotient of $q(M)$, so $L \sqsubseteq q(M)$. Conversely, if $L \sqsubseteq q(M)$, then $L \cong L^@$ is a subquotient of $q(M)^@$, which is a subquotient of L. So $q(M)^@ \cong L$.

(2) First if S is a finite p-group with $S^@ \sqsubset L$, then $L \not\sqsubseteq S$, for otherwise $L \sqsubseteq S^@ \sqsubset L$, a contradiction. Conversely, let S be a finite p-group such that $L \not\sqsubseteq S$, or equivalently $L \not\sqsubseteq S^@$. By (2.7), any element of $RB(Q,S)B(S,P)$ is a linear combination of (Q,P)-bisets of the form $(Q \times P)/(M * N)$, for $M \leq (Q \times S)$ and $N \leq (S \times P)$. This biset $(Q \times P)/(M * N)$ also factors through $T = q(M * N)$, by 2.6. Moreover T is a subquotient of $q(M)$ and $q(N)$, by Lemma 2.3.22 of [7], hence a subquotient of Q, S, and P. Hence $T^@ \sqsubseteq Q^@ \cong L$, and $T^@ \not\cong L$, since $L \not\sqsubseteq S^@$. Hence $T^@ \sqsubset L$. We observe that conversely, any transitive biset $(Q \times P)/N$, with $q(N)^@ \sqsubset L$, factors through

$q(N)$, so it lies in the sum $\sum_{S^@ \sqsubset L} RB(Q,S)B(S,P)$. Hence this sum is equal to the set of linear combinations of bisets $(Q \times P)/N$, with $q(N)^@ \sqsubset L$.

(3) The (images of the) elements $(Q \times P)/M$, where M is a subgroup of $(Q \times P)$ such that $q(M)^@ \cong L$ (up to conjugation), clearly generate $\mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}(P,Q)$. Moreover, they are linearly independent, since the transitive (Q,P)-bisets of the form $(Q \times P)/M$, for $q(M)^@ \cong L$, generate a supplement in $RB(Q,P)$ of the sum $\sum_{S^@ \sqsubset L} RB(Q,S)B(S,P)$, by the observation at the end of the proof of Assertion 2. □

7.17. Remark: If G is an R-linear functor from $R\mathcal{C}_p^{\sharp L}$ to the category R-Mod of R-modules, we can extend G to an R-linear functor from $R\mathcal{C}_p^L$ to R-Mod by setting $G(P) = \{0\}$ if P is a finite p-group such that $P^@$ is a proper subquotient of L. Conversely, an R-linear functor from $R\mathcal{C}_p^L$ to R-Mod which vanishes on p-groups P such that $P^@ \not\cong L$ can be viewed as an R-linear functor from $R\mathcal{C}_p^{\sharp L}$ to R-Mod. In the sequel, we will freely identify those two types of functors, and consider $\mathsf{Fun}_R\big(R\mathcal{C}_p^{\sharp L}, R\text{-}\mathsf{Mod}\big)$ as the full subcategory of $\mathsf{Fun}_R\big(R\mathcal{C}_p^L, R\text{-}\mathsf{Mod}\big)$ consisting of functors which vanish on p-groups P such that $P^@ \not\cong L$.

7.18. Theorem: $[\, p \in R^\times]$ *Let L be an atoric p-group.*

1. *If F is a p-biset functor over R such that $F = \widehat{b}_L F$, and P is a finite p-group such that $L \not\sqsubseteq P$, then $F(P) = \{0\}$.*
2. *If G is an R-linear functor from $R\mathcal{C}_p^{\sharp L}$ to R-Mod, then $\widehat{b}_L \mathcal{R}_{\mathcal{Y}_L}(G) = \mathcal{R}_{\mathcal{Y}_L}(G)$.*
3. *The forgetful functor $\mathcal{O}_{\mathcal{Y}_L}$ and its right adjoint $\mathcal{R}_{\mathcal{Y}_L}$ restrict to quasi-inverse equivalences of categories*

$$\widehat{b}_L \mathcal{F}_{p,R} \underset{\mathcal{R}_{\mathcal{Y}_L}}{\overset{\mathcal{O}_{\mathcal{Y}_L}}{\rightleftarrows}} \mathsf{Fun}_R\big(R\mathcal{C}_p^{\sharp L}, R\text{-}\mathsf{Mod}\big) \;\; .$$

Proof : (1) Since $\widehat{b}_L F = F$, then in particular $F(b_L^P)F(P) = F(P)$. If $L \not\sqsubseteq P$, then there is no minimal section (T,S) of P with $(T/S)^@ \cong L$, thus $b_L^P = 0$, and $F(P) = \{0\}$.

(2) Let G be an R-linear functor from $R\mathcal{C}_p^{\sharp L}$ to R-Mod, in other words an R-linear functor from $\mathcal{F}\mathcal{C}_p^L$ to R-Mod which vanishes on p-groups P such that $P^@$ is a proper subquotient of L. By Theorem 7.12, we have $\widehat{b}_L^+ \mathcal{R}_{\mathcal{Y}_L}(G) =$

$\mathcal{R}_{\mathcal{Y}_L}(G)$. If H is an atoric p-group which is a proper subquotient of L, then G vanishes over any subquotient Q of H, since $Q^@ \sqsubseteq H \sqsubset L$ if $Q \sqsubseteq H$. In particular b_H^P acts by 0 on $\mathcal{R}_{\mathcal{Y}_L}(G)(P)$, for any finite p-group P: indeed b_H^P is a linear combination of terms of the form $\mathrm{Indinf}_{X/M}^P\mathrm{Defres}_{X/M}^P$, where (X,M) is a section of P such that $S \leq M \leq \Phi(T) \leq X \leq T$, for some section (T,S) of P with $(T/S)^@ \cong H$. For such a section (X,M) of P, we have $(X/M)^@ \sqsubseteq (T/S)^@ \sqsubseteq H$, thus G vanishes on any subquotient of X/M, so $\mathcal{R}_{\mathcal{Y}_L}(G)(X/M) = \{0\}$, hence b_H^P acts by 0 on $\mathcal{R}_{\mathcal{Y}_L}(G)(P)$, as claimed. It follows that $\widehat{b}_H\mathcal{R}_{\mathcal{Y}_L}(G) = 0$, hence the equality $\widehat{b}_L^+\mathcal{R}_{\mathcal{Y}_L}(G) = \mathcal{R}_{\mathcal{Y}_L}(G)$ reduces to $\widehat{b}_L\mathcal{R}_{\mathcal{Y}_L}(G) = \mathcal{R}_{\mathcal{Y}_L}(G)$.

(3) This is a straightforward consequence of (1) and (2), by Theorem 7.12, using Remark 7.17. □

7.19. Corollary: $[\ p \in R^\times]$ *The category $\mathcal{F}_{p,R}$ of p-biset functors over R is equivalent to the direct product of the categories $\mathsf{Fun}_R\big(R\mathcal{C}_p^{\sharp L}, R\text{-}\mathsf{Mod}\big)$ of R-linear functors from $R\mathcal{C}_p^{\sharp L}$ to R-Mod, for $L \in [\mathcal{A}t_p]$.*

Proof : This follows from Theorem 7.18, using Equivalence (7.6) of Corollary 7.5. □

8. L-enriched bisets

8.1. Notation: *Let G and H be finite groups. If U is an (H,G)-biset, and $u \in U$, let $(H,G)_u$ denote the stabilizer of u in $(H \times G)$, i.e.*

$$(H,G)_u = \{(h,g) \in (H \times G) \mid hu = ug\} \enspace .$$

Let $H_u = k_1\big((H,G)_u\big)$ denote the stabilizer of u in H, and ${}_uG = k_2\big((H,G)_u\big)$ denote the stabilizer of u in G. Set moreover

$$q(u) = q\big((H,G)_u\big) = (H,G)_u/(H_u \times {}_uG) \enspace .$$

8.2. Definition: *Let L be a finite group. For two finite groups G and H, an L-*enriched *(H,G)-biset is an $(H \times L, G \times L)$-biset U such that $L \sqsubseteq q(u)$, for any $u \in U$. A morphism of L-enriched (H,G)-bisets is a morphism of $(H \times L, G \times L)$-bisets.*

The disjoint union of two L-enriched (H,G)-bisets is again an L-

enriched (H,G)-biset. Let $B[L](H,G)$ denote the Grothendieck group of finite L-enriched (H,G)-bisets for relations given by disjoint union decompositions. The group $B[L](H,G)$ is called the Burnside group of L-enriched (H,G)-bisets.

8.3. Lemma: *Let G,H,L be finite groups, and U be an $(H\times L, G\times L)$-biset. Let $U^{\sharp L}$ denote the set of elements $u\in U$ such that $L\sqsubseteq q(u)$. Then $U^{\sharp L}$ is the largest L-enriched (H,G)-sub-biset of U.*

Proof : It suffices to show that $U^{\sharp L}$ is a $(H\times L, G\times L)$-sub-biset of U, for then it is clearly the largest L-enriched (H,G)-sub-biset of U. And this is straightforward, since for any $(u,g,h,x,y)\in(U\times G\times H\times L\times L)$, if $v=(h,y)u(g,x)$, then

$$(H\times L, G\times L)_v = {}^{\big((h,y),(g,x)^{-1}\big)}(H\times L, G\times L)_u \ \ ,$$

and this conjugation induces a group isomorphism $q(v)\cong q(u)$. □

8.4. Lemma: *Let G,H,L be finite groups.*

1. *Let U be an L-enriched (H,G)-biset. If V is an $(H\times L, G\times L)$-sub-biset of U, then V is an L-enriched (H,G)-biset.*
2. *The group $B[L](H,G)$ has a $\mathbb{Z}$-basis consisting of the transitive bisets $\big((H\times L)\times(G\times L)\big)/M$, where M is a subgroup of $\big((H\times L)\times(G\times L)\big)$ (up to conjugation) such that $L\sqsubseteq q(M)$.*

Proof : (1) This is straightforward.

(2) It follows from (1) that $B[L](H,G)$ has a basis consisting of the isomorphism classes of L-enriched (H,G)-bisets which are transitive $(H\times L, G\times L)$-bisets. These are of the form $U=\big((H\times L)\times(G\times L)\big)/M$, for some subgroup M of $\big((H\times L)\times(G\times L)\big)$. Now if u is the element $\big((1,1),(1,1)\big)M$ of U, the group $(H\times L, G\times L)_u$ is equal to M, hence $q(u)\cong q(M)$. □

8.5. Example: Let G,H,K,L be finite groups. The following can easily be checked:

1. For an (H,G)-biset U, endow $U\times L$ with the $(H\times L, G\times L)$-biset structure defined by

   $$\forall h\in H, \forall g\in G, \forall x,y,z\in L, \forall u\in U, \quad (h,x)(u,y)(g,z)=(hug,xyz)\ \ .$$

   Then $U\times L$ is an L-enriched (H,G)-biset.

2. In particular, for any finite group G, the identity biset of $G \times L$ is an L-enriched (G,G)-biset.
3. If U is an (H,G)-biset and V is a (K,H)-biset, then there is an isomorphism

$$(V \times L) \times_{(H\times L)} (U \times L) \cong (V \times_H U) \times L$$

of L-enriched (H,G)-bisets.

8.6. Notation: *Let G,H,K,L be finite groups. If U is an L-enriched (H,G)-biset and V is an L-enriched (K,H)-biset, let $V \overset{L}{\times}_H U$ denote the L-enriched (K,G)-biset defined by*

$$V \overset{L}{\times}_H U = \big(V \times_{(H\times L)} U\big)^{\sharp L} \quad .$$

8.7. Remark: The set $V \overset{L}{\times}_H U$ is in general a proper subset of $V \times_{(H\times L)} U$: for example if $K = G = \mathbf{1}$ and $H = L$, and if $V = \big((K \times L) \times (H \times L)\big)/N$ and $U = \big((H \times L) \times (G \times L)\big)/M$, where $N = \big\{\big(1,l),(l,1)\big) \mid l \in L\big\}$ and $M = \big\{\big(1,l),(1,l)\big) \mid l \in L\big\}$, then $p_2(N) = L \times \mathbf{1}$ and $k_2(N) = \mathbf{1} \times \mathbf{1}$, so $q(N) \cong (L \times L)/(L \times \mathbf{1}) \cong L$. Similarly $p_1(M) = \mathbf{1} \times L$ and $k_1(M) = \mathbf{1} \times \mathbf{1}$, so $q(M) \cong L$. However by 2.7, since $p_2(N)p_1(M) = H \times L$,

$$V \times_{(H\times L)} U = \big((K \times L) \times (G \times L)\big)/(N * M) \quad ,$$

and moreover $N*M = \big\{\big(1,l),(l,1)\big) \mid l \in L\big\} * \big\{\big(1,l),(1,l)\big) \mid l \in L\big\} = \mathbf{1}\times\mathbf{1}$, so $q(N * M) = \mathbf{1}$. It follows that $V \overset{L}{\times}_H U = \emptyset$ if L is non trivial.

8.8. Lemma: *Let G,H,J,K,L be finite groups.*

1. *If V is a $(K \times L, H \times L)$-biset and U is an $(H \times L, G \times L)$-biset, then*

$$(V \times_{(H\times L)} U)^{\sharp L} = V^{\sharp L} \overset{L}{\times}_H U^{\sharp L} \quad .$$

In particular, if V and U are L-enriched bisets, so is $V \overset{L}{\times}_H U$.

2. *If U and U' are L-enriched (H,G)-bisets, if V,V' are L-enriched (K,H)-bisets, then there are isomorphisms*

$$V \overset{L}{\times}_H (U \sqcup U') \;\cong\; (V \overset{L}{\times}_H U) \sqcup (V \overset{L}{\times}_H U')$$

$$(V \sqcup V')\overset{L}{\times}_H U \;\cong\; (V\overset{L}{\times}_H U) \sqcup (V'\overset{L}{\times}_H U)$$

of L-enriched (K, G)-bisets.

3. *If moreover W is an L-enriched (J, K)-biset, then there is a canonical isomorphism*
$$(W\overset{L}{\times}_K V)\overset{L}{\times}_H U \cong W\overset{L}{\times}_K (V\overset{L}{\times}_H U)$$
of L-enriched (J, G)-bisets.

Proof : (1) Denote by $[v, u]$ the image in $V\times_{(H\times L)}U$ of a pair $(v, u) \in (V\times U)$. By Lemma 2.3.20 of [7],

$$(K \times L, G \times L)_{[v,u]} = (K \times L, H \times L)_v * (H \times L, G \times L)_u \ \ ,$$

so by Lemma 2.3.22 of [7], the group $q\big([v, u]\big)$ is a subquotient of $q(v)$ and $q(u)$. So if $[v, u] \in (V \times_{(H\times L)} U)^{\sharp L}$, then L is a subquotient of $q\big([v, u]\big)$, hence it is a subquotient of $q(v)$ and $q(u)$, that is $v \in V^{\sharp L}$ and $u \in U^{\sharp L}$. Hence

$$(V \times_{(H\times L)} U)^{\sharp L} \subseteq (V^{\sharp L} \times_{(H\times L)} U^{\sharp L})^{\sharp L} = V^{\sharp L}\overset{L}{\times}_H U^{\sharp L} \ \ ,$$

and the reverse inclusion $(V^{\sharp L} \times_{(H\times L)} U^{\sharp L})^{\sharp L} \subseteq (V \times_{(H\times L)} U)^{\sharp L}$ is obvious. Hence $(V \times_{(H\times L)} U)^{\sharp L} = V^{\sharp L}\overset{L}{\times}_H U^{\sharp L}$. If V and U are L-enriched bisets, i.e. if $V = V^{\sharp L}$ and $U = U^{\sharp L}$, this gives $(V \times_{(H\times L)} U)^{\sharp L} = V\overset{L}{\times}_H U$, so $V\overset{L}{\times}_H U$ is an L-enriched biset.

(2) This is straightforward.

(3) With the above notation, there is a canonical isomorphism

$$\alpha : (W \times_{(K\times L)} V) \times_{(H\times L)} U \to W \times_{(K\times L)} (V \times_{(H\times L)} U)$$

sending $\big[[w, v], u\big]$ to $\big[w, [v, u]\big]$. Hence

$$\begin{aligned}(W\overset{L}{\times}_K V)\overset{L}{\times}_H U &= \big((W\overset{L}{\times}_K V) \times_{(H\times L)} U\big)^{\sharp L}\\ &= \big((W \times_{(K\times L)} V)^{\sharp L} \times_{(H\times L)} U\big)^{\sharp L}\\ &= \big((W \times_{(K\times L)} V) \times_{(H\times L)} U\big)^{\sharp L} \quad \text{[by (1)]}\end{aligned}$$

Similarly

$$\begin{aligned}W\overset{L}{\times}_K (V\overset{L}{\times}_H U) &= \big(W \times_{(K\times L)} (V\overset{L}{\times}_H U)\big)^{\sharp L}\\ &= \big(W \times_{(K\times L)} (V \times_{(H\times L)} U)^{\sharp L}\big)^{\sharp L}\\ &= \big(W \times_{(K\times L)} (V \times_{(H\times L)} U)\big)^{\sharp L} \quad \text{[by (1)]} \ .\end{aligned}$$

Hence α induces an isomorphism $(W \overset{L}{\times}_K V) \overset{L}{\times}_H U \cong W \overset{L}{\times}_K (V \overset{L}{\times}_H U)$. □

8.9. Definition: *Let L be a finite group, and R be a commutative ring. The* L-enriched biset category *$R\mathcal{C}[L]$ of finite groups over R is defined as follows:*

- *The objects of $R\mathcal{C}[L]$ are the finite groups.*
- *For finite groups G and H,*

$$\mathrm{Hom}_{R\mathcal{C}[L]}(G, H) = R \otimes_{\mathbb{Z}} B[L](H, G) = RB[L](H, G)$$

  is the R-linear extension of the Burnside group of L-enriched (H, G)-bisets.
- *The composition in $R\mathcal{C}[L]$ is the R-linear extension of the product $(V, U) \mapsto V \overset{L}{\times}_H U$ defined in 8.6.*
- *The identity morphism of the group G is (the image in $RB[L](G, G)$ of) the identity biset of $G \times L$, viewed as an L-enriched (G, G)-biset.*

The category $R\mathcal{C}[L]$ is R-linear. An L-enriched biset functor *over R is an R-linear functor from $R\mathcal{C}[L]$ to R-Mod. The category of L-enriched biset functors over R is denoted by $\mathcal{F}_R[L]$. It is an abelian R-linear category.*

8.10. Theorem: *Let p be a prime number, and R be a commutative ring.*

1. *If L is an atoric p-group, the category $R\mathcal{C}_p^{\sharp L}$ of Definition 7.14 is equivalent to the full subcategory $R\mathcal{E}l_p[L]$ of $R\mathcal{C}[L]$ consisting of elementary abelian p-groups.*
2. *If $p \in R^\times$, the category $\mathcal{F}_{p,R}$ of p-biset functors over R is equivalent to the direct product of the categories $\mathsf{Fun}_R\big(R\mathcal{E}l_p[L], R\text{-}\mathsf{Mod}\big)$ of R-linear functors from $R\mathcal{E}l_p[L]$ to R-Mod, for $L \in [\mathcal{A}t_p]$.*

Proof : (1) Let E be an elementary abelian p-group. Then $(E \times L)^{@} \cong L$, so $E \times L$ is an object of $R\mathcal{C}_p^{\sharp L}$. Set $\mathcal{I}(E) = E \times L$. If E and F are elementary abelian p-groups, and if U is a finite L-enriched (F, E)-biset, then U is in particular an $(F \times L, E \times L)$-biset, and we can consider its image $\mathcal{I}(U)$ in the quotient $\mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}(E \times L, F \times L)$ of $RB(F \times L, E \times L)$. This yields a unique R-linear map $RB[L](F, E) \to \mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}(E \times L, F \times L)$, still denoted by $\mathcal{I}$.

We claim that these assignments define a functor $\mathcal{I}$ from $R\mathcal{E}l_p[L]$ to $R\mathcal{C}_p^{\sharp L}$: indeed, the identity $(E \times L, E \times L)$-biset is clearly mapped to the identity morphism of $\mathcal{I}(E)$. Moreover, if G is an elementary abelian p-group, if V

is an L-enriched (G,F)-biset and U is an L-enriched (F,E)-biset, it is clear that

$$\mathcal{I}\big(V\overset{L}{\times}_F U\big)=\mathcal{I}(V)\circ\mathcal{I}(U)\ \ ,$$

where the right hand side composition is in the category $R\mathcal{C}_p^{\sharp L}$: indeed, the transitive bisets $\big((G\times L)\times(E\times L)\big)/M$ with $q(M)^@\sqsubset L$ appearing in the product $V\times_{(F\times L)}U$ are exactly those vanishing in $\mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}\big(\mathcal{I}(E),\mathcal{I}(F)\big)$, by Lemma 7.16. Hence $\mathcal{I}$ induces an isomorphism

$$\mathcal{I}:RB[L](F,E)\to\mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}\big(\mathcal{I}(E),\mathcal{I}(F)\big)\ \ .$$

In other words $\mathcal{I}$ is a fully faithful functor from $R\mathcal{E}l_p[L]$ to $R\mathcal{C}_p^{\sharp L}$. Moreover, by Proposition 6.7, if P is a finite p-group with $P^@\cong L$, there exists an elementary abelian p-group E such that P is isomorphic to $E\times L$, hence P is isomorphic to $E\times L$ in the category $R\mathcal{C}_p^{\sharp L}$.

It follows that the functor $\mathcal{I}$ is fully faithful and essentially surjective, so it is an equivalence of categories.

(2) This is a straightforward consequence of (1), Assertion 5 of Corollary 7.5, and Assertion 3 of Theorem 7.18. □

8.11. Remark: Let E and F be elementary abelian p-groups. In view of Theorem 8.10, it is interesting to give some detail on the hom set from E to F in the category $R\mathcal{E}l_p[L]$, in other words to describe the subgroups M of $(F\times L)\times(E\times L)$ such that $q(M)^@\cong L$. One can show that they are exactly those subgroups M such that

$$p_{1,2}(M)=p_{2,2}(M)=L\quad\text{and}\quad k_{1,2}(M)=k_{2,2}(M)=\mathbf{1}\ \ ,$$

where $p_{1,2}$ and $p_{2,2}$ are the morphisms from $\big((H\times L)\times(G\times L)\big)$ to L defined by $p_{1,2}\big((h,x),(g,y)\big)=x$ and $p_{2,2}\big((h,x),(g,y)\big)=y$, and

$$\begin{aligned}k_{1,2}(M)&=\{x\in L\mid \big((1,x),(1,1)\big)\in M\}\ \ ,\\ k_{2,2}(M)&=\{x\in L\mid \big((1,1),(1,x)\big)\in M\}\ \ .\end{aligned}$$

9. The category $\widehat{b}_L\mathcal{F}_{p,R}$, for an atoric p-group L ($p\in R^\times$)

Let L be a fixed atoric p-group. In this section, we give some detail on the structure of the category $\widehat{b}_L\mathcal{F}_{p,R}$ of p-biset functors invariant by the idempotent $\widehat{b}_L$. We return to the initial definition of this category given in Assertion 5 of Corollary 7.5, and we do not use the equivalent category $\mathsf{Fun}_R\big(R\mathcal{E}l_p[L], R\text{-}\mathsf{Mod}\big)$ of Theorem 8.10.

We start by straightforward consequences of Theorem 7.18. For a finite p-group P, we denote by $\Sigma_{\sharp L}(P)$ the subset of $\Sigma_L(P)$ consisting of sections (X, M) of P such that $(X/M)^@ \cong L$. When G is an R-linear functor from $R\mathcal{C}_p^{\sharp L}$ to R-Mod, we first extend it to a functor defined on $R\mathcal{C}_p^L$ by setting $G(P) = \{0\}$ if $P^@ \sqsubset L$, as in Remark 7.17. Then we compute $\mathcal{R}_{\mathcal{Y}_L}(G)$ at P by restricting the inverse limit of 7.8 to the subset $\Sigma_{\sharp L}(P)$, i.e. by

$$\mathcal{R}_{\mathcal{Y}_L}(G)(P) = \lim_{\substack{\longleftarrow \\ (X,M)\in\Sigma_{\sharp L}(P)}} G(X/M) \ .$$

9.1. Proposition: [$p \in R^\times$] *Let L be an atoric p-group. If F is a p-biset functor in $\widehat{b}_L\mathcal{F}_{p,R}$, and P is a finite p-group, then*

$$\begin{aligned} F(P) &\cong \lim_{\substack{\longleftarrow \\ (X,M)\in\Sigma_{\sharp L}(P)}} F(X/M) \ , \\ &\cong \bigoplus_{\substack{(T,S)\in[\mathcal{M}(P)] \\ (T/S)^@\cong L}} \delta_\Phi F(T/S)^{N_P(T,S)/T} \ . \end{aligned}$$

Proof : The isomorphism $F(P) \cong \lim\limits_{\substack{\longleftarrow \\ (X,M)\in\Sigma_{\sharp L}(P)}} F(X/M)$ is Assertion 3 of Theorem 7.18. The second isomorphism follows from Theorem 5.4, which implies that for $(T, S) \in \mathcal{M}(P)$

$$\delta_\Phi F(T/S)^{N_P(T,S)/T} \cong F(\epsilon_{T,S}^P)\big(F(P)\big) \ .$$

Moreover $F(b_L^P)F(P) = F(P)$ since $F \in \widehat{b}_L\mathcal{F}_{p,R}$, and

$$F(\epsilon_{T,S}^P)F(b_L^P) = F(\epsilon_{T,S}^P b_L^P) = 0$$

unless $(T/S)^@ \cong L$. Thus $\delta_\Phi F(T/S)^{N_P(T,S)/T} = \{0\}$ unless $(T/S)^@ \cong L$, which completes the proof. □

The decomposition of the category $\mathcal{F}_{p,R}$ of p-biset functors stated in Corollary 7.5 leads to the following natural definition:

9.2. Definition: [$p \in R^\times$] *Let F be an indecomposable p-biset functor over R. There exists a unique atoric p-group L (up to isomorphism) such that $F = \widehat{b}_L F$. The group L is called the* vertex *of F.*

9.3. Remark:

1. It follows in particular from this definition that if F and F' are indecomposable p-biset functors over R with non-isomorphic vertices, then $\mathrm{Ext}^*_{\mathcal{F}_{p,R}}(F,F') = \{0\}$.
2. It may happen that an indecomposable p-biset functor F with vertex L vanishes at L (see e.g. the case of a simple functor $F = S_{Q,V}$ of Corollary 9.5, when $Q \not\cong Q^{@}$).

9.4. Theorem: $[\, p \in R^\times]$ *Let F be an indecomposable p-biset functor over R and let L be a vertex of F. If Q is a finite p-group such that $F(Q) \neq \{0\}$, but F vanishes on any proper subquotient of Q, then $L \cong Q^{@}$.*

Proof : Let Q be a finite p-group such that $F(Q) \neq \{0\}$ and $F(Q') = \{0\}$ for any proper subquotient Q' of Q. By Proposition 4.7, if (T,S) is a minimal section of Q, then

$$\epsilon^Q_{T,S} = \frac{1}{|N_Q(T,S)|} \sum_{\substack{X \le T, M \trianglelefteq T \\ S \le M \le \Phi(T) \le X \le T}} |X|\mu(X,T)\mu_{\trianglelefteq T}(S,M)\,\mathrm{Indinf}^Q_{X/M}\,\mathrm{Defres}^Q_{X/M} \quad .$$

Now if X/M is a proper subquotient of Q, i.e. if $X \neq Q$ or $M \neq \mathbf{1}$, then $F(X/M) = \{0\}$, and $F(\mathrm{Indinf}^Q_{X/M}\,\mathrm{Defres}^Q_{X/M}) = 0$. Hence $F(\epsilon^Q_{T,S}) = 0$ unless $T = Q$ and $S = \mathbf{1}$, and moreover

$$F(\epsilon^Q_{Q,\mathbf{1}}) = \frac{1}{|Q|}|Q|\mu(Q,Q)\mu_{\trianglelefteq Q}(\mathbf{1},\mathbf{1})F(\mathrm{Indinf}^Q_{Q/\mathbf{1}}\mathrm{Defres}^Q_{Q/\mathbf{1}}) = \mathrm{Id}_{F(Q)} \quad .$$

Since $\widehat{b}_L F = F$, then in particular $F(b^Q_L)$ is equal to the identity map of $F(Q)$. This can only occur if the idempotent $\epsilon^Q_{Q,\mathbf{1}}$ appears in the sum defining b^Q_L, in other words if $(Q/\mathbf{1})^{@} \cong L$, i.e. $Q^{@} \cong L$. □

We assume from now on that $R = k$ is a field. Recall ([7] Chapter 4) that the simple p-biset functors $S_{Q,V}$ over k are indexed by pairs (Q,V) consisting of a p-group Q and a simple $k\mathrm{Out}(Q)$-module V. Also recall that if P is a finite p-group and if $Q \not\sqsubseteq P$, then $S_{Q,V}(P) = \{0\}$. Moreover $S_{Q,V}(Q) \cong V$.

9.5. Corollary: *Let k be a field of characteristic different from p.*

1. *If Q is a finite p-group, and V is a simple $k\mathrm{Out}(Q)$-module, then the vertex of the simple p-biset functor $S_{Q,V}$ is isomorphic to $Q^{@}$.*

2. *Let Q (resp. Q') be a finite p-group, and V (resp. V') be a simple $k\mathrm{Out}(Q)$-module (resp. a simple $k\mathrm{Out}(Q')$-module). If $Q^{@} \not\cong Q'^{@}$, then $\mathrm{Ext}^*_{\mathcal{F}_{p,k}}(S_{Q,V}, S_{Q',V'}) = \{0\}$.*

Proof : (1) Indeed Q is a minimal group for $S_{Q,V}$, so $S_{Q,V}(Q) \neq \{0\}$, but $S_{Q,V}$ vanishes on any proper subquotient of Q.

(2) Follows from (1) and Remark 9.3. □

9.6. Definition: *Let F be a p-biset functor. A non zero functor S is a* subquotient *of F (notation $S \sqsubseteq F$) if there exist subfunctors $F_2 < F_1 \leq F$ such that $F_1/F_2 \cong S$. A* composition factor *of F is a simple subquotient of F.*

9.7. Lemma: *Let k be a field, and F be a p-biset functor over k.*

1. *If F is non zero, then F admits a composition factor.*
2. *If $\mathcal{S}$ is a family of simple p-biset functors over k, there exists a greatest subfunctor of F all composition factors of which belong to $\mathcal{S}$.*

Proof : (1) Let P be a finite p-group such that $F(P) \neq \{0\}$. Then $F(P)$ is a $kB(P,P)$-module. Choose $m \in F(P) - \{0\}$, and consider the $kB(P,P)$-submodule M of $F(P)$ generated by m. Since $kB(P,P)$ is finite dimensional over k, the module M is also finite dimensional over k, hence it contains a simple submodule V. By Proposition 3.1 of [8], there exists a simple p-biset functor S such that $S(P) \cong V$ as $kB(P,P)$-module. Then $S(P)$ is a subquotient of $F(P)$, so by Proposition 3.5 of [8], there exists a subquotient of F isomorphic to S.

(2) Observe first that if M, N are subfunctors of F, then any composition factor of $M + N$ is a composition factor of M or a composition factor of N: indeed, if S is a composition factor of $M + N$, let $F_2 < F_1 \leq M + N$ with $S \cong F_2/F_1$, and consider the images F_1' and F_2' of F_1 and F_2, respectively, in the quotient $(M + N)/N \cong M/(M \cap N)$. If $F_1' \neq F_2'$, that is if $F_1 + N \neq F_2 + N$, then $F_1'/F_2' \cong (F_1 + N)/(F_2 + N) \cong F_1/F_2 \cong S$ is a subquotient of $(M + N)/N \cong M/(M \cap N)$, hence S is a subquotient of M. Otherwise $F_1+N = F_2+N$, so $F_1 = F_2+(F_1\cap N)$, hence $S \cong F_1/F_2 \cong (F_1\cap N)/(F_2\cap N)$ is a subquotient of N. It follows by induction that any composition factor S of a finite sum $\sum_{M\in\mathcal{I}} M$ of subfunctors of F is a composition factor of some $M \in \mathcal{I}$.

The latter also holds when $\mathcal{I}$ is infinite: let $\Sigma = \sum_{M\in\mathcal{I}} M$ be an arbitrary sum of subfunctors of F, and S be a composition factor of Σ. Let $F_2 < F_1$ be subfunctors of Σ such that $S \cong F_1/F_2$. If P is a p-group such that $S(P) \cong F_1(P)/F_2(P) \neq 0$, let U be a finite subset of $F_1(P)$ such that $F_1(P)/F_2(P)$ is generated as a $kB(P,P)$-module by the images of the elements of U (such a set exists because $S(P)$ is finite dimensional over k, for any P). If V is the $kB(P,P)$-submodule of $F_1(P)$ generated by U, then V maps surjectively on the module $F_1(P)/F_2(P)$, so there is a $kB(P,P)$-submodule W of V such that $V/W \cong S(P)$. Now since U is finite, there exists a finite subset $\mathcal{J}$ of $\mathcal{I}$ such that $U \subseteq \sum_{M\in\mathcal{J}} M(P)$. Setting $\Sigma_1 = \sum_{M\in\mathcal{J}} M$, it follows that $V/W \cong S(P)$ is a subquotient of $\Sigma_1(P)$, so by Proposition 3.5 of [8], there exists a subquotient of Σ_1 isomorphic to S. By the observation above S is a subquotient of some $M \in \mathcal{J} \subseteq \mathcal{I}$.

Now let $\mathcal{I}$ be the set of subfunctors M of F such that all the composition factors of M belong to $\mathcal{S}$, and $N = \sum_{M\in\mathcal{I}} M$. The above discussion shows that $N \in \mathcal{I}$, so N is the greatest element of $\mathcal{I}$. □

9.8. Theorem: *Let k be a field of characteristic different from p, and L be an atoric p-group. Let $\mathcal{F}_{p,k}[L]$ be the full subcategory of $\mathcal{F}_{p,k}$ consisting of functors whose composition factors all have vertex L, i.e. are all isomorphic to $S_{P,V}$, for some p-group P such that $P^{@} \cong L$, and some simple $k\mathrm{Out}(P)$-module V.*

1. *If F is a p-biset functor, then $\widehat{b}_L F$ is the greatest subfunctor of F which belongs to $\mathcal{F}_{p,k}[L]$.*
2. *In particular $\widehat{b}_L \mathcal{F}_{p,k} = \mathcal{F}_{p,k}[L]$.*

Proof : (1) Let F be a p-biset functor over k, and let $F_1 = \widehat{b}_L F$. If S is a composition factor of F_1, then $S = \widehat{b}_L S$, as S is a subquotient of F_1. Hence S has vertex L, by Definition 9.2. It follows that F_1 is contained in the greatest subfunctor F_2 of F which belongs to $\mathcal{F}_{p,k}[L]$ (such a subfunctor exists by Lemma 9.7).

Conversely, we know that $F_2 = \mathop{\oplus}_{Q\in[\mathcal{A}t_p]} \widehat{b}_Q F_2$. For $Q \in [\mathcal{A}t_p]$, any composition factor S of $\widehat{b}_Q F_2$ has vertex Q, by Definition 9.2. But S is also a composition factor of F_2, so $Q \cong L$. It follows that if $Q \ncong L$, then $\widehat{b}_Q F_2$ has no composition factor, so $\widehat{b}_Q F_2 = \{0\}$, by Lemma 9.7. In other words $F_2 = \widehat{b}_L F_2$, hence $F_2 \leq F_1$, and $F_2 = F_1$, as was to be shown.

(2) Let F be a p-biset functor. Then $F \in \widehat{b}_L \mathcal{F}_{p,k}$ if and only if $F = \widehat{b}_L F$, i.e. by (1) if and only if all the composition factors of F have vertex L. □

9.9. Example: the Burnside functor. Let k be a field of characteristic $q \neq p$ ($q \geq 0$). It was shown in [10] Theorem 8.2 (see also [7] 5.6.9) that the Burnside functor kB restricted to the class of p-groups (hence an object of $\mathcal{F}_{p,k}$) is uniserial, hence indecomposable. As $kB(\mathbf{1}) \neq 0$, the vertex of kB is the trivial group, by Theorem 9.4, thus kB is an object of $\widehat{b}_{\mathbf{1}} \mathcal{F}_{p,k} = \mathcal{F}_{p,k}[\mathbf{1}]$. It means that all the composition factors of kB have to be of form $S_{Q,V}$, where $Q^{@} = \mathbf{1}$, i.e. Q is elementary abelian. And indeed by [10] Theorem 8.2, the composition factors of kB are all of the form $S_{Q,k}$, where Q runs through a specific set of elementary abelian p-groups which depends on the order of p modulo q (suitably interpreted when $q = 0$).

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