Hochschild constructions for Green functors

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1. Introduction

Let G be a finite group, and R be a commutative ring. This paper provides a possible generalization to any Green functor for G over R of the construction of the Hochschild cohomology ring $HH^*(G,R)$ of the group algebra RG from the ordinary cohomology functor $H^*(-,R)$. This construction also involves as a special case the construction of the crossed Burnside ring of G from the ordinary Burnside functor.

The general abstract setting is the following : let A be a Green functor for the group G. Let G^c denote the group G, on which G acts by conjugation. Suppose Γ is a crossed G-monoid, i.e. that Γ is a G-monoid over the G-group G^c. Then the Mackey functor A_Γ obtained from A by Dress construction has a natural structure of Green functor. In particular $A_\Gamma(G)$ is a ring.

In the case where Γ is the crossed G-monoid G^c, and A is the cohomology functor (with trivial coefficients R), the ring $A_\Gamma(G)$ is the Hochschild cohomology ring of G over R. If A is the Burnside functor for G over R, then the ring $A_\Gamma(G)$ is the crossed Burnside ring of G over R.

This article presents some properties of those Green functors A_Γ, and the functorial relations between the corresponding categories of modules. Instead of exposing first the general framework, and since the Green functor setting is rather abstract, the first sections only deal with the two above special cases.

In particular, they provide a new proof for a result of S. Siegel and S. Witherspoon ([6]), which was conjectured by C. Cibils ([3]) and C. Cibils and A. Solotar ([4]). This result describes the multiplicative structure of the Hochschild cohomology ring of a finite group in terms of cup products, transfers and restrictions on the ordinary cohomology.

The case of the crossed Burnside ring can be viewed as a conceptual reason for the existence of a ring homomorphism from the crossed Burnside

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ring to the center of the Mackey algebra, which was used in [1] to determine the p-blocks of the Mackey algebra.

The next sections present the general theoretic setting : first the definition of crossed G-monoids, and then the construction of the Green functor structure on A_Γ, for a Green functor A and a crossed G-monoid Γ. A general formula for the product in the ring $A_\Gamma(G)$ is stated. Here one more example is built : it is possible to attach to any normal subgroup N of a finite group G a kind of “relative” Hochschild cohomology ring $HH^*(G, N, R)$, which is the ordinary cohomology ring for $N = 1$, and the usual Hochschild cohomology ring for $N = G$.

The next section is an exposition of the functorial relations between categories of A-modules and A_Γ-modules. These relations explain in particular the existence of a natural ring homomorphism from the crossed Burnside ring to the center of the Mackey algebra.

2. Green functors and G-sets

When dealing with Mackey and Green functors, two different (but equivalent) points of view are possible : in the first one, a Mackey functor for G over R is a collection of R-modules $M(H)$, indexed by the subgroups H of G, together with transfer maps $t_H^K : M(H) \to M(K)$, with restrictions maps $r_H^K : M(K) \to M(H)$, and conjugation maps $c_{x,H} : M(H) \to M({}^xH)$, whenever $H \subseteq K$ are subgroups of G, and $x \in G$. These maps are subject to a list of conditions : transitivity, commutation, triviality, and Mackey formula.

In the second point of view, a Mackey functor for G over R is a bivariant functor from the category of finite G-sets to the category of R-modules, which transforms disjoint unions into direct sums, and has some compatibility property with cartesian squares (see [2] 1.1.2 for details).

Similarly, a Green functor A is a Mackey functor “with a compatible ring structure” : each evaluation $A(H)$ has a ring structure (with unit), for a product $(a, b) \mapsto a.b$, restrictions maps and conjugation maps are homomorphisms of rings (with unit), and the Frobenius relations on products and transfers hold. There is also a definition of Green functors in terms of G-sets ([2] 2.2) :

Definition 2.1 : *Let R be a commutative ring. A Green functor A (over R) for the group G is a Mackey functor (over R) endowed for any G-sets X and Y with bilinear maps*

$$A(X) \times A(Y) \to A(X \times Y)$$

denoted by $(a, b) \mapsto a \times b$ *which are bifunctorial, associative, and unitary, in the following sense:*

- *(Bifunctoriality) If* $f : X \to X'$ *and* $g : Y \to Y'$ *are morphisms of* G*-sets, then the squares*

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ {\scriptstyle A_*(f) \times A_*(g)} \downarrow & & \downarrow {\scriptstyle A_*(f \times g)} \\ A(X') \times A(Y') & \xrightarrow[\times]{} & A(X' \times Y') \end{array}$$

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ {\scriptstyle A^*(f) \times A^*(g)} \uparrow & & \uparrow {\scriptstyle A^*(f \times g)} \\ A(X') \times A(Y') & \xrightarrow[\times]{} & A(X' \times Y') \end{array}$$

  are commutative.

- *(Associativity) If* X*,* Y *and* Z *are* G*-sets, then the square*

$$\begin{array}{ccc} A(X) \times A(Y) \times A(Z) & \xrightarrow{Id_{A(X)} \times (\times)} & A(X) \times A(Y \times Z) \\ {\scriptstyle (\times) \times Id_{A(Z)}} \downarrow & & \downarrow {\scriptstyle \times} \\ A(X \times Y) \times A(Z) & \xrightarrow[\times]{} & A(X \times Y \times Z) \end{array}$$

  is commutative, up to identifications $(X \times Y) \times Z \simeq X \times Y \times Z \simeq X \times (Y \times Z)$.

- *(Unitarity) If* $\bullet$ *denotes the* G*-set with one element, there exists an element* $\varepsilon_A \in A(\bullet)$ *such that for any* G*-set* X *and for any* $a \in A(X)$

$$A_*(p_X)(a \times \varepsilon_A) = a = A_*(q_X)(\varepsilon_A \times a)$$

  denoting by p_X *(resp.* q_X*) the (bijective) projection from* $X \times \bullet$ *(resp. from* $\bullet \times X$*) to* X.

If A *and* B *are Green functors for the group* G*, a morphism* f *(of Green functors) from* A *to* B *is a morphism of Mackey functors such that for any* G*-sets* X *and* Y*, the square*

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ {\scriptstyle f_X \times f_Y} \downarrow & & \downarrow {\scriptstyle f_{X \times Y}} \\ B(X) \times B(Y) & \xrightarrow[\times]{} & B(X \times Y) \end{array}$$

is commutative. The composition of morphisms of Green functors is defined in the obvious way. The category of Green functors for G over R is denoted by $\mathsf{Green}_R(G)$.

Remark 2.2 : One can pass from the second definition of Green functors to the first one by the following procedure : suppose that A is a Green functor in the second sense. Then setting $A(H) = A(G/H)$, for a subgroup H of G, one can define the maps t_H^K and r_H^K, for $H \subseteq K$ by

$$t_H^K = A_*(p_H^K) \qquad r_H^K = A^*(p_H^K)$$

where $p_H^K : G/H \to G/K$ is the natural projection. If $x \in G$, then the conjugation map $c_{x,H}$ is defined by

$$c_{x,H} = A_*(\gamma_{x,H})$$

where $\gamma_{x,H} : G/H \to G/{}^xH$ maps gH to $gx^{-1}\,{}^xH$. One can also define a product $(a, b) \mapsto a.b$ on $A(H)$ by

$$a.b = A^*(\delta_{G/H})(a \times b)$$

where $\delta_{G/H}$ is the diagonal inclusion from G/H to $(G/H) \times (G/H)$.

Remark 2.3 : One can pass from the first definition to the second by the following construction ([2] 2.3) : suppose that A is a Green functor in the first sense. Then setting $A(G/H) = A(H)$ leads by linearity to a definition of $A(X)$ for any finite G-set X : one can set

$$A(X) = \left(\bigoplus_{x\in X} A(G_x)\right)^G$$

where the exponent denotes fixed points under the natural action of G on $\bigoplus_{x\in X} A(G_x)$ by permutation of the components, and G_x is the stabilizer of x in G. If $f : X \to Y$ is a map of G-sets, then the maps $A_*(f)$ and $A^*(f)$ are defined by

$$\begin{aligned} A_*(f)(u)_y &= \sum_{x\in[G_y\backslash f^{-1}(y)]} t_{G_x}^{G_y}(u_x) && \text{for } u \in A(X) \text{ and } y \in Y \\ A^*(f)(v)_x &= r_{G_x}^{G_{f(x)}} v_{f(x)} && \text{for } v \in A(Y) \text{ and } x \in X \end{aligned}$$

where G_y is the stabilizer of y in G, and $[G_y\backslash f^{-1}(y)]$ is a set of representatives of the orbits of G_y on $f^{-1}(y)$. The summation in the formula does not depend on the choice of such a set of representatives.

The product of elements $a \in A(X)$ and $b \in A(Y)$ is defined as the element of $A(X \times Y)$ whose component in $A(G_{(x,y)})$, for $(x, y) \in X \times Y$, is equal to

$$(a \times b)_{x,y} = r^{G_x}_{G_{(x,y)}}(a_x).r^{G_y}_{G_{(x,y)}}(b_y) \quad .$$

One can also express the previous formulae after choosing sets of representatives of orbits of G on X, Y, and $X \times Y$. For instance, there is an isomorphism

$$A(X) \cong \bigoplus_{x \in [G\backslash X]} A(G_x)$$

where $[G\backslash X]$ is a set of representatives of orbits of G on X. Now if H and K are subgroups of G, then there is an isomorphism of G-sets

$$(G/H) \times (G/K) \cong \bigsqcup_{x \in [H\backslash G/K]} G/(H \cap {}^xK)$$

hence an isomorphism

$$A\Big((G/H) \times (G/K)\Big) \cong \bigoplus_{x \in [H\backslash G/K]} A(H \cap {}^xK) \quad .$$

and the product $a \times b$ of $a \in A(G/H)$ and $b \in A(G/K)$ is the element

$$\bigoplus_{x \in [H\backslash G/K]} r^H_{H \cap {}^xK} a \,.\, r^{{}^xK}_{H \cap {}^xK} {}^xb \;\in\; \bigoplus_{x \in [H\backslash G/K]} A(H \cap {}^xK)$$

where ${}^xb = c_{x,K}(b)$.

Remark 2.4 : There is an alternative procedure to pass from the first definition of Green functors to the second one, which is often easier to handle for computations. It uses co-invariants rather than invariants : if A is a Green functor for G in the first sense, and X is a finite G-set, then one can set

$$A(X) = \left(\bigoplus_{x \in X} A(G_x)\right)_G$$

where the subscript denotes co-invariants (i.e. the largest quotient with trivial G-action). If $x \in X$ and $u \in A(G_x)$, denote by $[x, u]_G$ the image in $A(X)$ of the element u of $A(G_x)$. Then if $f : X \to Y$ is a map of G-sets

$$\begin{aligned} A_*(f)([x,u]_G) &= [f(x), t^{G_{f(x)}}_{G_x} u]_G && \text{for } x \in X \text{ and } u \in A(G_x) \\ A^*(f)([y,v]_G) &= \sum_{x \in G_y\backslash f^{-1}(y)} [x, r^{G_y}_{G_x} v]_G && \text{for } y \in Y \text{ and } v \in A(G_y) \quad . \end{aligned}$$

The product of the elements $[x,u]_G$ and $[y,v]_G$ is given by

$$[x,u]_G \times [y,v]_G = \sum_{w\in[G_x\backslash G/G_y]} [(x,wy), r^{G_x}_{G_x\cap G_{wy}} u . r^{G_{wy}}_{G_x\cap G_{wy}} {}^{w}v]_G \quad .$$

Example 2.5 : Let X and Y be finite G-sets, and set

$$H^{\oplus}(G,RX) = \bigoplus_{n=0}^{\infty} H^n(G,RX) \quad .$$

This is usually denoted by $H^*(G,RX)$, but this notation could be confusing here with the Mackey functor formalism. Then the cup-product on cohomology gives maps

$$H^{\oplus}(G,RX) \times H^{\oplus}(G,RY) \to H^{\oplus}(G,RX\otimes_R RY)$$

and identifying $RX\otimes_R RY$ with $R(X\times Y)$, this gives cross product maps

$$H^{\oplus}(G,RX) \times H^{\oplus}(G,RY) \to H^{\oplus}(G,R(X\times Y)) \quad .$$

It is easy to check that this gives $H^{\oplus}(G,-)$ a Green functor structure, and that if K is a subgroup of G, the induced ring structure on

$$H^{\oplus}(G,R(G/K)) \cong H^{\oplus}(K,R)$$

coincides with the ordinary ring structure of $H^{\oplus}(K,R)$ for cup-products.

Example 2.6 : Let B denote the Burnside functor. If X is a finite G-set, then $B(X)$ is the Grothendieck group of the category of G-sets over X. The obvious product

$$\left(\begin{array}{ccc} Z & & T \\ \downarrow & , & \downarrow \\ X & & Y \end{array}\right) \mapsto \begin{array}{c} Z\times T \\ \downarrow \\ X\times Y \end{array}$$

extends linearly to a cross product $B(X)\times B(Y) \to B(X\times Y)$, which gives B its structure of Green functor ([2] 2.4).

3. The Dress construction

The Dress construction is a fundamental endofunctor of the category of Mackey functors for G over R, defined as follows. Let Γ be a fixed finite

G-set. If M is a Mackey functor for G over R, then the Mackey functor M_Γ is the bivariant functor defined on the finite G-set Y by

$$M_\Gamma(Y) = M(Y \times \Gamma) \quad .$$

If $f\ : Y \to Z$ is a map of G-sets, then

$$(M_\Gamma)_*(f) = M_*(f \times Id_\Gamma) \qquad (M_\Gamma)^*(f) = M^*(f \times Id_\Gamma) \quad .$$

One checks easily ([2] 1.2) that M_Γ is a Mackey functor for G over R.

It follows from the definitions that the evaluation of M_Γ at the trivial G-set $\bullet = G/G$ is equal to

$$M_\Gamma(\bullet) = M(\Gamma) \quad .$$

Suppose now that Γ is a finite G-monoid, i.e. that Γ is a finite monoid on which G acts by monoid automorphisms. Let A be a Green functor for G over R. Apart from the product on $A(\Gamma)$ defined by the Green functor structure, defined by

$$(a,b) \in A(\Gamma) \mapsto a.b = A^* \begin{pmatrix} \gamma \\ \downarrow \\ \gamma,\gamma \end{pmatrix} (a \times b) \quad ,$$

where $\begin{pmatrix} \gamma \\ \downarrow \\ \gamma,\gamma \end{pmatrix}$ denotes the diagonal inclusion of Γ into $\Gamma \times \Gamma$, there is another natural product on $A(\Gamma)$, defined by

$$(a,b) \in A(\Gamma) \mapsto a \times_\Gamma b = A_* \begin{pmatrix} \gamma_1,\gamma_2 \\ \downarrow \\ \gamma_1\gamma_2 \end{pmatrix} (a \times b)$$

where $A_* \begin{pmatrix} \gamma_1,\gamma_2 \\ \downarrow \\ \gamma_1\gamma_2 \end{pmatrix}$ denotes the image by A_* of the multiplication $\Gamma \times \Gamma \to \Gamma$. Let $\begin{pmatrix} \bullet \\ \downarrow \\ 1_\Gamma \end{pmatrix}$ denote the map from $\bullet$ to Γ sending the unique element of $\bullet$ to the unit of Γ. Set $\varepsilon_{A_\Gamma} = A_* \begin{pmatrix} \bullet \\ \downarrow \\ 1_\Gamma \end{pmatrix} (\varepsilon_A)$.

Lemma 3.1 : *The above product gives $A(\Gamma)$ a ring structure, with unit ε_{A_Γ}.*

Proof : This will follow from theorem 5.1 below, since $A(\Gamma)$ will be the evaluation at the trivial G-set of the Green functor A_{Γ^u}. □

Notation 3.2 : *In the sequel, for a Green functor A for G over R, and a G-monoid Γ, the ring $A(\Gamma)$ will always be understood as the R-module $A(\Gamma)$, equipped with the product $(a,b) \mapsto a \times_\Gamma b$.*

Example 3.3 : Let A denote the cohomology functor $H^{\oplus}(G,-)$, and $\Gamma = G^c$ denote the group G, on which G acts by conjugation. Then

$$A(\Gamma) = A(G^c) = H^{\oplus}(G, RG^c)$$

is isomorphic to the Hochschild cohomology $HH^{\oplus}(G, R)$, as an R-module. But Proposition 3.1 of [6] shows that this isomorphism is actually a ring isomorphism.

Example 3.4 : Let B denote the Burnside functor for G, and $\Gamma = G^c$ denote the group G, on which G acts by conjugation. Then

$$A(\Gamma) = B(G^c)$$

is isomorphic to the crossed Burnside ring $B^c(G)$ of G, as a $\mathbb{Z}$-module. It is clear moreover from the definitions of the ring structure on $B^c(G)$ ([1] 2.1) that this is actually a ring isomorphism.

Example 3.5 : Let k be a commutative ring, and let $M_k(G)$ denote the Grothendieck group of the category of finitely generated kG-modules, for relations given by direct sum decompositions. The usual operations of induction and restriction of modules endow M_k with a structure of Mackey functor, and the tensor product of modules (over k) gives a structure of Green functor on M_k (for G, over $\mathbb{Z}$). The ring $M_k(G)$ is usually called the Green ring of kG-modules.

Let $\Gamma = G^c$ denote the group G, on which G acts by conjugation, as in the previous example. Then the ring $M_k(\Gamma)$ is isomorphic to the Grothendieck ring of Hopf bimodules for the Hopf algebra kG. This was originally a question of Cibils, and follows easily from his theorems 2.1 and 3.1 in [3], using the product formula of theorem 6.1.

4. Crossed G-monoids

Definition 4.1 : *Let G be a finite group. A crossed G-monoid (Γ, φ) is a pair consisting of a finite monoid Γ with a left action of G by monoid automorphisms (denoted by $(g,\gamma) \mapsto g\gamma$ or $(g,\gamma) \mapsto {}^g\gamma$, for $g \in G$ and $\gamma \in \Gamma$), and a map of G-monoids φ from Γ to G^c (i.e. a map φ which is both* *a map of monoids and a map of G-sets). A morphism of crossed G-monoids from (Γ, φ) to (Γ', φ') is a map of G-monoids $\theta : \Gamma \to \Gamma'$ such that $\varphi' \circ \theta = \varphi$. A crossed G-group (Γ, φ) is a crossed G-monoid for which Γ is a group.*

Remark 4.2 : Generally the map $\varphi : \Gamma \to G^c$ will be clear from context, and will be understood in the notation.

Proposition 4.3 : *Let (Γ, φ) be a crossed G-monoid. If X is any G-set, there is a natural monoid action of Γ on X, denoted by*

$$(\gamma, x) \in \Gamma \times X \mapsto \gamma.x \in X$$

and defined by

$$\gamma.x = \varphi(\gamma)x \quad .$$

This action of Γ has the following properties :

1. *If X is a G-set, if $g \in G$, if $\gamma \in \Gamma$ and $x \in X$, then*

$$g(\gamma.x) = g\gamma.gx \quad .$$

2. *If $f \; : X \to Y$ is a morphism of G-sets, then for all $\gamma \in \Gamma$ and all $x \in X$*

$$f(\gamma.x) = \gamma.f(x) \quad .$$

Moreover the square

$$\begin{array}{ccc} \Gamma \times X & \xrightarrow{Id_\Gamma \times f} & \Gamma \times Y \\ \Big\downarrow & & \Big\downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

is cartesian, where the vertical arrows are given by the actions of Γ on X and Y.

3. *If X is a G-set and if $\theta : (\Gamma, \varphi) \to (\Gamma', \varphi')$ is a morphism of crossed G-monoids, then for all $\gamma \in \Gamma$ and all $x \in X$*

$$\theta(\gamma).x = \gamma.x \quad .$$

Proof : The only non-obvious point concerns the cartesian square in 2). Suppose $x \in X$ and $(\gamma, y) \in \Gamma \times Y$ are such that $f(x) = \gamma.y$. Then setting $z = \varphi(\gamma)^{-1}x$ gives an element $(\gamma, z) \in \Gamma \times X$ such that $(Id_\Gamma \times f)(\gamma, z) = (\gamma, y)$

and $\gamma.z = x$. And if (γ', z') is another pair in $\Gamma \times X$ satisfying these two conditions, then $\gamma = \gamma'$ and $\gamma.z = \gamma'.z$, thus $\varphi(\gamma)z = \varphi(\gamma)z'$, and $z = z'$. □

Remark 4.4 : Property 1) is equivalent to saying that the action $(\gamma, x) \mapsto \gamma.x$ of Γ on X is a map of G-sets from $\Gamma \times X$ to X. Property 2) implies in particular that if X and Y are two G-sets, if $(x, y) \in X \times Y$ and $\gamma \in \Gamma$, then $\gamma.(x, y) = (\gamma.x, \gamma.y)$.

Example 4.5 : Let H be a normal subgroup of G, and φ be the inclusion homomorphism from H to G. Then $H^c = (H, \varphi)$ is a crossed G-group.

Example 4.6 : Let Γ be any G-monoid (i.e. any monoid with a left action of G by monoid automorphisms). Let u be the trivial monoid homomorphism from Γ to G. Then $\Gamma^u = (\Gamma, u)$ is a crossed G-monoid.

Example 4.7 : Let (Γ, φ) be a crossed G-monoid. Then $\varphi(\Gamma)$ is a normal subgroup of G, and $\varphi^{-1}(1)$ is a G-submonoid of Γ. There is a natural inclusion of crossed G-monoids from $\varphi^{-1}(1)^u$ to (Γ, φ), and a natural surjection from (Γ, φ) to $\varphi(\Gamma)^c$.

Example 4.8 : Let $\mathbb{E}$ be a group of cardinality 1, with trivial G-action. Let $u : \mathbb{E} \to G^c$ be the map sending the unique element of $\mathbb{E}$ to the unit of G. Then $(\mathbb{E}, u)$ is an initial object in the category of crossed G-monoids. On the other hand the crossed G-monoid $G^c = (G, Id_G)$ is a final object in the category of crossed G-monoids.

5. The Green functor structure on A_Γ

Let R be a commutative ring, and Γ be a crossed G-monoid. If A is a Green functor for G over R, then the Dress construction gives a Mackey functor A_Γ defined on the G-set X by

$$A_\Gamma(X) = A(X \times \Gamma) \quad .$$

If X and Y are finite G-sets, define maps

$$A_\Gamma(X) \otimes_R A_\Gamma(Y) \to A_\Gamma(X \times Y) : a \otimes b \mapsto a \times_\Gamma b$$

by

$$a \times_\Gamma b = A_*\begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle x,\gamma_1.y,\gamma_1\gamma_2} \end{pmatrix}(a \times b) \quad .$$

The notation $A_*\begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle x,\gamma_1.y,\gamma_1\gamma_2} \end{pmatrix}$ means $A_*(f)$, where $f = \begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle x,\gamma_1.y,\gamma_1\gamma_2} \end{pmatrix}$ is the map from $X \times \Gamma \times Y \times \Gamma$ to $X \times Y \times \Gamma$ mapping $(x, \gamma_1, y, \gamma_2)$ to $(x, \gamma_1.y, \gamma_1\gamma_2)$.

This definition makes sense, since the map f is a map of G-sets if Γ is a crossed G-monoid. Moreover if $a \in A(X \times \Gamma)$ and $b \in A(Y \times \Gamma)$, then $a \times b \in A(X \times \Gamma \times Y \times \Gamma)$, hence $a \times_\Gamma b \in A(X \times Y \times \Gamma) = A_\Gamma(X \times Y)$.

Let moreover ε_{A_Γ} denote the element $A_*\begin{pmatrix} \bullet \\ \downarrow \\ {\scriptstyle 1_\Gamma} \end{pmatrix}(\varepsilon_A)$ of $A(\Gamma) = A_\Gamma(\bullet)$.

Theorem 5.1 : *The functor A_Γ is a Green functor for G over R,with unit ε_{A_Γ}. Moreover the correspondence $A \mapsto A_\Gamma$ is an endo-functor of the category $\mathsf{Green}_R(G)$.*

Proof : First A_Γ is a Mackey functor for G over R. Moreover, the product $\times_\Gamma$ is bifunctorial : suppose that $f\ : X \to X'$ and $g\ : Y \to Y'$ are maps of G-sets. I must check that for any $a \in A_\Gamma(X)$ and $b \in A_\Gamma(Y)$

$$(A_\Gamma)_*(f)(a) \times_\Gamma (A_\Gamma)_*(g)(b) = (A_\Gamma)_*(f \times g)(a \times_\Gamma b) \quad .$$

This is equivalent to

$$A_*(f \times Id_\Gamma)(a) \times_\Gamma A_*(g \times Id_\Gamma)(b) = A_*(f \times g \times Id_\Gamma)(a \times_\Gamma b) \qquad \mathbf{(5.2)}$$

The right hand side of this equation is equal to

$$A_*(f \times g \times Id_\Gamma)A_*\begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle x,\gamma_1.y,\gamma_1\gamma_2} \end{pmatrix}(a \times b) = A_*\begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle f(x),g(\gamma_1.y),\gamma_1\gamma_2} \end{pmatrix}(a \times b) \quad .$$

The left hand side of equation (5.2) is equal to

$$A_*\begin{pmatrix} {\scriptstyle x',\gamma_1,y',\gamma_2} \\ \downarrow \\ {\scriptstyle x',\gamma_1.y',\gamma_1\gamma_2} \end{pmatrix}\Big(A_*(f \times Id_\Gamma)(a) \times A_*(g \times Id_Y)(b)\Big) \quad .$$

Since A is a Green functor, this is equal to

$$A_*\begin{pmatrix} {\scriptstyle x',\gamma_1,y',\gamma_2} \\ \downarrow \\ {\scriptstyle x',\gamma_1.y',\gamma_1\gamma_2} \end{pmatrix} A_*\begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle f(x),\gamma_1,g(y),\gamma_2} \end{pmatrix}(a \times b) = \ldots$$

$$\ldots = A_*\begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle f(x),\gamma_1.g(y),\gamma_1\gamma_2} \end{pmatrix}(a \times b) \quad .$$

It follows that equation (5.2) holds, since $\gamma_1.g(y) = g(\gamma_1.y)$ for any $\gamma_1 \in \Gamma$ and $y \in Y$.

Similarly, I must check that for any $a' \in A_\Gamma(X')$ and $b' \in A_\Gamma(Y')$

$$A^*(f \times Id_\Gamma)(a') \times_\Gamma A^*(g \times Id_\Gamma)(b') = A^*(f \times g \times Id_\Gamma)(a' \times_\Gamma b') \qquad \textbf{(5.3)}$$

The left hand side of this equation is equal to

$$A_*\begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle x,\gamma_1.y,\gamma_1\gamma_2} \end{pmatrix} \Big(A^*(f \times Id_\Gamma)(a') \times A^*(g \times Id_\Gamma)(b') \Big) \quad .$$

Since A is a Green functor, this is equal to

$$A_*\begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle x,\gamma_1.y,\gamma_1\gamma_2} \end{pmatrix} A^*\begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle f(x),\gamma_1,g(y),\gamma_2} \end{pmatrix} (a' \times b') \qquad \textbf{(5.4)}$$

The right hand side of equation (5.3) is equal to

$$A^*(f \times g \times Id_\Gamma)A_*\begin{pmatrix} {\scriptstyle x',\gamma_1,y',\gamma_2} \\ \downarrow \\ {\scriptstyle x',\gamma_1.y',h_1'h_2'} \end{pmatrix} (a' \times b') \qquad \textbf{(5.5)}$$

Now the square

$$\begin{array}{ccc} X\Gamma & \xrightarrow{\begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle f(x),\gamma_1,g(y),\gamma_2} \end{pmatrix}} & X'\Gamma Y'\Gamma \\ {\scriptstyle \begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle x,\gamma_1.y,\gamma_1\gamma_2} \end{pmatrix}} \Big\downarrow & & \Big\downarrow {\scriptstyle \begin{pmatrix} {\scriptstyle x',\gamma_1,y',\gamma_2} \\ \downarrow \\ {\scriptstyle x',\gamma_1.y',\gamma_1\gamma_2} \end{pmatrix}} \\ XY\Gamma & \xrightarrow[f \times g \times Id_\Gamma]{} & X'Y'\Gamma \end{array}$$

is cartesian if Γ is a crossed G-monoid. It follows that expression (5.5) is equal to expression (5.4), and equation (5.3) holds.

Next I have to check that the product on A_Γ is associative. In other words, if X, Y, and Z are finite G-sets, if $a \in A_\Gamma(X)$, if $b \in A_\Gamma(Y)$ and $c \in A_\Gamma(Z)$, I must show that

$$a \times_\Gamma (b \times_\Gamma c) = (a \times_\Gamma b) \times_\Gamma c \qquad \textbf{(5.6)}$$

in $A_\Gamma(X \times Y \times Z)$. The left hand side is equal to

$$A_*\begin{pmatrix} {\scriptstyle x,\gamma_1,y,z,\gamma_2} \\ \downarrow \\ {\scriptstyle x,\gamma_1.y,\gamma_1.z,\gamma_1\gamma_2} \end{pmatrix} \left(a \times A_*\begin{pmatrix} {\scriptstyle y,\gamma_1,z,\gamma_2} \\ \downarrow \\ {\scriptstyle y,\gamma_1.z,\gamma_1\gamma_2} \end{pmatrix} (b \times c) \right) \quad .$$

Since A is a Green functor, this is also equal to

$$A_*\begin{pmatrix} {\scriptstyle x,\gamma_1,y,z,\gamma_2} \\ \downarrow \\ {\scriptstyle x,\gamma_1.y,\gamma_1.z,\gamma_1\gamma_2} \end{pmatrix} A_*\begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2,z,\gamma_3} \\ \downarrow \\ {\scriptstyle x,\gamma_1,y,\gamma_2.z,\gamma_2\gamma_3} \end{pmatrix} (a \times b \times c)$$

which is also equal to

$$A_*\begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2,z,\gamma_3} \\ \downarrow \\ {\scriptstyle x,\gamma_1.y,\gamma_1.(\gamma_2.z),\gamma_1(\gamma_2\gamma_3)} \end{pmatrix} (a \times b \times c) \quad . \qquad \textbf{(5.7)}$$

The right hand side of equation (5.6) is equal to

$$A_*\left(\begin{smallmatrix} x,y,\gamma_1,z,\gamma_2 \\ \downarrow \\ x,y,\gamma_1.z,\gamma_1\gamma_2 \end{smallmatrix}\right)\Big(A_*\left(\begin{smallmatrix} x,\gamma_1,y,\gamma_2 \\ \downarrow \\ x,\gamma_1.y,\gamma_1\gamma_2 \end{smallmatrix}\right)(a\times b)\times c\Big) \quad .$$

Since A is a Green functor, this is also

$$A_*\left(\begin{smallmatrix} x,y,\gamma_1,z,\gamma_2 \\ \downarrow \\ x,y,\gamma_1.z,\gamma_1\gamma_2 \end{smallmatrix}\right)A_*\left(\begin{smallmatrix} x,\gamma_1,y,\gamma_2,z,\gamma_3 \\ \downarrow \\ x,\gamma_1.y,\gamma_1\gamma_2,z,\gamma_3 \end{smallmatrix}\right)(a\times b\times c)$$

which is also equal to

$$A_*\left(\begin{smallmatrix} x,\gamma_1,y,\gamma_2,z,\gamma_3 \\ \downarrow \\ x,\gamma_1.y,(\gamma_1\gamma_2).z,(\gamma_1\gamma_2)\gamma_3 \end{smallmatrix}\right)(a\times b\times c) \quad .$$

This is equal to (5.7). Hence A_Γ is associative.

Finally I have to check that ε_{A_Γ} is a unit for A_Γ. Let X be a finite G-set, and let $a\in A_\Gamma(X)$. Then

$$\begin{aligned}
\varepsilon_{A_\Gamma}\times_\Gamma a &= A_*\left(\begin{smallmatrix} \gamma_1,x,\gamma_2 \\ \downarrow \\ \gamma_1.x,\gamma_1\gamma_2 \end{smallmatrix}\right)\Big(A_*\left(\begin{smallmatrix} \bullet \\ \downarrow \\ 1_\Gamma \end{smallmatrix}\right)(\varepsilon_A)\times a\Big)\\
&= A_*\left(\begin{smallmatrix} \gamma_1,x,\gamma_2 \\ \downarrow \\ \gamma_1.x,\gamma_1\gamma_2 \end{smallmatrix}\right)A_*\left(\begin{smallmatrix} x,\gamma \\ \downarrow \\ 1,x,\gamma \end{smallmatrix}\right)(a)\\
&= A_*\left(\begin{smallmatrix} x,\gamma \\ \downarrow \\ 1_\Gamma.x,1_\Gamma\gamma \end{smallmatrix}\right)(a)=a \quad .
\end{aligned}$$

One checks similarly that $a\times_\Gamma\varepsilon_{A_\Gamma}=a$.

To complete the proof of theorem 5.1, it remains to check that the construction $A\mapsto A_\Gamma$ is an endo-functor of $\mathsf{Green}_R(G)$. So let $\theta:A\to A'$ be a morphism of Green functors. Then θ is in particular a morphism of Mackey functors, and it induces a morphism of Mackey functors θ_Γ from $A_\Gamma\to A'_\Gamma$, whose evaluation at the finite G-set X is the map

$$(\theta_\Gamma)_X=\theta_{X\times\Gamma}:A_\Gamma(X)=A(X\times\Gamma)\to A'(X\times\Gamma)=A_{\Gamma'}(X) \quad .$$

It is clear moreover that with obvious notation $(\theta\circ\theta')_\Gamma=\theta_\Gamma\circ\theta'_\Gamma$, and that if θ is the identity endomorphism of A, then θ_Γ is the identity of A_Γ.

It remains to check that θ_Γ is a morphism of Green functors, i.e. that it is compatible with the product. So let X and Y be finite G-sets, let $a\in A_\Gamma(X)$ and $b\in A_\Gamma(Y)$. Then

$$\begin{aligned}
(\theta_\Gamma)_{X\times Y}(a\times_\Gamma b) &= \theta_{X\times Y\times\Gamma}\Big(A_*\left(\begin{smallmatrix} x,\gamma_1,y,\gamma_2 \\ \downarrow \\ x,\gamma_1.y,\gamma_1\gamma_2 \end{smallmatrix}\right)(a\times b)\Big)\\
&= A'_*\left(\begin{smallmatrix} x,\gamma_1,y,\gamma_2 \\ \downarrow \\ x,\gamma_1.y,\gamma_1\gamma_2 \end{smallmatrix}\right)(\theta_{X\times\Gamma\times Y\times\Gamma}(a\times b))\\
&\qquad \text{(because } \theta:A\to A' \text{ is a morphism of Mackey functors)}\\
&= A'_*\left(\begin{smallmatrix} x,\gamma_1,y,\Gamma_2 \\ \downarrow \\ x,\gamma_1.y,\Gamma_1\gamma_2 \end{smallmatrix}\right)(\theta_{X\times\Gamma}(a)\times\theta_{Y\times\Gamma}(b))\\
&\qquad \text{(because } \theta \text{ is a morphism of Green functors)}\\
&= \theta_{X\times\Gamma}(a)\times_\Gamma\theta_{Y\times\Gamma}(b)\\
&= (\theta_\Gamma)_X(a)\times_\Gamma(\theta_\Gamma)_Y(b) \quad .
\end{aligned}$$

This shows that θ_Γ is a morphism of Green functors from A_Γ to A'_Γ, and completes the proof of theorem 5.1. □

Remark 5.8 : The evaluation at the trivial G-set of the Green functor A_Γ is

$$A_\Gamma(\bullet) = A(\bullet \times \Gamma) \cong A(\Gamma)$$

and with this identification the product on $A(\Gamma)$ is given by

$$(a,b) \in A(\Gamma) \times A(\Gamma) \mapsto A_* \begin{pmatrix} {\scriptstyle \gamma_1,\gamma_2} \\ \downarrow \\ {\scriptstyle \gamma_1\gamma_2} \end{pmatrix} (a \times b) \quad .$$

This product coincides with the product defined in section 3, and justifies lemma 3.1.

Proposition 5.9 : *Let $f : (\Gamma, \varphi) \to (\Gamma', \varphi')$ be a morphism of crossed G-monoids. For any G-set X, denote by $A_{f,X}$ the map $A_*(Id_X \times f)$ from $A_\Gamma(X)$ to $A_{\Gamma'}(X)$. Then these maps $A_{f,X}$ define a morphism of Green functors A_f from A_Γ to $A_{\Gamma'}$. Moreover, if f is injective, then A_f is a split injection of Mackey functors.*

Proof : The maps $A_{f,X}$ define a morphism of Mackey functors from A_Γ to $A_{\Gamma'}$ ([2] 1.2). Moreover clearly

$$A_{f,\bullet}(\varepsilon_{A_\Gamma}) = A_*(f) A_* \begin{pmatrix} \bullet \\ \downarrow \\ {\scriptstyle 1_\Gamma} \end{pmatrix} (\varepsilon_A) = A_* \begin{pmatrix} \bullet \\ \downarrow \\ {\scriptstyle 1_{\Gamma'}} \end{pmatrix} (\varepsilon_A) = \varepsilon_{A_{\Gamma'}} \quad .$$

Moreover if X and Y are finite G-sets, if $a \in A_\Gamma(X)$ and $b \in A_\Gamma(Y)$, then

$$A_* \begin{pmatrix} {\scriptstyle x,y,\gamma} \\ \downarrow \\ {\scriptstyle x,y,f(\gamma)} \end{pmatrix} (a \times_\Gamma b) = A_* \begin{pmatrix} {\scriptstyle x,\gamma} \\ \downarrow \\ {\scriptstyle x,f(\gamma)} \end{pmatrix} (a) \times_{\Gamma'} A_* \begin{pmatrix} {\scriptstyle y,\gamma} \\ \downarrow \\ {\scriptstyle y,f(\gamma)} \end{pmatrix} (b) \quad .$$

Indeed, the left hand side is equal to

$$A_* \begin{pmatrix} {\scriptstyle x,y,\gamma} \\ \downarrow \\ {\scriptstyle x,y,f(\gamma)} \end{pmatrix} A_* \begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle x,\gamma_1\cdot y,\gamma_1\gamma_2} \end{pmatrix} (a \times b) = A_* \begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle x,\gamma_1\cdot y,f(\gamma_1\gamma_2)} \end{pmatrix} (a \times b)$$

whereas the right hand side is equal to

$$A_* \begin{pmatrix} {\scriptstyle x,\gamma'_1,y,\gamma'_2} \\ \downarrow \\ {\scriptstyle x,\gamma'_1\cdot y,\gamma'_1\gamma'_2} \end{pmatrix} \left(A_* \begin{pmatrix} {\scriptstyle x,\gamma} \\ \downarrow \\ {\scriptstyle x,f(\gamma)} \end{pmatrix} (a) \times A_* \begin{pmatrix} {\scriptstyle y,\gamma} \\ \downarrow \\ {\scriptstyle y,f(\gamma)} \end{pmatrix} (b) \right)$$

or

$$A_* \begin{pmatrix} {\scriptstyle x,\gamma'_1,y,\gamma'_2} \\ \downarrow \\ {\scriptstyle x,\gamma'_1\cdot y,\gamma'_1\gamma'_2} \end{pmatrix} A_* \begin{pmatrix} {\scriptstyle x,\gamma_1,y,\gamma_2} \\ \downarrow \\ {\scriptstyle x,f(\gamma_1),y,f(\gamma_2)} \end{pmatrix} (a \times b) = \ldots$$

$$\ldots = A_* \left({}^{x,\gamma_1,y,\gamma_2}_{x,f(\gamma_1).y,f(\gamma_1)f(\gamma_2)} \right) (a \times b) \quad .$$

Now $f(\gamma_1).y = \gamma_1.y$ and $f(\gamma_1)f(\gamma_2) = f(\gamma_1\gamma_2)$ for all $y \in Y$ and $\gamma_1, \gamma_2 \in \Gamma$, since f is a morphism of crossed G-monoids.

Finally if f is injective, then the maps $A^f_X = A^*(Id_X \times f)$ define a morphism of Mackey functors A^f from $A_{\Gamma'}$ to A_Γ ([2] 1.2), which is a section to A_f, i.e. such that $A^f \circ A_f = Id_{A_\Gamma}$. The proposition follows. □

Corollary 5.10 : *Let Γ be a crossed G-monoid. For any finite G-set X, denote by ι_X the map $A_* \left({}^{x}_{x,1_\Gamma} \right)$ from $A(X)$ to $A(X \times \Gamma) = A_\Gamma(X)$. Then ι_X is a split injective map of R-modules, and the maps ι_X define an injective morphism of Green functors from A to A_Γ.*

Proof : This is the special case of the proposition, where f is the (unique) morphism of crossed G-monoids from $(\mathbb{E}, u)$ to (Γ, φ) (see example 4.8), sending the unique element of $\mathbb{E}$ to 1_Γ. □

6. The product formula

This section states a product formula for the ring $A_\Gamma(G)$, which generalizes theorem 5.1 of [6].

Theorem 6.1 : *Let A be a Green functor for G over R, and Γ be a crossed G-monoid. Then*

$$A_\Gamma(G) = A(\Gamma) = \left(\bigoplus_{\gamma \in \Gamma} A(G_\gamma) \right)^G$$

and for $\gamma \in \Gamma$, the γ-component of the product of the elements a and b of $A(\Gamma)$ is given by

$$(a \times_\Gamma b)_\gamma = \sum_{\substack{(\alpha,\beta) \in G_\gamma \backslash (\Gamma \times \Gamma) \\ \alpha\beta = \gamma}} t^{G_\gamma}_{G_{(\alpha,\beta)}} \left(r^{G_\alpha}_{G_{(\alpha,\beta)}} a_\alpha . r^{G_\beta}_{G_{(\alpha,\beta)}} b_\beta \right) \quad .$$

Corollary 6.2 : *Taking first sets of orbit representatives, there is an isomorphism of R-modules*

$$A(\Gamma) \cong \bigoplus_{\gamma \in [G \backslash \Gamma]} A(G_\gamma)$$

where $[G\backslash\Gamma]$ *is a set of representatives of the orbits of* G *in* Γ*. With this notation, the product of* $a \in A(G_\gamma)$ *and* $b \in A(G_\delta)$ *is equal to*

$$\bigoplus_{\varepsilon\in[G\backslash\Gamma]} \bigoplus_{w\in[G_\gamma\backslash G/G_\delta]} t_{G_{g(w,\varepsilon)\gamma}\cap G_{g(w,\varepsilon)w\delta}}^{G_\varepsilon}\,{}^{g(w,\varepsilon)}\Big(r^{G_\gamma}_{G_\gamma\cap G_{w\delta}} a . r^{G_{w\delta}}_{G_\gamma\cap G_{w\delta}}\,{}^{w}b\Big)$$

where $g(w,\varepsilon)$ *is an element of the unique class* $G_\varepsilon g$ *in* $G_\varepsilon\backslash G$ *such that* $g(\gamma\, w\delta) = \varepsilon$*.*

Proof : This follows from the formulae given in remark 2.3 : the product $a \times b$ is the element of $A(\Gamma\times\Gamma)$ whose component (α,β) is equal to

$$r^{G_\alpha}_{G_{(\alpha,\beta)}} a_\alpha . r^{G_\beta}_{G_{(\alpha,\beta)}} b_\beta$$

and $a \times_\Gamma b$ is the image of $a\times b$ by $A_*(\mu)$, where $\mu : \Gamma \to \Gamma$ is the product in Γ. The formula of the theorem follows.

The formula in the corollary is a translation using sets of representatives, and is a generalization of the product formula of [6] theorem 5.1. □

Remark 6.3 : There is a similar formula for the product in $A(\Gamma)$, if one uses co-invariants rather than invariants, as in remark 2.4. There is an isomorphism of R-modules

$$A(\Gamma) \cong \Big(\bigoplus_{\gamma\in\Gamma} A(G_\gamma)\Big)_G$$

and denoting by $[\gamma, a]_G$ the image in $A(G_\Gamma)$ of the element a of $A(\gamma)$, for $\gamma\in\Gamma$, the product of $[\gamma,a]_G$ and $[\delta,b]_G$ (for $\delta\in\Gamma$ and $b\in A(G_\delta)$) is equal to

$$[\gamma,a]_G \times_\Gamma [\delta,b]_G = \sum_{w\in[G_\gamma\backslash G/G_\delta]} [\gamma\,{}^{w}\delta, t^{G_{\gamma\,{}^{w}\delta}}_{G_\gamma\cap G_{{}^{w}\delta}}\Big(r^{G_\gamma}_{G_\gamma\cap G_{{}^{w}\delta}} a . r^{G_{{}^{w}\delta}}_{G_\gamma\cap G_{{}^{w}\delta}}\,{}^{w}b\Big)]_G \quad .$$

This kind of formula was used in [1].

Corollary 6.4 : *Let* H *be a normal subgroup of* G*. Suppose that* A *is a (graded) commutative Green functor. If for any subgroup* K *of* G*, the group* $H\cap C_G(K)$ *acts trivially on* $A(K)$*, then the ring* $A(H^c)$ *is (graded) commutative.*

Proof : Let γ be an element of H, and consider the G_γ-set P_γ of pairs (α,β) of elements of H such that $\alpha\beta=\gamma$. The correspondence

$$\sigma : (\alpha,\beta) \mapsto (\beta, \alpha^\beta)$$

is a permutation of P, which commutes with the action of G_γ. Moreover

$$G_\alpha \cap G_\beta = G_\beta \cap G_{\alpha^\beta} \quad .$$

Let a and b be elements of $A(H^c)$. If A is graded, suppose moreover that a and b are homogeneous. The γ-component of the product of a and b is equal to

$$(a \times_\Gamma b)_\gamma = \sum_{(\alpha,\beta)\in G_\gamma \backslash P_\gamma} t_{G_{(\alpha,\beta)}}^{G_\gamma} \left(r_{G_{(\alpha,\beta)}}^{G_\alpha} a_\alpha . r_{G_{(\alpha,\beta)}}^{G_\beta} b_\beta \right) \quad .$$

Since A is (graded) commutative, this is also equal (up to sign if A is graded) to

$$(a \times_\Gamma b)_\gamma = \sum_{(\alpha,\beta)\in G_\gamma \backslash P_\gamma} t_{G_{(\alpha,\beta)}}^{G_\gamma} \left(r_{G_{(\alpha,\beta)}}^{G_\beta} b_\beta . r_{G_{(\alpha,\beta)}}^{G_\alpha} a_\alpha \right) \quad .$$

Changing the order of summation via the permutation σ gives

$$\begin{aligned}(a \times_\Gamma b)_\gamma &= \sum_{(\beta,\alpha)\in G_\gamma \backslash P_\gamma} t_{G_{(\beta,\alpha)}}^{G_\gamma} \left(r_{G_{(\beta,\alpha)}}^{G_\beta} b_\beta . r_{G_{(\alpha,\beta)}}^{{}^\beta G_\alpha} a_{\beta\alpha} \right) \\ &= \sum_{(\beta,\alpha)\in G_\gamma \backslash P_\gamma} t_{G_{(\beta,\alpha)}}^{G_\gamma} \left(r_{G_{(\beta,\alpha)}}^{G_\beta} b_\beta . {}^\beta r_{G_{(\alpha,\beta)}}^{G_\alpha} a_\alpha \right) \quad .\end{aligned}$$

Now for $(\beta,\alpha) \in P_\gamma$, set $K = G_{\beta,\alpha} = C_G(<\beta,\alpha>)$. Then $\beta \in H \cap C_G(K)$, thus β acts trivially on $A(K)$. Hence ${}^\beta r_{G_{(\alpha,\beta)}}^{G_\alpha} a_\alpha = r_{G_{(\alpha,\beta)}}^{G_\alpha} a_\alpha$ and

$$(a \times_\Gamma b)_\gamma = \sum_{(\beta,\alpha)\in G_\gamma \backslash P_\gamma} t_{G_{(\beta,\alpha)}}^{G_\gamma} \left(r_{G_{(\beta,\alpha)}}^{G_\beta} b_\beta . r_{G_{(\alpha,\beta)}}^{G_\alpha} a_\alpha \right) = (b \times_\Gamma a)_\gamma$$

and $A(H^c)$ is (graded) commutative. □

Remark 6.5 : Corollary 6.4 shows in particular that the crossed Burnside ring of G is commutative. Similarly, the Hochschild cohomology ring of G is graded commutative. This was first proved by Gerstenhaber ([5]).

Remark 6.6 : Let N be a normal subgroup of G, viewed as a crossed G-monoid via the inclusion morphism from N to G. Then taking for A the cohomology functor $H^\oplus(-,R)$ leads to a ring $A_N(G)$. This ring is isomorphic to the cohomology ring $H^\oplus(G,R)$ if N is trivial, and to the Hochschild cohomology ring $HH^\oplus(G,R)$ if $N = G$. It is a kind of "relative" Hochschild cohomology, and could be denoted by $HH^\oplus(G,N,R)$. As an R-module, it is isomorphic to

$$HH^\oplus(G,N,R) \cong \bigoplus_{\substack{n\in N \\ mod.\, G}} H^\oplus(C_G(n),R)$$

In other words $HH^{\oplus}(G,N,R) \cong H^{\oplus}(G,RN)$, where the action of G on RN comes from the conjugation action of G on N. Corollary 6.4 shows that $HH^{\oplus}(G,N,R)$ is graded commutative.

7. Semi-direct products of crossed G-monoids

Theorem 5.1 shows that the correspondence $A \mapsto A_\Gamma$ is an endo-functor of $\mathsf{Green}_R(G)$. It is natural to compose those endo-functors, and this leads to the notion of semi-direct product of crossed G-monoids.

Proposition 7.1 : *Let (Γ,φ) and (Γ',φ') be crossed G-monoids. Let Γ" denote the direct product $\Gamma'\times\Gamma$, with diagonal G-action. Define the following multiplication on Γ" :*

$$(\gamma'_1,\gamma_1)(\gamma'_2,\gamma_2) = \Big(\gamma'_1(\gamma_1.\gamma'_2),\gamma_1\gamma_2\Big) \quad \forall \gamma_1,\gamma_2\in\Gamma,\ \forall\gamma'_1,\gamma'_2\in\Gamma' \quad .$$

Define $\varphi" : \Gamma" \to G^c$ by $\varphi"(\gamma',\gamma) = \varphi'(\gamma')\varphi(\gamma)$ for all $\gamma\in\Gamma$ and $\gamma'\in\Gamma'$.

Then $(\Gamma",\varphi")$ is a crossed G-monoid, with unit $(1_{\Gamma'},1_\Gamma)$.

Proof : This is a series of straightforward verifications. □

Definition 7.2 : *The crossed G-monoid $(\Gamma",\varphi")$ of proposition 7.1 is called the semi-direct product of the crossed G-monoids (Γ',φ') and (Γ,φ), and it is denoted by $(\Gamma',\varphi')\rtimes(\Gamma,\varphi)$, or $\Gamma'\rtimes\Gamma$ for short.*

Proposition 7.3 : *Let A be a Green functor for G over R. If Γ and Γ' are crossed G-monoids, then the Green functor $(A_\Gamma)_{\Gamma'}$ is naturally isomorphic to $A_{\Gamma'\rtimes\Gamma}$.*

Proof : Let X be a finite G-set. Then

$$(A_\Gamma)_{\Gamma'}(X) = A_\Gamma(X\times\Gamma') = A(X\times\Gamma'\times\Gamma)$$

and this induces clearly an isomorphism of Mackey functors

$$(A_\Gamma)_{\Gamma'} \cong A_{\Gamma'\times\Gamma}$$

where $\Gamma'\times\Gamma$ is the direct product of Γ' and Γ, with diagonal G-action, i.e. the underlying G-set of $\Gamma'\rtimes\Gamma$.

Now let X and Y be finite G-sets, let $a \in A(X \times \Gamma' \times \Gamma)$, viewed as $(A_\Gamma)_{\Gamma'}(X)$, and $b \in A(Y \times \Gamma' \times \Gamma)$, viewed as $(A_\Gamma)_{\Gamma'}(Y)$. Then the product of a and b in $(A_\Gamma)_{\Gamma'}(X \times Y)$ is equal to

$$\begin{aligned}
a \times_{\Gamma,\Gamma'} b &= A_{\Gamma *}\begin{pmatrix} x,\gamma'_1,y,\gamma'_2 \\ \downarrow \\ x,\gamma'_1.y,\gamma'_1\gamma'_2 \end{pmatrix} (a \times_\Gamma b) \\
&= A_*\begin{pmatrix} x,\gamma'_1,y,\gamma'_2,\gamma \\ \downarrow \\ x,\gamma'_1.y,\gamma'_1\gamma'_2,\gamma \end{pmatrix} A_*\begin{pmatrix} x,\gamma'_1,\gamma_1,y,\gamma'_2,\gamma_2 \\ \downarrow \\ x,\gamma'_1,\gamma_1.y,\gamma_1.\gamma'_2,\gamma_1\gamma_2 \end{pmatrix} (a \times b) \\
&= A_*\begin{pmatrix} x,\gamma'_1,\gamma_1,y,\gamma'_2,\gamma_2 \\ \downarrow \\ x,\gamma'_1.(\gamma_1.y),\gamma'_1(\gamma_1.\gamma'_2),\gamma_1\gamma_2 \end{pmatrix} (a \times b) \\
&= a \times_{\Gamma' \rtimes \Gamma} b
\end{aligned}$$

since moreover $\gamma'_1.(\gamma_1.y) = \varphi'(\gamma')\varphi(\gamma)y = \varphi"(\gamma',\gamma)y = (\gamma',\gamma).y$.

It follows that the previous isomorphism of Mackey functors is compatible with the product. Moreover, the unit of $(A_\Gamma)_{\Gamma'}$ is by definition equal to the following element of $A(\Gamma' \times \Gamma)$:

$$\begin{aligned}
\varepsilon_{(A_\Gamma)_{\Gamma'}} &= A_{\Gamma *}\begin{pmatrix} \bullet \\ \downarrow \\ 1_{\Gamma'} \end{pmatrix} (\varepsilon_{A_\Gamma}) \\
&= A_*\begin{pmatrix} \gamma \\ \downarrow \\ 1_{\Gamma'},\gamma \end{pmatrix} A_*\begin{pmatrix} \bullet \\ \downarrow \\ 1_\Gamma \end{pmatrix} (\varepsilon_A) \\
&= A_*\begin{pmatrix} \bullet \\ \downarrow \\ 1_{\Gamma'},1_\Gamma \end{pmatrix} (\varepsilon_A) \\
&= \varepsilon_{A_{\Gamma' \rtimes \Gamma}}
\end{aligned}$$

and the proposition follows. □

8. From A-modules to A_Γ-modules

It follows in particular from corollary 5.10 that there is a functor of restriction r_Γ along the Green functor homomorphism $\iota : A \to A_\Gamma$, from the category A_Γ-Mod of A_Γ-modules to the category A-Mod. This section describes a functor i_Γ from A-Mod to A_Γ-Mod.

Notation 8.1 : *Let A be a Green functor for G over R, and M be an A-module. If X and Y are finite G-sets, if $a \in A_\Gamma(X)$ and $m \in M(Y)$, denote by $a \times_\Gamma m$ the element of $M(X \times Y)$ defined by*

$$a \times_\Gamma m = M_*\begin{pmatrix} x,\gamma,y \\ \downarrow \\ x,\gamma.y \end{pmatrix} (a \times m) \in M(X \times Y) \quad .$$

Theorem 8.2 : *Let Γ be a crossed G-monoid. If A is a Green functor for G over R, and if M is an A-module, then the product*

$$(a,m)\in A_\Gamma(X)\times M(Y)\mapsto a\times_\Gamma m\in M(X\times Y)$$

endows M with a structure of A_Γ-module, denoted by $\mathsf{i}_\Gamma(M)$. If $f:M\to N$ is a morphism of A-modules, then the maps $f_X:M(X)\to N(X)$ define a morphism $\mathsf{i}_\Gamma(f)$ of A_Γ-modules from $\mathsf{i}_\Gamma(M)$ to $\mathsf{i}_\Gamma(N)$.

Proof : Again I have to check that this product is bifunctorial, associative, and that ε_{A_Γ} is a left unit. Let $f\ :X\to X'$ and $g\ :Y\to Y'$ be maps of G-sets. Let $a\in A_\Gamma(X)$ and $m\in M(Y)$. I must check that

$$A_*(f)(a)\times_\Gamma M_*(g)(m)=M_*(f\times g)(a\times_\Gamma m)\quad .$$

From the definitions, this amounts to checking that

$$M_*\begin{pmatrix}x',\gamma,y'\\ \downarrow\\ x',\gamma.y'\end{pmatrix}(A_*(f)(a)\times M_*(g)(m))=M_*(f\times g)M_*\begin{pmatrix}x,\gamma,y\\ \downarrow\\ x,\gamma.y\end{pmatrix}(a\times m)$$

or equivalently that

$$M_*\begin{pmatrix}x',\gamma,y'\\ \downarrow\\ x',\gamma.y'\end{pmatrix}M_*\begin{pmatrix}x,\gamma,y\\ \downarrow\\ f(x),\gamma,g(y)\end{pmatrix}(a\times m)=\ldots$$

$$\ldots=M_*\begin{pmatrix}x,y\\ \downarrow\\ f(x),g(y)\end{pmatrix}M_*\begin{pmatrix}x,\gamma,y\\ \downarrow\\ x,\gamma.y\end{pmatrix}(a\times m)\quad .$$

This holds, since both sides are equal to

$$M_*\begin{pmatrix}x,\gamma,y\\ \downarrow\\ f(x),\gamma.g(y)\end{pmatrix}(a\times m)\quad .$$

I must also check that for $a'\in A_\Gamma(X')$ and $m'\in M(Y')$

$$A^*(f)(a')\times_\Gamma M^*(g)(m')=M^*(f\times g)(a'\times_\Gamma m')\quad .$$

By similar transformations, this amounts to checking that

$$M_*\begin{pmatrix}x,\gamma,y\\ \downarrow\\ x,\gamma.y\end{pmatrix}M^*\begin{pmatrix}x,\gamma,y\\ \downarrow\\ f(x),\gamma,g(y)\end{pmatrix}(a'\times m')=\ldots$$

$$\ldots=M^*\begin{pmatrix}x,y\\ \downarrow\\ f(x),g(y)\end{pmatrix}M_*\begin{pmatrix}x',\gamma,y'\\ \downarrow\\ x',\gamma.y'\end{pmatrix}(a'\times m')\quad .$$

This holds because M is a Mackey functor, and because the square

$$\begin{array}{ccc}
X\Gamma Y & \xrightarrow{\begin{pmatrix}x,\gamma,y\\ \downarrow\\ f(x),\gamma,g(y)\end{pmatrix}} & X'\Gamma Y' \\
{\scriptstyle\begin{pmatrix}x,\gamma,y\\ \downarrow\\ x,\gamma.y\end{pmatrix}}\Big\downarrow & & \Big\downarrow{\scriptstyle\begin{pmatrix}x',\gamma,y'\\ \downarrow\\ x',\gamma.y'\end{pmatrix}} \\
XY & \xrightarrow[\begin{pmatrix}x,y\\ \downarrow\\ f(x),g(y)\end{pmatrix}]{} & X'Y'
\end{array}$$

is a cartesian square of G-sets if Γ is a crossed G-monoid. This shows that the product $(a,m)\mapsto a\times_\Gamma m$ is bifunctorial.

It is moreover associative : let X, Y, Z be finite G-sets, and let $a\in A_\Gamma(X)$, $b\in A_\Gamma(Y)$, and $m\in M(Z)$. Then I must check that

$$a\times_\Gamma(b\times_\Gamma m)=(a\times_\Gamma b)\times_\Gamma m \qquad \textbf{(8.3)}$$

in $M_\Gamma(X\times Y\times Z)$. The left hand side is equal to

$$M_*\begin{pmatrix}x,\gamma,y,z\\ \downarrow\\ x,\gamma.y,\gamma.z\end{pmatrix}\left(a\times M_*\begin{pmatrix}y,\gamma,z\\ \downarrow\\ y,\gamma.z\end{pmatrix}(b\times m)\right)\quad .$$

Since M is a module for the Green functor A, this is also equal to

$$M_*\begin{pmatrix}x,\gamma,y,z\\ \downarrow\\ x,\gamma.y,\gamma.z\end{pmatrix}M_*\begin{pmatrix}x,\gamma_1,y,\gamma_2,z\\ \downarrow\\ x,\gamma_1,y,\gamma_2.z\end{pmatrix}(a\times b\times m)$$

which is also equal to

$$M_*\begin{pmatrix}x,\gamma_1,y,\gamma_2,z\\ \downarrow\\ x,\gamma_1.y,\gamma_1.(\gamma_2.z)\end{pmatrix}(a\times b\times m)\quad . \qquad \textbf{(8.4)}$$

The right hand side of equation (8.3) is equal to

$$M_*\begin{pmatrix}x,y,\gamma,z\\ \downarrow\\ x,y,\gamma.z\end{pmatrix}\left(A_*\begin{pmatrix}x,\gamma_1,y,\gamma_2\\ \downarrow\\ x,\gamma_1.y,\gamma_1\gamma_2\end{pmatrix}(a\times b)\times m\right)\quad .$$

Since M is a module for the Green functor A, this is also

$$M_*\begin{pmatrix}x,y,\gamma,z\\ \downarrow\\ x,y,\gamma.z\end{pmatrix}M_*\begin{pmatrix}x,\gamma_1,y,\gamma_2,z\\ \downarrow\\ x,\gamma_1.y,\gamma_1\gamma_2,z\end{pmatrix}(a\times b\times m)\quad .$$

This is equal to

$$M_*\begin{pmatrix}x,\gamma_1,y,\gamma_2,z\\ \downarrow\\ x,\gamma_1.y,(\gamma_1\gamma_2).z\end{pmatrix}(a\times b\times m)$$

which is equal to (8.4). Hence the product is associative.

Finally, if X is any G-set and $m \in M(X)$, then

$$\begin{aligned}\varepsilon_{A_\Gamma} \times_\Gamma m &= M_*\begin{pmatrix}\gamma,x\\ \downarrow\\ \gamma.x\end{pmatrix}\left(A_*\begin{pmatrix}\bullet\\ \downarrow\\ 1\end{pmatrix}(\varepsilon_A)\times m\right)\\ &= M_*\begin{pmatrix}\gamma,x\\ \downarrow\\ \gamma.x\end{pmatrix} M_*\begin{pmatrix}x\\ \downarrow\\ 1,x\end{pmatrix}(m)\\ &= M_*\begin{pmatrix}x\\ \downarrow\\ 1.x\end{pmatrix}(m) = m \quad .\end{aligned}$$

This shows that $\mathsf{i}_\Gamma(M)$ is an A_Γ-module.

Now if $f : M \to N$ is a morphism of A-modules, if X and Y are finite G-sets, if $a \in A_\Gamma(X)$ and $m \in M(Y)$, then

$$\begin{aligned}f_{X\times Y}(a \times_\Gamma m) &= f_{X\times Y} M_*\begin{pmatrix}x,\gamma,y\\ \downarrow\\ x,\gamma.y\end{pmatrix}(a\times m)\\ &= N_*\begin{pmatrix}x,\gamma,y\\ \downarrow\\ x,\gamma.y\end{pmatrix} f_{X\times\Gamma\times Y}(a\times m)\\ &\quad \text{(because } f \text{ is a morphism of Mackey functors)}\\ &= N_*\begin{pmatrix}x,\gamma,y\\ \downarrow\\ x,\gamma.y\end{pmatrix}(a\times f_Y(m))\\ &\quad \text{(because } f \text{ is a morphism of } A\text{-modules)}\\ &= a\times_\Gamma f_Y(m) \quad .\end{aligned}$$

Hence f defines a morphism of A_Γ-modules from $\mathsf{i}_\Gamma(M)$ to $\mathsf{i}_\Gamma(N)$, and the proof of the theorem is complete. □

Proposition 8.5 : *Let Γ be a crossed G-monoid, and A be a Green functor for G over R. If M is an A-module, then the composition $\mathsf{r}_\Gamma \circ \mathsf{i}_\Gamma$ is isomorphic to the identity functor of A-*Mod.

Proof : Let M be an A-module. If X is a finite G-set, then

$$\mathsf{r}_\Gamma \circ \mathsf{i}_\Gamma(M)(X) = M(X)$$

thus $\mathsf{r}_\Gamma \circ \mathsf{i}_\Gamma(M)$ is isomorphic to M as a Mackey functor. Moreover if $f : M \to N$ is a morphism of A-modules, then $\mathsf{r}_\Gamma \circ \mathsf{i}_\Gamma(f) = f$. It remains to check that $\mathsf{r}_\Gamma \circ \mathsf{i}_\Gamma(M)$ is isomorphic to M as an A-module, i.e. that if X and Y are finite G-sets, if $a \in A(X)$ and $m \in M(Y)$, then

$$a\times m = \iota_X(a)\times_\Gamma m \quad .$$

This is equivalent to

$$\begin{aligned}a\times m &= M_*\begin{pmatrix}x,\gamma,y\\ \downarrow\\ x,\gamma.y\end{pmatrix}\left(A_*\begin{pmatrix}x\\ \downarrow\\ x,1_\Gamma\end{pmatrix}(a)\times m\right)\\ &= M_*\begin{pmatrix}x,\gamma,y\\ \downarrow\\ x,\gamma.y\end{pmatrix} M_*\begin{pmatrix}x,y\\ \downarrow\\ x,1_\Gamma,y\end{pmatrix}(a\times m)\\ &= M_*\begin{pmatrix}x,y\\ \downarrow\\ x,1_\gamma.y\end{pmatrix}(a\times m)\\ &= a\times m\end{aligned}$$

which proves the proposition. □

Corollary 8.6 : *The functor* i_Γ *is a full embedding of* A-Mod *into* A_Γ-Mod.

Proof : Since $\mathsf{r}_\Gamma \circ \mathsf{i}_\Gamma$ is isomorphic to the identity functor, the functor i_Γ is faithful. Moreover, the functor r_Γ is clearly faithful : if $f : M \to N$ is a morphism of A_Γ-modules, then for any finite G-set X, the map $\mathsf{r}_\Gamma(f)_X : M(X) \to N(X)$ is equal to the map f_X. Now the functors i_Γ and r_Γ induce maps

$$\mathrm{Hom}_A(M,N) \stackrel{\mathsf{i}_\Gamma}{\to} \mathrm{Hom}_{A_\Gamma}\Big(\mathsf{i}_\Gamma(M), \mathsf{i}_\Gamma(N)\Big) \stackrel{\mathsf{r}_\Gamma}{\to} \mathrm{Hom}_A(M,N)$$

whose composition is the identity. Since the map r_Γ is moreover injective, it follows that both maps are bijections, and the corollary follows. □

9. Centres and centralizers

Recall the following definitions from [2] 6.5 and 12.2 :

Definition 9.1 : *Let* A *be a Green functor for* G *over* R. *If* X *and* Y *are finite* G*-sets, if* $a \in A(X)$ *and* $b \in A(Y)$, *set*

$$a \times^{op} b = A_*\begin{pmatrix} {\scriptstyle y,x} \\ \downarrow \\ {\scriptstyle x,y} \end{pmatrix}(b \times a) \in A(X \times Y) \quad .$$

If M *is a Mackey subfunctor of* A, *define the commutant of* M *in* A *by*

$$C_A(M)(X) = \{a \in A(X) \mid \forall Y,\ \forall m \in M(Y),\ a \times m = a \times^{op} m\}$$

Also define $\zeta_A(X)$ *as the set of natural transformations from the identity functor* $\mathcal{I}$ *of* A-Mod *to the endofunctor* $\mathcal{I}_X$ *of* A-Mod *given by the Dress construction associated to* X.

The commutant of M in A is a Green subfunctor of A ([2] 6.5.3). In section 12.2 of [2], it is shown that ζ_A has a natural structure of Green functor. Its evaluation at the trivial G-set is the center of the category A-Mod, i.e. the set of natural transformations from the identity functor of A-Mod to itself.

Theorem 9.2 : *Let Γ be a crossed G-monoid, and A be a Green functor. Let $C(A,\Gamma)$ denote the commutant of $\iota(A)$ in A_Γ. If X and Y are finite G-sets, if M is an A-module, and if $\alpha \in C(A,\Gamma)(X)$, define a map $z_X(\alpha)_{M,Y} : M(Y) \to M(Y \times X)$ by*

$$z_X(\alpha)_{M,Y}(m) = M_*\begin{pmatrix} x,y \\ \downarrow \\ y,x \end{pmatrix} (\alpha \times_\Gamma m) \quad .$$

Then :

1. *For given X, α and M, the maps $z_X(\alpha)_{M,Y}$ define a morphism of A-modules $z_X(\alpha)_M$ from M to M_X.*
2. *For given X and α, these morphisms $z_X(\alpha)_M$ define an element $z_X(\alpha)$ of $\zeta_A(X)$.*
3. *The maps z_X define a morphism of Green functors z from $C(A,\Gamma)$ to ζ_A.*

Proof : Let $f : Y \to Z$ be a map of G-sets. Then for $m \in M(Y)$

$$\begin{aligned} z_X(\alpha)_{M,Z}\Big(M_*(f)(m)\Big) &= M_*\begin{pmatrix} x,z \\ \downarrow \\ z,x \end{pmatrix} \Big(\alpha \times_\Gamma M_*(f)(m)\Big) \\ &= M_*\begin{pmatrix} x,z \\ \downarrow \\ z,x \end{pmatrix} M_*\begin{pmatrix} x,y \\ \downarrow \\ x,f(y) \end{pmatrix} (\alpha \times_\Gamma m) \\ &\quad \text{(because } \mathsf{i}_\Gamma(M) \text{ is an } A_\Gamma\text{-module)} \\ &= M_*\begin{pmatrix} y,x \\ \downarrow \\ f(y),x \end{pmatrix} M_*\begin{pmatrix} x,y \\ \downarrow \\ y,x \end{pmatrix} (\alpha \times_\Gamma m) \\ &= (M_X)_*(f)\Big(z_X(\alpha)_{M,Y}(m)\Big) \quad . \end{aligned}$$

A similar computation, using the fact that $M_*\begin{pmatrix} x,y \\ \downarrow \\ y,x \end{pmatrix} = M^*\begin{pmatrix} y,x \\ \downarrow \\ x,y \end{pmatrix}$, shows that

$$z_X(\alpha)_{M,Y} \circ M^*(f) = (M_X)^*(f) \circ z_X(\alpha)_{M,Z}$$

hence that the maps $z_X(\alpha)_{M,Y}$ define a morphism of Mackey functors $z_X(\alpha)_M$ from M to M_X.

Now let Y and Z be finite G-sets, let $a \in A(Z)$ and $m \in M(Y)$. Then

$$\begin{aligned} a \times z_X(\alpha)_{M,Y}(m) &= a \times M_*\begin{pmatrix} x,y \\ \downarrow \\ y,x \end{pmatrix} (\alpha \times_\Gamma m) \\ &= M_*\begin{pmatrix} z,x,y \\ \downarrow \\ z,y,x \end{pmatrix} \Big(a \times (\alpha \times_\Gamma m)\Big) \\ &\quad \text{(because } M \text{ is an } A\text{-module)} \\ &= M_*\begin{pmatrix} z,x,y \\ \downarrow \\ z,y,x \end{pmatrix} \Big(\iota_Z(a) \times_\Gamma (\alpha \times_\Gamma m)\Big) \\ &\quad \text{(by proposition 8.5)} \\ &= M_*\begin{pmatrix} z,x,y \\ \downarrow \\ z,y,x \end{pmatrix} \Big((\iota_Z(a) \times_\Gamma \alpha) \times_\Gamma m\Big) \end{aligned}$$

$$
\begin{aligned}
&= M_*\binom{z,x,y}{\downarrow \atop z,y,x}\Big((\iota_Z(a)\times_\Gamma^{op}\alpha)\times_\Gamma m\Big)\\
&\qquad \text{(because } \alpha\in C_{A_\Gamma}(\iota(A))(X))\\
&= M_*\binom{z,x,y}{\downarrow \atop z,y,x}\Big((A_\Gamma)_*\binom{x,z}{\downarrow \atop z,x}(\alpha\times_\Gamma \iota_Z(a))\times_\Gamma m\Big)\\
&= M_*\binom{z,x,y}{\downarrow \atop z,y,x}M_*\binom{x,z,y}{\downarrow \atop z,x,y}\Big((\alpha\times_\Gamma \iota_Z(a))\times_\Gamma m\Big)\\
&\qquad \text{(because } \mathsf{i}_\Gamma(M) \text{ is an } A_\Gamma\text{-module)}\\
&= M_*\binom{z,x,y}{\downarrow \atop z,y,x}M_*\binom{x,z,y}{\downarrow \atop z,x,y}\Big(\alpha\times_\Gamma (\iota_Z(a)\times_\Gamma m)\Big)\\
&= M_*\binom{x,z,y}{\downarrow \atop z,y,x}\Big(\alpha\times_\Gamma (a\times m)\Big)\\
&\qquad \text{(by proposition 8.5)}\\
&= z_X(\alpha)_{M,Z\times Y}(a\times m)
\end{aligned}
$$

which proves assertion 1).

To prove assertion 2), I must check that if $f : M \to N$ is a morphism of A-modules, then for any finite G-set Y, the square

$$
\begin{array}{ccc}
M(Y) & \xrightarrow{f_Y} & N(Y)\\
{\scriptstyle z_X(\alpha)_{M,Y}}\downarrow & & \downarrow{\scriptstyle z_X(\alpha)_{N,Y}}\\
M(Y\times X) & \xrightarrow[f_{Y\times X}]{} & N(Y\times X)
\end{array}
$$

is commutative. But for $m \in M(Y)$

$$
\begin{aligned}
f_{Y\times X}\circ z_X(\alpha)_{M,Y}(m) &= f_{Y\times X}M_*\binom{x,y}{\downarrow \atop y,x}(\alpha\times_\Gamma m)\\
&= N_*\binom{x,y}{\downarrow \atop y,x}f_{X\times Y}(\alpha\times_\Gamma m)\\
&\qquad \text{(because } f \text{ is a morphism of Mackey functors)}\\
&= N_*\binom{x,y}{\downarrow \atop y,x}\Big(\alpha\times_\Gamma f_Y(m)\Big)\\
&\qquad \text{(because } \mathsf{i}_\Gamma(f) \text{ is a morphism of } A_\Gamma\text{-modules)}\\
&= z_X(\alpha)_{N,Y}\circ f_Y(m)
\end{aligned}
$$

as was to be shown.

For assertion 3), first recall the Mackey functor structure on ζ_A ([2] 12.2.1) : if X is a finite G-set and θ is a natural transformation from the identity functor $\mathcal{I}$ of A-Mod to $\mathcal{I}_X$, then θ is characterized by maps of A-modules $\theta_M : M \to M_X$, for each A-module M, i.e. by maps of R-modules $\theta_{M,Y} : M(Y) \to M(Y \times X)$, for any finite G-set Y. Now if $f : X \to Z$ is a map of finite G-sets, then $(\zeta_A)_*(f)(\theta)$ is the natural transformation from $\mathcal{I}$ to $\mathcal{I}_Z$ characterized by the maps

$$(\zeta_A)_*(f)(\theta)_{M,Y} = M_*(Id_Y \times f) \circ \theta_{M,Y} \quad .$$

Similarly, if $\theta' \in \zeta_A(Z)$, then $\zeta_A^*(f)(\theta')$ is the element of $\zeta_A(X)$ defined by

$$\zeta_A^*(f)(\theta') = M^*(Id_Y \times f) \circ \theta'_{M,Y} \quad .$$

Let C denote the Green functor $C(A,\Gamma)$. To show that the maps $z_X : C(X) \to \zeta_A(X)$ define a morphism of Mackey functors, I must check that for any $m \in M(Y)$

$$M_*(Id_Y \times f) \circ z_X(\alpha)_{M,Y}(m) = z_Z\Big(C_*(f)(\alpha)\Big)_{M,Y}(m) \qquad \mathbf{(9.3)}$$

and similarly

$$M^*(Id_Y \times f) \circ z_Z(\alpha)_{M,Y}(m) = z_X\Big(C^*(f)(\alpha)\Big)_{M,Y}(m) \quad . \qquad \mathbf{(9.4)}$$

The left hand side of equation 9.3 is equal to

$$\begin{aligned} M_*(Id_Y \times f) \circ z_X(\alpha)_{M,Y}(m) &= M_*(Id_Y \times f) M_* \left(\begin{smallmatrix} x,y \\ \downarrow \\ y,x \end{smallmatrix}\right) (\alpha \times_\Gamma m) \\ &= M_* \left(\begin{smallmatrix} x,y \\ \downarrow \\ y,f(x) \end{smallmatrix}\right) (\alpha \times_\Gamma m) \\ &= M_* \left(\begin{smallmatrix} z,y \\ \downarrow \\ y,z \end{smallmatrix}\right) M_* \left(\begin{smallmatrix} x,y \\ \downarrow \\ f(x),y \end{smallmatrix}\right) (\alpha \times_\Gamma m) \\ &= M_* \left(\begin{smallmatrix} z,y \\ \downarrow \\ y,z \end{smallmatrix}\right) \Big((A_\Gamma)_*(f)(\alpha) \times_\Gamma m\Big) \\ &\qquad \text{(because } \mathsf{i}_\Gamma(M) \text{ is an } A_\Gamma\text{-module)} \\ &= z_Z\Big(C_*(f)(\alpha)\Big)(m) \end{aligned}$$

as was to be shown, since $C_*(f) = (A_\Gamma)_*(f)$, because C is a subfunctor of A_Γ. The proof of equation 9.4 is similar, using the fact that $M_* \left(\begin{smallmatrix} x,y \\ \downarrow \\ y,x \end{smallmatrix}\right) = M^* \left(\begin{smallmatrix} y,x \\ \downarrow \\ x,y \end{smallmatrix}\right)$.

It remains to check that the maps z_X define a morphism of Green functors from $C = C(A,\Gamma)$ to ζ_A. Recall from [2] 12.2.1 that if X and Y are finite G-sets, if $\theta \in \zeta_A(X)$ and $\psi \in \zeta_A(Y)$, then the product $\theta \times \psi$ is the element of $\zeta_A(X \times Y)$ determined by the maps

$$(\theta \times \psi)_{M,Z} = M_* \left(\begin{smallmatrix} z,y,x \\ \downarrow \\ z,x,y \end{smallmatrix}\right) \circ \theta_{M,Z\times Y} \circ \psi_{M,Z}$$

for any A-module M and any finite G-set Z.

So let $\alpha \in C(X)$ and $\beta \in C(Y)$. Then for $m \in M(Z)$

$$\Big(z_X(\alpha) \times z_Y(\beta)\Big)_{M,Z}(m) = M_*\begin{pmatrix} z,y,x \\ \downarrow \\ z,x,y \end{pmatrix} \circ z_X(\alpha)_{M,Z\times Y} \circ z_Y(\beta)_{M,Z}(m) = \dots$$

$$\begin{aligned} \dots &= M_*\begin{pmatrix} z,y,x \\ \downarrow \\ z,x,y \end{pmatrix} M_*\begin{pmatrix} x,z,y \\ \downarrow \\ z,y,x \end{pmatrix} \Big[\alpha \times_\Gamma M_*\begin{pmatrix} y,z \\ \downarrow \\ z,y \end{pmatrix} (\beta \times_\Gamma m)\Big] \\ &= M_*\begin{pmatrix} x,z,y \\ \downarrow \\ z,x,y \end{pmatrix} \Big[M_*\begin{pmatrix} x,y,z \\ \downarrow \\ x,z,y \end{pmatrix} (\alpha \times_\Gamma \beta \times_\Gamma m)\Big] \\ &\quad \text{(because } \mathsf{i}_\Gamma(M) \text{ is an } A_\Gamma\text{-module)} \\ &= M_*\begin{pmatrix} x,y,z \\ \downarrow \\ z,x,y \end{pmatrix} (\alpha \times_\Gamma \beta \times_\Gamma m) \\ &= z_{X\times Y}(\alpha \times_\Gamma \beta)_{M,Z}(m) \end{aligned}$$

This shows that $z_X(\alpha)\times z_Y(\beta) = z_{X\times Y}(\alpha\times_\Gamma\beta)$. The last verification concerns units : for any $m \in M(Z)$

$$\begin{aligned} z_\bullet(\varepsilon_{A_\Gamma})_{M,Z}(m) &= M_*\begin{pmatrix} \bullet,y \\ \downarrow \\ y,\bullet \end{pmatrix} (\varepsilon_{A_\Gamma} \times_\Gamma m) \\ &= m \end{aligned}$$

since ε_{A_Γ} is the unit of A_Γ. Hence $z_\bullet(\varepsilon_{A_\Gamma})$ is the identity transformation of the identity functor of A-Mod. This completes the proof of the theorem. □

Remark 9.5 : Theorem 9.2 provides in particular a natural ring homomorphism from $C(A,\Gamma)(\bullet)$ to the center of the category A-Mod. One can check from the definitions that

$$C(A,\Gamma)(\bullet) = \{\alpha \in A(\Gamma) \mid \forall X,\, \forall a \in A(X),\, a \times \alpha = A_*\begin{pmatrix} \gamma,x \\ \downarrow \\ \gamma\cdot x,\gamma \end{pmatrix} (\alpha \times x)\}$$

as a subring of $A(\Gamma)$. If A is the Burnside functor B, and $\Gamma = G^c$, then actually $C(A,\Gamma) = A_\Gamma$, and the previous ring homomorphism is the natural morphism from the crossed Burnside ring of G over R to the center of the Mackey algebra of G over R. This morphism leads in particular to a description of the block idempotents of the Mackey algebra ([1]).

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