Green functors

Recall that the letter $R$ denotes a commutative and associative ring with unit, and $G$ denotes a finite group.

1. Two equivalent definitions

There are (at least) two equivalent definitions of Green functors for $G$ over $R$.

1.1. Using the poset of subgroups of $G$. This definition of Green functors uses the poset of subgroups of $G$. It goes back to Green ([7])

Definition 1.1.1 : A Green functor $A$ for $G$ over $R$ is a Mackey functor for $G$ over $R$, together with an $R$-algebra structure on $A(H)$, for each subgroup $H$ of $G$. The Mackey structure and the algebras structures have to be compatible in the following sense :

- If $H \subseteq K$ are subgroups of $G$, and if $x \in G$, the maps $r^K_H$ and $c_{x,H}$ are maps of $R$-algebras.
- (Frobenius relations) If $H \subseteq K$ are subgroups of $G$, if $a \in A(H)$ and $b \in A(K)$, then

$$b(t^K_H a) = t^K_H (r^K_H b) a \quad (t^K_H a) b = t^K_H (a (r^K_H b)) .$$

A morphism of Green functors $f : A \rightarrow B$ is a morphism of Mackey functors such that for each subgroup $H$ of $G$, the map $f_H : A(H) \rightarrow B(H)$ is a map of $R$-algebras.

Remark 1.1.2 : The $R$-algebras considered here are always supposed unital, and the maps of $R$-algebras must preserve identity elements.
1.2. **Using the category of G-sets.** The following definition of Green functors ([3] Section 2.2) is analogous to the Dress definition of Mackey functors.

**Definition 1.2.1 :** A Green functor $A$ for $G$ over $R$ is a Mackey functor for $G$ over $R$ endowed for any $G$-sets $X$ and $Y$ with bilinear maps

$$A(X) \times A(Y) \to A(X \times Y)$$

denoted by $(a, b) \mapsto a \times b$ which are bifunctorial, associative, and unitary, in the following sense:

- **(Bifunctoriality)** If $f : X \to X'$ and $g : Y \to Y'$ are morphisms of $G$-sets, then the squares

$$
\begin{array}{ccc}
A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\
A_*(f) \times A_*(g) & \downarrow & A_*(f \times g) \\
A(X') \times A(Y') & \xrightarrow{\times} & A(X' \times Y')
\end{array}
$$

are commutative.

- **(Associativity)** If $X, Y$ and $Z$ are $G$-sets, then the square

$$
\begin{array}{ccc}
A(X) \times A(Y) \times A(Z) & \xrightarrow{\text{Id}_{A(X)} \times (\times)} & A(X) \times A(Y \times Z) \\
(\times) \times \text{Id}_{A(Z)} & \downarrow & \times \\
A(X \times Y) \times A(Z) & \xrightarrow{\times} & A(X \times Y \times Z)
\end{array}
$$

is commutative, up to identifications $(X \times Y) \times Z \simeq X \times Y \times Z \simeq X \times (Y \times Z)$.

- **(Unitarity)** If $\bullet$ denotes the $G$-set with one element, there exists an element $\varepsilon_A \in A(\bullet)$ such that for any $G$-set $X$ and for any $a \in A(X)$

$$A_*(p_X)(a \times \varepsilon_A) = a = A_*(q_X)(\varepsilon_A \times a)$$

denoting by $p_X$ (resp. $q_X$) the (bijective) projection from $X \times \bullet$ (resp. from $\bullet \times X$) to $X$. 

2
If $A$ and $B$ are Green functors for the group $G$, a morphism of Green functors $f : A \to B$ is a morphism of Mackey functors such that for any $G$-sets $X$ and $Y$, the square

$$
\begin{array}{c}
A(X) \times A(Y) \times X & \to & A(X \times Y) \\
\downarrow f_X \times f_Y & & \downarrow f_{X \times Y} \\
B(X) \times B(Y) \times X & \to & B(X \times Y)
\end{array}
$$

is commutative. The composition of morphisms of Green functors is the composition of morphisms of Mackey functors. The category of Green functors for $G$ over $R$ is denoted by $\text{Green}_R(G)$.

**Remark 1.2.2**: The most concise way to express this definition (Street [9]) uses the monoidal structures on $G\text{-set}$ and $R\text{-Mod}$, given respectively by direct product of $G$-sets and tensor product of $R$-modules. More precisely, let $C : G\text{-set} \times G\text{-set} \to G\text{-set}$ denote the direct product functor, and $T : R\text{-Mod} \times R\text{-Mod} \to R\text{-Mod}$ denote the tensor product functor. A Green functor for $G$ over $R$ is just a Mackey functor for $G$ over $R$, viewed as a bivariant functor from $G\text{-set}$ to $RG\text{-Mod}$, which is monoidal, i.e. endowed with a natural transformation $T \circ A \to A \circ C$ of bivariant functors, which has to be compatible with the various isomorphisms corresponding to the associativity and unit objects of the monoidal structures.

1.3. Equivalence of the definitions.

• $[1 \to 2]$ If $A$ is a Green functor for the first definition, recall that the corresponding Mackey functor in the sense of Dress is defined for a finite $G$-set $X$ by

$$
A(X) = \left( \bigoplus_{x \in X} A(G_x) \right)^G.
$$

If $Y$ is another finite $G$-set, define a product map $A(X) \times A(Y) \to A(X \times Y)$ in the following way: if $u = (u_x)_{x \in X} \in A(X)$ and $v = (v_y)_{y \in Y} \in A(Y)$, then the product $u \times v$ is defined by

$$(u \times v)_{x,y} = r^{G_x}_{G_{x,y}}(u_x) r^{G_y}_{G_{x,y}}(v_y),$$

for $x \in X$ and $y \in Y$, where $G_{x,y} = G_x \cap G_y$. The identity element $\varepsilon$ of $A$ is the identity element of the algebra $A(G) = A(\bullet)$. 

3
• $[2 \to 1]$ If $A$ is a Green functor in the second sense, then recall that $A(H)$ is defined by $A(H) = A(G/H)$ for a subgroup $H$ of $G$. Then the product $A(H) \times A(H) \to A(H)$ is defined by

$$a.b = A^*(\delta_{G/H})(a \times b),$$

where $\delta_{G/H}$ is the diagonal inclusion $G/H \to (G/H) \times (G/H)$. The identity element $\varepsilon_H$ of $A(H)$ is equal to $r_{H}^{G} \varepsilon$, where $\varepsilon$ is the identity element of $A$.

## 2. Examples

### 2.1. Representations rings.

The representation groups associated to subgroups of $G$ have generally a natural ring structure, for which they can be viewed as Green functors:

• If $k$ is a field, then the tensor product of $RH$-modules (over $k$) induces a ring structure on $R_k(H)$, for $H \subseteq G$. The identity element is the image of the trivial module $k$. One can check that $R_k$ is a Green functor for $G$ over $\mathbb{Z}$.

• If $H$ is a subgroup of $G$, then the direct product of $H$-sets gives a ring structure on the Burnside group $B(H)$. Its identity element is the image of the trivial $H$-set $\bullet$. This endows the Burnside functor $B$ with a Green functor structure.

Recall that if $X$ is a finite $G$-set, then $B(X)$ is the Grothendieck group of the category of finite $G$-sets over $X$. If $Y$ is another finite $G$-set, then the product $B(X) \times B(Y) \to B(X \times Y)$ is induced by the obvious product sending a $G$-set $U$ over $X$ and a $G$-set $V$ over $Y$ to their direct product $U \times V$ over $X \times Y$.

More generally, the functor $RB$ is a Green functor for $G$ over $R$. It is an initial object in the category $\text{Green}_R(G)$.

### 2.2. Cohomology rings.

Let $K$ be a subgroup of $G$. The cup product in cohomology defines a ring structure on

$$H^\otimes(K, R) = \bigoplus_{i=0}^{\infty} H^i(K, R),$$

for which $H^\otimes(-, R)$ becomes a Green functor. The subfunctor $H^0(-, R)$ is also a Green functor, usually denoted by $FP_R$. More generally, if $A$ is a $G$-algebra over $R$, the functor $FP_A$ is a Green functor for $G$ over $R$. 

4
3. Modules over a Green functor

Green functors can be viewed as generalized \( R \)-algebras: a Green functor for the trivial group over \( R \) is nothing but an \( R \)-algebra. Similarly, there are (at least) two equivalent definitions of the notion of module over a Green functor:

3.1. Using the poset of subgroups of \( G \).

Definition 3.1.1: Let \( A \) be a Green functor for \( G \) over \( R \). A module \( M \) over \( A \) (or an \( A \)-module) is a Mackey functor for \( G \) over \( R \), together with a structure of \( A(H) \)-module on \( M(H) \), for each subgroup \( H \) of \( G \). The Mackey structure and the module structures have to be compatible in the following sense:

- If \( H \subseteq K \) are subgroups of \( G \), and if \( x \in G \), then
  \[
  \forall a \in A(K), \forall m \in M(K), \quad r^K_H(a)(am) = r^K_H(a)r^K_H(m),
  \]
  \[
  \forall a \in A(H), \forall a \in M(H), \quad c_{x,H}(am) = c_{x,H}(a)c_{x,H}(m).
  \]

- (Frobenius relations) If \( H \subseteq K \) are subgroups of \( G \), then
  \[
  \forall a \in A(K), \forall m \in M(H), \quad a(t^K_H(m)) = t^K_H\left((r^K_Ha)m\right),
  \]
  \[
  \forall a \in A(H), \forall m \in M(K), \quad (t^K_Ha)m = t^K_H\left(a(r^K_Hm)\right).
  \]

A morphism of \( A \)-modules Green functors \( f : M \to N \) is a morphism of Mackey functors such that for each subgroup \( H \) of \( G \), the map \( f_H : M(H) \to N(H) \) is a map of \( A(H) \)-modules.

3.2. Using the category of \( G \)-sets.

Definition 3.2.1: A module \( M \) over the Green functor \( A \) for \( G \) over \( R \) is a Mackey functor for \( G \) over \( R \), endowed for any \( G \)-sets \( X \) and \( Y \) with bilinear maps

\[
A(X) \times M(Y) \to M(X \times Y)
\]

denoted by \((a, m) \mapsto a \times m\) which are bifunctorial, associative, and unitary, in the following sense:
• (Bifunctoriality) If $f : X \to X'$ and $g : Y \to Y'$ are morphisms of $G$-sets, then the squares

$$
\begin{array}{ccc}
A(X) \times M(Y) & \times & M(X \times Y) \\
A_*(f) \times M_*(g) & \downarrow & M_*(f \times g) \\
A(X') \times M(Y') & \times & M(X' \times Y') \\
\end{array}
$$

are commutative.

• (Associativity) If $X$, $Y$ and $Z$ are $G$-sets, then the square

$$
\begin{array}{cc}
A(X) \times A(Y) \times M(Z) & \xrightarrow{(\times)} A(X) \times M(Y \times Z) \\
(\times) \times Id_{A(Z)} & \downarrow \\
A(X \times Y) \times M(Z) & \times M(X \times Y \times Z) \\
\end{array}
$$

is commutative, up to identifications $(X \times Y) \times Z \simeq X \times Y \times Z \simeq X \times (Y \times Z)$.

• (Unitality) For any $G$-set $X$ and for any $m \in M(X)$

$$A_*(q_X)(\varepsilon_A \times m) = m$$

denoting by $q_X$ the (bijective) projection from $\bullet \times X$ to $X$.

A morphism of $A$-modules $f : M \to N$ is a morphism of Mackey functors such that for any $G$-sets $X$ and $Y$, the square

$$
\begin{array}{ccc}
A(X) \times M(Y) & \times & M(X \times Y) \\
Id \times f_Y & \downarrow & f_{XXY} \\
A(X) \times N(Y) & \times & N(X \times Y) \\
\end{array}
$$

is commutative. The composition of morphisms of $A$-modules is the composition of morphisms of Mackey functors. The category of $A$-modules is denoted by $A$-Mod.
3.3. Examples.

- Let $V$ be an $RG$-module. Then the functor $FP_V$ is a module over the Green functor $FP_R$. More generally, if $l \in \mathbb{N}$, the cohomology functor $H^l(-, V)$ and the homology functor $H_1(-, V)$ are $FP_R$-modules.

Conversely, it has been observed by Puig that the $FP_R$-modules can be characterized among the Mackey functor over $R$ as the cohomological Mackey functors (recall that it means that $t^K_{R}^{H}$ is equal to the multiplication by the index $|K : H|$, for any subgroups $H \subseteq K$ of $G$).

- Let $M$ and $N$ be Mackey functors for $G$ over $R$. If $X$ is a finite $G$-set, define

$$\mathcal{H}(M, N)(X) = \text{Hom}_{Mack_R(G)}(M, N_X) .$$

This definition can be extended to give a natural Mackey functor structure on $\mathcal{H}(M, N)$. In the case $M = N$, there is similarly a natural Green functor structure on $\mathcal{H}(M, M)$. If $A$ is any Green functor for $G$ over $R$, it is equivalent to give $M$ a structure of $A$-module, or to give a morphism of Green functors $A \to \mathcal{H}(M, M)$ ([3] 2.1.2).

Since the Green functor $RB$ is an initial object in the category $\text{Green}_R(G)$, it follows that the category of Mackey functors for $G$ over $R$ is equivalent to the category of $RB$-modules.

- The previous example shows that the category $Mack_R(G)$ admits an internal Hom construction. There is also an internal tensor product in $Mack_R(G)$: if $M$ and $N$ are Mackey functors for $G$ over $R$, this construction gives another Mackey functor $M \hat{\otimes} N$ for $G$ over $R$. The value of $M \hat{\otimes} N$ at the finite $G$-set $X$ can be defined as

$$(M \hat{\otimes} N)(X) = \hat{M} \otimes_{\mu_R(G)} \hat{N}_X ,$$

where $\hat{M}$ is the $\mu_R(G)$-module associated to $M$, viewed as a right $\mu_R(G)$-module using the anti-automorphism of the Mackey algebra.

This construction is functorial in $M$ and $N$, and there is an adjunction

$$\text{Hom}_{Mack_R(G)}(M \hat{\otimes} N, P) \cong \text{Hom}_{Mack_R(G)}(N, \mathcal{H}(M, P)) .$$

- If $A$ is a Green functor for $G$ over $R$, the opposite Green functor $A^{op}$ is the Mackey functor $A$, equipped with the opposite product $\times^{op}$, defined for finite $G$-sets $X$ and $Y$ by

$$\forall a \in A(X), \forall b \in A(Y), a \times^{op} b = A_* \left( \frac{y \cdot x}{z} \right) (b \times a) .$$
where \( \left( \frac{y,x}{x,y} \right) \) is the map from \( Y \times X \) to \( X \times Y \) sending \((y,x)\) to \((x,y)\). The identity element of \( A^{op} \) is the identity element of \( A \).

There is an obvious notion of right module over a Green functor, and the category of right modules over \( A \) is equivalent to the category of left modules over \( A^{op} \).

If \( A \) and \( B \) are Green functors for \( G \) over \( R \), there is also an obvious notion of \((A,B)\)-bimodule, or \( A \)-module-\( B \). Moreover the tensor product \( A \widehat{\otimes} B^{op} \) has a natural structure of Green functor, with canonical Green functor homomorphisms \( A \rightarrow A \widehat{\otimes} B^{op} \) and \( B^{op} \rightarrow A \widehat{\otimes} B^{op} \), and the category of \( A \)-modules-\( B \) is equivalent to the category of \( A \widehat{\otimes} B^{op} \)-modules.

- Let \( A \) be a Green functor. Then \( A \) is an \( A \)-module-\( A \). If \( M \) is an \( A \)-module, and if \( X \) is a finite \( G \)-set, there is a natural structure of \( A \)-module on the Dress construction \( M_X \). This gives an endofunctor of the category \( A \text{-Mod} \), which is self adjoint. Moreover, if \( M \) is an \( A \)-module, then

\[
\text{Hom}_{A \text{-Mod}}(A_X, M) \cong M(X) \ .
\]

In particular the modules \( A_X \) are projective \( A \)-modules, and the module \( A_\Omega \) is a progenator of the category \( A \text{-Mod} \).

### 4. Associated categories and algebras

#### 4.1. The category associated to a Green functor

Let \( A \) be a Green functor for \( G \) over \( R \). Let \( C_A \) denote the following category :

- The objects of \( C_A \) are the finite \( G \)-sets.
- If \( X \) and \( Y \) are finite \( G \)-sets, then

\[
\text{Hom}_{C_A}(X, Y) = A(Y \times X) \ .
\]

- If \( X, Y, \) and \( Z \) are finite \( G \)-sets, then the composition of the morphisms \( f \in A(Y \times X) \) and \( g \in A(Z \times Y) \) in \( C_A \) is the element \( g \circ f \) of \( A(Z \times X) \) defined by

\[
g \circ f = A_{*} \left( \frac{z,y,x}{z,x} \right) A^{*} \left( \frac{z,y,x}{z,x,y} \right) (g \times f) \ ,
\]

where \( \left( \frac{z,y,x}{z,x} \right) \) is the map from \( Z \times Y \times X \) to \( Z \times X \) sending \( (z,y,x) \) to \( (z,x) \), and \( \left( \frac{z,y,x}{z,y} \right) \) is the map from \( Z \times Y \times X \) to \( Z \times Y \times Y \times X \) sending \( (z,y,x) \) to \( (z,y,y,x) \).
• The identity morphism of the finite $G$-set $X$ is the element

$$A_* \left( \begin{array}{c} x \\ x \end{array} \right) A^* \left( \begin{array}{c} x \\ x \end{array} \right) (\varepsilon)$$

of $A(X \times X)$, where $\left( \begin{array}{c} x \\ x \end{array} \right)$ is the diagonal inclusion from $X$ to $X \times X$, and $\left( \begin{array}{c} x \\ x \end{array} \right)$ is the unique map from $X$ to the trivial $G$-set $\mathbf{1}$.

**Proposition 4.1.1**: [[3] 3.3.5] The category of $A$-modules is equivalent to the category of $R$-linear functors from $C_A$ to $R$-Mod.

4.2. **The algebra associated to a Green functor.** Since $A_\Omega$ is a pro-generator of the category $A$-Mod, it follows that $A$-Mod is equivalent to the category of modules over the algebra

$$\mu(A) = \text{End}_{A\text{-Mod}}(A_\Omega),$$

which is also isomorphic to the algebra $\text{End}_{c_A}(\Omega) = A(\Omega \times \Omega)$. The algebra $\mu(A)$ can also be defined by generators and relations using the following result:

**Proposition 4.2.1**: Let $A$ be a Green functor for the group $G$. Then $\mu(A)$ is isomorphic to the $R$-algebra defined by the following generators and relations:

- **The generators are:**
  - The elements $t_k^H$ and $r_K^H$, for $K \subseteq H \subseteq G$.
  - The elements $c_{x,H}$ for $x \in G$ and $H \subseteq G$.
  - The elements $\lambda_{K,a}$ for $K \subseteq G$ and $a \in A(K)$.

- **The relations are:**
  - The relations of the Mackey algebra for $r_K^H$, $t_K^H$, and $c_{x,H}$, i.e.
    $$t_K^H t_L^H = t_L^H, \quad r_K^H r_L^H = r_L^H \quad \forall L \subseteq K \subseteq H$$
    $$c_{y,z,H} c_{x,H} = c_{y,z,H} \quad \forall x, y, H$$
    $$t_K^H = r_K^H = c_{h,H} \quad \forall h \in H$$
    $$c_{x,H} t_K^H = t_K^H c_{x,K}, \quad c_{x,H} r_K^H = r_K^H c_{x,H} \quad \forall x, K, H$$
    $$\sum_H t_K^H = \sum_H r_K^H = 1$$
    $$r_K^H t_L^H = \sum_{x \in K \cap L/L} t_K^{K \cap L} c_{x,K \cap L} r_L^{K \cap L} \quad \forall K \subseteq H \supseteq L$$

the other products of $r_K^H$, $t_K^H$ and $c_{g,H}$ being zero.
5. Simple modules and simple Green functors

5.1. Simple modules. The classification and description of simple Mackey functors can be generalized to an arbitrary Green functor, in the following form ([3] Chapter 11):

**Notation 5.1.1:** Let $A$ be a Green functor for $G$ over $R$. If $H$ is a subgroup of $G$, denote by $\overline{A}(H)$ the Brauer quotient of $A$ in $H$, defined by

$$\overline{A}(H) = A(H)/\sum_{K \subseteq H} t^K_A(K),$$

and denote by $a \mapsto \overline{a}$ the projection map from $A(H)$ to $\overline{A}(H)$. This map induces an $R$-algebra structure on $\overline{A}(H)$, together with a natural action of the group $\overline{N}_G(H)$. Denote by $\hat{A}(H)$ the semi-direct product $\overline{A}(H) \otimes R \overline{N}_G(H)$.

Let $V$ be a simple $\hat{A}(H)$-module. If $X$ is a finite $G$-set, then $R(X^H)$ is a $\overline{N}_G(H)$-module. Set

$$S_{H,V}(X) \cong \mathcal{T}_{\overline{N}_G(H)}^{\overline{A}(H)} \left( \text{Hom}_R(R(X^H), V) \right)$$

If $f : X \rightarrow Y$ is a morphism of finite $G$-sets, then for $\alpha \in S_{H,V}(X)$ and $y \in Y^H$, set

$$S_{H,V}(f)(\alpha)(y) = \sum_{x \in X^H, f(x) = y} \alpha(x)$$

If $g : Y \rightarrow X$ is a morphism of finite $G$-sets, then for $\alpha \in S_{H,V}(X)$ and $y \in Y^H$, set

$$S_{H,V}^*(g)(\alpha)(y) = \alpha(g(y))$$
Finally if \( a \in A(X) \) and \( f \in S_{H,V}(Y) \), then define a morphism \( a \times f \) from \( R\left((X \times Y)^H\right) = R(X^H \times Y^H) \) to \( V \) by

\[
(a \times f)(x, y) = \left( A^*(m_x)(a) \otimes 1 \right) f(y)
\]

where \( m_x \) is the morphism of \( G \)-sets from \( G/H \) to \( X \) defined by \( m_x(uH) = ux \).

One can show that these definitions give an \( A \)-module structure on \( S_{H,V} \). Moreover:

**Proposition 5.1.2:** Let \( A \) be a Green functor for the group \( G \).

1. If \( S \) is a simple \( A \)-module, and \( H \) is a minimal subgroup for \( S \), then \( V = S(H) \) is a simple \( A(H) \)-module, and \( S \) is isomorphic to \( S_{H,V} \).

2. Conversely, if \( H \) is a subgroup of \( G \), and \( V \) is a simple \( A(H) \)-module, then \( S_{H,V} \) is a simple \( A \)-module, the group \( H \) is minimal for \( S_{H,V} \), and moreover \( S_{H,V}(H) \cong V \).

3. Let \( H \) and \( K \) be subgroups of \( G \). If \( V \) is a simple \( A(H) \)-module, and if \( W \) is a simple \( A(K) \)-module, then the modules \( S_{H,V} \) and \( S_{K,W} \) are isomorphic if and only if the pairs \( (H,V) \) and \( (K,W) \) are conjugate under \( G \).

**5.2. Simple Green functors.** A simple Green functor is a Green functor \( A \) for \( G \) over \( R \), such that \( A \) is simple as \( A \)-module-\( A \) (or \( A \otimes A^{op} \)-module). This is equivalent to requiring that \( A \) has no non-trivial functorial two-sided ideal, in the sense of Thévenaz ([10]). The simple Green functors can be described using the following result (for details, see [3] 11.5):

**Proposition 5.2.1:** [Thévenaz [10] Theorem 12.11]

1. Let \( A \) be a simple Green functor for \( G \). Then there exists a subgroup \( M \) of \( G \), a normal subgroup \( H \) of \( M \), and a simple algebra \( S \) on which \( M/H \) acts projectively, such that

\[
A \simeq \text{Ind}_M^G \text{Inf}_{M/H}^M FP_S
\]

The triple \((M,H,S)\) is unique up to conjugation by \( G \) (and up to isomorphism of \( M/H \)-algebras for \( S \)).

2. Conversely, if \( H \trianglelefteq M \) are subgroups of \( G \), if \( S \) is a simple algebra on which \( M/H \) acts projectively, then \( \text{Ind}_M^G \text{Inf}_{M/H}^M FP_S \) is a simple Green functor.
6. An example of related construction


**Definition 6.1.1**: A $G$-monoid is a monoid endowed with a left $G$-action by monoid automorphisms. A $G$-monoid which is a group is called a $G$-group. A morphism of $G$-monoids is a $G$-equivariant monoid homomorphism.

**Example 6.1.2**: Denote by $G^c$ the set $G$ on which the group $G$ acts by conjugation. Then the multiplication map $G^c \times G^c \to G^c$ endows $G^c$ with a structure of $G$-group. More generally, if $N$ is a normal subgroup of $G$, then $G$ acts on $N$ by conjugation, and $N$ is a $G$-group for this action.

**Definition 6.1.3**: A crossed $G$-monoid is a pair $(\Gamma, \varphi)$, where $\Gamma$ is a $G$-monoid, and $\varphi : \Gamma \to G^c$ is a morphism of $G$-monoids. A morphism $\theta : (\Gamma, \varphi) \to (\Gamma', \varphi')$ of crossed $G$-monoids is a morphism of $G$-monoids $\theta : \Gamma \to \Gamma'$ such that $\varphi' \circ \theta = \varphi$.

6.2. Associated Green functors.

**Proposition 6.2.1**: [4,1] Let $(\Gamma, \varphi)$ be a crossed $G$-monoid. If $A$ is a Green functor for $G$ over $R$, let $A_{\Gamma}$ denote the Mackey functor obtained by the Dress construction from the $G$-set $\Gamma$. If $X$ and $Y$ are finite $G$-sets, define a product map $\times_{\Gamma} : A_{\Gamma}(X) \otimes_R A_{\Gamma}(Y) \to A_{\Gamma}(X \times Y)$ by

$$\forall a \in A_{\Gamma}(X), \forall b \in A_{\Gamma}(Y), \quad a \otimes b \mapsto a \times_{\Gamma} b = A_{\ast} \left( \begin{array}{c} x, \varphi(y), y' \\ x, \varphi(y), y' \end{array} \right) \left( a \times b \right),$$

where $\left( \begin{array}{c} x, \varphi(y), y' \\ x, \varphi(y), y' \end{array} \right)$ is the map from $X \times \Gamma \times Y \times \Gamma$ to $X \times Y \times \Gamma$ sending $(x, \gamma, y, \gamma')$ to $(x, \varphi(\gamma) y, \gamma')$. Let moreover $\varepsilon_{A_{\Gamma}}$ denote the element $A_{\ast} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) (\varepsilon_A)$ of $A(\Gamma) \cong A_{\Gamma}(\bullet)$, where $\left( \begin{array}{c} 1 \\ 1 \end{array} \right)$ is the map sending the unique element of $\bullet$ to the identity element of $\Gamma$.

Then $A_{\Gamma}$ is a Green functor for $G$ over $R$, and the correspondence $A \mapsto A_{\Gamma}$ is an endo-functor of the category $\text{Green}_R(G)$.

In particular, the evaluation $A_{\Gamma}(\bullet) \cong A(\Gamma)$ of $A_{\Gamma}$ at the trivial $G$-set is an $R$-algebra. One can express the product formula for this algebra in the decomposition $A(\Gamma) \cong \left( \bigoplus_{\gamma \in \Gamma} A(G_{\gamma}) \right)^G$.

12
Proposition 6.2.2: Let $a$ and $b$ be elements of $A(\Gamma)$. Then for $\gamma \in \Gamma$, the $\gamma$ component of $a \times_\Gamma b$ is given by

$$(a \times_\Gamma b)_{\gamma} = \sum_{(a, b) \in G_\gamma \setminus \Gamma \times \Gamma} \epsilon_{G_{\alpha, \beta}}^{G_{\gamma}} (r_{G_{\alpha, \beta}} a \cdot r_{G_{\alpha, \beta}} b)_{\beta}.$$ 

6.3. Examples.

- Let $B$ denote the Burnside Green functor, and let $\Gamma = G^c$. Then the ring $B(\Gamma) = B(G^c)$ is called the crossed Burnside ring. It is the Grothendieck ring of the monoidal category of crossed $G$-sets, i.e. $G$-sets over $G^c$. This ring has been studied by Yoshida (see also [2]).

- Let $A$ denote the cohomology Green functor $H^\ast (\cdot, R)$, and let $\Gamma = G^c$. Then one can show that the algebra $A(\Gamma)$ is isomorphic to the Hochschild cohomology algebra of the group algebra $RG$. In this case, the above product formula has been conjectured by Cibils ([5]) and Cibils and Solotar ([6]), and proved by Siegel and Witherspoon ([8]).

- (Cibils) Let $A$ denote the Grothendieck ring of the category of finitely generated $RG$-modules, for relations given by direct sum decompositions. If $\Gamma = G^c$, then the ring $A(\Gamma)$ is isomorphic to the Grothendieck ring of Hopf bimodules for the Hopf algebra $RG$.

References


