Title : Gluing endo-permutation modules

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Abstract : In this paper, I show that if \( p \) is an odd prime, and if \( P \) is a finite \( p \)-group, then there exists an exact sequence of abelian groups

\[
0 \rightarrow T(P) \rightarrow D(P) \rightarrow \lim_{1 < Q \leq P} D(N_P(Q)/Q) \rightarrow H^1(A_{\geq 2}(P), \mathbb{Z})^{(P)},
\]

where \( D(P) \) is the Dade group of \( P \) and \( T(P) \) is the subgroup of endo-trivial modules.

Here \( \lim_{1 < Q \leq P} D(N_P(Q)/Q) \) is the group of sequences of compatible elements in the Dade groups \( D(N_P(Q)/Q) \) for non trivial subgroups \( Q \) of \( P \). The poset \( A_{\geq 2}(P) \) is the set of elementary abelian subgroups of rank at least 2 of \( P \), ordered by inclusion.

The group \( H^1(A_{\geq 2}(P), \mathbb{Z})^{(P)} \) is the subgroup of \( H^1(A_{\geq 2}(P), \mathbb{Z}) \) consisting of classes of \( P \)-invariant 1-cocycles.

A key result to prove that the above sequence is exact is a characterization of elements of \( 2D(P) \) by sequences of integers, indexed by sections \( (T, S) \) of \( P \) such that \( T/S \cong (\mathbb{Z}/p\mathbb{Z})^2 \), fulfilling certain conditions associated to subquotients of \( P \) which are either elementary abelian of rank 3, or extraspecial of order \( p^3 \) and exponent \( p \).

AMS Subject classification : 20C20

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where $D(P)$ is the Dade group of $P$ and $T(P)$ is the subgroup of endo-trivial modules. Here $\lim_{1<Q \leq P} D(N_P(Q)/Q)$ is the group of sequences of compatible elements in the Dade groups $D(N_P(Q)/Q)$ for non trivial subgroups $Q$ of $P$. The poset $A_{\geq 2}(P)$ is the set of elementary abelian subgroups of rank at least 2 of $P$, ordered by inclusion. The group $H^1\left(A_{\geq 2}(P), \mathbb{Z}\right)^{(P)}$ is the subgroup of $H^1\left(A_{\geq 2}(P), \mathbb{Z}\right)$ consisting of classes of $P$-invariant 1-cocycles.

A key result to prove that the above sequence is exact is a characterization of elements of $2D(P)$ by sequences of integers, indexed by sections $(T, S)$ of $P$ such that $T/S \cong (\mathbb{Z}/p\mathbb{Z})^2$, fulfilling certain conditions associated to subquotients of $P$ which are either elementary abelian of rank 3, or extraspecial of order $p^3$ and exponent $p$.

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1. Introduction

The classification of all endo-permutation modules for finite $p$-groups has been completed recently, thanks to the work of several authors (see in particular [1], [11], [12], [5], [3]). This paper addresses the question of gluing arbitrary endo-permutation modules, and it is intended to be a complement to our previous joint work with Jacques Thévenaz ([7]), where the case of torsion endo-permutation modules was handled.

The gluing problem is the following: let $p$ be an odd prime, let $P$ be a finite $p$-group, and let $k$ be a field of characteristic $p$. If $v$ is an element of the Dade group $D(P)$ of endo-permutation $kP$-modules, and if $Q$ is a non trivial subgroup of $P$, denote by $v_Q$ the image of $v$ by the deflation-restriction map $\text{Defres}^P_{N_P(Q)/Q}$. Then the $v_Q$'s are subject to some obvious compatibility conditions. Conversely, if $Q$ is a non-trivial subgroup of $P$, let $u_Q$ be an element of the Dade group $D_k\left(N_P(Q)/Q\right)$, and assume that these compatibility conditions between the $u_Q$'s are fulfilled. Is there an element $u \in D(P)$ such that for any non trivial subgroup $Q$ of $P$

$$\text{Defres}^P_{N_P(Q)/Q}(u) = u_Q?$$

Such an element $u$ is called a solution to the gluing problem for the gluing data $(u_Q)_{1<Q\leq P}$. When $P$ is abelian, the gluing problem was completely solved by Puig [16] (see also Lemma 2.3 below), and he used the result to construct suitable stable equivalences between blocks.

The main result of the present paper is that if $p$ is an odd prime, and if $P$ is a finite $p$-group, then there exists an exact sequence of abelian groups

$$0 \rightarrow T(P) \rightarrow D(P) \rightarrow \lim_{1<Q \leq P} D(N_P(Q)/Q) \xrightarrow{h_P} H^1\left(A_{\geq 2}(P), \mathbb{Z}\right)^{(P)},$$
where \( D(P) \) is the Dade group of \( P \) and \( T(P) \) is the subgroup of endo-trivial modules. Here \( \lim_{1 \leq Q \leq P} D(N_P(Q)/Q) \) is the group of gluing data for \( P \), i.e. the group of sequences of compatible elements in the Dade groups \( D(N_P(Q)/Q) \) for non trivial subgroups \( Q \) of \( P \). The poset \( A_{\geq 2}(P) \) is the set of elementary abelian subgroups of rank at least 2 of \( P \), ordered by inclusion. The group \( H^1(A_{\geq 2}(P), \mathbb{Z})^{(P)} \) is the subgroup of \( H^1(A_{\geq 2}(P), \mathbb{Z}) \) consisting of classes of \( P \)-invariant 1-cocycles.

The main consequence of this result is that if \( H^1(A_{\geq 2}(P), \mathbb{Z}) = \{0\} \), then the gluing problem always has a solution. Unfortunately, the map \( h_P \) is not surjective in general, so when \( H^1(A_{\geq 2}(P), \mathbb{Z}) \neq \{0\} \), much can be said at the time for the gluing problem. In Section 6, the example of the extraspecial group of order \( p^5 \) and exponent \( p \) is described in details. In this case, the group \( H^1(A_{\geq 2}(P), \mathbb{Z})^{(P)} \) is a free group of rank \( p^4 \), and the image of \( h_P \) has finite index in this group. In particular it is non zero, and the gluing problem does not always have a solution.

It could be true in general that \( h_P \) always has finite cokernel, and this would be enough to show that if \( H^1(A_{\geq 2}(P), \mathbb{Z}) \neq \{0\} \), then the image of \( h_P \) is non zero, hence that the gluing problem does not always have a solution : it is known indeed that the group \( H^1(A_{\geq 2}(P), \mathbb{Z})^{(P)} \) is a free abelian group, since the poset \( A_{\geq 2}(P) \) has the homotopy type of a wedge of spheres (see [8]).

1.1. Notation. Throughout this paper, the symbol \( p \) denotes an odd prime number, and \( P \) denotes a finite \( p \)-group. Inclusion of subgroups will be denoted by \( \leq \), and strict inclusion by \( < \). Inclusion up to \( P \)-conjugation will be denoted by \( \leq_P \).

A section \((T, S)\) of \( P \) is a pair of subgroups of \( P \) with \( S \leq T \). The factor group \( T/S \) is the corresponding subquotient of \( P \). If \((T, S)\) is a section of \( P \), then \( N_P(T, S) \) denotes \( N_P(T) \cap N_P(S) \).

A class \( \mathcal{Y} \) of \( p \)-groups is said to be closed under taking subquotients if for any \( Y \in \mathcal{Y} \) and any section \((T, S)\) of \( Y \), any group isomorphic to \( T/S \) belongs to \( \mathcal{Y} \). If \( \mathcal{Y} \) is such a class, and \( P \) is a finite \( p \)-group, let \( \mathcal{Y}(P) \) be the set of sections \((T, S)\) of \( P \) such that \( T/S \in \mathcal{Y} \).

The symbol \( X_p^3 \) denotes an extraspecial \( p \)-group of order \( p^5 \) and exponent \( p \). The symbol \( X_3 \) denotes the class of \( p \)-groups which are either elementary abelian of rank at most 3, or isomorphic to \( X_p^3 \). Thus, the symbol \( X_3(P) \) denotes the set of sections \((T, S)\) of \( P \) such that \( T/S \) is elementary abelian of rank at most 3, or isomorphic to \( X_p^3 \). Let moreover \( E_3^2(P) \) denote the set of sections \((T, S)\) of \( P \) such that \( T/S \cong (\mathbb{Z}/p\mathbb{Z})^2 \).

If \( P \) is a finite \( p \)-group, and \( k \) is a field of characteristic \( p \), let \( D(P) \) denote the Dade group of endo-permutation \( kP \)-modules. The field \( k \) does not appear in this notation, because it turns out that \( D(P) \) is independent of \( k \), at least when \( p \) is odd (see [3] Theorem 9.5 for details).

When \((T, S)\) is a section of \( P \), there is a deflation-restriction map \( \text{Defres}^P_{T/S} : D(P) \to D(T/S) \), which is the group homomorphism obtained by composing the restriction map \( \text{Res}^P_T : D(P) \to D(T) \), followed by the deflation map \( \text{Def}^P_{T/S} : D(T) \to D(T/S) \).

Recall that if \( X \) is a finite \( P \)-set, there is a corresponding element \( \Omega_X \) of the Dade group of \( P \), called the syzygy of the trivial module relative to \( X \) (or the \( X \)-relative syzygy for short) : it is defined as the class of the kernel of the augmentation map \( kX \to k \) when this does make sense, and by 0 otherwise (see e.g. [2] for details). When \( X \) is the set \( P \) itself, on which \( P \) acts by multiplication, the corresponding element will be denoted by \( \Omega_{P/1} \) or \( \Omega_P \).
1.2. Contents. This paper is organized as follows:

- In Section 2, I state the main theorem (Theorem 2.15), and this requires in particular the definition of some objects and maps between them.
- Section 3 recalls some notation on biset functors, forgetful functors between categories of biset functors, and corresponding adjoint functors.
- Section 4 is devoted to the main tool (Theorem 4.5) used in the proof of Theorem 2.15, namely a characterization by linear equations of the image of the group $2D(P)$ by the deflation-restriction maps to all subquotients $T/S$ of $P$ which are elementary abelian of rank 2. This characterization may be a result of independent interest.
- Section 5 exposes the proof of Theorem 2.15.
- Finally, Section 6 focuses on the example of the extraspecial $p$-group of order $p^5$ and exponent $p$: the reason for choosing this particular group is twofold: it is one of the smallest $p$-groups $P$ for which $H^1(A_{\geq 2}(P), Z) \neq \{0\}$, and moreover the Dade group of this $p$-group is rather well known, thanks to our joint work with Nadia Mazza ([5]).

2. Statement of the main theorem

2.1. Notation. If $P$ is a finite $p$-group, then $A_{\geq 2}(P)$ denotes the poset of elementary abelian subgroups of $P$ of rank at least 2. Let $A_{=}2(P)$ denote the set of elementary abelian subgroups of rank 2 of $P$.

Recall that if the $p$-rank of $P$ is at least equal to 3, then all the elementary abelian subgroup of $P$ of rank at least 3 are in the same connected component of $A_{\geq 2}(P)$. This component is called the big component. It is obviously invariant under $P$-conjugation. Each of the other connected components, if there are any, consists of a single maximal elementary abelian subgroup of rank 2.

2.2. Notation. Denote by $\lim_{\leftarrow} \frac{D(NP(Q)/Q)}{1<Q\leq P}$ indexed by non trivial subgroups of $P$, where $u_Q \in D(NP(Q)/Q)$, such that:

- If $x \in P$, then $xu_Q = u_{xQ}$.
- If $Q \not\cong R$, then $\text{Defres}_{NP(Q,R)/R}^{NP(Q,R)}u_Q = \text{Res}_{NP(Q,R)/R}^{NP(R,R)}u_R$.

Denote by $\tau_P : D(P) \rightarrow \lim_{1<Q\leq P} D(NP(Q)/Q)$ the map sending $v \in D(P)$ to the sequence $(\text{Defres}_{NP(Q)/Q}^{NP(Q,R)}u_{Q})_{1<Q\leq P}$.

If $E$ is an abelian $p$-group, denote by $\sigma_E$ the map $\lim_{1<F\leq E} D(E/F) \rightarrow D(E)$ defined by $\sigma_E(u) = - \sum_{1<F\leq E} \mu(1,F) \text{Inf}_{E/F}^{E}u_F$, where $\mu$ is the Möbius function of the poset of subgroups of $P$.

It has been shown by Puig ([15] 2.1.2) that the kernel of $\tau_P$ is equal to the group $T(P)$ of endo-trival modules. Moreover, when $E$ is an abelian group, the map $\tau_E$ is surjective ([16] Proposition 3.6). More precisely:
2.3. Lemma. Let $E$ be an abelian $p$-group. Then $\sigma_E$ is a section of $r_E$, i.e. $r_E\sigma_E$ is equal to the identity map of $\lim_{1<F \leq E} D(E/F)$.

**Proof.** Let $1 < G \leq E$. Then

$$\text{Def}_{E/G}^{E} \sigma_E(u) = - \sum_{1<F \leq E} \mu(1, F) \text{Def}_{E/G}^{E} \text{Inf}_{E/F}^{E} u_{F}$$

$$= - \sum_{1<F \leq E} \mu(1, F) \text{Inf}_{E/FG}^{E} \text{Def}_{E/G}^{E} u_{FG}$$

$$= - \sum_{R \leq E} \text{Inf}_{E/R}^{E} \text{Def}_{E/G}^{E} u_{R} \cdot \sum_{G \leq R \leq E} \mu(1, F)\mu(1, F) = -1.$$

Now if $G < R$

$$\sum_{1<F \leq R} \mu(1, F) = \sum_{1<F \leq R} \mu(1, F),$$

and this is equal to zero, by a classical combinatorial lemma, since $G \neq 1$. And if $G = R$

$$\sum_{1<F \leq R} \mu(1, F) = -\mu(1, 1) + \sum_{1<F \leq R} \mu(1, F) = -1.$$

Thus $\text{Def}_{E/G}^{E} \sigma_E(u) = \text{Inf}_{E/G}^{E} u_G = u_G$, as was to be shown. 

2.4. Lemma. Let $E$ be an elementary abelian group of rank at least 2. Then the map $r_E$ is surjective, and its kernel is the free abelian group of rank one generated by $\Omega_{1}^{E}$. 

**Proof.** The kernel of $r_E$ is the group $T(E)$ of endo-trivial modules. Since $E$ is elementary abelian, this group is free of rank one, generated by $\Omega_{1}^{E}$, by Dade’s Theorem ([13] [14]). The surjectivity of $r_E$ follows from Lemma 2.3.

2.5. Restriction and conjugation. The following construction has been introduced in [7], for the torsion subgroup of the Dade group, but it works as well for the whole Dade group: let $P$ be a finite $p$-group, and $H$ be a subgroup of $P$. If $u \in \lim_{1<Q \leq P} D(N_{P}(Q)/Q)$, then the sequence $(v_{Q})_{1<Q \leq H}$ defined by

$$v_{Q} = \text{Res}_{N_{H}(Q)/Q}^{N_{P}(Q)/Q} u_{Q}$$

is an element of $\lim_{1<Q \leq H} D(N_{H}(Q)/Q)$, denoted by $\text{Res}_{H}^{P} u$. The map $u \mapsto \text{Res}_{H}^{P} u$ is a linear map $\lim_{1<Q \leq P} D(N_{P}(Q)/Q) \to \lim_{1<Q \leq Q} D(N_{Q}(Q)/Q)$. The following is the analogue of Lemma 2.4 of [7]:
2.6. Lemma. Let $H$ be a subgroup of $P$. The diagram

$$
\begin{array}{ccc}
D(P) & \xrightarrow{r_P} & \lim_{1 < Q \leq P} D(N_P(Q)/Q) \\
\downarrow \text{Res}_H^P & & \downarrow \text{Res}_H^P \\
D(H) & \xrightarrow{r_H} & \lim_{1 < Q \leq H} D(N_H(Q)/Q)
\end{array}
$$

is commutative.

Proof. This is straightforward.

Similarly, if $x \in P$, denote by $c_{x,H} : D(H) \to D(xH)$ the conjugation by $x$, sending $v$ to $xv$. If $u \in \lim_{1 < Q \leq H} D(N_H(Q)/Q)$, then the sequence $(u_R)_{1 < R \leq xH}$ defined by $v_R = c_{x,R}(u_R)$ is an element of $\lim_{1 < R \leq xH} D(N_H(R)/R)$, that will be denoted by $xu$. The assignment $u \mapsto xu$ is a linear map from $\lim_{1 < Q \leq H} D(N_H(Q)/Q)$ to $\lim_{1 < R \leq xH} D(N_H(R)/R)$, also denoted by $c_{x,H}$.

2.7. Lemma. Let $H$ be a subgroup of $P$, and let $E$ be an abelian subgroup of $P$. The following diagrams are commutative:

$$
\begin{array}{ccc}
D(H) & \xrightarrow{r_H} & \lim_{1 < Q \leq H} D(N_H(Q)/Q) \\
\downarrow \text{c}_{x,H} & & \downarrow \text{c}_{x,E} \\
D(xH) & \xrightarrow{r_{xH}} & \lim_{1 < R \leq xH} D(N_{xH}(R)/R)
\end{array}
\quad
\begin{array}{ccc}
D(E) & \xrightarrow{\sigma_E} & \lim_{1 < R \leq E} D(E/R) \\
\downarrow \text{c}_{x,E} & & \downarrow \text{c}_{x,E} \\
D(xE) & \xrightarrow{\sigma_{xE}} & \lim_{1 < R \leq xE} D(E/R)
\end{array}
$$

Proof. This is also straightforward.

2.8. Construction of a map. Let $E$ and $F$ be elements of $A_{\geq 2}(P)$ such that $E < F$. If $v \in \lim_{1 < Q \leq P} D(N_P(Q)/Q)$, consider the element

$$
d_{E,F} = \text{Res}_E^F \sigma_F \text{Res}_F^P v - \sigma_E \text{Res}_E^P v
$$

of $D(E)$. Then by Lemma 2.4 and Lemma 2.3

$$
r_E(d_{E,F}) = r_E \text{Res}_E^F \sigma_F \text{Res}_F^P v - r_E \sigma_E \text{Res}_E^P v \\
= \text{Res}_E^F \sigma_F \text{Res}_F^P v - r_E \sigma_E \text{Res}_E^P v \\
= \text{Res}_E^F \text{Res}_F^P v - \text{Res}_E^P v = 0.
$$

By Lemma 2.4, there exists a unique integer $w_{E,F}$ such that

$$
d_{E,F} = w_{E,F} \cdot \Omega_{E/1}.
$$
If \( x \in P \), then it is clear from Lemma 2.7 that \( x d_{E,F} = d_{E,F} \), and it follows that \( w_{E,F} = w_{E,F} \). Moreover, if \( E, F, G \in A_{2}(P) \) with \( E < F < G \), then
\[
d_{E,F} + \text{Res}_{E}^{F}d_{F,G} = d_{E,G},
\]
hence \( w_{E,F} + w_{F,G} = w_{E,G} \). In other words the function sending the pair \((E, F)\) of elements of \( A_{2}(P) \), with \( E < F \), to \( w_{E,F} \), is a \( P \)-invariant 1-cocycle on \( A_{2}(P) \), with values in \( \mathbb{Z} \):

**2.9. Notation.** Let \( P \) be a finite \( p \)-group. A \( P \)-invariant 1-cocycle on \( A_{2}(P) \), with values in \( \mathbb{Z} \), is a function sending a pair \((E, F)\) of elements of \( A_{2}(P) \), with \( E < F \), to an integer \( w_{E,F} \), with the following two properties:

1. If \( x \in P \) and \( E < F \) in \( A_{2}(P) \), then \( w_{x,E,F} = w_{E,F} \).

2. If \( E < F < G \) in \( A_{2}(P) \), then \( w_{E,F} + w_{F,G} = w_{E,G} \).

The set \( \left( Z^{1}(A_{2}(P)) \right)^{P} \) of \( P \)-invariants 1-cocycles is a group for addition of functions.

Denote by \( \left( B^{1}(A_{2}(P)) \right)^{P} \) the subgroup of \( \left( Z^{1}(A_{2}(P)) \right)^{P} \) consisting of cocycles \( w \) for which there exists a \( P \)-invariant function \( E \mapsto m_{E} \) from \( A_{2}(P) \) to \( \mathbb{Z} \) such that
\[
\forall E < F \in A_{2}(P), \ w_{E,F} = m_{F} - m_{E}.
\]
Denote by \( H^{1}(A_{2}(P), \mathbb{Z})^{(P)} \) the factor group \( \left( Z^{1}(A_{2}(P)) \right)^{P} / \left( B^{1}(A_{2}(P)) \right)^{P} \).

**2.10. Remark:** One can show that the group \( \left( B^{1}(A_{2}(P)) \right)^{P} \) is also equal to the set of elements \( w \in \left( Z^{1}(A_{2}(P)) \right)^{P} \) for which there exists a (not necessarily \( P \)-invariant) function \( E \mapsto m_{E} \) such that \( w_{E,F} = m_{F} - m_{E} \) for any \( E < F \) in \( A_{2}(P) \). This is because if \( E < F \) in \( A_{2}(P) \), then \( E \) and \( F \) are the “big component”, which is \( P \)-invariant. Since \( w \) is \( P \)-invariant, it follows that the function
\[
x \in P \mapsto m_{x} - m_{E}
\]
does not depend on the choice of \( E \), and that it is a group homomorphism from \( P \) to \( \mathbb{Z} \) (i.e. an element of \( H^{1}(P, \mathbb{Z}) \)). There are no non zero such homomorphisms, so \( m \) is actually \( P \)-invariant.

**2.11. Remark:** On the other hand, one can consider the ordinary first cohomology group \( H^{1}(A_{2}(P), \mathbb{Z}) \) of \( A_{2}(P) \) over \( \mathbb{Z} \), which is defined similarly to \( H^{1}(A_{2}(P), \mathbb{Z})^{(P)} \), but forgetting all conditions of \( P \)-invariance. Then the group \( P \) acts on \( H^{1}(A_{2}(P), \mathbb{Z}) \), and it follows from Remark 2.10 that \( H^{1}(A_{2}(P), \mathbb{Z})^{(P)} \) is a subgroup of the group \( H^{1}(A_{2}(P), \mathbb{Z}) \) of \( P \)-invariant elements in \( H^{1}(A_{2}(P), \mathbb{Z}) \). It might happen however that this inclusion is proper : an argument similar to the one used in Remark 2.10 yields an element in \( H^{2}(P, \mathbb{Z}) \), and this group need not be zero.

**2.12. Notation.** Let \( P \) be a finite \( p \)-group. Denote by
\[
h_{P} : \lim_{1 \leq Q \leq P} D(N_{P}(Q)/Q) \to H^{1}(A_{2}(P), \mathbb{Z})^{(P)}
\]
the map sending \( v \in \lim_{1 \leq Q \leq P} D(N_P(Q)/Q) \) to the class of the 1-cocycle \( w \) defined by the 
following equality, for \( E < F \) in \( \mathcal{A}_{\geq 2}(P) \):

\[
(2.13) \quad w_{E,F} \cdot \Omega_{E/1} = \text{Res}_E^F \sigma_F \text{Res}_F^P v - \sigma_E \text{Res}_E^P v.
\]

2.14. Proposition. Let \( P \) be a finite \( p \)-group. Then \( h_P \) is a group homomorphism, 
and the composition

\[
D(P) \xrightarrow{r_P} \lim_{1 < Q \leq P} D(N_P(Q)/Q) \xrightarrow{h_P} H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)}
\]

is equal to 0.

Proof. Clearly, the definition of \( h_P \) implies that it is a group homomorphism. Observe 
next that for any \( E \in \mathcal{A}_{\geq 2}(P) \), since \( r_E \sigma_E \) is the identity map, 
the image of the map \( \sigma_E r_E - \text{Id}_{D(E)} \) is contained in the kernel of \( r_E \). By Lemma 2.4, it follows that there 
is a unique linear form \( s_E \) on \( D(E) \), with values in \( \mathbb{Z} \), such that 

\[
\sigma_E r_E(u) = u + s_E(u) \cdot \Omega_{E/1},
\]

for any \( u \in D(E) \). By Lemma 2.7, this definition clearly implies that if \( x \in P \), then 
\( s_E(x^u) = s_E(u) \), for any \( u \in D(E) \).

Now if \( E < F \) in \( \mathcal{A}_{\geq 2}(P) \), and if \( v = r_P(t) \), for \( t \in D(P) \), Equation 2.13 becomes

\[
w_{E,F} \cdot \Omega_{E/1} = \text{Res}_E^F \sigma_F \text{Res}_F^P r_P(t) - \sigma_E \text{Res}_E^P r_P(t)
\]

\[
= \text{Res}_E^F \sigma_F \text{Res}_F^P t - \sigma_E \text{Res}_E^P t \quad \text{(by Lemma 2.6)}
\]

\[
= \text{Res}_E^F (\text{Res}_F^P t + s_F(\text{Res}_F^P t) \cdot \Omega_{F/1}) - (\text{Res}_E^P t + s_E(\text{Res}_E^P t) \cdot \Omega_{E/1})
\]

\[
= (s_F(\text{Res}_F^P t) - s_E(\text{Res}_E^P t)) \cdot \Omega_{E/1}.
\]

Setting \( m_E = s_E(\text{Res}_E^P t) \), for \( E \in \mathcal{A}_{\geq 2}(P) \), yields

\[
w_{E,F} = m_F - m_E,
\]

hence \( w \in \left( B^1(\mathcal{A}_{\geq 2}(P)) \right)^P \) (the \( P \)-invariance of \( m \) follows easily from the above remark, 
and from Remark 2.10). Thus \( h_P r_P(u) = 0 \), as was to be shown.

The main theorem of this paper is the following:

2.15. Theorem. Let \( P \) be a finite \( p \)-group. Then the sequence of abelian groups

\[
0 \longrightarrow T(P) \longrightarrow D(P) \xrightarrow{r_P} \lim_{1 < Q \leq P} D(N_P(Q)/Q) \xrightarrow{h_P} H^1(\mathcal{A}_{\geq 2}(P), \mathbb{Z})^{(P)}
\]

is exact.

The key point in this theorem is to show that the kernel of \( h_P \) is equal to the image 
of \( r_P \). This will be done in two steps: first take an element \( u \in \ker h_P \), and show that 
\( 2u \in r_P(2D(P)) \). This amounts to replacing \( D \) by \( 2D \), which is easier to handle, since 
it is torsion free. Next, write \( 2u = r_P(2v) \), for some \( v \in D(P) \). Then \( u - r_P(v) \) is an 
element in \( \lim_{1 < Q \leq P} D_t(N_P(Q)/Q) \), and it has been shown in [7] that such a sequence 
of compatible torsion elements can always be glued (i.e. it always lies in \( r_P(D(P)) \)), 
though possibly not in \( r_P(D_t(P)) \).
3. Biset functors

The main ingredient in the proof of Theorem 2.15 is the formalism of biset functors. A short exposition of the notation and main results on this subject can be found in Section 2 of [9], Section 3 of [4], or Section 3 of [3].

Recall in particular that if \((T, S)\) is a section of the group \(P\), and if \(M\) is a biset functor, then the set \(P/S\) is a \((P, T/S)\)-biset, and the corresponding induction-inflation morphism

\[
M(P/S) : M(T/S) \to M(P)
\]

is denoted by \(\text{Ind}_{P \times T/S} M\). Similarly, the set \(S \setminus P\) is a \((T/S, P)\)-biset, and the corresponding deflation-restriction map

\[
M(S \setminus P) : M(P) \to M(T/S)
\]

is denoted by \(\text{Def}_{P \times T/S} M\).

This notation was already used for the Dade group, and it is coherent: it was shown more generally in [6] that, if \(P\) and \(Q\) are finite \(p\)-groups, and \(U\) is a finite \((Q, P)\)-biset, one can define a natural group homomorphism \(\text{Dade} : \text{Dade}(U) \to \text{Dade}(Q)\). This construction yields a structure of biset functor on the correspondence sending a \(p\)-group \(P\) to the subgroup \(\text{Dade}(P)\) of \(\text{Dade}(P)\) generated by all the relative syzygies \(\Omega_X\) obtained for various \(P\)-sets \(X\). In the case \(p \neq 2\), it was shown in [3] that \(\text{Dade} = \text{Dade}(\Omega_X)\), so \(\text{Dade}\) is a biset functor in this case. For \(p = 2\), the biset functor structure on \(\text{Dade}(\Omega_X)\) cannot in general be extended to the whole of \(\text{Dade}\), because of Frobenius twists (also called Galois twists) (see [6] or [3] for details).

The following additional notation was introduced in [9]: a class \(Y\) of \(p\)-groups is said to be closed under taking subquotients if for any \(Y \in Y\) and any section \((T, S)\) of \(Y\), the corresponding subquotient \(T/S\) belongs to \(Y\). If \(Y\) is such a class, and \(P\) is a finite \(p\)-group, denote by \(Y(P)\) the set of sections \((T, S)\) of \(P\) such that \(T/S \in Y\). One can consider biset functors defined only on \(Y\), with values in abelian groups. Let \(\mathcal{F}_Y\) be the category of all such functors, and let \(\mathcal{F}\) denote the category of functors defined on all finite \(p\)-groups. These categories are abelian categories.

The obvious forgetful functor

\[
\mathcal{O}_Y : \mathcal{F} \to \mathcal{F}_Y
\]

admits left and right adjoints

\[
\mathcal{L}_Y : \mathcal{F}_Y \to \mathcal{F} \quad \text{and} \quad \mathcal{R}_Y : \mathcal{F}_Y \to \mathcal{F},
\]

whose evaluations can be computed as follows (cf. [9] Theorem 1.2 for details, in particular on the direct and inverse limits appearing in this statement):

**3.1. Theorem.** With the notation above, for any functor \(M \in \mathcal{F}_Y\), we have:

\[
\mathcal{L}_Y M(P) \cong \lim_{(T, S) \in Y(P)} M(T/S) \quad \text{and} \quad \mathcal{R}_Y M(P) \cong \lim_{(T, S) \in Y(P)} M(T/S).
\]

Moreover ([9] Corollary 6.17), for any biset functor \(M\) and any \(p\)-group \(P\), the unit map

\[
\eta_{M,P} : M(P) \to \mathcal{R}_Y M(P) = \lim_{(T, S) \in Y(P)} M(T/S)
\]

is given by

\[
\eta_{M,P}(u)_{T,S} = \text{Def}_{T/S}^P u,
\]

for any \(u \in M(P)\) and any \((T, S) \in Y(P)\).
4. Image in subquotients of rank two

4.1. Lemma. Let $X$ be a non-empty class of finite $p$-groups, closed under taking subquotients. Let $A$ be any abelian group, and denote by $B$ the functor $\text{Hom}_\mathbb{Z}(B(-), A)$, where $B$ is the Burnside functor. Then the unit map

$$\beta : \hat{B} \to \mathcal{R}_X \mathcal{O}_X \hat{B}$$

is an isomorphism.

Proof. Indeed if $P$ is a $p$-group, if $\varphi \in \text{Hom}_\mathbb{Z}(B(P), A)$, if $(T, S)$ is a section of $P$ and $X$ is a subgroup such that $S \leq X \leq T$, then

$$(\text{Defres}_{P/(S \varphi)}((T/S)/(X/S)) = \varphi(\text{Indinf}_{T/(S/X)}(T/S)/(X/S)) = \varphi(P/X).$$

Suppose that $\varphi \in \text{Ker} \beta$. Since the class $X$ is non empty and closed under taking subquotients, then it contains the trivial group, and for any subgroup $X$ of $P$, the section $(X, X)$ is in $X(P)$. It follows in particular that

$$0 = (\text{Defres}_{X/X}^P((X/X)/(X/X)) = \varphi(P/X).$$

Thus $\varphi = 0$, and $\beta$ is injective.

Conversely, let $\psi = (\psi_{T,S}(T,S)_{(T,S)\in X(P)}$ be an element of $\varprojlim_{X(P)} \text{Hom}_\mathbb{Z}(B(T/S), A)$.

Equivalently, for each section $(T, S) \in X(P)$ and each subgroup $X$ with $S \leq X \leq T$, we have an element $\psi_{T,S}(T/S)/(X/S)$ of $A$, fulfilling the following two conditions:

i) If $x \in P$, then $\psi_{xT,S}(x^2T/(x^2S)/(x^2S)) = \psi_{T,S}(T/S)/(X/S))$.

ii) If $(T, S) \in X(P)$ and $(T', S') \in X(P)$, and if $X$ is a subgroup of $P$ such that $S \leq S' \leq X \leq T' \leq T$, then $\psi_{T,S}(T/S)/(X/S)) = \psi_{T',S'}(T'/S')/(X/S')$.

Condition i) implies in particular that the element $\psi_{X,X}((X/X)/(X/X)))$ is constant on the conjugacy class of $X$ in $P$. Hence we can define an element $\varphi \in \text{Hom}_\mathbb{Z}(B(P), A)$ by setting

$$\varphi(P/X) = \psi_{X,X}((X/X)/(X/X))),$$

for any subgroup $X$ of $P$. Now if $(T, S) \in X(P)$ and if $S \leq X \leq T$

$$(\text{Defres}_{T/S}^P((T/S)/(X/S)) = \varphi(P/X) = \psi_{X,X}((X/X)/(X/X))$$

$$= (\text{Defres}_{X/X}^{T=S}((X/X)/(X/X))$$

$$= \psi_{T,S}(\text{Indinf}_{X/X}(T/S)/(X/X))$$

$$= \psi_{T,S}((T/S)/(X/S))).$$

Thus $\text{Defres}_{T/S}^P = \psi_{T,S}$ for any $(T, S) \in X(P)$. Equivalently $\beta(\varphi) = \psi$, so $\beta$ is surjective, hence it is an isomorphism.

4.2. Lemma. When $p$ is odd, the map

$$\varepsilon = \prod_{(T, S) \in X_3(P)} \text{Defres}_{T/S}^P : 2D(P) \longrightarrow \varprojlim_{(T, S) \in X_3(P)} 2D(T/S)$$

is an isomorphism.
**Proof.** Consider the short exact sequence of $p$-biset functors

$$0 \rightarrow 2D \rightarrow D \rightarrow \mathbb{F}_2D \rightarrow 0.$$  

Applying the functor $\lim_{\mathcal{X}_3(P)}$ yields the bottom line of the following commutative diagram with exact lines

$$0 \rightarrow 2D(P) \rightarrow D(P) \rightarrow \mathbb{F}_2D(P) \rightarrow 0$$

(4.3)

$$0 \rightarrow \lim_{\mathcal{X}_3(P)} 2D \rightarrow \lim_{\mathcal{X}_3(P)} D \rightarrow \lim_{\mathcal{X}_3(P)} \mathbb{F}_2D$$

where the map $\delta$ is an isomorphism, by Theorem 1.1 of [9]. Moreover, by Corollary 1.5 of [10], there is an exact sequence of $p$-biset functors

$$0 \rightarrow B^\times \rightarrow \mathbb{F}_2B^\star \rightarrow \mathbb{F}_2D^\Omega \rightarrow 0,$$

where $B^\times$ is the functor of units of the Burnside ring, which is isomorphic to the constant functor $\Gamma_{\mathcal{X}_2}$ for $p$ odd. Moreover $D^\Omega = D$ in this case. Applying the functor $\lim_{\mathcal{X}_3(P)}$ to this sequence yields the bottom line of the following commutative diagram with exact lines

(4.4)

$$0 \rightarrow \mathbb{F}_2 \rightarrow \mathbb{F}_2B^\star \rightarrow \mathbb{F}_2D \rightarrow 0$$

$$0 \rightarrow \lim_{\mathcal{X}_3(P)} \Gamma_{\mathcal{X}_2} \rightarrow \lim_{\mathcal{X}_3(P)} \mathbb{F}_2B^\star \rightarrow \lim_{\mathcal{X}_3(P)} \mathbb{F}_2D$$

Since $\mathbb{F}_2B^\star$ is naturally isomorphic to $\text{Hom}_2(B(-), \mathbb{F}_2)$, the map $\beta$ is an isomorphism, by Lemma 4.1. Now the group $\lim_{\mathcal{X}_3(P)} \Gamma_{\mathcal{X}_2}$ is the set of sequences $(u_{T,S})_{(T,S)\in \mathcal{X}_3(P)}$ fulfilling the two following conditions:

1. If $(T, S) \in \mathcal{X}_3(P)$ and $x \in P$, then $u_{xT,xS} = u_{T,S}$.
2. If $(T, S) \in \mathcal{X}_3(P)$ and $(T', S') \in \mathcal{X}_3(P)$ are such that $S \leq S' \leq T' \leq T$, then $u_{T',S'} = u_{T,S}$.

Applying this for the case $T' = S' = S$, and next for the case $T = T' = S'$, it follows that $u_{T,S} = u_{S,S} = u_{T,T}$ for any $(T, S) \in \mathcal{X}_3(P)$. Thus $u_{T,T} = u_{S,S}$ if $T/S \in \mathcal{X}_3$. Since $\mathcal{X}_3$ contains the cyclic group of order $p$, and since $P$ is a $p$-group, it follows that $u_{T,T} = u_{1,1}$ for any subgroup $T$ of $P$. Hence $u_{T,S}$ is constant, and the map $\alpha$ is an isomorphism.

Now the Snake’s Lemma, applied to Diagram 4.4, shows that the map $\gamma$ is injective. And another application of this Lemma to Diagram 4.3 shows that the map $\varepsilon$ is an isomorphism.

Recall that if $E$ is an elementary abelian group of rank 2, then $2D(E)$ is free of rank one, generated by $2\Omega_{E/1}$. Thus if $u \in 2D(P)$, and if $(T, S) \in E^\times_2(P)$, then $\text{Defres}_{T/S}^pu$ is a multiple of $2\Omega_{T/S}$. The following theorem characterizes the sequences of integers $(u_{T,S})_{(T,S)\in E^\times_2(P)}$ which can be obtained that way from an element of $2D(P)$:
4.5. Theorem. Let $P$ be a $p$-group (with $p > 2$). The map

$$ \mathcal{D}_P : 2D(P) \to \prod_{(T,S) \in E_2^3(P)} \mathbb{Z} $$

sending an element $u \in 2D(P)$ to the sequence $\mathcal{D}_P(u)_{T,S}$ of integers defined by

$$ \text{Defres}^P_{T/S}(u) = \mathcal{D}_P(u)_{T,S} \cdot 2\Omega_{T/S}, $$

is injective, and its image is equal to the set of sequences $(v_{T,S})_{(T,S) \in E_2^3(P)}$ fulfilling the following conditions:

1. If $(T, S) \in E_2^3(P)$ and $x \in P$, then $v_{T,S} = v_{T,S}.x$.
2. If $(T, S)$ and $(T', S)$ are in $E_2^3(P)$, if $T' \leq N_P(T)$, and $TT'/S \cong (\mathbb{Z}/p\mathbb{Z})^3$, then
   $$ v_{T,S} + \sum_{S < X < T} v_{TT'X} = v_{T',S} + \sum_{S < X < T'} v_{TT'X}. $$
3. If $(T, S)$ and $(T', S)$ are in $E_2^3(P)$, if $T' \leq N_P(T)$, and $TT'/S \cong X_p$, then
   $$ v_{T,S} \equiv v_{T',S} \pmod{p}. $$

Proof. Let $u \in \text{Ker} \mathcal{D}_P$. Then $\text{Defres}^P_{T/S}u = 0$ for any $(T, S) \in E_2^3(P)$. Moreover $\text{Defres}^P_{T/S}u \in 2D(\mathbb{Z}/p\mathbb{Z}) = \{0\}$ when $(T, S)$ is a section of $P$ with $T/S \cong \mathbb{Z}/p\mathbb{Z}$. The detection theorem of Carlson and Thévenaz ([11] Theorem 13.1) shows that $u = 0$. Hence $\mathcal{D}_P$ is injective.

Now Condition 1 of Theorem 4.5 holds obviously for the elements of the image of $\mathcal{D}_P$. To check that Conditions 2 and 3 also hold, suppose that $(T, S), (T', S) \in E_2^3(P)$, with $T' \leq N_P(T)$ and $|TT'/P| = p^3$, and observe that setting $R = TT'$, the diagram

$$ \begin{array}{ccc}
D(P) & \xrightarrow{\mathcal{D}_P} & \prod_{(T''',S''') \in E_2^3(P)} \mathbb{Z} \\
\text{Defres}^P_{R/S} \downarrow \quad & & \downarrow \pi_{R,S} \\
D(R/S) & \xrightarrow{\mathcal{D}_P} & \prod_{(T''',S'''/S) \in E_2^3(R/S)} \mathbb{Z}
\end{array} $$

is commutative, where $\pi_{R,S}$ is the projection map obtained by identifying sections $(T''', S''')$ of $P$ such that $S \leq S''' \leq T''' \leq R$ with sections $(T''/S, S''''/S)$ of $R/S$. This shows that it is enough to suppose that the group $P$ is either elementary abelian of rank $3$, or isomorphic to $X_p$. These special cases are detailed below.

To prove that conversely, Conditions 1, 2, and 3 characterize the image of $\mathcal{D}_P$, observe first that by Theorem 1.1 of [9], the map

$$ \prod_{(T,S)} \text{Defres}^P_{T/S} : D(P) \to \lim_{(T,S)} D(T/S) $$

is an isomorphism, where $T/S$ runs through all sections of $P$ which are either elementary abelian $p$-groups of rank $\leq 3$ or extraspecial groups of order $p^3$ and exponent $p$. 
Suppose that Theorem 4.5 is true when \( P \) is elementary abelian of rank at most 3, or extraspecial of exponent \( p \). Let \( P \) be an arbitrary \( p \)-group, and consider a sequence \( v = (v_{T,S})_{(T,S) \in \mathcal{E}_2^3(P)} \) fulfilling the conditions of Theorem 4.5.

If \((V,U) \in X_3(P)\), then the correspondence \((T,S) \mapsto (T/U,S/U)\) is a one to one correspondence between the set of elements \((T,S)\) of \( \mathcal{E}_2^3(P) \) such that \( U \leq S \leq T \leq V \), and \( \mathcal{E}_2^3(V/U) \). Through this bijection, the sequence of integers \( v_{T,S} \), for \( U \leq S \leq T \leq V \), yields a sequence of integers fulfilling the conditions of Theorem 4.5 for the group \( V/U \). In other words, there is a unique element \( w_{V,U} \in 2D(V/U) \) such that

\[
\text{Defres}_{T/S}^{V/U} w_{V,U} = v_{T,S} \cdot 2\Omega_{T/S},
\]

for all \((T,S) \in \mathcal{E}_2^3(P)\) with \( U \leq S \leq T \leq V \).

Now the uniqueness of \( w_{V,U} \) shows that \( w_{V,x,U} = w_{V,U} \) for any \( x \in P \), and that \( \text{Defres}_{V/U} \cdot w_{V,U} = w_{V',U'} \) whenever \((V,U)\) and \((V',U')\) are in \( X_3(P)\) and \( U \leq U' \leq V' \leq V \). In other words, the sequence \( (w_{V,U})_{(V,U) \in X_3(P)} \) is an element of \( \text{lim} \leftarrow X_3(P) \). By Lemma 4.2, there exists an element \( t \in 2D(P) \) such that

\[
w_{V,U} = \text{Defres}_{V/U}^P t,
\]

for any \((V,U) \in X_3(P)\). Then obviously \( D_P(t) = v \), and \( v \) lies in the image of \( D_P \), as was to be shown.

So the only thing left to check is that Theorem 4.5 holds when \( P \) is elementary abelian of rank at most 3, or isomorphic to \( X_{p^3} \). This is a case by case verification, using the following lemma:

**4.6. Lemma.** Let \( P \) be a finite \( p \)-group, and \( X \) be a finite \( P \)-set.

1. If \( T/S \) is a section of \( P \), then

\[
\text{Defres}_{T/S}^P \Omega_X = \Omega_{X^S},
\]

where \( X^S \) denotes the set of fixed points of \( S \) on \( X \), viewed as a \( T/S \)-set.

2. If moreover \((T,S) \in \mathcal{E}_2^3(P)\), then

\[
\text{Defres}_{T/S}^P(2\Omega_X) = \left( \sum_{S \leq V \leq T \atop X^V \neq \emptyset} \mu(S,V) \right) 2\Omega_{T/S}.
\]

In other words \( D_P(2\Omega_X)_{T,S} = \sum_{S \leq V \leq T \atop X^V \neq \emptyset} \mu(S,V). \)

**Proof.** Assertion 1 follows from Section 4 of [2]. For Assertion 2, note that by Assertion 1 and Lemma 5.2.3 of [2], since \( T/S \) is abelian,

\[
\text{Defres}_{T/S}^P 2\Omega_X = \sum_{S \leq U \leq V \leq T \atop X^V \neq \emptyset} \mu(U,V) \cdot 2\Omega_{T/U},
\]

and that \( 2\Omega_{T/U} = 0 \) in \( D(T/S) \) unless \( U = S \).
Now there are four cases:

- If $|P| \leq p$, there is nothing to do, since the map $D_P$ is an isomorphism $\{0\} \to \{0\}$.

- If $P \cong (\mathbb{Z}/p\mathbb{Z})^2$, then $2D(P) \cong \mathbb{Z}$, and $E_2^2(P) = \{(P, 1)\}$. In this case, there is no condition on the image of $D_P$, and $D_P$ is an isomorphism $\mathbb{Z} \to \mathbb{Z}$. So Theorem 4.5 holds in this case.

- If $P \cong (\mathbb{Z}/p\mathbb{Z})^3$, then $E_2^2(P)$ consists of $p^2 + p + 1$ sections $(P, R)$, for $|R| = p$, and $p^2 + p + 1$ sections $(Q, 1)$, for $|Q| = p^2$. The group $2D(P)$ is a free abelian group, with basis

$$\{2\Omega_{P/1} \cup \{2\Omega_{P/R} \mid |R| = p\} \}.$$

The following arrays gives the values of the sequence $v = D_P(u)$ for the element $u$ in its first column on the left:

<table>
<thead>
<tr>
<th>$2\Omega_{P/1}$</th>
<th>$v_{P,R}$</th>
<th>$v_{Q,1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$v_{P,R}$</td>
<td>$v_{Q,1}$</td>
</tr>
<tr>
<td></td>
<td>$m_R$</td>
<td>$m_1 + \sum_{R \leq Q} m_R$</td>
</tr>
</tbody>
</table>

The image of the element $u = m_1 \cdot 2\Omega_{P/1} + \sum_{|R| = p} m_R \cdot 2\Omega_{P/R}$ by the map $D_P$ is equal to the sequence $v = (v_{T,S})$, where

$$v_{P,R} = m_R \quad v_{Q,1} = m_1 + \sum_{R \leq Q} m_R$$

If $Q \neq Q'$ are subgroups of order $p^2$ of $P$, then $QQ' = P$, and

$$v_{Q,1} + \sum_{1 < X < Q} v_{P,X} = m_1 + \sum_{|X| = p} m_X = v_{Q',1} + \sum_{1 < X < Q'} v_{P,X},$$

so Condition 2 of 4.5 holds for the sections $(Q, 1)$ and $(Q', 1)$ of $P$. Since $P$ is abelian, Conditions 1 and 3 of 4.5 are obviously satisfied.

Conversely, suppose that Condition 2 hold for a sequence $v = (v_{T,S})_{(T,S) \in E_2^2(P)}$. This sequence is in the image of $D_P$ if and only if there exist integers $m_1$, $m_R$, for $|R| = p$, such that 4.7 hold.

The first equation gives $m_R = v_{P,R}$, and then the second one gives

$$m_1 = v_{Q,1} - \sum_{R \leq Q} v_{P,R}.$$

This is consistent if the right hand side does not depend on $Q$, i.e. if for any subgroups $Q \neq Q'$ of order $p^2$ of $P$

$$v_{Q,1} - \sum_{R \leq Q} v_{P,R} = v_{Q',1} - \sum_{R \leq Q'} v_{P,R},$$

or equivalently

$$v_{Q,1} + \sum_{R < Q} v_{P,R} = v_{Q',1} + \sum_{R < Q'} v_{P,R}.$$

This is precisely Condition 2 of 4.5 for the section $(Q, 1)$ and $(Q', 1)$, since $QQ' = P$ in this case. Thus Theorem 4.5 holds for $P \cong (\mathbb{Z}/p\mathbb{Z})^3$. 

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• If \( P \cong X_{p^2} \), then \( E_2^0(P) \) consists of the section \( (P, Z) \), where \( Z \) is the centre of \( P \), and of \( p + 1 \) sections \( (Q, 1) \), where \( Q \) is a subgroup of index \( p \) in \( P \). The group \( D(P) \) is equal to \( D^0(P) \), since \( p \neq 2 \), so it is generated by the elements \( \Omega_{P/1}, \Omega_{P/X} \), for \( |X| = p \), and \( \Omega_{P/Q} \), for \( |Q| = p^2 \), which have order 2 in \( D(P) \). Thus \( 2D(P) \) is generated by the elements \( 2\Omega_{P/1} \) and \( 2\Omega_{P/X} \), for \( |X| = p \). The following array gives the values of the sequence \( \nu = D_P(u) \) for the element \( u \) in its first column on the left, where \( R \) denotes a non central subgroup of order \( p \) of \( P \) :

\[
\begin{array}{c|cc}
   v_{P, Z} & v_{Q, 1} \\
\hline
   2\Omega_{P/1} & 0 & 1 \\
   2\Omega_{P/Z} & 1 & 0 \\
   2\Omega_{P/R} & 0 & \begin{cases} 
   1 & \text{if } R \not< Q \\
   1 - p & \text{if } R < Q
   \end{cases}
\end{array}
\]

The values in this table can be computed using Lemma 4.6: for example

\[
\text{Res}_Q^P 2\Omega_{P/R} = \left( \sum_{1 \leq V, R \leq Q, V \leq p} \mu(1, V) \right) 2\Omega_{Q/1}.
\]

If \( R \not< Q \), then there is only one term in the summation, for \( V = 1 \), and \( \mu(1, V) = 1 \) in this case. And if \( R \leq Q \), then there are \( p \) additional terms, obtained for the \( p \) distinct conjugates \( V \) of \( R \) in \( P \), and \( \mu(1, V) = -1 \) for each of them. This gives the value \( 1 - p \) in this case.

Now if \( u = m_1 \cdot 2\Omega_{P/1} + m_2 \cdot 2\Omega_{P/Z} + \sum_{[R]} m_R \cdot 2\Omega_{P/R} \) (where the brackets around \( R \) mean that \( R \) runs through a set of representatives of conjugacy classes of non central subgroups of order \( p \) of \( P \)), then the sequence \( \nu = D_P(u) \) is given by :

\[
(4.8) \quad v_{P, Z} = m_Z \quad v_{Q, 1} = m_1 + \sum_{[R]} m_R + (1 - p) \sum_{[R]} m_R.
\]

The second equation is equivalent to

\[
(4.9) \quad v_{Q, 1} = m_1 + \sum_{[R]} m_R - p \sum_{[R]} m_R.
\]

It follows that \( v_{Q, 1} = v_{Q', 1} \) (mod \( p \)), for any subgroups \( Q \) and \( Q' \) of order \( p^2 \) in \( P \). This shows that Condition 3 of 4.5 holds for the sections \( (Q, 1) \) and \( (Q', 1) \) of \( P \). Condition 2 is obviously satisfied in this case, since \( P \) has no subquotient isomorphic to \( (\mathbb{Z}/p\mathbb{Z})^3 \).

Suppose now conversely that a sequence \( v = (v_{T, S})_{(T, S) \in E_2^1(P)} \) is given, and that Conditions 1 and 3 of 4.5 hold. Then \( v \) lies in the image of \( D_P \) if and only if there exist integers \( m_1, m_Z, m_R \) (invariant by \( P \)-conjugation), such that Equations 4.8 hold.

The first equation in 4.8 gives \( m_Z = v_{P, Z} \), and the second one gives

\[
m_1 - v_{Q, 1} + \sum_{[R]} m_R = p \sum_{[R]} m_R.
\]

All subgroups of order \( p \) of \( Q \) different from \( Z \) are conjugate in \( P \). Denoting by \( R_Q \) one of them, this equation becomes

\[
(4.10) \quad m_1 - v_{Q, 1} + \sum_{[R]} m_R = pm_{R_Q}.
\]
Summing this relation over $Q$ yields

$$(p + 1)m_1 = \sum_Q v_{Q,1} + (p + 1) \sum_{[R]} m_R = p \sum_{[R]} m_R,$$

thus

$$\sum_{[R]} m_R = \sum_Q v_{Q,1} - (p + 1)m_1.$$

Now 4.10 yields

$$pm_{R_0} = \sum_{Q \neq Q'} v_{Q',1} - pm_1.$$

By Condition 3, the sum $\sum_{Q \neq Q'} v_{Q',1}$ is congruent to $pv_{Q,1}$ modulo $p$, i.e. to 0. Since $Q = R_QZ$, this gives finally

$$m_R = \frac{1}{p} (\sum_{Q \neq RZ} v_{Q,1}) - m_1.$$

Conversely, if this holds for any $R$, then equation 4.9 holds : indeed, in this case

$$\sum_{[R]} m_R = \sum_Q v_{Q,1} - (p + 1)m_1,$$

thus

$$m_1 + \sum_{[R]} m_R - p \sum_{[R] < Q} m_R = m_1 + \sum_Q v_{Q,1} - (p + 1)m_1 - pm_{R_0}$$

$$= \sum_Q v_{Q,1} - pm_1 - (\sum_{Q' \neq Q} v_{Q',1}) + pm_1$$

$$= v_{Q,1}.$$

Thus 4.9 holds, and Theorem 4.5 also, when $P = X_p^3$.

5. Proof of Theorem 2.15

Let $P$ be a finite $p$-group. Clearly $T(P)$ is the kernel of $r_P$, and $\text{Im} r_P \leq \text{Ker} h_P$, by Proposition 2.14. So the only thing to show is that this inclusion is an equality.

Let $u \in \text{Ker} h_P$. It means that there exists a $P$-invariant function $E \mapsto m_E$ from $A_{\geq 2}(P)$ to $\mathbb{Z}$ such that for any $E < F$ in $A_{\geq 2}(P)$

$$w_{E,F} = m_E - m_F,$$

where the integer $w_{E,F}$ is defined by the equality

$$w_{E,F} \cdot \Omega_{E/1} = \text{Res}_E^F \sigma_F \text{Res}_F^P u - \sigma_E \text{Res}_E^P u.$$

In other words

$$\text{Res}_E^F(m_F \cdot \Omega_{F/1} + \sigma_F \text{Res}_F^P u) = m_E \cdot \Omega_{E/1} + \sigma_E \text{Res}_E^P u.$$
Set \( w_E = m_E \cdot \Omega_{E/1} + \sigma_E \text{Res}_P^E u \), for \( E \in A_{\geq 2}(P) \). Then \( \text{Res}_P^E w_E = w_E \), for any \( E < F \) in \( A_{\geq 2}(P) \), and \( \bar{x}(w_E) = w \cdot x_E \) for any \( x \in P \) and \( E \in A_{\geq 2}(P) \). Moreover, for any \( E \in A_{\geq 2}(P) \)

\[
\tau_E(w_E) = r_E \sigma_E \text{Res}_E^P u = \text{Res}_E^P u,
\]

since \( r_E(\Omega_{E/1}) = 0 \), and since \( \sigma_E \) is a section of \( r_E \). It means that for any subgroup \( Y \neq 1 \) of \( E \)

\[
\text{Def}_{E/Y}^E w_E \equiv \text{Res}_{E/Y}^{N_P(Y)/Y} u_Y.
\]

If \((T,S) \in \mathcal{E}^T_2(P)\), define an integer \( v_{T,S} \) by

\[
(5.1) \quad \left\{ \begin{array}{ll}
\text{Res}_{T/S}^{N_P(S)/S}(2u_S) = v_{T,S} \cdot 2\Omega_{T/S} & \text{if } S \neq 1 \\
2u_T = v_{T,1} \cdot 2\Omega_{T/1} & \text{if } S = 1
\end{array} \right.
\]

This sequence of integers \( (v_{T,S})_{(T,S) \in \mathcal{E}^T_2(P)} \) satisfies some of the conditions of Theorem 4.5. Indeed:

- If \( x \in P \) and \((T,S) \in \mathcal{E}^T_2(P)\), then \( v_{T,x} = v_{T,S} \) : this is because \( \bar{x}(w_E) = w \cdot x_E \) for any \( E \in A_{\geq 2}(P) \), and because \( \bar{x}(u_Q) = u \cdot x \) for any subgroup \( Q \neq 1 \) of \( P \). Thus Condition 1 of Theorem 4.5 holds.

- Suppose that \((T,S) \) and \((T',S) \) are elements of \( \mathcal{E}^T_2(P) \) such that \( T \leq N_P(T') \). There are two cases to consider:

  (a) If \( S \neq 1 \), then for any section \((V,U) \in \mathcal{E}^T_2(N_P(S)/S)\)

\[
v_{V,U} \cdot \Omega_{V/U} = \text{Res}_{V/U}^{N_P(U)/U}(2u_U)
= \text{Res}_{V/U}^{N_P(S,U)/U} \text{Res}_{N_P(S)/U}^{N_P(U)/U}(2u_U)
= \text{Res}_{V/U}^{N_P(S,U)/U} \text{Defres}_{N_P(S)/U}^{N_P(S)/S}(2u_S)
= \text{Defres}_{V/U}^{N_P(S)/S}(2u_S).
\]

It follows that the sequence \( (v_{V,U})_{(V,U) \in \mathcal{E}^T_2(N_P(S)/S)} \) is equal to \( \mathcal{D}_{N_P(S)/S}(2u_S) \),

hence it is in the image of the map \( \mathcal{D}_{N_P(S)/S} \). Thus if \( TT'/S \cong (Z/pZ)^3 \), then Condition 2 of Theorem 4.5 holds for the sections \((T,S) \) and \((T',S) \) of \( N_P(S)/S \).

And if \( TT'/S \cong X_r \), then Condition 3 of Theorem 4.5 holds, for a similar reason.

(b) If \( S = 1 \), then set \( F = TT' \). If \( F \cong (Z/pZ)^3 \), then consider a section \((V,U) \in \mathcal{E}^T_2(F) \). If \( U = 1 \), then

\[
\text{Defres}_{V/U}^F 2w_F = \text{Res}_{V/U}^F 2w_F = 2w_V = v_{V,1} \cdot 2\Omega_{V/1}.
\]

And if \( U \neq 1 \), then

\[
\text{Defres}_{V/U}^F 2w_F = \text{Res}_{V/U}^F \text{Defres}_{V/U}^F 2w_F
= \text{Res}_{V/U}^F \text{Res}_{F/U}^{N_F(U)/U} 2u_U
= \text{Res}_{V/U}^{N_F(U)/U} 2u_U
= v_{V,U} \cdot 2\Omega_{V/U}.
\]

It follows that the sequence \( (v_{V,U})_{(V,U) \in \mathcal{E}^T_2(F)} \) is equal to \( \mathcal{D}_F(2w_F) \). In particular, Condition 2 of Theorem 4.5 is fulfilled for the sections \((T,1) \) and \((T',1) \) of \( F \).

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Hence the sequence \((v_{T,S})_{(T,S)\in \mathcal{E}_2^3(P)}\) fulfills all the conditions of Theorem 4.5, except possibly Condition 3 for sections \((T,1)\) and \((T',1)\) such that \(T \leq N_P(T')\) and \(TT' \cong X_{p^3}\). This situation is handled by the following lemma:

5.2. Lemma. Let \(P\) be a finite p-group, and \((v_{T,S})_{(T,S)\in \mathcal{E}_2^3(P)}\) be a sequence of integers such that:

1. If \(x \in P\) and \((T,S) \in \mathcal{E}_2^3(P)\), then \(v_{T,S} = v_{T,S}^x\).
2. If \((T,S)\) and \((T',S)\) are in \(\mathcal{E}_2^3(P)\), if \(T \leq N_P(T')\) and if \(TT'/S \cong (\mathbb{Z}/p\mathbb{Z})^3\), then
   \[
   v_{T,S} + \sum_{S < X < T} v_{TT',X} = v_{T',S} + \sum_{S < X < T'} v_{TT',X}.
   \]
3. If \(S \neq 1\), if \((T,S)\) and \((T',S)\) are in \(\mathcal{E}_2^3(P)\), if \(T \leq N_P(T')\) and if \(TT'/S \cong X_{p^3}\), then
   \[
   v_{T,S} \equiv v_{T',S} \pmod{p}.
   \]

Then:

(i) If \(T,T' \in A_{=2}(P)\), if \(T\) and \(T'\) are in the same connected component of \(A_{\geq 2}(P)\), if \(T \leq N_P(T')\) and \(TT' \cong X_{p^3}\), then \(v_{T,1} \equiv v_{T',1} \pmod{p}\).

(ii) There exists a sequence of integers \((y_T)_{T \in A_{=2}(P)}\) such that
   
   (a) If \(x \in P\) and \(T \in A_{=2}(P)\), then \(y_T = y_{T,x}\).
   
   (b) If \(T,T' \in A_{=2}(P)\), if \(T \leq N_P(T')\) and \(TT' \cong (\mathbb{Z}/p\mathbb{Z})^3\), then \(y_T = y_{T'}\).
   
   (c) If \(T,T' \in A_{=2}(P)\), if \(T \leq N_P(T')\) and \(TT' \cong X_{p^3}\), then
   \[
   y_T + v_{T,1} \equiv y_{T',1} + v_{T',1} \pmod{p}.
   \]

Proof. The proof of Assertion (i) goes by induction on \(|P|\), starting with the case where \(P\) is cyclic, where there is nothing to prove. Assume then that Hypotheses 1), 2), and 3) imply Assertion 1, for any \(p\)-group of order strictly smaller than \(|P|\). Let \(T\) and \(T'\) be elementary abelian subgroups of rank 2 of \(P\), such that \(T \leq N_P(T')\) and \(TT' \cong X_{p^3}\). Set \(X = TT'\), and denote by \(Z\) the centre of \(X\).

If there is a proper subgroup \(Q\) of \(P\) containing \(X\), and such that \(T\) and \(T'\) are in the same connected component of \(A_{\geq 2}(Q)\), then \(v_{T,1} \equiv v_{T',1} \pmod{p}\), by induction, since Hypotheses 1), 2), and 3) obviously hold for \(Q\) if they hold for \(P\). It is the case in particular if \(A_{\geq 2}(Q)\) is connected.

Suppose that there exists a subgroup \(C\) of order \(p\) in \(C_P(X)\), not contained in \(X\) (i.e., different from \(Z\)). Then the center \(T'' = C \times Z\) of the subgroup \(Q = C \times X\) of \(P\) is not cyclic. Hence \(A_{\geq 2}(Q)\) is connected, and \(Q\) contains \(T\) and \(T'\). Thus I can suppose that \(Q = P\), and then \(T''\) is equal to the centre of \(P\). It is elementary abelian of rank 2. Moreover \(TT'' \cong (\mathbb{Z}/p\mathbb{Z})^3\), since \(T\) and \(T''\) are elementary abelian of rank 2 and centralize each other, and since \(T \cap T'' = T \cap X \cap T'' = Z\). Hypothesis 2, applied to the sections \((T,1)\) and \((T',1)\) of \(P\) yields

\[
(5.3) \quad v_{T,1} - v_{T'',1} = \sum_{1 < F < T''} v_{TT'',F} - \sum_{1 < F < T} v_{TT',F}.
\]

Now \(TT'' \subseteq P\) since \(|P : TT''| = p\), and \(T \subseteq P\), since \(T \subseteq X\) and \(C \leq C_P(X)\). Hence \(P\) acts by conjugation on the set of subgroups \(F\) such that \(1 < F < T\), and \(F = Z\) is the
unique fixed point under this action. Now Hypothesis 1 implies that

$$\sum_{1 \leq F < T} v_{T', F} \equiv v_{T', Z} \pmod{p},$$

and Equation 5.3 yields

$$(5.4) \quad v_{T', 1} - v_{T'', 1} \equiv \sum_{1 \leq F < T'' \atop F \neq Z} v_{T''', F} \pmod{p}.$$  

The same argument applies with $T'$ instead of $T$, so

$$(5.5) \quad v_{T', 1} - v_{T'', 1} \equiv \sum_{1 \leq F < T'' \atop F \neq Z} v_{T''', F} \pmod{p}.$$  

Now if $1 < F < T''$ and $F \neq Z$, the group $P/F$ has order $p^3$ and exponent $p$ (since $P$ has exponent $p$), and it is non abelian (since $F \not\subseteq [P, P] = Z$). Hence $P/F \cong X_{p^3}$.

Since $P = (TT'')(T'T'')$, Hypothesis 3, applied to the sections $(TT'', F)$ and $(T'T'', F)$ yields $v_{T'', F} \equiv v_{T''', F} \pmod{p}$. This shows that the right hand sides of 5.4 and 5.5 are congruent modulo $p$. So are the left hand sides, thus $v_{T', 1} - v_{T'', 1} \equiv v_{T', 1} - v_{T'', 1} \pmod{p}$, and $v_{T', 1} \equiv v_{T'', 1} \pmod{p}$.

Hence I can suppose that $Z$ is the only subgroup of order $p$ of $C_P(X)$. In particular, the centre of $P$ is cyclic, and $Z$ is the only subgroup of order $p$ in this centre. Moreover, since $T \neq T'$ and $T, T'$ are in the same connected component of $A_{22}(P)$, the groups $T$ and $T'$ are not maximal elementary abelian subgroups, thus $P$ has $p$-rank at least equal to 3, and $T$ and $T'$ are in the big component $C$ of $A_{22}(P)$.

In this case, there is a normal subgroup $T_0$ of $P$ which is elementary abelian of rank $2$, and $T_0 \in C$. Moreover $T_0 > Z$.  

If $T_0 \not\subseteq X$, then $T_0 \cap X = Z$. Then $|T_0X| = p^4$, and $|T_0X : X| = p$. Thus $T_0$ normalizes $X$. Moreover, if $Y$ is a subgroup of index $p$ of $X$, then $Y > Z$, and $[T_0Y : Y] = p$, and $T_0$ normalizes $Y$. It follows that the image of $T_0$ in the group Out($X$) of outer automorphisms of $X$, which is isomorphic to $GL_2(\mathbb{F}_p)$, is a $p$-subgroup stabilizing every line. So this image is trivial, and $T_0$ acts on $X$ by inner automorphisms. Let $t \in T_0 - X$. Then there exists $y \in X$ such that $y^{-1}t \in C_P(X)$. In particular $y^{-1}t$ centralizes $y$, so $t$ centralizes $y$, and then $(y^{-1}t)^p = (y^{-1})^p |_{\mathbb{F}_p} = 1$. Moreover $y \neq t$, since $t \notin X$. Hence $y^{-1}t$ has order $p$. Since $Z$ is the only subgroup of order $p$ of $C_P(X)$, it follows that $y^{-1}t \in Z$, so $t \subseteq X$. This contradiction shows that $T_0 \subseteq X$.

Since the congruences $v_{T', 1} \equiv v_{T_0, 1} \pmod{p}$ and $v_{T', 1} \equiv v_{T_0, 1} \pmod{p}$ imply the congruence $v_{T', 1} \equiv v_{T', 1} \pmod{p}$, it is enough to suppose that $T_0 = T$, thus $T \subseteq P$. Let $F$ be an elementary abelian subgroup of rank 3 of $P$ containing $T'$ : such a subgroup exists, since $T'$ is not the maximal element of $A_{22}(P)$. Set $T'' = C_F(T)$. Then $|F : T''|$ divide $p$, since $F/T''$ is a $p$-subgroup of Aut($T$) $\cong GL_2(\mathbb{F}_p)$. Moreover $F \not\subseteq C_F(T)$, since $F > T'$. Thus $|F : T''| = p$, and $T'' \cong (\mathbb{Z}/p\mathbb{Z})^2$. Moreover $T' \neq T''$, since $T' \not\subseteq C_F(T)$, thus $F = T'T''$.

Now $F$ centralizes $T'$, and normalizes $T$. Thus $F$ normalizes $TT' = X$. Moreover $F \cap X = T'$, since $T' \leq F \cap X$, and since $F$ and $X$ are distinct subgroups of order $p^3$, for $F$ is abelian and $X$ is not. Hence $|FX : F| = p$, so $FX$ normalizes $F$. Thus $X$ normalizes $F$, and $X$ also normalizes $C_F(T)$ since $T \not\subseteq P$. It follows that $X$ normalizes $F \cap C_F(T) = C_F(T) = T''$. Obviously $X$ also normalizes its subgroup $T'$.  

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Thus \( v_{T',1} - v_{T'',1} = \sum_{1 < Y < T''} v_{T',Y} - \sum_{1 < Y < T'} v_{T'',Y} \).

Since \( X \) normalizes \( T' \) and \( T'' \), and since any subgroup of order \( p \) normalized by \( X \) is centralized by \( X \), Hypothesis 1 yields

\[
\sum_{1 < Y < T''} v_{T',Y} \equiv \sum_{1 < Y < T' \leq C_P(X)} v_{T'',Y} \pmod{p}.
\]

But \( T'' \cap C_P(X) = Z \), since \( Z \leq T'' \cap C_P(X) \), and since \( Z \) is the only subgroup of order \( p \) of \( C_P(X) \). Thus

\[
\sum_{1 < Y < T''} v_{T',Y} \equiv v_{T',Z} \pmod{p}.
\]

The same argument, applied with \( T' \) instead of \( T'' \), since \( T'' \cap C_P(X) = Z \), yields

\[
\sum_{1 < Y < T'} v_{T',Y} \equiv v_{T',Z} \pmod{p}.
\]

Now it follows from 5.6, 5.7 and 5.8 that \( v_{T',1} - v_{T'',1} \equiv 0 \pmod{p} \), i.e.

\[
v_{T',1} \equiv v_{T'',1} \pmod{p}.
\]

Now the group \( TT'' \) is also elementary abelian of rank 3: indeed, the group \( T'' \) centralizes \( T \), and \( T'' \cap T = Z \) since \( T'' \cap T \geq Z \) and \( T'' \neq T \) for \( T \neq F \leq C_P(T') \). Then Hypothesis 2, for the sections \((T, 1)\) and \((T', 1)\), yields

\[
v_{T,1} - v_{T'',1} = \sum_{1 < Y < T''} v_{T',Y} - \sum_{1 < Y < T} v_{T'',Y}.
\]

The group \( X \) normalizes \( T \) and \( T'' \), and \( T \cap C_P(X) = Z = T'' \cap C_P(X) \). The same argument as above yields

\[
v_{T,1} \equiv v_{T'',1} \pmod{p}.
\]

Thus \( v_{T,1} \equiv v_{T',1} \pmod{p} \), by 5.9 and 5.11, and this completes the proof of Assertion (i).

For Assertion 2, there is nothing to do if \( P \) has no normal subgroup \( T_0 \cong (\mathbb{Z}/p\mathbb{Z})^2 \), since then \( P \) is cyclic, and \( A_{\geq 2}(P) = \emptyset \). If \( P \) is not cyclic, fix such a normal subgroup \( T_0 \) of \( P \), and denote by \( \mathcal{C} \) the connected component of \( T_0 \) in \( A_{\geq 2}(P) \). Thus \( \mathcal{C} \) is the big component if \( P \) has \( p \)-rank at least 3, and \( \mathcal{C} = \{ T_0 \} \) otherwise. Define the sequence \( (y_T)_{T \in A_{\geq 2}(P)} \) by

\[
y_T = \begin{cases} 
0 & \text{if } T \in \mathcal{C} \\
v_{T,1} - v_{T',1} & \text{otherwise}
\end{cases}
\]

This sequence obviously fulfills Condition (a) of Lemma 5.2, by Hypothesis 1, and since \( \mathcal{C} \) is invariant by \( P \)-conjugation. Now it \( T, T' \in A_{\geq 2}(P) \), if \( T \leq N_P(T') \) and \( TT' \cong (\mathbb{Z}/p\mathbb{Z})^3 \), it follows that \( P \) has \( p \)-rank at least 3, that \( \mathcal{C} \) is the big component, and that \( T, T' \in \mathcal{C} \). Thus \( y_T = y_{T'} = 0 \), so Condition (b) of Lemma 5.2 holds. Finally, if \( T, T' \in A_{\geq 2}(P) \), if \( T \leq N_P(T') \) and \( TT' \cong X_{p^3} \), then there are three cases:
• if $T$ and $T'$ are in $C$, then $y_T = y_{T'} = 0$, thus $y_T + v_{T,1} = v_{T,1}$, and $y_{T'} + v_{T',1} = v_{T',1}$. But $v_{T,1} \equiv v_{T',1} \pmod{p}$ in this case, by Assertion 1. Thus Condition (c) holds in this case.

• if $T \in C$ and $T' \notin C$, then $y_T + v_{T,1} = v_{T,1}$, and $y_{T'} + v_{T',1} = v_{T_0,1}$. But now $T$ and $T_0$ are both in $C$, so $v_{T,1} \equiv v_{T_0,1} \pmod{p}$ in this case, by Assertion 1. Thus Condition (c) holds in this case also. The case $T \notin C$ and $T' \in C$ is similar.

• if $T \notin C$ and $T' \notin C$, then $y_T + v_{T,1} = v_{T_0,1} = y_{T'} + v_{T',1}$, so Condition (c) holds in this case also.

This completes the proof of Lemma 5.2.

**End of the proof of Theorem 2.15:** In the beginning of the proof of Theorem 2.15, I started with an element $u \in \lim_{1 < Q \leq P} D(N_P(Q)/Q)$ such that $h_P(u) = 0$. From this data, in 5.1, I built a sequence of integers $(v_{T,S})_{(T,S) \in \mathcal{E}_2(P)}$ fulfilling Hypothesis 1, 2 and 3 of Lemma 5.2. Let $(y_T)_{T \in \mathcal{A}_{\geq 2}(P)}$ denote the sequence of integers provided by this lemma, and define a new sequence of integers $(v'_{T,S})_{(T,S) \in \mathcal{E}_2(P)}$ by

$$v'_{T,S} = \begin{cases} v_{T,S} & \text{if } S \neq 1 \\ y_T + v_{T,S} & \text{if } S = 1 \end{cases}.$$  

Then this sequence fulfills Conditions 1, 2, and 3 of Theorem 4.5: indeed, the new sequence is clearly invariant by conjugation, so Condition 1 is fulfilled. Conditions 2 and 3 for sections $(T,S)$ and $(T',S)$ with $S \neq 1$ are obviously fulfilled, since they are for the sequence $(v_{T,S})$, and since $v_{T,S} = v'_{T,S}$ when $S \neq 1$.

Now if $T, T' \in \mathcal{A}_{\geq 2}(P)$, if $T \leq N_P(T')$ and $TT' \equiv (\mathbb{Z}/p\mathbb{Z})^3$, then $T$ and $T'$ are in the same connected component of $\mathcal{A}_{\geq 2}(P)$, and $y_T = y_{T'}$. Thus

$$v'_{T,1} + \sum_{1 < Y < T} v'_{TTY,Y} = y_T + v_{T,1} + \sum_{1 < Y < T} v_{TTY,Y}$$

$$= y_{T'} + v_{T',1} + \sum_{1 < Y < T'} v_{TTY,Y}$$

$$= v'_{T',1} + \sum_{1 < Y < T'} v'_{TTY,Y},$$

so Condition 2 is fulfilled.

Finally if $T, T' \in \mathcal{A}_{\geq 2}(P)$, if $T \leq N_P(T')$ and $TT' \equiv X_{p^3}$, then

$$v'_{T,1} = y_T + v_{T,1} \equiv y_{T'} + v_{T',1} \pmod{p},$$

hence Condition 3 is fulfilled, since $y_{T'} + v_{T',1} = v'_{T',1}$.

By Theorem 4.5, there exists $n \in D(P)$ such that $D_P(2n) = (v'_{T,S})$. In other words, for any $(T,S) \in \mathcal{E}_2(P)$

$$\text{Defres}_{T,S}(2n) = v'_{T,S} \cdot 2\Omega_{T/S}.$$

Thus if $S \neq 1$

$$\text{Defres}_{T,S}(2n) = v_{T,S} \cdot 2\Omega_{T/S} = \text{Res}_{T/S}^{N_P(S)/S}(2u_S).$$
Set $t_S = \text{Defres}_{N_P(S)/S}^F(2n) - 2u_S$. Now for any $(V,U) \in \mathcal{E}_1^2(N_P(S)/S)$

$$\text{Defres}_{V/U}^{N_P(S)/S}(t_S) = \text{Defres}_{V/U}^F(2n) - \text{Defres}_{V/U}^{N_P(S)/S}(2u_S)$$

$$= \text{Defres}_{V/U}^F(2n) - \text{Res}_{V/U}^{N_P(S,U)/U} \text{Defres}_{N_P(S,U)/U}^F(2u_S)$$

$$= \text{Defres}_{V/U}^F(2n) - \text{Res}_{V/U}^{N_P(S,U)/U} \text{Res}_{N_P(S,U)/U}^F(2u_S)$$

$$= \text{Defres}_{V/U}^F(2n) - \text{Res}_{V/U}^{N_P(U)/U}(2u_U) = 0.$$  

It follows that $t_S$ is a torsion element of $D(N_P(S)/S)$, which is also in $2D(N_P(S)/S)$. Since the latter is torsion free, it follows that $t_S = 0$, i.e. that $2u_S = \text{Defres}_{N_P(S)/S}^F(2n)$, for any $S \neq 1$. Equivalently $2u = r_p(2n)$, or $2(u - r_p(n)) = 0$.

Now $u - r_p(n)$ is an element of $\text{lim}_{1 \leq Q \leq P} D_{\text{tors}}(N_P(Q)/Q)$. By Proposition 5.5 of [7], there exists an element $m \in D(P)$ such that $r_p(m) = u - r_p(n)$. It follows that $u = r_p(m + n)$, as was to be shown. This completes the proof of Theorem 2.15. ☐

6. Example : the group $X_{p^5}$

Let $P$ be an extraspecial group of order $p^5$ and exponent $p$. The centre $Z$ of $P$ is cyclic of order $p$, and it is equal to the Frattini subgroup of $P$. The commutator $P \times P \to Z$ induces a non degenerate symplectic $F_p$-valued scalar product on the factor group $E = P/Z \cong (F_p)^4$, and the map $Q \mapsto Q/Z$ is a poset isomorphism from the poset of elementary abelian subgroups of $P$ strictly containing $Z$ to the poset $\mathcal{E}$ of non zero totally isotropic subspaces of $E$. There are $e = \frac{p^4 - 1}{p - 1}$ isotropic lines in $E$, and the same number of totally isotropic 2-dimensional subspaces. It follows that $|\mathcal{E}|$ is equal to $2e$.

There is a commutative diagram

$$
\begin{array}{ccc}
D(P/Z) & \xrightarrow{d} & \text{lim}_{Q \geq Z} D(P/Q) \\
\text{Def}_{P/Z}^P & \downarrow & \\
0 \longrightarrow T(P) & \longrightarrow & D(P) \xrightarrow{r_p} \text{lim}_{1 \leq Q \leq P} D(N_P(Q)/Q) \xrightarrow{h_P} H^1(A_{\geq 2}(P), Z)^{(P)}
\end{array}
$$

In this diagram, the group $\text{lim}_{Q \geq Z} D(P/Q)$ is the group of sequences $(u_Q)_{Z \leq Q \leq P}$, where $u_Q \in D(P/Q)$ (note that $Q \leq P$ if $Q \geq Z$), such that

$$\forall R \geq Q \geq Z, \text{ Def}_{P/Q}^{P,R} u_Q = u_R,$$

and the map $\pi$ is the projection map on the components $Q \geq Z$. The map $d$ is the product of the deflation maps $\text{Def}_{P/Z}^P$, for $Q \geq Z$. It is an isomorphism, since the sequence $(u_Q)_{Q \geq Z}$, where $u_Q \in D(P/Q)$, is in the group $\text{lim}_{Q \geq Z} D(P/Q)$ if and only if

$u_Q = \text{Def}_{P/Q}^{P,Z} u_Z$ for any $Q \geq Z$.  

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The kernel of the map $\text{Def}_{P/Z}^C$ is the set of faithful elements of $D(P)$, and it was denoted by $\partial D(P)$ in [5]. It was shown in that paper (Theorem 9.1) that
\[
\partial D(P) \cong \mathbb{Z}^{2e} \oplus \mathbb{Z}/2\mathbb{Z}.
\]
The kernel of $\pi$ consists of the sequences $(u_Q)_{1 < Q \leq P}$ in $\lim_{1 < Q \leq P} D(N_P(Q)/Q)$ for which $u_Q = 0$ if $Q \geq Z$. It was shown in [5] that this is also the group $\lim_{1 < Q \leq P} \partial D(N_P(Q)/Q)$.

All these facts show that there is an exact sequence
\[
0 \longrightarrow T(P) \longrightarrow \partial D(P) \overset{r_P}{\longrightarrow} \lim_{1 < Q \leq P} \partial D(N_P(Q)/Q) \overset{h_P}{\longrightarrow} H^1(A_{\geq 2}(P), \mathbb{Z})^{(P)},
\]
where $r_P$ and $h_P$ are the restrictions of the previously defined maps with the same names to the corresponding subgroups.

If $Q$ is a subgroup of $P$ such that $Q \cap Z = 1$, then $Q$ is elementary abelian of rank at most 2 (see [5] for details). If $Q$ has order $p$, then $N_P(Q)/Q \cong X_p$, thus $\partial D(N_P(Q)/Q) \cong \mathbb{Z}^{p+1} \oplus \mathbb{Z}/2\mathbb{Z}$ ([5] Theorem 9.1 or Section 11). If $Q$ has order $p^2$, then $N_P(Q)/Q \cong \mathbb{Z}/p\mathbb{Z}$, thus $\partial D(N_P(Q)/Q) \cong \mathbb{Z}/2\mathbb{Z}$.

It follows easily that the group $\lim_{1 < Q \leq P} \partial D(N_P(Q)/Q)$ has free rank at least equal to $e(p+1)$, since it contains the group $\bigoplus_{|Q|=p, \ Q \neq Z} 2\partial D(N_P(Q)/Q)$.

Now the group $T(P)$ is free of rank one, generated by $\Omega_{P/1}$, by Corollary 1.3 of [11], and the group $H^1(A_{\geq 2}(P), \mathbb{Z})^{(P)}$ is isomorphic to $H^1(E, \mathbb{Z})$. An easy computation, using e.g. Section 6 of [5], shows that this group is free of rank $p^4$.

Now the free rank of the image of $h_P$ in the exact sequence 6.1 is at least equal to
\[
1 - 2e + e(p+1) = 1 + e(p-1) = p^4,
\]
and since this is equal to the free rank of $H^1(A_{\geq 2}(P), \mathbb{Z})^{(P)}$, it follows that the free rank of the image of $h_P$ is actually equal to $p^4$, and that the free rank of $\lim_{1 < Q \leq P} \partial D(N_P(Q)/Q)$ is equal to $e(p+1)$.

Moreover the map $h_P$ has finite cokernel, and this shows that in this case, the gluing problem does not always have a solution.

6.2. Remark: In this case, a precise description of the map $h_P$ shows that its cokernel is a non trivial finite $p$-group.

6.3. Acknowledgements: I wish to thank Nadia Mazza and Jacques Thévenaz for careful reading of an early version of this paper, and for many suggestions, comments, and stimulating discussions about it.

References


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