Title : **Gluing endo-permutation modules**

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Abstract : In this paper, I show that if p is an odd prime, and if P is a finite p-group, then there exists an exact sequence of abelian groups

$$0 \to T(P) \to D(P) \to \varprojlim_{\mathbf{1}<Q\leq P} D\big(N_P(Q)/Q\big) \to H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big)^{(P)} \ ,$$

where $D(P)$ is the Dade group of P and $T(P)$ is the subgroup of endo-trivial modules.

Here $\varprojlim_{\mathbf{1}<Q\leq P} D\big(N_P(Q)/Q\big)$ is the group of sequences of compatible elements in the Dade groups $D\big(N_P(Q)/Q\big)$ for non trivial subgroups Q of P. The poset $\mathcal{A}_{\geq 2}(P)$ is the set of elementary abelian subgroups of rank at least 2 of P, ordered by inclusion. The group $H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big)^{(P)}$ is the subgroup of $H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big)$ consisting of classes of P-invariant 1-cocycles.

A key result to prove that the above sequence is exact is a characterization of elements of $2D(P)$ by sequences of integers, indexed by sections (T,S) of P such that $T/S \cong (\mathbb{Z}/p\mathbb{Z})^2$, fulfilling certain conditions associated to subquotients of P which are either elementary abelian of rank 3, or extraspecial of order p^3 and exponent p.

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1. Introduction

The classification of all endo-permutation modules for finite p-groups has been completed recently, thanks to the work of several authors (see in particular [1], [11], [12], [5], [3]). This paper addresses the question of *gluing arbitrary endo-permutation modules*, and it is intended to be a complement to our previous joint work with Jacques Thévenaz ([7]), where the case of torsion endo-permutation modules was handled.

The gluing problem is the following : let p be an odd prime, let P be a finite p-group, and let k be a field of characteristic p. If v is an element of the Dade group $D(P)$ of endo-permutation kP-modules, and if Q is a non trivial subgroup of P, denote by v_Q the image of v by the deflation-restriction map $\mathrm{Defres}^P_{N_P(Q)/Q}$. Then the v_Q's are subject to some obvious compatibility conditions. Conversely, if Q is a non-trivial subgroup of P, let u_Q be an element of the Dade group $D_k\big(N_P(Q)/Q\big)$, and assume that these compatibility conditions between the u_Q's are fulfilled. Is there an element $u \in D(P)$ such that for any non trivial subgroup Q of P

$$\mathrm{Defres}^P_{N_P(Q)/Q}(u) = u_Q \quad ?$$

Such an element u is called a *solution to the gluing problem* for the *gluing data* $(u_Q)_{1<Q\leq P}$.

When P is abelian, the gluing problem was completely solved by Puig [16] (see also Lemma 2.3 below), and he used the result to construct suitable stable equivalences between blocks.

The main result of the present paper is that if p is an odd prime, and if P is a finite p-group, then there exists an exact sequence of abelian groups

$$0 \longrightarrow T(P) \longrightarrow D(P) \longrightarrow \lim_{\substack{\longleftarrow \\ \mathbf{1}<Q\leq P}} D\big(N_P(Q)/Q\big) \xrightarrow{h_P} H^1\big(\mathcal{A}_{\geq 2}(P), \mathbb{Z}\big)^{(P)} ,$$

where $D(P)$ is the Dade group of P and $T(P)$ is the subgroup of endo-trivial modules. Here $\lim\limits_{\longleftarrow \atop \mathbf{1}<Q\leq P} D\big(N_P(Q)/Q\big)$ is the group of gluing data for P, i.e. the group of sequences of compatible elements in the Dade groups $D\big(N_P(Q)/Q\big)$ for non trivial subgroups Q of P. The poset $\mathcal{A}_{\geq 2}(P)$ is the set of elementary abelian subgroups of rank at least 2 of P, ordered by inclusion. The group $H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big)^{(P)}$ is the subgroup of $H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big)$ consisting of classes of P-invariant 1-cocycles.

The main consequence of this result is that if $H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big) = \{0\}$, then the gluing problem always has a solution. Unfortunately, the map h_P is not surjective in general, so when $H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big) \neq \{0\}$, not much can be said at the time for the gluing problem. In Section 6, the example of the extraspecial group of order p^5 and exponent p is described in details. In this case, the group $H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big)^{(P)}$ is a free group of rank p^4, and the image of h_P has finite index in this group. In particular it is non zero, and the gluing problem does not always have a solution.

It could be true in general that h_P always has finite cokernel, and this would be enough to show that if $H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big)^{(P)} \neq \{0\}$, then the image of h_P is non zero, hence that the gluing problem does not always have a solution : it is known indeed that the group $H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big)^{(P)}$ is a free abelian group, since the poset $\mathcal{A}_{\geq 2}(P)$ has the homotopy type of a wedge of spheres (see [8]).

1.1. Notation. Throughout this paper, the symbol p denotes an odd prime number, and P denotes a finite p-group. Inclusion of subgroups will be denoted by $\leq$, and strict inclusion by $<$. Inclusion up to P-conjugation will be denoted by $\leq_P$.

A *section* (T,S) of P is a pair of subgroups of P with $S \trianglelefteq T$. The factor group T/S is the corresponding *subquotient* of P. If (T,S) is a section of P, then $N_P(T,S)$ denotes $N_P(T)\cap N_P(S)$.

A class $\mathcal{Y}$ of p-groups is said to be *closed under taking subquotients* if for any $Y\in\mathcal{Y}$ and any section (T,S) of Y, any group isomorphic to T/S belongs to $\mathcal{Y}$. If $\mathcal{Y}$ is such a class, and P is a finite p-group, let $\mathcal{Y}(P)$ be the set of sections (T,S) of P such that $T/S\in\mathcal{Y}$.

The symbol X_{p^3} denotes an extraspecial p-group of order p^3 and exponent p. The symbol $\mathcal{X}_3$ denotes the class of p-groups which are either elementary abelian of rank at most 3, or isomorphic to X_{p^3}. Thus, the symbol $\mathcal{X}_3(P)$ denotes the set of sections (T,S) of P such that T/S is elementary abelian of rank at most 3, or isomorphic to X_{p^3}. Let moreover $\mathcal{E}_2^\sharp(P)$ denote the set of sections (T,S) of P such that $T/S\cong(\mathbb{Z}/p\mathbb{Z})^2$.

If P is a finite p-group, and k is a field of characteristic p, let $D(P)$ denote the Dade group of endo-permutation kP-modules. The field k does not appear in this notation, because it turns out that $D(P)$ is independent of k, at least when p is odd (see [3] Theorem 9.5 for details).

When (T,S) is a section of P, there is a *deflation-restriction map* $\mathrm{Defres}^P_{T/S}$: $D(P)\to D(T/S)$, which is the group homomorphism obtained by composing *the restriction map* $\mathrm{Res}^P_T : D(P)\to D(T)$, followed by the *deflation map* $\mathrm{Def}^T_{T/S} : D(T)\to D(T/S)$.

Recall that if X is a finite P-set, there is a corresponding element Ω_X of the Dade group of P, called the syzygy of the trivial module relative to X (or the *X-relative syzygy* for short) : it is defined as the class of the kernel of the augmentation map $kX\to k$ when this does make sense, and by 0 otherwise (see e.g. [2] for details). When X is the set P itself, on which P acts by multiplication, the corresponding element will be denoted by $\Omega_{P/\mathbf{1}}$ or Ω_P.

1.2. Contents. This paper is organized as follows :

- In Section 2, I state the main theorem (Theorem 2.15), and this requires in particular the definition of some objects and maps between them.
- Section 3 recalls some notation on biset functors, forgetful functors between categories of biset functors, and corresponding adjoint functors.
- Section 4 is devoted to the main tool (Theorem 4.5) used in the proof of Theorem 2.15, namely a characterization by linear equations of the image of the group $2D(P)$ by the deflation-restriction maps to all subquotients T/S of P which are elementary abelian of rank 2. This characterization may be a result of independent interest.
- Section 5 exposes the proof of Theorem 2.15.
- Finally, Section 6 focuses on the example of the extraspecial p-group of order p^5 and exponent p : the reason for choosing this particular group is twofold : it is is one of the smallest p-groups P for which $H^1\big(\mathcal{A}_{\geq 2}(P), \mathbb{Z}\big) \neq \{0\}$, and moreover the Dade group of this p-group is rather well known, thanks to our joint work with Nadia Mazza ([5]).

2. Statement of the main theorem

2.1. Notation. *If P is a finite p-group, then $\mathcal{A}_{\geq 2}(P)$ denotes the poset of elementary abelian subgroups of P of rank at least 2. Let $\mathcal{A}_{=2}(P)$ denote the set of elementary abelian subgroups of rank 2 of P.*

Recall that if the p-rank of P is at least equal to 3, then all the elementary abelian subgroup of P of rank at least 3 are in the same connected component of $\mathcal{A}_{\geq 2}(P)$. This component is called *the big component*. It is obviously invariant under P-conjugation. Each of the other connected components, if there are any, consists of a single maximal elementary abelian subgroup of rank 2.

2.2. Notation. *Denote by* $\lim\limits_{\substack{\longleftarrow\\ \mathbf{1}<Q\leq P}} D\big(N_P(Q)/Q\big)$ *the set of sequences* $(u_Q)_{\mathbf{1}<Q\leq P}$*, indexed by non trivial subgroups of P, where $u_Q \in D\big(N_P(Q)/Q\big)$, such that :*

- *If $x \in P$, then ${}^x u_Q = u_{{}^xQ}$.*
- *If $Q \trianglelefteq R$, then* $\mathrm{Defres}^{N_P(Q)/Q}_{N_P(Q,R)/R} u_Q = \mathrm{Res}^{N_P(R)/R}_{N_P(Q,R)/R} u_R$.

Denote by $r_P : D(P) \to \lim\limits_{\substack{\longleftarrow\\ \mathbf{1}<Q\leq P}} D\big(N_P(Q)/Q\big)$ *the map sending $v \in D(P)$ to the sequence* $(\mathrm{Defres}^P_{N_P(Q)/Q} v)_{\mathbf{1}<Q\leq P}$.

If E is an abelian p-group, denote by σ_E the map $\lim\limits_{\substack{\longleftarrow\\ \mathbf{1}<F\leq E}} D(E/F) \to D(E)$ *defined by*

$$\sigma_E(u) = -\sum_{\mathbf{1}<F\leq E} \mu(\mathbf{1}, F)\mathrm{Inf}^E_{E/F} u_F \ ,$$

where μ is the Möbius function of the poset of subgroups of P.

It has been shown by Puig ([15] 2.1.2) that the kernel of r_P is equal to the group $T(P)$ of endo-trival modules. Moreover, when E is an abelian group, the map r_E is surjective ([16] Proposition 3.6). More precisely :

2.3. Lemma. *Let E be an abelian p-group. Then σ_E is a section of r_E, i.e. $r_E\sigma_E$ is equal to the identity map of* $\displaystyle\lim_{\substack{\longleftarrow\\ \mathbf{1}<F\leq E}} D(E/F)$.

Proof. Let $\mathbf{1} < G \leq E$. Then

$$\begin{aligned}
\mathrm{Def}^E_{E/G}\sigma_E(u) &= -\sum_{\mathbf{1}<F\leq E}\mu(\mathbf{1},F)\mathrm{Def}^E_{E/G}\mathrm{Inf}^G_{E/F}u_F\\
&= -\sum_{\mathbf{1}<F\leq E}\mu(\mathbf{1},F)\mathrm{Inf}^{E/G}_{E/FG}\mathrm{Def}^{E/F}_{E/FG}u_F\\
&= -\sum_{\mathbf{1}<F\leq E}\mu(\mathbf{1},F)\mathrm{Inf}^{E/G}_{E/FG}u_{FG}\\
&= -\sum_{G\leq R\leq E}\big(\sum_{\substack{\mathbf{1}<F\leq R\\ FG=R}}\mu(\mathbf{1},F)\big)\mathrm{Inf}^{E/G}_{E/R}u_R\ .
\end{aligned}$$

Now if $G < R$

$$\sum_{\substack{\mathbf{1}<F\leq R\\ FG=R}}\mu(\mathbf{1},F) = \sum_{\substack{\mathbf{1}\leq F\leq R\\ FG=R}}\mu(\mathbf{1},F)\ ,$$

and this is equal to zero, by a classical combinatorial lemma, since $G \neq \mathbf{1}$. And if $G = R$

$$\sum_{\substack{\mathbf{1}<F\leq R\\ FG=R}}\mu(\mathbf{1},F) = -\mu(\mathbf{1},\mathbf{1}) + \sum_{\mathbf{1}\leq F\leq R}\mu(\mathbf{1},F) = -1\ .$$

Thus $\mathrm{Def}^E_{E/G}\sigma_E(u) = \mathrm{Inf}^{E/G}_{E/G}u_G = u_G$, as was to be shown. □

2.4. Lemma. *Let E be an elementary abelian group of rank at least 2. Then the map r_E is surjective, and its kernel is the free abelian group of rank one generated by $\Omega_{E/\mathbf{1}}$.*

Proof. The kernel of r_E is the group $T(E)$ of endo-trivial modules. Since E is elementary abelian, this group is free of rank one, generated by $\Omega_{E/\mathbf{1}}$, by Dade's Theorem ([13] [14]). The surjectivity of r_E follows from Lemma 2.3. □

2.5. Restriction and conjugation. The following construction has been introduced in [7], for the torsion subgroup of the Dade group, but it works as well for the whole Dade group : let P be a finite p-group, and H be a subgroup of P. If $u \in \displaystyle\lim_{\substack{\longleftarrow\\ \mathbf{1}<Q\leq P}} D\big(N_P(Q)/Q\big)$, then the sequence $(v_Q)_{\mathbf{1}<Q\leq H}$ defined by

$$v_Q = \mathrm{Res}^{N_G(Q)/Q}_{N_H(Q)/Q}u_Q$$

is an element of $\displaystyle\lim_{\substack{\longleftarrow\\ \mathbf{1}<Q\leq H}} D\big(N_H(Q)/Q\big)$, denoted by $\mathrm{Res}^P_H u$. The map $u \mapsto \mathrm{Res}^P_H u$ is a linear map $\displaystyle\lim_{\substack{\longleftarrow\\ \mathbf{1}<Q\leq P}} D\big(N_P(Q)/Q\big)$ to $\displaystyle\lim_{\substack{\longleftarrow\\ \mathbf{1}<Q\leq Q}} D\big(N_Q(Q)/Q\big)$. The following is the analogue of Lemma 2.4 of [7] :

2.6. Lemma. *Let H be a subgroup of P. The diagram*

$$\begin{array}{ccc}
D(P) & \xrightarrow{r_P} & \lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<Q\leq P}} D\big(N_P(Q)/Q\big) \\
{\scriptstyle \mathrm{Res}^P_H}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Res}^P_H} \\
D(H) & \xrightarrow[r_H]{} & \lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<Q\leq H}} D\big(N_H(Q)/Q\big)
\end{array}$$

is commutative.

Proof. This is straightforward. □

Similarly, if $x \in P$, denote by $c_{x,H} : D(H) \to D({}^xH)$ the conjugation by x, sending v to ${}^xv$. If $u \in \lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<Q\leq H}} D\big(N_H(Q)/Q\big)$, then the sequence $(v_R)_{\mathbf{1}<R\leq {}^xH}$ defined by $v_R = c_{x,R^x}(u_{R^x})$ is an element of $\lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<R\leq {}^xH}} D\big(N_{{}^xH}(R)/R\big)$, that will be denoted by ${}^xu$. The assignment $u \mapsto {}^xu$ is a linear map from $\lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<Q\leq H}} D\big(N_H(Q)/Q\big)$ to $\lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<R\leq {}^xH}} D\big(N_{{}^xH}(R)/R\big)$, also denoted by $c_{x,H}$. Then :

2.7. Lemma. *Let H be a subgroup of P, and let E be an abelian subgroup of P. The following diagrams are commutative :*

$$\begin{array}{ccc}
D(H) & \xrightarrow{r_H} & \lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<Q\leq H}} D\big(N_H(Q)/Q\big) \\
{\scriptstyle c_{x,H}}\big\downarrow & & \big\downarrow{\scriptstyle c_{x,H}} \\
D({}^xH) & \xrightarrow{r_{{}^xH}} & \lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<R\leq {}^xH}} D\big(N_{{}^xH}(R)/R\big)
\end{array}
\qquad
\begin{array}{ccc}
D(E) & \xleftarrow{\sigma_E} & \lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<R\leq E}} D(E/R) \\
{\scriptstyle c_{x,E}}\big\downarrow & & \big\downarrow{\scriptstyle c_{x,E}} \\
D({}^xE) & \xleftarrow{\sigma_{{}^xE}} & \lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<R\leq {}^xE}} D(E/R)
\end{array}$$

Proof. This is also straightforward. □

2.8. Construction of a map. Let E and F be elements of $\mathcal{A}_{\geq 2}(P)$ such that $E < F$. If $v \in \lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<Q\leq P}} D\big(N_P(Q)/Q\big)$, consider the element

$$d_{E,F} = \mathrm{Res}^F_E \sigma_F \mathrm{Res}^P_F v - \sigma_E \mathrm{Res}^P_E v$$

of $D(E)$. Then by Lemma 2.4 and Lemma 2.3

$$\begin{aligned}
r_E(d_{E,F}) &= r_E \mathrm{Res}^F_E \sigma_F \mathrm{Res}^P_F v - r_E \sigma_E \mathrm{Res}^P_E v \\
&= \mathrm{Res}^F_E r_F \sigma_F \mathrm{Res}^P_F v - r_E \sigma_E \mathrm{Res}^P_E v \\
&= \mathrm{Res}^F_E \mathrm{Res}^P_F v - \mathrm{Res}^P_E v = 0 \ .
\end{aligned}$$

By Lemma 2.4, there exists a unique integer $w_{E,F}$ such that

$$d_{E,F} = w_{E,F} \cdot \Omega_{E/\mathbf{1}} \ .$$

If $x \in P$, then it is clear from Lemma 2.7 that ${}^x d_{E,F} = d_{{}^xE,{}^xF}$, and it follows that $w_{{}^xE,{}^xF} = w_{E,F}$. Moreover, if $E, F, G \in \mathcal{A}_{\geq 2}(P)$ with $E < F < G$, then

$$d_{E,F} + \mathrm{Res}_E^F d_{F,G} = d_{E,G} \ ,$$

hence $w_{E,F} + w_{F,G} = w_{E,G}$. In other words the function sending the pair (E, F) of elements of $\mathcal{A}_{\geq 2}(P)$, with $E < F$, to $w_{E,F}$, is a P-invariant 1-cocycle on $\mathcal{A}_{\geq 2}(P)$, with values in $\mathbb{Z}$:

2.9. Notation. *Let P be a finite p-group. A* P-invariant 1-cocycle *on $\mathcal{A}_{\geq 2}(P)$, with values in $\mathbb{Z}$, is a function sending a pair (E, F) of elements of $\mathcal{A}_{\geq 2}(P)$, with $E < F$, to an integer $w_{E,F}$, with the following two properties :*

1. *If $x \in P$ and $E < F$ in $\mathcal{A}_{\geq 2}(P)$, then $w_{{}^xE,{}^xF} = w_{E,F}$.*
2. *If $E < F < G$ in $\mathcal{A}_{\geq 2}(P)$, then $w_{E,F} + w_{F,G} = w_{E,G}$.*

The set $\Big(Z^1\big(\mathcal{A}_{\geq 2}(P)\big)\Big)^P$ of P-invariants 1-cocycles is a group for addition of functions.

Denote by $\Big(B^1\big(\mathcal{A}_{\geq 2}(P)\big)\Big)^P$ the subgroup of $\Big(Z^1\big(\mathcal{A}_{\geq 2}(P)\big)\Big)^P$ consisting of cocycles w for which there exists a P-invariant function $E \mapsto m_E$ from $\mathcal{A}_{\geq 2}(P)$ to $\mathbb{Z}$ such that

$$\forall E < F \in \mathcal{A}_{\geq 2}(P),\ w_{E,F} = m_F - m_E \ .$$

Denote by $H^1\big(\mathcal{A}_{\geq 2}(P), \mathbb{Z}\big)^{(P)}$ the factor group $\Big(Z^1\big(\mathcal{A}_{\geq 2}(P)\big)\Big)^P / \Big(B^1\big(\mathcal{A}_{\geq 2}(P)\big)\Big)^P$

2.10. Remark: One can show that the group $\Big(B^1\big(\mathcal{A}_{\geq 2}(P)\big)\Big)^P$ is also equal to the set of elements w of $\Big(Z^1\big(\mathcal{A}_{\geq 2}(P)\big)\Big)^P$ for which there exists a (not necessarily P-invariant) function $E \mapsto m_E$ such that $w_{E,F} = m_F - m_E$ for any $E < F$ in $\mathcal{A}_{\geq 2}(P)$. This is because if $E < F$ in $\mathcal{A}_{\geq 2}(P)$, then E and F are the "big component", which is P-invariant. Since w is P-invariant, it follows that the function

$$x \in P \mapsto m_{{}^xE} - m_E$$

does not depend on the choice of E, and that it is a group homomorphism from P to $\mathbb{Z}$ (i.e. an element of $H^1(P, \mathbb{Z})$). There are no non zero such homomorphisms, so m is actually P-invariant.

2.11. Remark: On the other hand, one can consider the ordinary first cohomology group $H^1\big(\mathcal{A}_{\geq 2}(P), \mathbb{Z}\big)$ of $\mathcal{A}_{\geq 2}(P)$ over $\mathbb{Z}$, which is defined similarly to $H^1\big(\mathcal{A}_{\geq 2}(P), \mathbb{Z}\big)^{(P)}$, but forgetting all conditions of P-invariance. Then the group P acts on $H^1\big(\mathcal{A}_{\geq 2}(P), \mathbb{Z}\big)$, and it follows from Remark 2.10 that $H^1\big(\mathcal{A}_{\geq 2}(P), \mathbb{Z}\big)^{(P)}$ is a subgroup of the group $H^1\big(\mathcal{A}_{\geq 2}(P), \mathbb{Z}\big)^P$ of P-invariant elements in $H^1\big(\mathcal{A}_{\geq 2}(P), \mathbb{Z}\big)$. It might happen however that this inclusion is proper : an argument similar to the one used in Remark 2.10 yields an element in $H^2(P, \mathbb{Z})$, and this group need not be zero.

2.12. Notation. *Let P be a finite p-group. Denote by*

$$h_P : \varprojlim_{\mathbf{1} < Q \leq P} D\big(N_P(Q)/Q\big) \to H^1\big(\mathcal{A}_{\geq 2}(P), \mathbb{Z}\big)^{(P)}$$

the map sending $v \in \varprojlim_{\mathbf{1}<Q\leq P} D\big(N_P(Q)/Q\big)$ *to the class of the 1-cocycle* w *defined by the following equality, for* $E < F$ *in* $\mathcal{A}_{\geq 2}(P)$ *:*

$$w_{E,F}\cdot\Omega_{E/\mathbf{1}} = \mathrm{Res}_E^F\sigma_F\mathrm{Res}_F^P v - \sigma_E\mathrm{Res}_E^P v\ . \tag{2.13}$$

2.14. Proposition. *Let* P *be a finite* p*-group. Then* h_P *is a group homomorphism, and the composition*

$$D(P)\xrightarrow{r_P}\varprojlim_{\mathbf{1}<Q\leq P} D\big(N_P(Q)/Q\big)\xrightarrow{h_P} H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big)^{(P)}$$

is equal to 0.

Proof. Clearly, the definition of h_P implies that it is a group homomorphism. Observe next that for any $E\in\mathcal{A}_{\geq 2}(P)$, since $r_E\sigma_E$ is the identity map, the image of the map $\sigma_E r_E - \mathrm{Id}_{D(E)}$ is contained in the kernel of r_E. By Lemma 2.4, it follows that there is a unique linear form s_E on $D(E)$, with values in $\mathbb{Z}$, such that

$$\sigma_E r_E(u) = u + s_E(u)\cdot\Omega_{E/\mathbf{1}}\ ,$$

for any $u\in D(E)$. By Lemma 2.7, this definition clearly implies that if $x\in P$, then $s_{{}^xE}({}^xu) = s_E(u)$, for any $u\in D(E)$.

Now if $E < F$ in $\mathcal{A}_{\geq 2}(P)$, and if $v = r_P(t)$, for $t\in D(P)$, Equation 2.13 becomes

$$\begin{aligned}
w_{E,F}\cdot\Omega_{E/\mathbf{1}} &= \mathrm{Res}_E^F\sigma_F\mathrm{Res}_F^P r_P(t) - \sigma_E\mathrm{Res}_E^P r_P(t)\\
&= \mathrm{Res}_E^F\sigma_F r_F\mathrm{Res}_F^P t - \sigma_E r_E\mathrm{Res}_E^P t \quad \text{(by Lemma 2.6)}\\
&= \mathrm{Res}_E^F\big(\mathrm{Res}_F^P t + s_F(\mathrm{Res}_F^P t)\cdot\Omega_{F/\mathbf{1}}\big) - \big(\mathrm{Res}_E^P t + s_E(\mathrm{Res}_E^P t)\cdot\Omega_{E/\mathbf{1}}\big)\\
&= \big(s_F(\mathrm{Res}_F^P t) - s_E(\mathrm{Res}_E^P t)\big)\cdot\Omega_{E/\mathbf{1}}\ .
\end{aligned}$$

Setting $m_E = s_E(\mathrm{Res}_E^P t)$, for $E\in\mathcal{A}_{\geq 2}(P)$, yields

$$w_{E,F} = m_F - m_E\ ,$$

hence $w\in\Big(B^1\big(\mathcal{A}_{\geq 2}(P)\big)\Big)^P$ (the P-invariance of m follows easily from the above remark, or from Remark 2.10). Thus $h_P r_P(u) = 0$, as was to be shown. □

The main theorem of this paper is the following :

2.15. Theorem. *Let* P *be a finite* p*-group. Then the sequence of abelian groups*

$$0\longrightarrow T(P)\longrightarrow D(P)\xrightarrow{r_P}\varprojlim_{\mathbf{1}<Q\leq P} D\big(N_P(Q)/Q\big)\xrightarrow{h_P} H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big)^{(P)}$$

is exact.

The key point in this theorem is to show that the kernel of h_P is equal to the image of r_P. This will be done in two steps : first take an element $u\in\mathrm{Ker}\ h_P$, and show that $2u\in r_P\big(2D(P)\big)$. This amounts to replacing D by $2D$, which is easier to handle, since it is torsion free. Next, write $2u = r_P(2v)$, for some $v\in D(P)$. Then $u - r_P(v)$ is an element in $\varprojlim_{\mathbf{1}<Q\leq P} D_t\big(N_P(Q)/Q\big)$, and it has been shown in [7] that such a sequence of compatible torsion elements can always be glued (i.e. it always lies in $r_P\big(D(P)\big)$, though possibly not in $r_P\big(D_t(P)\big)$).

3. Biset functors

The main ingredient in the proof of Theorem 2.15 is the formalism of *biset functors*. A short exposition of the notation and main results on this subject can be found in Section 2 of [9], Section 3 of [4], or Section 3 of [3].

Recall in particular that if (T,S) is a section of the group P, and if M is a biset functor, then the set P/S is a $(P,T/S)$-biset, and the corresponding *induction-inflation* morphism

$$M(P/S) : M(T/S) \to M(P)$$

is denoted by $\mathrm{Indinf}_{T/S}^{P}$. Similarly, the set $S\backslash P$ is a $(T/S,P)$-biset, and the corresponding *deflation-restriction* map

$$M(S\backslash P) : M(P) \to M(T/S)$$

is denoted by $\mathrm{Defres}_{T/S}^{P}$.

This notation was already used for the Dade group, and it is coherent : it was shown more generally in [6] that, if P and Q are finite p-groups, and U is a finite (Q,P)-biset, one can define a natural group homomorphism $D(U) : D(P) \to D(Q)$. This construction yields a structure of biset functor on the correspondence sending a p-group P to the subgroup $D^{\Omega}(P)$ of $D(P)$ generated by all the relative syzygies Ω_X obtained for various P-sets X. In the case $p \neq 2$, it was shown in [3] that $D = D^{\Omega}$, so D is a biset functor in this case. For $p = 2$, the biset functor structure on D^{Ω} cannot in general be extended to the whole of D, because of *Frobenius twists* (also called *Galois twists*) (see [6] or [3] for details).

The following additional notation was introduced in [9] : a class $\mathcal{Y}$ of p-groups is said to be *closed under taking subquotients* if for any $Y \in \mathcal{Y}$ and any section (T,S) of Y, the corresponding subquotient T/S belongs to $\mathcal{Y}$. If $\mathcal{Y}$ is such a class, and P is a finite p-group, denote by $\mathcal{Y}(P)$ the set of sections (T,S) of P such that $T/S \in \mathcal{Y}$. One can consider biset functors defined only on $\mathcal{Y}$, with values in abelian groups. Let $\mathcal{F}_{\mathcal{Y}}$ be the category of all such functors, and let $\mathcal{F}$ denote the category of functors defined on all finite p-groups. These categories are abelian categories.

The obvious forgetful functor

$$\mathcal{O}_{\mathcal{Y}} : \mathcal{F} \to \mathcal{F}_{\mathcal{Y}}$$

admits left and right adjoints

$$\mathcal{L}_{\mathcal{Y}} : \mathcal{F}_{\mathcal{Y}} \to \mathcal{F} \quad \text{and} \quad \mathcal{R}_{\mathcal{Y}} : \mathcal{F}_{\mathcal{Y}} \to \mathcal{F} \ ,$$

whose evaluations can be computed as follows (cf. [9] Theorem 1.2 for details, in particular on the direct and inverse limits appearing in this statement) :

3.1. Theorem. *With the notation above, for any functor $M \in \mathcal{F}_{\mathcal{Y}}$, we have :*

$$\mathcal{L}_{\mathcal{Y}}M(P) \cong \varinjlim_{(T,S)\in\mathcal{Y}(P)} M(T/S) \qquad \text{and} \qquad \mathcal{R}_{\mathcal{Y}}M(P) \cong \varprojlim_{(T,S)\in\mathcal{Y}(P)} M(T/S) \ .$$

Moreover ([9] Corollary 6.17), for any biset functor M and any p-group P, the unit map

$$\eta_{M,P} : M(P) \to \mathcal{R}_{\mathcal{Y}}\mathcal{O}_{\mathcal{Y}}(P) = \varprojlim_{(T,S)\in\mathcal{Y}(P)} M(T/S)$$

is given by

$$\eta_{M,P}(u)_{T,S} = \mathrm{Defres}_{T/S}^{P} u \ ,$$

for any $u \in M(P)$ and any $(T,S) \in \mathcal{Y}(P)$.

4. Image in subquotients of rank two

4.1. Lemma. *Let $\mathcal{X}$ be a non-empty class of finite p-groups, closed under taking subquotients. Let A be any abelian group, and denote by $\hat{B}$ the functor* $\mathrm{Hom}_{\mathbb{Z}}\big(B(-),A\big)$*, where B is the Burnside functor. Then the unit map*

$$\beta : \hat{B} \to \mathcal{R}_{\mathcal{X}}\mathcal{O}_{\mathcal{X}}\hat{B}$$

is an isomorphism.

Proof. Indeed if P is a p-group, if $\varphi \in \mathrm{Hom}_{\mathbb{Z}}\big(B(P),A\big)$, if (T,S) is a section of P and X is a subgroup such that $S \leq X \leq T$, then

$$(\mathrm{Defres}^P_{T/S}\varphi)\big((T/S)/(X/S)\big) = \varphi\big(\mathrm{Indinf}^P_{T/S}(T/S)/(X/S)\big) = \varphi(P/X)\ .$$

Suppose that $\varphi \in \mathrm{Ker}\ \beta$. Since the class $\mathcal{X}$ is non empty and closed under taking subquotients, then it contains the trivial group, and for any subgroup X of P, the section (X,X) is in $\mathcal{X}(P)$. It follows in particular that

$$0 = (\mathrm{Defres}^P_{X/X}\varphi)\big((X/X)/(X/X)\big) = \varphi(P/X)\ .$$

Thus $\varphi = 0$, and β is injective.

Conversely, let $\psi = (\psi_{T,S})_{(T,S)\in\mathcal{X}(P)}$ be an element of $\varprojlim_{\mathcal{X}(P)} \mathrm{Hom}_{\mathbb{Z}}\big(B(T/S),A\big)$. Equivalently, for each section $(T,S) \in \mathcal{X}(P)$ and each subgroup X with $S \leq X \leq T$, we have an element $\psi_{T,S}\big((T/S)/(X/S)\big)$ of A, fulfilling the following two conditions :

i) If $x \in P$, then $\psi_{^xT,\,^xS}\big((^xT/^xS)/(^xX/^xS)\big) = \psi_{T,S}\big((T/S)/(X/S)\big)$.

ii) If $(T,S) \in \mathcal{X}(P)$ and $(T',S') \in \mathcal{X}(P)$, and if X is a subgroup of P such that $S \leq S' \leq X \leq T' \leq T$, then $\psi_{T,S}\big((T/S)/(X/S)\big) = \psi_{T',S'}\big((T'/S')/(X/S')\big)$.

Condition *i)* implies in particular that the element $\psi_{X,X}\big((X/X)/(X/X)\big)$ is constant on the conjugacy class of X in P. Hence we can define an element $\varphi \in \mathrm{Hom}_{\mathbb{Z}}\big(B(P),A\big)$ by setting

$$\varphi(P/X) = \psi_{X,X}\big((X/X)/(X/X)\big)\ ,$$

for any subgroup X of P. Now if $(T,S) \in \mathcal{X}(P)$ and if $S \leq X \leq T$

$$\begin{aligned}(\mathrm{Defres}^P_{T/S}\varphi)\big((T/S)/(X/S)\big) &= \varphi(P/X) = \psi_{X,X}\big((X/X)/(X/X)\big)\\ &= (\mathrm{Defres}^{T/S}_{X/X}\psi_{T,S})\big((X/X)/(X/X)\big)\\ &= \psi_{T,S}\big(\mathrm{Indinf}^{T/S}_{(T/S)/(X/X)}(X/X)/(X/X)\big)\\ &= \psi_{T,S}\big((T/S)/(X/S)\big)\ .\end{aligned}$$

Thus $\mathrm{Defres}^P_{T/S}\varphi = \psi_{T,S}$ for any $(T,S) \in \mathcal{X}(P)$. Equivalently $\beta(\varphi) = \psi$, so β is surjective, hence it is an isomorphism. □

4.2. Lemma. *When p is odd, the map*

$$\varepsilon = \prod_{(T,S)\in\mathcal{X}_3(P)} \mathrm{Defres}^P_{T/S} :\ 2D(P) \longrightarrow \varprojlim_{(T,S)\in\mathcal{X}_3(P)} 2D(T/S)$$

is an isomorphism.

Proof. Consider the short exact sequence of p-biset functors

$$0 \to 2D \to D \to \mathbb{F}_2 D \to 0 \; .$$

Applying the functor $\displaystyle\lim_{\longleftarrow \atop \mathcal{X}_3(P)}$ yields the bottom line of the following commutative diagram with exact lines

(**4.3**)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & 2D(P) & \longrightarrow & D(P) & \longrightarrow & \mathbb{F}_2D(P) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \varepsilon} & & \Big\downarrow{\scriptstyle \delta} & & \Big\downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \displaystyle\lim_{\longleftarrow \atop \mathcal{X}_3(P)} 2D & \longrightarrow & \displaystyle\lim_{\longleftarrow \atop \mathcal{X}_3(P)} D & \longrightarrow & \displaystyle\lim_{\longleftarrow \atop \mathcal{X}_3(P)} \mathbb{F}_2D & &
\end{array}$$

where the map δ is an isomorphism, by Theorem 1.1 of [9]. Moreover, by Corollary 1.5 of [10], there is an exact sequence of p-biset functors

$$0 \to B^\times \to \mathbb{F}_2 B^* \to \mathbb{F}_2 D^\Omega \to 0 \; ,$$

where $B^\times$ is the functor of units of the Burnside ring, which is isomorphic to the constant functor $\Gamma_{\mathbb{F}_2}$ for p odd. Moreover $D^\Omega = D$ in this case. Applying the functor $\displaystyle\lim_{\longleftarrow \atop \mathcal{X}_3(P)}$ to this sequence yields the bottom line of the following commutative diagram with exact lines

(**4.4**)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{F}_2 & \longrightarrow & \mathbb{F}_2B^*(P) & \longrightarrow & \mathbb{F}_2D(P) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \displaystyle\lim_{\longleftarrow \atop \mathcal{X}_3(P)} \Gamma_{\mathbb{F}_2} & \longrightarrow & \displaystyle\lim_{\longleftarrow \atop \mathcal{X}_3(P)} \mathbb{F}_2B^* & \longrightarrow & \displaystyle\lim_{\longleftarrow \atop \mathcal{X}_3(P)} \mathbb{F}_2D & &
\end{array}$$

Since $\mathbb{F}_2B^*$ is naturally isomorphic to $\mathrm{Hom}_{\mathbb{Z}}\big(B(-), \mathbb{F}_2\big)$, the map β is an isomorphism, by Lemma 4.1. Now the group $\displaystyle\lim_{\longleftarrow \atop \mathcal{X}_3(P)} \Gamma_{\mathbb{F}_2}$ is the set of sequences $(u_{T,S})_{(T,S)\in\mathcal{X}_3(P)}$ fulfilling the two following conditions :

i) If $(T, S) \in \mathcal{X}_3(P)$ and $x \in P$, then $u_{{}^xT,{}^xS} = u_{T,S}$.

ii) If $(T, S) \in \mathcal{X}_3(P)$ and $(T', S') \in \mathcal{X}_3(P)$ are such that $S \leq S' \leq T' \leq T$, then $u_{T,S} = u_{T',S'}$.

Applying this for the case $T' = S' = S$, and next for the case $T = T' = S'$, it follows that $u_{T,S} = u_{S,S} = u_{T,T}$ for any $(T, S) \in \mathcal{X}_3(P)$. Thus $u_{T,T} = u_{S,S}$ if $T/S \in \mathcal{X}_3$. Since $\mathcal{X}_3$ contains the cyclic group of order p, and since P is a p-group, it follows that $u_{T,T} = u_{\mathbf{1},\mathbf{1}}$ for any subgroup T of P. Hence $u_{T,S}$ is constant, and the map α is an isomorphism.

Now the Snake's Lemma, applied to Diagram 4.4, shows that the map γ is injective. And another application of this Lemma to Diagram 4.3 shows that the map ε is an isomorphism. □

Recall that if E is an elementary abelian group of rank 2, then $2D(E)$ is free of rank one, generated by $2\Omega_{E/\mathbf{1}}$. Thus if $u \in 2D(P)$, and if $(T, S) \in \mathcal{E}_2^\sharp(P)$, then $\mathrm{Defres}^P_{T/S} u$ is a multiple of $2\Omega_{T/S}$. The following theorem characterizes the sequences of integers $(v_{T,S})_{(T,S)\in\mathcal{E}_2^\sharp(P)}$ which can be obtained that way from an element of $2D(P)$:

4.5. Theorem. *Let P be a p-group (with $p > 2$). The map*

$$\mathcal{D}_P : 2D(P) \to \prod_{(T,S)\in\mathcal{E}_2^\sharp(P)} \mathbb{Z}$$

sending an element $u \in 2D(P)$ to the sequence $\mathcal{D}_P(u)_{T,S}$ of integers defined by

$$\mathrm{Defres}^P_{T/S}(u) = \mathcal{D}_P(u)_{T,S} \cdot 2\Omega_{T/S} \ ,$$

is injective, and its image is equal to the set of sequences $(v_{T,S})_{(T,S)\in\mathcal{E}_2^\sharp(P)}$ fulfilling the following conditions :

1. *If $(T,S) \in \mathcal{E}_2^\sharp(P)$ and $x \in P$, then $v_{T,S} = v_{{}^xT,{}^xS}$.*
2. *If (T,S) and (T',S) are in $\mathcal{E}_2^\sharp(P)$, if $T' \leq N_P(T)$, and $TT'/S \cong (\mathbb{Z}/p\mathbb{Z})^3$, then*

$$v_{T,S} + \sum_{S<X<T} v_{TT',X} = v_{T',S} + \sum_{S<X<T'} v_{TT',X} \ .$$

3. *If (T,S) and (T',S) are in $\mathcal{E}_2^\sharp(P)$, if $T' \leq N_P(T)$, and $TT'/S \cong X_{p^3}$, then*

$$v_{T,S} \equiv v_{T',S} \pmod{p} \ .$$

Proof. Let $u \in \mathrm{Ker}\ \mathcal{D}_P$. Then $\mathrm{Defres}^P_{T/S}u = 0$ for any $(T,S) \in \mathcal{E}_2^\sharp(P)$. Moreover $\mathrm{Defres}^P_{T/S}u \in 2D(\mathbb{Z}/p\mathbb{Z}) = \{0\}$ when (T,S) is a section of P with $T/S \cong \mathbb{Z}/p\mathbb{Z}$. The detection theorem of Carlson and Thévenaz ([11] Theorem 13.1) shows that $u = 0$. Hence $\mathcal{D}_P$ is injective.

Now Condition 1 of Theorem 4.5 holds obviously for the elements of the image of $\mathcal{D}_P$. To check that Conditions 2 and 3 also hold, suppose that $(T,S), (T',S) \in \mathcal{E}_2^\sharp(P)$, with $T' \leq N_P(T)$ and $|TT' : P| = p^3$, and observe that setting $R = TT'$, the diagram

$$\begin{array}{ccc} D(P) & \xrightarrow{\mathcal{D}_P} & \prod_{(T'',S'')\in\mathcal{E}_2^\sharp(P)} \mathbb{Z} \\ \mathrm{Defres}^P_{R/S} \downarrow & & \downarrow \pi_{R,S} \\ D(R/S) & \xrightarrow{\mathcal{D}_P} & \prod_{(T''/S,S''/S)\in\mathcal{E}_2^\sharp(R/S)} \mathbb{Z} \end{array}$$

is commutative, where $\pi_{R,S}$ is the projection map obtained by identifying sections (T'',S'') of P such that $S \leq S'' \leq T'' \leq R$ with sections $(T''/S, S''/S)$ of R/S. This shows that it is enough to suppose that the group P is either elementary abelian of rank 3, or isomorphic to X_{p^3}. These special cases are detailed below.

To prove that conversely, Conditions 1, 2, and 3 characterize the image of $\mathcal{D}_P$, observe first that by Theorem 1.1 of [9], the map

$$\prod_{(T,S)} \mathrm{Defres}^P_{T/S} : \ D(P) \longrightarrow \varprojlim_{(T,S)} D(T/S)$$

is an isomorphism, where T/S runs through all sections of P which are either elementary abelian p-groups of rank ≤ 3 or extraspecial groups of order p^3 and exponent p.

Suppose that Theorem 4.5 is true when P is elementary abelian of rank at most 3, or extraspecial of exponent p. Let P be an arbitrary p-group, and consider a sequence $v = (v_{T,S})_{(T,S)\in\mathcal{E}_2^\sharp(P)}$ fulfilling the conditions of Theorem 4.5.

If $(V,U) \in \mathcal{X}_3(P)$, then the correspondence $(T,S) \mapsto (T/U, S/U)$ is a one to one correspondence between the set of elements (T,S) of $\mathcal{E}_2^\sharp(P)$ such that $U \leq S \leq T \leq V$, and $\mathcal{E}_2^\sharp(V/U)$. Through this bijection, the sequence of integers $v_{T,S}$, for $U \leq S \leq T \leq V$, yields a sequence of integers fulfilling the conditions of Theorem 4.5 for the group V/U, hence an element in the image of the map $\mathcal{D}_{V/U}$. In other words, there is a unique element $w_{V,U} \in 2D(V/U)$ such that

$$\mathrm{Defres}_{T/S}^{V/U} w_{V,U} = v_{T,S} \cdot 2\Omega_{T/S} \ ,$$

for all $(T,S) \in \mathcal{E}_2^\sharp(P)$ with $U \leq S \leq T \leq V$.

Now the uniqueness of $w_{V,U}$ shows that $w_{{}^xV,{}^xU} = {}^x w_{V,U}$ for any $x \in P$, and that $\mathrm{Defres}_{V'/U'}^{V/U} w_{V,U} = w_{V',U'}$ whenever (V,U) and (V',U') are in $\mathcal{X}_3(P)$ and $U \leq U' \leq V' \leq V$. In other words, the sequence $(w_{V,U})_{(V,U)\in\mathcal{X}_3(P)}$ is an element of $\varprojlim_{\mathcal{X}_3(P)} 2D$. By Lemma 4.2, there exists an element $t \in 2D(P)$ such that

$$w_{V,U} = \mathrm{Defres}_{V/U}^{P} t \ ,$$

for any $(V,U) \in \mathcal{X}_3(P)$. Then obviously $\mathcal{D}_P(t) = v$, and v lies in the image of $\mathcal{D}_P$, as was to be shown.

So the only thing left to check is that Theorem 4.5 holds when P is elementary abelian of rank at most 3, or isomorphic to X_{p^3}. This is a case by case verification, using the following lemma :

4.6. Lemma. *Let P be a finite p-group, and X be a finite P-set.*

1. *If T/S is a section of P, then*

$$\mathrm{Defres}_{T/S}^{P} \Omega_X = \Omega_{X^S} \ ,$$

where X^S denotes the set of fixed points of S on X, viewed as a T/S-set.

2. *If moreover $(T,S) \in \mathcal{E}_2^\sharp(P)$, then*

$$\mathrm{Defres}_{T/S}^{P}(2\Omega_X) = \Big(\sum_{\substack{S\leq V\leq T\\ X^V\neq\emptyset}} \mu(S,V)\Big) 2\Omega_{T/S} \ .$$

In other words $\mathcal{D}_P(2\Omega_X)_{T,S} = \sum_{\substack{S\leq V\leq T\\ X^V\neq\emptyset}} \mu(S,V)$.

Proof. Assertion 1 follows from Section 4 of [2]. For Assertion 2, note that by Assertion 1 and Lemma 5.2.3 of [2], since T/S is abelian,

$$\mathrm{Defres}_{T/S}^{P} 2\Omega_X = \sum_{\substack{S\leq U\leq V\leq T\\ X^V\neq\emptyset}} \mu(U,V) \cdot 2\Omega_{T/U} \ ,$$

and that $2\Omega_{T/U} = 0$ in $D(T/S)$ unless $U = S$. ∎

Now there are four cases :

• If $|P| \leq p$, there is nothing to do, since the map $\mathcal{D}_P$ is an isomorphism $\{0\} \to \{0\}$.

• If $P \cong (\mathbb{Z}/p\mathbb{Z})^2$, then $2D(P) \cong \mathbb{Z}$, and $\mathcal{E}_2^\sharp(P) = \{(P,\mathbf{1})\}$. In this case, there is no condition on the image of $\mathcal{D}_P$, and $\mathcal{D}_P$ is an isomorphism $\mathbb{Z} \to \mathbb{Z}$. So Theorem 4.5 holds in this case.

• If $P \cong (\mathbb{Z}/p\mathbb{Z})^3$, then $\mathcal{E}_2^\sharp(P)$ consists of p^2+p+1 sections (P,R), for $|R| = p$, and p^2+p+1 sections $(Q,\mathbf{1})$, for $|Q| = p^2$. The group $2D(P)$ is a free abelian group, with basis

$$\{2\Omega_{P/\mathbf{1}}\} \sqcup \{2\Omega_{P/R} \mid |R| = p\} \ .$$

The following arrays gives the values of the sequence $v = \mathcal{D}_P(u)$ for the element u in its first column on the left :

	$v_{P,R}$	$v_{Q,\mathbf{1}}$
$2\Omega_{P/\mathbf{1}}$	0	1
$2\Omega_{P/R'}$	$\begin{cases} 0 & \text{if } R' \neq R \\ 1 & \text{if } R' = R \end{cases}$	$\begin{cases} 0 & \text{if } R' < Q \\ 1 & \text{if } R' \not< Q \end{cases}$

The image of the element $u = m_{\mathbf{1}} \cdot 2\Omega_{P/\mathbf{1}} + \sum\limits_{|R'|=p} m_{R'} \cdot 2\Omega_{P/R'}$ by the map $\mathcal{D}_P$ is equal to the sequence $v = (v_{T,S})$, where

$$(\mathbf{4.7}) \qquad v_{P,R} = m_R \qquad v_{Q,\mathbf{1}} = m_{\mathbf{1}} + \sum_{R \not< Q} m_R$$

If $Q \neq Q'$ are subgroups of order p^2 of P, then $QQ' = P$, and

$$v_{Q,\mathbf{1}} + \sum_{\mathbf{1}<X<Q} v_{P,X} = m_{\mathbf{1}} + \sum_{|X|=p} m_X = v_{Q',\mathbf{1}} + \sum_{\mathbf{1}<X<Q'} v_{P,X} \ ,$$

so Condition 2 of 4.5 holds for the sections $(Q,\mathbf{1})$ and $(Q',\mathbf{1})$ of P. Since P is abelian, Conditions 1 and 3 of 4.5 are obviously satisfied.

Conversely, suppose that Condition 2 hold for a sequence $v = (v_{T,S})_{(T,S)\in\mathcal{E}_2^\sharp(P)}$. This sequence is in the image of $\mathcal{D}_P$ if and only if there exist integers $m_{\mathbf{1}}$, $m_{R'}$, for $|R'| = p$, such that 4.7 hold.

The first equation gives $m_R = v_{P,R}$, and then the second one gives

$$m_{\mathbf{1}} = v_{Q,\mathbf{1}} - \sum_{R \not< Q} v_{P,R} \ .$$

This is consistent if the right hand side does not depend on Q, i.e. if for any subgroups $Q \neq Q'$ of order p^2 of P

$$v_{Q,\mathbf{1}} - \sum_{R \not< Q} v_{P,R} = v_{Q',\mathbf{1}} - \sum_{R \not< Q'} v_{P,R} \ ,$$

or equivalently

$$v_{Q,\mathbf{1}} + \sum_{R<Q} v_{P,R} = v_{Q',\mathbf{1}} + \sum_{R<Q'} v_{P,R} \ .$$

This is precisely Condition 2 of 4.5 for the section $(Q,\mathbf{1})$ and $(Q',\mathbf{1})$, since $QQ' = P$ in this case. Thus Theorem 4.5 holds for $P \cong (\mathbb{Z}/p\mathbb{Z})^3$.

• If $P \cong X_{p^3}$, then $\mathcal{E}_2^\sharp(P)$ consists of the section (P,Z), where Z is the centre of P, and of $p+1$ sections $(Q,\mathbf{1})$, where Q is a subgroup of index p in P. The group $D(P)$ is equal to $D^\Omega(P)$, since $p \neq 2$, so it is generated by the elements $\Omega_{P/\mathbf{1}}$, $\Omega_{P/X}$, for $|X| = p$, and $\Omega_{P/Q}$, for $|Q| = p^2$, which have order 2 in $D(P)$. Thus $2D(P)$ is generated by the elements $2\Omega_{P/\mathbf{1}}$ and $2\Omega_{P/X}$, for $|X| = p$. The following array gives the values of the sequence $v = \mathcal{D}_P(u)$ for the element u in its first column on the left, where R denotes a non central subgroup of order p of P :

	$v_{P,Z}$	$v_{Q,\mathbf{1}}$
$2\Omega_{P/\mathbf{1}}$	0	1
$2\Omega_{P/Z}$	1	0
$2\Omega_{P/R}$	0	$\begin{cases} 1 & \text{if } R \not< Q \\ 1-p & \text{if } R < Q \end{cases}$

The values in this table can be computed using Lemma 4.6 : for example

$$\mathrm{Res}^P_Q 2\Omega_{P/R} = \Big(\sum_{\substack{\mathbf{1} \leq V \leq Q \\ V \leq_P R}} \mu(\mathbf{1}, V)\Big) 2\Omega_{Q/\mathbf{1}} .$$

If $R \not\leq Q$, then there is only one term in the summation, for $V = \mathbf{1}$, and $\mu(\mathbf{1}, V) = 1$ in this case. And if $R \leq Q$, then there are p additional terms, obtained for the p distinct conjugates V of R in P, and $\mu(\mathbf{1}, V) = -1$ for each of them. This gives the value $1-p$ in this case.

Now if $u = m_{\mathbf{1}} \cdot 2\Omega_{P/\mathbf{1}} + m_Z \cdot 2\Omega_{P/Z} + \sum_{[R]} m_R \cdot 2\Omega_{P/R}$ (where the brackets around R mean that R runs through a set of representatives of conjugacy classes of non central subgroups of order p of P), then the sequence $v = \mathcal{D}_P(u)$ is given by :

(4.8) $$v_{P,Z} = m_Z \qquad v_{Q,\mathbf{1}} = m_{\mathbf{1}} + \sum_{[R] \not< Q} m_R + (1-p) \sum_{[R] < Q} m_R ,$$

The second equation is equivalent to

(4.9) $$v_{Q,\mathbf{1}} = m_{\mathbf{1}} + \sum_{[R]} m_R - p \sum_{[R]<Q} m_R .$$

It follows that $v_{Q,\mathbf{1}} \equiv v_{Q',\mathbf{1}} \pmod{p}$, for any subgroups Q and Q' of order p^2 in P. This shows that Condition 3 of 4.5 holds for the sections $(Q,\mathbf{1})$ and $(Q',\mathbf{1})$ of P. Condition 2 is obviously satisfied in this case, since P has no subquotient isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$.

Suppose now conversely that a sequence $v = (v_{T,S})_{(T,S) \in \mathcal{E}_2^\sharp(P)}$ is given, and that Conditions 1 and 3 of 4.5 hold. Then v lies in the image of $\mathcal{D}_P$ if and only if there exist integers $m_{\mathbf{1}}$, m_Z, m_R (invariant by P-conjugation), such that Equations 4.8 hold.

The first equation in 4.8 gives $m_Z = v_{P,Z}$, and the second one gives

$$m_{\mathbf{1}} - v_{Q,\mathbf{1}} + \sum_{[R]} m_R = p \sum_{[R]<Q} m_R .$$

All subgroups of order p of Q different from Z are conjugate in P. Denoting by R_Q one of them, this equation becomes

(4.10) $$m_{\mathbf{1}} - v_{Q,\mathbf{1}} + \sum_{[R]} m_R = p m_{R_Q} .$$

Summing this relation over Q yields

$$(p+1)m_{\mathbf{1}} - \sum_{Q} v_{Q,\mathbf{1}} + (p+1)\sum_{[R]} m_R = p\sum_{[R]} m_R \ ,$$

thus

$$\sum_{[R]} m_R = \sum_{Q} v_{Q,\mathbf{1}} - (p+1)m_{\mathbf{1}} \ .$$

Now 4.10 yields

$$pm_{R_Q} = \sum_{Q' \neq Q} v_{Q',\mathbf{1}} - pm_{\mathbf{1}} \ .$$

By Condition 3, the sum $\sum\limits_{Q' \neq Q} v_{Q',\mathbf{1}}$ is congruent to $pv_{Q,\mathbf{1}}$ modulo p, i.e. to 0. Since $Q = R_Q Z$, this gives finally

$$m_R = \frac{1}{p}\Big(\sum_{Q' \neq RZ} v_{Q',\mathbf{1}}\Big) - m_{\mathbf{1}} \ .$$

Conversely, if this holds for any R, then equation 4.9 holds : indeed, in this case

$$\sum_{[R]} m_R = \sum_{Q} v_{Q,\mathbf{1}} - (p+1)m_{\mathbf{1}} \ ,$$

thus

$$\begin{aligned} m_{\mathbf{1}} + \sum_{[R]} m_R - p\sum_{[R]<Q} m_R &= m_{\mathbf{1}} + \sum_{Q} v_{Q,\mathbf{1}} - (p+1)m_{\mathbf{1}} - pm_{R_Q} \\ &= \sum_{Q} v_{Q,\mathbf{1}} - pm_{\mathbf{1}} - \Big(\sum_{Q' \neq Q} v_{Q',\mathbf{1}}\Big) + pm_{\mathbf{1}} \\ &= v_{Q,\mathbf{1}} \ . \end{aligned}$$

Thus 4.9 holds, and Theorem 4.5 also, when $P = X_{p^3}$. □

5. Proof of Theorem 2.15

Let P be a finite p-group. Clearly $T(P)$ is the kernel of r_P, and Im $r_P \leq$ Ker h_P, by Proposition 2.14. So the only thing to show is that this inclusion is an equality.

Let $u \in$ Ker h_P. It means that there exists a P-invariant function $E \mapsto m_E$ from $\mathcal{A}_{\geq 2}(P)$ to $\mathbb{Z}$ such that for any $E < F$ in $\mathcal{A}_{\geq 2}(P)$

$$w_{E,F} = m_E - m_F \ ,$$

where the integer $w_{E,F}$ is defined by the equality

$$w_{E,F} \cdot \Omega_{E/\mathbf{1}} = \mathrm{Res}_E^F \sigma_F \mathrm{Res}_F^P u - \sigma_E \mathrm{Res}_E^P u \ .$$

In other words

$$\mathrm{Res}_E^F(m_F \cdot \Omega_{F/\mathbf{1}} + \sigma_F \mathrm{Res}_F^P u) = m_E \cdot \Omega_{E/\mathbf{1}} + \sigma_E \mathrm{Res}_E^P u \ .$$

Set $w_E = m_E \cdot \Omega_{E/\mathbf{1}} + \sigma_E \mathrm{Res}^P_E u$, for $E \in \mathcal{A}_{\geq 2}(P)$. Then $\mathrm{Res}^F_E w_F = w_E$, for any $E < F$ in $\mathcal{A}_{\geq 2}(P)$, and ${}^x(w_E) = w_{{}^xE}$ for any $x \in P$ and $E \in \mathcal{A}_{\geq 2}(P)$. Moreover , for any $E \in \mathcal{A}_{\geq 2}(P)$

$$r_E(w_E) = r_E \sigma_E \mathrm{Res}^P_E u = \mathrm{Res}^P_E u \ ,$$

since $r_E(\Omega_{E/\mathbf{1}}) = 0$, and since σ_E is a section of r_E. It means that for any subgroup $Y \neq \mathbf{1}$ of E

$$\mathrm{Def}^E_{E/Y} w_E = \mathrm{Res}^{N_P(Y)/Y}_{E/Y} u_Y \ .$$

If $(T,S) \in \mathcal{E}_2^\sharp(P)$, define an integer $v_{T,S}$ by

$$\textbf{(5.1)} \qquad \begin{cases} \mathrm{Res}^{N_P(S)/S}_{T/S}(2u_S) = v_{T,S} \cdot 2\Omega_{T/S} & \text{if } S \neq \mathbf{1} \\ 2w_T = v_{T,\mathbf{1}} \cdot 2\Omega_{T/\mathbf{1}} & \text{if } S = \mathbf{1} \end{cases}$$

This sequence of integers $(v_{T,S})_{(T,S)\in\mathcal{E}_2^\sharp(P)}$ satisfies some of the conditions of Theorem 4.5. Indeed :

- If $x \in P$ and $(T,S) \in \mathcal{E}_2^\sharp(P)$, then $v_{{}^xT,{}^xS} = v_{T,S}$: this is because ${}^x(w_E) = w_{{}^xE}$ for any $E \in \mathcal{A}_{\geq 2}(P)$, and because ${}^x(u_Q) = u_{{}^xQ}$ for any subgroup $Q \neq \mathbf{1}$ of P. Thus Condition 1 of Theorem 4.5 holds.
- Suppose that (T,S) and (T',S) are elements of $\mathcal{E}_2^\sharp(P)$ such that $T \leq N_P(T')$. There are two cases to consider :

(a) If $S \neq \mathbf{1}$, then for any section $(V,U) \in \mathcal{E}_2^\sharp\big(N_P(S)/S\big)$

$$\begin{aligned} v_{V,U} \cdot \Omega_{V/U} &= \mathrm{Res}^{N_P(U)/U}_{V/U}(2u_U) \\ &= \mathrm{Res}^{N_P(S,U)/U}_{V/U} \mathrm{Res}^{N_P(U)/U}_{N_P(S,U)/U}(2u_U) \\ &= \mathrm{Res}^{N_P(S,U)/U}_{V/U} \mathrm{Defres}^{N_P(S)/S}_{N_P(S,U)/U}(2u_S) \\ &= \mathrm{Defres}^{N_P(S)/S}_{V/U}(2u_S) \ . \end{aligned}$$

It follows that the sequence $(v_{V,U})_{(V,U)\in\mathcal{E}_2^\sharp\left(N_P(S)/S\right)}$ is equal to $\mathcal{D}_{N_P(S)/S}(2u_S)$, hence it is in the image of the map $\mathcal{D}_{N_P(S)/S}$. Thus if $TT'/S \cong (\mathbb{Z}/p\mathbb{Z})^3$, then Condition 2 of Theorem 4.5 holds for the sections (T,S) and (T',S) of $N_P(S)/S$. And if $TT'/S \cong X_{p^3}$, then Condition 3 of Theorem 4.5 holds, for a similar reason.

(b) If $S = \mathbf{1}$, then set $F = TT'$. If $F \cong (\mathbb{Z}/p\mathbb{Z})^3$, then consider a section $(V,U) \in \mathcal{E}_2^\sharp(F)$. If $U = \mathbf{1}$, then

$$\mathrm{Defres}^F_{V/U} 2w_F = \mathrm{Res}^F_V 2w_F = 2w_V = v_{V,\mathbf{1}} \cdot 2\Omega_{V/\mathbf{1}} \ .$$

And if $U \neq \mathbf{1}$, then

$$\begin{aligned} \mathrm{Defres}^F_{V/U} 2w_F &= \mathrm{Res}^{F/U}_{V/U} \mathrm{Def}^F_{F/U} 2w_F \\ &= \mathrm{Res}^{F/U}_{V/U} \mathrm{Res}^{N_P(U)/U}_{F/U} 2u_U \\ &= \mathrm{Res}^{N_P(U)/U}_{V/U} 2u_U \\ &= v_{V,U} \cdot 2\Omega_{V/U} \ . \end{aligned}$$

It follows that the sequence $(v_{V,U})_{(V,U)\in\mathcal{E}_2^\sharp(F)}$ is equal to $\mathcal{D}_F(2w_F)$. In particular, Condition 2 of Theorem 4.5 is fulfilled for the sections $(T,\mathbf{1})$ and $(T',\mathbf{1})$ of F.

Hence the sequence $(v_{T,S})_{(T,S)\in\mathcal{E}_2^\sharp(P)}$ fulfills all the conditions of Theorem 4.5, except possibly Condition 3 for sections $(T,\mathbf{1})$ and $(T',\mathbf{1})$ such that $T\leq N_P(T')$ and $TT'\cong X_{p^3}$. This situation is handled by the following lemma :

5.2. Lemma. *Let P be a finite p-group, and $(v_{T,S})_{(T,S)\in\mathcal{E}_2^\sharp(P)}$ be a sequence of integers such that :*

1. *If $x\in P$ and $(T,S)\in\mathcal{E}_2^\sharp(P)$, then $v_{{}^xT,{}^xS}=v_{T,S}$.*
2. *If (T,S) and (T',S) are in $\mathcal{E}_2^\sharp(P)$, if $T\leq N_P(T')$ and if $TT'/S\cong(\mathbb{Z}/p\mathbb{Z})^3$, then*
$$v_{T,S}+\sum_{S<X<T}v_{TT',X}=v_{T',S}+\sum_{S<X<T'}v_{TT',X}\;.$$
3. *If $S\neq\mathbf{1}$, if (T,S) and (T',S) are in $\mathcal{E}_2^\sharp(P)$, if $T\leq N_P(T')$ and if $TT'/S\cong X_{p^3}$, then*
$$v_{T,S}\equiv v_{T',S}\ (\mathrm{mod}\ p)\;.$$

Then :

(i) If $T,T'\in\mathcal{A}_{=2}(P)$, if T and T' are in the same connected component of $\mathcal{A}_{\geq2}(P)$, if $T\leq N_P(T')$ and $TT'\cong X_{p^3}$, then $v_{T,\mathbf{1}}\equiv v_{T',\mathbf{1}}$ (mod p).

(ii) There exists a sequence of integers $(y_T)_{T\in\mathcal{A}_{=2}(P)}$ such that

(a) If $x\in P$ and $T\in\mathcal{A}_{=2}(P)$, then $y_{{}^xT}=y_T$.
(b) If $T,T'\in\mathcal{A}_{=2}(P)$, if $T\leq N_P(T')$ and $TT'\cong(\mathbb{Z}/p\mathbb{Z})^3$, then $y_T=y_{T'}$.
(c) If $T,T'\in\mathcal{A}_{=2}(P)$, if $T\leq N_P(T')$ and $TT'\cong X_{p^3}$, then
$$y_T+v_{T,\mathbf{1}}\equiv y_{T'}+v_{T',\mathbf{1}}\ (\mathrm{mod}\ p)\;.$$

Proof. The proof of Assertion (i) goes by induction on $|P|$, starting with the case where P is cyclic, where there is nothing to prove. Assume then that Hypotheses 1), 2), and 3) imply Assertion 1, for any p-group of order strictly smaller than $|P|$. Let T and T' be elementary abelian subgroups of rank 2 of P, such that $T\leq N_P(T')$ and $TT'\cong X_{p^3}$. Set $X=TT'$, and denote by Z the centre of X.

If there is a proper subgroup Q of P containing X, and such that T and T' are in the same connected component of $\mathcal{A}_{\geq2}(Q)$, then $v_{T,\mathbf{1}}\equiv v_{T',\mathbf{1}}$ (mod p), by induction, since Hypotheses 1), 2), and 3) obviously hold for Q is they hold for P. It is the case in particular if $\mathcal{A}_{\geq2}(Q)$ is connected.

Suppose that there exists a subgroup C of order p in $C_P(X)$, not contained in X (i.e. different from Z). Then the center $T''=C\times Z$ of the subgroup $Q=C\times X$ of P is not cyclic. Hence $\mathcal{A}_{\geq2}(Q)$ is connected, and Q contains T and T'. Thus I can suppose that $Q=P$, and then T'' is equal to the centre of P. It is elementary abelian of rank 2. Moreover $TT''\cong(\mathbb{Z}/p\mathbb{Z})^3$, since T and T'' are elementary abelian of rank 2 and centralize each other, and since $T\cap T''=T\cap X\cap T''=Z$. Hypothesis 2, applied to the sections $(T,\mathbf{1})$ and $(T',\mathbf{1})$ of P yields

$$v_{T,\mathbf{1}}-v_{T'',\mathbf{1}}=\sum_{\mathbf{1}<F<T''}v_{TT'',F}-\sum_{\mathbf{1}<F<T}v_{TT'',F}\;.\qquad\textbf{(5.3)}$$

Now $TT''\trianglelefteq P$ since $|P:TT''|=p$, and $T\trianglelefteq P$, since $T\trianglelefteq X$ and $C\leq C_P(X)$. Hence P acts by conjugation on the set of subgroups F such that $\mathbf{1}<F<T$, and $F=Z$ is the

unique fixed point under this action. Now Hypothesis 1 implies that

$$\sum_{\mathbf{1}<F<T} v_{TT'',F} \equiv v_{TT'',Z} \pmod{p} ,$$

and Equation 5.3 yields

$$(\mathbf{5.4}) \qquad v_{T,\mathbf{1}} - v_{T'',\mathbf{1}} \equiv \sum_{\substack{\mathbf{1}<F<T'' \\ F\neq Z}} v_{TT'',F} \pmod{p} .$$

The same argument applies with T' instead of T, so

$$(\mathbf{5.5}) \qquad v_{T',\mathbf{1}} - v_{T'',\mathbf{1}} \equiv \sum_{\substack{\mathbf{1}<F<T'' \\ F\neq Z}} v_{T'T'',F} \pmod{p} .$$

Now if $\mathbf{1} < F < T''$ and $F \neq Z$, the group P/F has order p^3 and exponent p (since P has exponent p), and it is non abelian (since $F \not\leq [P,P] = Z$). Hence $P/F \cong X_{p^3}$. Since $P = (TT'')(T'T'')$, Hypothesis 3, applied to the sections (TT'', F) and $(T'T'', F)$ yields $v_{TT'',F} \equiv v_{T'T'',F} \pmod{p}$. This shows that the right hand sides of 5.4 and 5.5 are congruent modulo p. So are the left hand sides, thus $v_{T,\mathbf{1}} - v_{T'',\mathbf{1}} \equiv v_{T',\mathbf{1}} - v_{T'',\mathbf{1}} \pmod{p}$, and $v_{T,\mathbf{1}} \equiv v_{T',\mathbf{1}} \pmod{p}$.

Hence I can suppose that Z is the only subgroup of order p of $C_P(X)$. In particular, the centre of P is cyclic, and Z is the only subgroup of order p in this centre. Moreover, since $T \neq T'$ and T, T' are in the same connected component of $\mathcal{A}_{\geq 2}(P)$, the groups T and T' are not maximal elementary abelian subgroups, thus P has p-rank at least equal to 3, and T and T' are in the big component $\mathcal{C}$ of $\mathcal{A}_{\geq 2}(P)$.

In this case, there is a normal subgroup T_0 of P which is elementary abelian of rank 2, and $T_0 \in \mathcal{C}$. Moreover $T_0 > Z$.

If $T_0 \not\leq X$, then $T_0 \cap X = Z$. Then $|T_0X| = p^4$, and $|T_0X : X| = p$. Thus T_0 normalizes X. Moreover, if Y is a subgroup of index p of X, then $Y > Z$, and $|T_0Y| = p^3$, thus $|T_0Y : Y| = p$, and T_0 normalizes Y. It follows that the image of T_0 in the group $\mathrm{Out}(X)$ of outer automorphisms of X, which is isomorphic to $GL_2(\mathbb{F}_p)$, is a p-subgroup stabilizing every line. So this image is trivial, and T_0 acts on X by inner automorphisms. Let $t \in T_0 - X$. Then there exists $y \in X$ such that $y^{-1}t \in C_P(X)$. In particular $y^{-1}t$ centralizes y, so t centralizes y, and then $(y^{-1}t)^p = (y^{-1})^p t^p = 1$. Moreover $y \neq t$, since $t \notin X$. Hence $y^{-1}t$ has order p. Since Z is the only subgroup of order p of $C_P(X)$, it follows that $y^{-1}t \in Z$, so $t \in X$. This contradiction shows that $T_0 \leq X$.

Since the congruences $v_{T,\mathbf{1}} \equiv v_{T_0,\mathbf{1}} \pmod{p}$ and $v_{T',\mathbf{1}} \equiv v_{T_0,\mathbf{1}} \pmod{p}$ imply the congruence $v_{T,\mathbf{1}} \equiv v_{T',\mathbf{1}} \pmod{p}$, it is enough to suppose that $T_0 = T$, thus $T \trianglelefteq P$. Let F be an elementary abelian subgroup of rank 3 of P containing T' : such a subgroup exists, since T' is not a maximal element of $\mathcal{A}_{\geq 2}(P)$. Set $T'' = C_F(T)$. Then $|F : T''|$ divide p, since F/T'' is a p-subgroup of $\mathrm{Aut}(T) \cong GL_2(\mathbb{F}_p)$. Moreover $F \not\leq C_P(T)$, since $F > T'$. Thus $|F : T''| = p$, and $T'' \cong (\mathbb{Z}/p\mathbb{Z})^2$. Moreover $T' \neq T''$, since $T' \not\leq C_P(T)$, thus $F = T'T''$.

Now F centralizes T', and normalizes T. Thus F normalizes $TT' = X$. Moreover $F \cap X = T'$, since $T' \leq F \cap X$, and since F and X are distinct subgroups of order p^3 of P, for F is abelian and X is not. Hence $|FX : F| = p$, so FX normalizes F. Thus X normalizes F, and X also normalizes $C_P(T)$ since $T \trianglelefteq P$. It follows that X normalizes $F \cap C_P(T) = C_F(T) = T''$. Obviously X also normalizes its subgroup T'.

Hypothesis 2, applied to the sections $(T',\mathbf{1})$ and $(T'',\mathbf{1})$ of P, yields

$$(\mathbf{5.6})\qquad v_{T',\mathbf{1}}-v_{T'',\mathbf{1}}=\sum_{1<Y<T''}v_{T'T'',Y}-\sum_{\mathbf{1}<Y<T'}v_{T'T'',Y}\ .$$

Since X normalizes T' and T'', and since any subgroup of order p normalized by X is centralized by X, Hypothesis 1 yields

$$\sum_{1<Y<T''}v_{T'T'',Y}\equiv\sum_{\substack{1<Y<T''\\ Y\leq C_P(X)}}v_{T'T'',Y}\ (\mathrm{mod}\ p)\ .$$

But $T''\cap C_P(X)=Z$, since $Z\leq T''\cap C_P(X)$, and since Z is the only subgroup of order p of $C_P(X)$. Thus

$$(\mathbf{5.7})\qquad \sum_{1<Y<T''}v_{T'T'',Y}\equiv v_{T'T'',Z}\ (\mathrm{mod}\ p)\ .$$

The same argument, applied with T' instead of T'', since $T'\cap C_P(X)=Z$, yields

$$(\mathbf{5.8})\qquad \sum_{1<Y<T'}v_{T'T'',Y}\equiv v_{T'T'',Z}\ (\mathrm{mod}\ p)\ .$$

Now it follows from 5.6, 5.7 and 5.8 that $v_{T',\mathbf{1}}-v_{T'',\mathbf{1}}\equiv 0\ (\mathrm{mod}\ p)$, i.e.

$$(\mathbf{5.9})\qquad v_{T',\mathbf{1}}\equiv v_{T'',\mathbf{1}}\ (\mathrm{mod}\ p)\ .$$

Now the group TT'' is also elementary abelian of rank 3 : indeed, the group T'' centralizes T, and $T''\cap T=Z$ since $T''\cap T\geq Z$ and $T''\neq T$ for $T\not\leq F\leq C_P(T')$. Then Hypothesis 2, for the sections $(T,\mathbf{1})$ and $(T'',\mathbf{1})$, yields

$$(\mathbf{5.10})\qquad v_{T,\mathbf{1}}-v_{T'',\mathbf{1}}=\sum_{1<Y<T''}v_{TT'',Y}-\sum_{\mathbf{1}<Y<T}v_{TT'',Y}\ .$$

The group X normalizes T and T'', and $T\cap C_P(X)=Z=T''\cap C_P(X)$. The same argument as above yields

$$(\mathbf{5.11})\qquad v_{T,\mathbf{1}}\equiv v_{T'',\mathbf{1}}\ (\mathrm{mod}\ p)\ .$$

Thus $v_{T,\mathbf{1}}\equiv v_{T',\mathbf{1}}\ (\mathrm{mod}\ p)$, by 5.9 and 5.11, and this completes the proof of Assertion (i).

For Assertion 2, there is nothing to do if P has no normal subgroup $T_0\cong(\mathbb{Z}/p\mathbb{Z})^2$, since then P is cyclic, and $\mathcal{A}_{\geq 2}(P)=\emptyset$. If P is not cyclic, fix such a normal subgroup T_0 of P, and denote by $\mathcal{C}$ the connected component of T_0 in $\mathcal{A}_{\geq 2}(P)$. Thus $\mathcal{C}$ is the big component if P has p-rank at least 3, and $\mathcal{C}=\{T_0\}$ otherwise. Define the sequence $(y_T)_{T\in\mathcal{A}_{=2}(P)}$ by

$$y_T=\begin{cases}0 & \text{if } T\in\mathcal{C}\\ v_{T_0,\mathbf{1}}-v_{T,1} & \text{otherwise}\end{cases}$$

This sequence obviously fulfills Condition (a) of Lemma 5.2, by Hypothesis 1, and since $\mathcal{C}$ is invariant by P-conjugation. Now it $T,T'\in\mathcal{A}_{=2}(P)$, if $T\leq N_P(T')$ and $TT'\cong(\mathbb{Z}/p\mathbb{Z})^3$, it follows that P has p-rank at least 3, that $\mathcal{C}$ is the big component, and that $T,T'\in\mathcal{C}$. Thus $y_T=y_{T'}=0$, so Condition (b) of Lemma 5.2 holds. Finally, if $T,T'\in\mathcal{A}_{=2}(P)$, if $T\leq N_P(T')$ and $TT'\cong X_{p^3}$, then there are three cases :

- if T and T' are in $\mathcal{C}$, then $y_T = y_{T'} = 0$, thus $y_T + v_{T,\mathbf{1}} = v_{T,\mathbf{1}}$, and $y_{T'} + v_{T',\mathbf{1}} = v_{T',\mathbf{1}}$. But $v_{T,\mathbf{1}} \equiv v_{T',\mathbf{1}} \pmod{p}$ in this case, by Assertion 1. Thus Condition (c) holds in this case.
- if $T \in \mathcal{C}$ and $T' \notin \mathcal{C}$, then $y_T + v_{T,\mathbf{1}} = v_{T,\mathbf{1}}$, and $y_{T'} + v_{T',\mathbf{1}} = v_{T_0,\mathbf{1}}$. But now T and T_0 are both in $\mathcal{C}$, so $v_{T,\mathbf{1}} \equiv v_{T_0,\mathbf{1}} \pmod{p}$ in this case, by Assertion 1. Thus Condition (c) holds in this case also. The case $T \notin \mathcal{C}$ and $T' \in \mathcal{C}$ is similar.
- if $T \notin \mathcal{C}$ and $T' \notin \mathcal{C}$, then $y_T + v_{T,\mathbf{1}} = v_{T_0,\mathbf{1}} = y_{T'} + v_{T',\mathbf{1}}$, so Condition (c) holds in this case also.

This completes the proof of Lemma 5.2. □

End of the proof of Theorem 2.15 : In the beginning of the proof of Theorem 2.15, I started with an element $u \in \varprojlim_{\mathbf{1}<Q\leq P} D\big(N_P(Q)/Q\big)$ such that $h_P(u) = 0$. From this data, in 5.1, I built a sequence of integers $(v_{T,S})_{(T,S)\in\mathcal{E}_2^\sharp(P)}$ fulfilling Hypothesis 1, 2 and 3 of Lemma 5.2. Let $(y_T)_{T\in\mathcal{A}_{=2}(P)}$ denote the sequence of integers provided by this lemma, and define a new sequence of integers $(v'_{T,S})_{(T,S)\in\mathcal{E}_2^\sharp(P)}$ by

$$v'_{T,S} = \begin{cases} v_{T,S} & \text{if } S \neq \mathbf{1} \\ y_T + v_{T,S} & \text{if } S = \mathbf{1} \end{cases} .$$

Then this sequence fulfills Conditions 1, 2, and 3 of Theorem 4.5 : indeed, the new sequence is clearly invariant by conjugation, so Condition 1 is fulfilled. Conditions 2 and 3 for sections (T,S) and (T',S) with $S \neq \mathbf{1}$ are obviously fulfilled, since they are for the sequence $(v_{T,S})$, and since $v_{T,S} = v'_{T,S}$ when $S \neq \mathbf{1}$.

Now if $T, T' \in \mathcal{A}_{=2}(P)$, if $T \leq N_P(T')$ and $TT' \cong (\mathbb{Z}/p\mathbb{Z})^3$, then T and T' are in the same connected component of $\mathcal{A}_{\geq 2}(P)$, and $y_T = y_{T'}$. Thus

$$\begin{aligned} v'_{T,\mathbf{1}} + \sum_{\mathbf{1}<Y<T} v'_{TT',Y} &= y_T + v_{T,\mathbf{1}} + \sum_{\mathbf{1}<Y<T} v_{TT',Y} \\ &= y_{T'} + v_{T',\mathbf{1}} + \sum_{\mathbf{1}<Y<T'} v_{TT',Y} \\ &= v'_{T',\mathbf{1}} + \sum_{\mathbf{1}<Y<T'} v'_{TT',Y} , \end{aligned}$$

so Condition 2 is fulfilled.

Finally if $T, T' \in \mathcal{A}_{=2}(P)$, if $T \leq N_P(T')$ and $TT' \cong X_{p^3}$, then

$$v'_{T,\mathbf{1}} = y_T + v_{T,\mathbf{1}} \equiv y_{T'} + v_{T',\mathbf{1}} \pmod{p} ,$$

hence Condition 3 is fulfilled, since $y_{T'} + v_{T',\mathbf{1}} = v'_{T',\mathbf{1}}$.

By Theorem 4.5, there exists $n \in D(P)$ such that $\mathcal{D}_P(2n) = (v'_{T,S})$. In other words, for any $(T,S) \in \mathcal{E}_2^\sharp(P)$

$$\mathrm{Defres}_{T/S}^P(2n) = v'_{T,S} \cdot 2\Omega_{T/S} .$$

Thus if $S \neq \mathbf{1}$

$$\mathrm{Defres}_{T/S}^P(2n) = v_{T,S} \cdot 2\Omega_{T/S} = \mathrm{Res}_{T/S}^{N_P(S)/S}(2u_S) .$$

Set $t_S = \mathrm{Defres}^P_{N_P(S)/S}(2n) - 2u_S$. Now for any $(V,U) \in \mathcal{E}_2^\sharp\big(N_P(S)/S\big)$

$$\begin{aligned}
\mathrm{Defres}^{N_P(S)/S}_{V/U}(t_S) &= \mathrm{Defres}^P_{V/U}(2n) - \mathrm{Defres}^{N_P(S)/S}_{V/U}(2u_S)\\
&= \mathrm{Defres}^P_{V/U}(2n) - \mathrm{Res}^{N_P(S,U)/U}_{V/U}\mathrm{Defres}^{N_P(S)/S}_{N_P(S,U)/U}(2u_S)\\
&= \mathrm{Defres}^P_{V/U}(2n) - \mathrm{Res}^{N_P(S,U)/U}_{V/U}\mathrm{Res}^{N_P(U)/U}_{N_P(S,U)/U}(2u_U)\\
&= \mathrm{Defres}^P_{V/U}(2n) - \mathrm{Res}^{N_P(U)/U}_{V/U}(2u_U) = 0\ .
\end{aligned}$$

It follows that t_S is a torsion element of $D\big(N_P(S)/S\big)$, which is also in $2D\big(N_P(S)/S\big)$. Since the latter is torsion free, it follows that $t_S = 0$, i.e. that $2u_S = \mathrm{Defres}^P_{N_P(S)/S}(2n)$, for any $S \neq \mathbf{1}$. Equivalently $2u = r_P(2n)$, or $2\big(u - r_P(n)\big) = 0$.

Now $u - r_P(n)$ is an element of $\varprojlim_{\mathbf{1}<Q\leq P} D_{tors}\big(N_P(Q)/Q\big)$. By Proposition 5.5 of [7], there exists an element $m \in D(P)$ such that $r_P(m) = u - r_P(n)$. It follows that $u = r_P(m+n)$, as was to be shown. This completes the proof of Theorem 2.15. $\square$

6. Example : the group X_{p^5}

Let P be an extraspecial group of order p^5 and exponent p. The centre Z of P is cyclic of order p, and it is equal to the Frattini subgroup of P. The commutator $P \times P \to Z$ induces a non degenerate symplectic $\mathbb{F}_p$-valued scalar product on the factor group $E = P/Z \cong (\mathbb{F}_p)^4$, and the map $Q \mapsto Q/Z$ is a poset isomorphism from the poset of elementary abelian subgroups of P strictly containing Z to the poset $\mathcal{E}$ of non zero totally isotropic subspaces of E. There are $e = \dfrac{p^4-1}{p-1}$ isotropic lines in E, and the same number of totally isotropic 2-dimensional subspaces. It follows that $|\mathcal{E}|$ is equal to $2e$.

There is a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & D(P/Z) & \xrightarrow[\cong]{d} & \varprojlim_{Q\geq Z} D(P/Q) & & \\
 & & & & \uparrow{\scriptstyle \mathrm{Def}^P_{P/Z}} & & \uparrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & T(P) & \longrightarrow & D(P) & \xrightarrow{r_P} & \varprojlim_{\mathbf{1}<Q\leq P} D\big(N_P(Q)/Q\big) & \xrightarrow{h_P} & H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big)^{(P)}
\end{array}$$

In this diagram, the group $\varprojlim_{Q\geq Z} D(P/Q)$ is the group of sequences $(u_Q)_{Z\leq Q\leq P}$, where $u_Q \in D(P/Q)$ (note that $Q \trianglelefteq P$ if $Q \geq Z$), such that

$$\forall R \geq Q \geq Z,\ \mathrm{Def}^{P/Q}_{P/R} u_Q = u_R\ ,$$

and the map π is the projection map on the components $Q \geq Z$. The map d is the product of the deflation maps $\mathrm{Def}^{P/Z}_{P/Q}$, for $Q \geq Z$. It is an isomorphism, since the sequence $(u_Q)_{Q\geq Z}$, where $u_Q \in D(P/Q)$, is in the group $\varprojlim_{Q\geq Z} D(P/Q)$ if and only if $u_Q = \mathrm{Def}^{P/Z}_{P/Q} u_Z$ for any $Q \geq Z$.

The kernel of the map $\mathrm{Def}^P_{P/Z}$ is the set of faithful elements of $D(P)$, and it was denoted by $\partial D(P)$ in [5]. It was shown in that paper (Theorem 9.1) that

$$\partial D(P) \cong \mathbb{Z}^{2e} \oplus \mathbb{Z}/2\mathbb{Z} .$$

The kernel of π consists of the sequences $(u_Q)_{\mathbf{1}<Q\leq P}$ in $\varprojlim_{\mathbf{1}<Q\leq P} D\big(N_P(Q)/Q\big)$ for which $u_Q = 0$ if $Q \geq Z$. It was shown in [5] that this is also the group $\varprojlim_{\substack{\mathbf{1}<Q\leq P\\ Q\cap Z=\mathbf{1}}} \partial D\big(N_P(Q)/Q\big)$.

All these facts show that there is an exact sequence

$$(\mathbf{6.1})\qquad 0 \longrightarrow T(P) \longrightarrow \partial D(P) \xrightarrow{r_P} \varprojlim_{\substack{\mathbf{1}<Q\leq P\\ Q\cap Z=\mathbf{1}}} \partial D\big(N_P(Q)/Q\big) \xrightarrow{h_P} H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big)^{(P)} ,$$

where r_P and h_P are the restrictions of the previously defined maps with the same names to the corresponding subgroups.

If Q is a subgroup of P such that $Q \cap Z = \mathbf{1}$, then Q is elementary abelian of rank at most 2 (see [5] for details). If Q has order p, then $N_P(Q)/Q \cong X_{p^3}$, thus $\partial D\big(N_P(Q)/Q\big) \cong \mathbb{Z}^{p+1} \oplus \mathbb{Z}/2\mathbb{Z}$ ([5] Theorem 9.1 or Section 11). If Q has order p^2, then $N_P(Q)/Q \cong \mathbb{Z}/p\mathbb{Z}$, thus $\partial D\big(N_P(Q)/Q\big) \cong \mathbb{Z}/2\mathbb{Z}$.

It follows easily that the group $\varprojlim_{\substack{\mathbf{1}<Q\leq P\\ Q\cap Z=\mathbf{1}}} \partial D\big(N_P(Q)/Q\big)$ has free rank at least equal to $e(p+1)$, since it contains the group $\bigoplus_{\substack{|Q|=p\\ Q\neq Z}} 2\partial D\big(N_P(Q)/Q\big)$.

Now the group $T(P)$ is free of rank one, generated by $\Omega_{P/\mathbf{1}}$, by Corollary 1.3 of [11], and the group $H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big)^{(P)}$ is isomorphic to $H^1(\mathcal{E},\mathbb{Z})$. An easy computation, using e.g. Section 6 of [5], shows that this group is free of rank p^4.

Now the free rank of the image of h_P in the exact sequence 6.1 is at least equal to

$$1 - 2e + e(p+1) = 1 + e(p-1) = p^4 ,$$

and since this is equal to the free rank of $H^1\big(\mathcal{A}_{\geq 2}(P),\mathbb{Z}\big)^{(P)}$, it follows that the free rank of the image of h_P is actually equal to p^4, and that the free rank of $\varprojlim_{\substack{\mathbf{1}<Q\leq P\\ Q\cap Z=\mathbf{1}}} \partial D\big(N_P(Q)/Q\big)$ is equal to $e(p+1)$.

Moreover the map h_P has finite cokernel, and this shows that in this case, the gluing problem does not always have a solution.

6.2. Remark: In this case, a precise description of the map h_P shows that its cokernel is a non trivial finite p-group.

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