Germs in a poset

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Abstract: Motivated by the theory of correspondence functors, we introduce the notion of *germ* in a finite poset, and the notion of *germ extension* of a poset. We show that any finite poset admits a largest germ extension called its *germ closure*. We say that a subset U of a finite lattice T is *germ extensible* in T if the germ closure of U naturally embeds in T. We show that any for any subset S of a finite lattice T, there is a unique germ extensible subset U of T such that $U \subseteq S \subseteq \overline{G}(U)$, where $\overline{G}(U) \subseteq T$ is the embedding of the germ closure of U.

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1. Introduction

In a series of papers with Jacques Thévenaz ([1], [2], [5], [4], [3]), we develop the theory of *correspondence functors* over a commutative ring k, i.e. linear representations over k of the category of finite sets, where morphisms are correspondences instead of maps. In this theory, finite lattices and finite posets play a crucial role, at various places.

In particular, we show ([1], Theorem 4.7) that the simple correspondence functors are parametrized by triples (E, R, V), where E is a finite set, R is a partial order relation on E - that is, (E, R) is a finite poset - and V is a simple $k\mathrm{Aut}(E, R)$-module. Moreover, the evaluation at a finite set X of the simple functor $S_{E,R,V}$ parametrized by the triple (E, R, V) can be completely described ([3], Theorem 6.6 and Theorem 7.9). It follows ([3], Theorem 8.2) that when k is a field, the dimension of $S_{E,R,V}(X)$ is given by

$$(1.1)\qquad \dim_k S_{E,R,V}(X) = \frac{\dim_k V}{|\mathrm{Aut}(E,R)|}\sum_{i=0}^{|E|}(-1)^i\binom{|E|}{i}\big(|G|-i\big)^{|X|}\ .$$

The main consequence of these results is a complete description of the simple modules for the algebra over k of the monoid of all relations on X ([3], Section 8).

Formula 1.1 is obtained by first choosing a finite lattice T such that the poset $\mathrm{Irr}(T)$ of join-irreducible elements of T is isomorphic to the opposite poset (E, R^{op}), and then constructing a specific subset $G = G_T$ of T (see (4.1) for a precise definition of G_T), which appears in the right hand side. Now for a given X, the left hand side of (1.1) only depends on the poset (E, R) and the simple $k\mathrm{Aut}(E, R)$-module V, whereas the right hand side depends in addition on the cardinality of the set G, which *a priori* depends on T, and not just on (E, R). We have checked ([3] Corollary 6.7) that $|G|$ indeed only depends on (E, R). A natural question is then to ask if the *subposet* G of T only depends on (E, R), up to isomorphism, and not really on T itself.

One of the aims of the present paper is to answer this question. In fact, the main aim is to introduce various structural results on posets and lattices, which appear to be new. The first notion we introduce is the notion of *germ* of a finite poset. A germ of a finite poset S is an element of S with specific properties (Definition 2.1).

A related notion is the following: When U is a subset of S, the poset S is called a *germ extension* of U if any element of $S - U$ is a germ of S. The main result of the paper (Theorem 2.22) is that conversely, being given a finite poset U, there is a (explicitly defined) finite poset $G(U)$, containing U as a full subposet, which is the largest germ extension of U, in the following sense: First $G(U)$ is a germ extension of U, and moreover, if S is a finite poset containing U as a full subposet, and such that S is a germ extension of U, then there is a unique full poset embedding $S \to G(U)$ which restricts to the identity map of U. For this reason, the poset $G(U)$ will be called the *germ closure* of U.

This poset $G(U)$ can be viewed as a structural answer to the above question: In the case of a finite lattice T with poset (E, R) of join-irreducible elements, we show (Theorem 4.2) that the poset set G identifies canonically with $G(E, R)$, and in particular, it only depends on the poset (E, R).

In Section 3, we consider *germ extensible* subsets of a finite lattice. For any full subposet U of a finite lattice T, the inclusion map $U \hookrightarrow T$ extends to a canonical map of posets $\nu : G(U) \to T$. We say that U is germ extensible in T if this map ν is injective, and in this case, we denote by $\overline{G}(U) \subseteq T$ its image. We give a characterization of germ extensible subsets of a lattice (Theorem 3.2), and then show (Theorem 3.4) that for any subset S of T, there exists a unique germ extensible subset U of T such that $U \subseteq S \subseteq \overline{G}(U)$.

In other words, the poset of subsets of T is partitioned by the intervals $[U, \overline{G}(U)]$, where U is a germ extensible subset of T. In a forthcoming paper, we will show how this rather surprising result yields a natural filtration of the correspondence functor F_T associated to T ([2], Definition 4.1), by fundamental functors indexed by germ extensible subsets of T.

The last section of the paper (Section 5) lists some examples of germs, germ closures, and germ extensible subsets of lattices.

2. Germs in a poset

Throughout the paper, we use the symbol $\subseteq$ for inclusion of sets, and the symbol $\subset$ for proper inclusion. We denote by $\sqcup$ the disjoint union of sets.

If $(U, \leq)$ is a poset, and u, v are elements of U, we set

$$\begin{array}{ll} [u,v]_U = \{w \in U \mid u \leq w \leq v\}, & [u,v[_U = \{w \in U \mid u \leq w < v\}, \\ ]u,v]_U = \{w \in U \mid u < w \leq v\}, &]u,v[_U = \{w \in U \mid u < w < v\}, \\ ]\,\cdot\,,v]_U = \{w \in U \mid w \leq v\}, &]\,\cdot\,,v[_U = \{w \in U \mid w < v\}, \\ [v,\,\cdot\,[_U = \{w \in U \mid w \geq v\}, &]v,\,\cdot\,[_U = \{w \in U \mid w > v\}. \end{array}$$

When V is a subset of a poset U, we denote by $\mathrm{Sup}_U V$ the least upper bound of V in U, when it exists. Similarly, we denote by $\mathrm{Inf}_U V$ the greatest lower bound of V in U, when it exists. For $u \in U$, when we write $u = \mathrm{Sup}_U V$ (resp. $u = \mathrm{Inf}_U V$), we mean that $\mathrm{Sup}_U V$ exists (resp. that $\mathrm{Inf}_U V$ exists) and is equal to u.

2.1. Definition: *Let $(U, \leq)$ be a finite poset. A* germ *of U is an element $u \in U$ such that there exists an element $v \geq u$ in U for which the following properties hold:*

1. $u = \mathrm{Sup}_U\,]\,\cdot\,,u[_U$ *and* $v = \mathrm{Inf}_U]v,\,\cdot\,[_U$.
2. $[u,\,\cdot\,[_U = [u,v]_U \sqcup]v,\,\cdot\,[_U$ *and* $]\,\cdot\,,v]_U =]\,\cdot\,,u[_U \sqcup [u,v]_U$.
3. $[u,v]_U$ *is totally ordered.*

Before giving some examples of germs (Examples 2.5, 2.6, 2.7, Proposition 2.8 - see also Section 5), we prove the following lemma motivating the subsequent definition of *cogerm*:

2.2. Lemma: *Let $(U, \leq)$ be a finite poset, and u be a germ of U. Then there exists a unique element $v \geq u$ in U with the properties of Definition 2.1.*

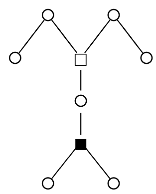
Proof : Let $v' \geq u$ be another element of U with the properties of the element v of Definition 2.1. Then $v' \in [u,\,\cdot\,[_U = [u,v]_U \sqcup]v,\,\cdot\,[_U$. In particular $u \leq v' \leq v$ or $v' \geq v$. If $v' \not\geq v$, then $u \leq v' < v$, and $]v',\,\cdot\,[_U =]v',v] \sqcup]v,\,\cdot\,[_U$. Moreover $]v',v]_U$ has a smallest element w, since $[u,v]_U$ is totally ordered. Then w is also the smallest element of $]v',\,\cdot\,[_U$, thus $w = \mathrm{Inf}_U]v',\,\cdot\,[_U \neq v'$, contradicting Property 1 of v' in Definition 2.1. It follows that $v' \geq v$. Exchanging the roles of v and v' gives $v \geq v'$, thus $v = v'$. □

2.3. Definition and Notation: *Let $(U, \leq)$ be a finite poset.*

1. *Let u be a germ of U. The unique element $v \geq u$ of U with the properties of Definition 2.1 is called the* cogerm *of u in U.*
2. *The set of germs of U is denoted by* $\mathrm{Grm}(U)$.

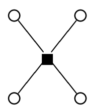
2.4. Remark: Clearly, if u is a germ of the poset U, then its cogerm v in U is a germ in the opposite poset U^{op}, and the cogerm of v in U^{op} is u. The correspondence $u \leftrightarrow v$ is a bijection between the germs of U and the germs of U^{op}.

2.5. Example: In the poset with the following Hasse diagram



the black square element is the only germ. The white square above it is its cogerm.

2.6. Example: Let u be an element of U such that $u = \mathrm{Sup}_U\,]\cdot, u[_U$ and $u = \mathrm{Inf}_U]u, \cdot[_U$. Then u is a germ of U, and u is its own cogerm in U. For example, the black square in the following poset



is the only germ, equal to its cogerm.

2.7. Example: Let $U = T_1 \times \ldots \times T_k$ be the direct product of $k \geq 2$ totally ordered sets T_i of cardinality at least 2. Then one can show that a germ of U is always equal to its cogerm. Moreover, an element $(t_1, \ldots, t_k)$ of U, different from the smallest and the greatest element of U, is a germ of U if and only if there are at least two indices $i \in \{1, \ldots, k\}$ such that $t_i \neq \mathrm{Inf}(T_i)$, and two indices $j \in \{1, \ldots, k\}$ such that $t_j \neq \mathrm{Sup}(T_j)$. In particular:

- If U is the poset of subsets of a finite set of cardinality $n \geq 2$, ordered by inclusion of subsets, then $\mathrm{Grm}(U)$ is the set of subsets of cardinality different from 1 and $n - 1$.

- If U is the set of divisors of an integer n which is not a prime power, ordered by divisibility, then $\mathrm{Grm}(U)$ is the union of $\{1,n\}$ with the set of divisors i of n such that i and n/i admit at least two distinct prime divisors.

2.8. Proposition: *Let $(U,\leq)$ be a finite poset.*

1. *If u is a minimal element of U which is also a germ of U, then u is the smallest element of U. In particular, a finite set of cardinality at least 2, ordered by the equality relation, admits no germs at all.*
2. *Conversely, suppose that U admits a smallest element u. Set $u_0 = u$, and define inductively a sequence $u_0 < u_1 < \ldots < u_n$, where each u_i is the smallest element of $]u_{i-1},\cdot[_U$, for $i \geq 1$, and $]u_n,\cdot[_U$ has no smallest element (possibly $n = 0$). Then $u = u_0$ is a germ of U, with cogerm u_n.*

Proof : Assertion 1 follows from the fact that $]\cdot,u[_U = \emptyset$ if u is minimal in U. Then $u = \mathrm{Sup}_U(\emptyset)$, as u is a germ of U, and for any poset U, if $\mathrm{Sup}_U(\emptyset)$ exists, it is the smallest element of U. Finally, in a set of cardinality at least 2 ordered by equality, all the elements are minimal, but there is no smallest element. So there are no germs.

For Assertion 2, we have first that $u = \mathrm{Sup}_U\,]\cdot,u[_U$, since $]\cdot,u[_U = \emptyset$ and $\mathrm{Sup}_U(\emptyset)$ is the smallest element of U. Moreover if $x \in U$, then either $x = u_i$ for some i, or $x > u_n$. And if $x \leq y$ for all $y \in]u_n,\cdot[_U$, then $x \notin]u_n,\cdot[_U$, since $]u_n,\cdot[_U$ has no smallest element. Thus $x \leq u_n$, hence $u_n = \mathrm{Inf}_U]u_n,\cdot[_U$. So u is a germ of U, with cogerm u_n, as was to be shown. □

2.9. Lemma: *Let $(U,\leq)$ be a finite poset, let u and u' be distinct germs of U, with respective cogerms v and v' in U.*

1. *If $u' > u$, then $v' \geq u' > v \geq u$.*
2. *If $u' \leq v$, then $u' \leq v' < u \leq v$.*

Proof : 1) If $u' > u$, then $u' \in]u,\cdot[_U =]u,v]_U \sqcup]v,\cdot[_U$. Moreover if $u' \leq v$, then $]\cdot,u'[_U =]\cdot,u[_U \sqcup [u,u'[_U$, and $[u,u'[_U$ has a greatest element w since $[u,v]_U$ is totally ordered. Then w is also the greatest element of $]\cdot,u'[_U$, hence $w = \mathrm{Sup}_U\,]\cdot,u'[_U \neq u'$, contradicting the first property of u' in Definition 2.1. It follows that $u' > v$.

2) If $u' \leq v$, then either $u' \leq u$ or $u' \in [u,v]_U$. If $u' \leq u$, then $u' < u$ since $u' \neq u$, and then $u' \leq v' < u \leq v$ by Assertion 1 (exchanging the roles of

(u, v) and (u', v')). Now if $u' \in [u, v]_U$, then $u' > u$ since $u \neq u'$, and then $u' > v$ by Assertion 1, contradicting the assumption $u' \leq v$. □

2.10. Definition and Notation: *Let $(S, \leq)$ be a finite poset, and let U be a subset of S. For $B \subseteq S$, set $U_{\leq B} = \{u \in U \mid u \leq b \ \forall b \in B\}$, and for $s \in S$, set $U_{\leq s} = U_{\leq \{s\}} =]\,.\,, s]_S \cap U$.*

1. *U is said to* detect *S if*

$$\forall s, t \in S, \quad s \leq t \quad \Longleftrightarrow \quad U_{\leq s} \subseteq U_{\leq t} \ .$$

2. *S is called a* germ extension *of U if $S = U \cup \mathrm{Grm}(S)$.*

2.11. Lemma: *Let $(S, \leq)$ be a finite poset, and let U be a subset of S. If S is a germ extension of U, then U detects S.*

Proof : Clearly if $s, t \in S$ and $s \leq t$, then $U_{\leq s} \subseteq U_{\leq t}$. Conversely, for $t \in S$, set

$$\Omega_t = \{s \in S \mid \ U_{\leq s} \subseteq U_{\leq t} \text{ but } s \not\leq t\} \ .$$

If $\Omega_t \neq \emptyset$, let s be a minimal element of Ω_t. If $s \in U$, then $s \in U_{\leq s}$, thus $s \in U_{\leq t}$, hence $s \leq t$, contradicting the assumption $s \in \Omega_t$. Hence $s \notin U$, thus $s \in \mathrm{Grm}(S)$, and in particular $s = \mathrm{Sup}_S\,]\,.\,, s[_S$. Now for $x \in]\,.\,, s[_S$, we have $U_{\leq x} =]\,.\,, x]_S \cap U \subseteq]\,.\,, s]_S \cap U = U_{\leq s} \subseteq U_{\leq t}$, hence $x \leq t$ by minimality of s. It follows that $t \geq \mathrm{Sup}_S\,]\,.\,, s[_S = s$, a contradiction. Hence $\Omega_t = \emptyset$ for any $t \in S$, as was to be shown. □

2.12. Theorem: *Let $(S, \leq)$ be a finite poset, and let U be a full subposet of S such that S is a germ extension of U. Let $s \in S$. Then one and only one of the following holds:*

1. *there exists a subset B of U such that $U_{\leq s} = U_{\leq B}$.*
2. *there exists a germ r of U such that $U_{\leq s} =]\,.\,, r[_U$.*

Proof : Let us first prove that one of the assertions 1 or 2 holds. If $s \in U$, then $U_{\leq s} =]\,.\,, s]_U$, so 1 holds for $B = \{s\}$. Now if $s \notin U$, then $s \in \mathrm{Grm}(S)$. Let $\hat{s}$ be the cogerm of s in S. The totally ordered poset $[s, \hat{s}]$ is of the form

$$[s, \hat{s}]_S = \{s = s_0 < s_1 < \ldots < s_n = \hat{s}\} \ .$$

Consider first an element $u \in U$ such that $u \leq u'$ for any $u' \in]\hat{s}, .\,[_S \cap U$. Suppose that there exists $t \in S$ such that $\hat{s} < t$ but $u \not\leq t$, and let t be maximal with this property. Then in particular $t \notin U$. Hence $t \in \mathrm{Grm}(S)$. Moreover $u \leq x$ for any $x \in]t, .[_S$, by maximality of t. Let $\hat{t}$ be the cogerm of t in S.

If $t = \hat{t}$, then $u \leq \mathrm{Inf}_S]t, .[_S = t$, a contradiction. Thus $t < \hat{t}$, and $[t, \hat{t}] = \{t = t_0 < t_1 < \ldots < t_m = \hat{t}\}$, for some $m \geq 1$. Then $t_i \in U$ for $i \geq 1$, for otherwise t_i is a germ of S, and $t_i > t = t_0$, thus $t_i > \hat{t} = t_m$ by Lemma 2.9, a contradiction. In particular $u \leq t_1$. If $u < t_1$, then $u \leq t_0 = t$, a contradiction. Thus $u = t_1$. It follows that t_1 is the smallest element of $]\hat{s}, .[_S \cap U$. Since t is the greatest element of $]\, ., t_1[_S$, it follows that t is unique. In other words the set $\{x \in S \mid x > \hat{s},\ u \not\leq x\}$ has a unique maximal element, i.e. it has a greatest element $t < t_1 = u$. Thus

$$]\hat{s}, .[_S =]\hat{s}, t]_S \sqcup [u, .[_S \quad .$$

Suppose that $]\hat{s}, t[_S \neq \emptyset$, and let t' be a maximal element of that poset. Then $t' \notin U$, for if $t' \in U$, then $t' \geq u > t$, since $t' > \hat{s}$. Thus $t' \in \mathrm{Grm}(S)$, and $t' < t$. By Lemma 2.9, it follows that $t' \leq \hat{t'} < t$, where $\hat{t'}$ is the cogerm of t' in S. Hence $t' = \hat{t'}$. But $]t', .[_S =]t', t]_S \sqcup [u, .[_S = \{t\} \sqcup [u, .[_S$, and $t = \mathrm{Inf}_S]t', .[_S$, a contradiction since $t' = \hat{t'} = \mathrm{Inf}_S]t', .[_S$.

Thus $]\hat{s}, t[_S = \emptyset$. It follows that $]\hat{s}, .[_S = \{t\} \sqcup [u, .[_S$, hence $\mathrm{Inf}]\hat{s}, .[_S = t \neq \hat{s}$, a contradiction.

Therefore $u \leq t$ for any $t \in]\hat{s}, .[_S$, hence $u \leq \hat{s} = \mathrm{Inf}_S]\hat{s}, .[_S$. This shows that for $u \in U$

$$u \leq u' \ \ \forall u' \in]\hat{s}, .[_S \cap U \iff u \leq \hat{s} \quad . \tag{2.13}$$

Now there are two cases:

- if $s = \hat{s}$, then by (2.13)

  $$u \leq s = \hat{s} \iff u \leq u' \ \ \forall u' \in]s, .[_S \cap U \quad ,$$

  so we are in Case 1 of the theorem, for $B =]s, .[_S \cap U$.

- if $s < \hat{s}$, that is $n \geq 1$. Then for $i \in \{1, \ldots, n\}$, the element s_i is in U: indeed if $s_i \notin U$ then $s_i \in \mathrm{Grm}(S)$, and $s_i > s = s_0$. Hence $s_i > \hat{s} = s_n$ by Lemma 2.9, a contradiction.

  Now if $u \in U$, and $u \geq u'$ for any $u' \in]\, ., s_1[_U$, then $u \geq u'$ for any $u' \in]\, ., s_0]_S \cap U$, that is $U_{\leq s_0} \subseteq U_{\leq u}$, hence $s_0 \leq u$ by Lemma 2.11. Thus $s_0 < u$, i.e. $s_1 \leq u$, showing that

  $$s_1 = \mathrm{Sup}_U]\, ., s_1[_U \quad . \tag{2.14}$$

Moreover

(2.15)

$$[s_1, \,.\, [_U = [s_0, \,.\, [_S \cap U = \big([s_0, \hat{s}]_S \cap U\big) \sqcup]\hat{s}, \,.\, [_U = [s_1, \hat{s}]_U \sqcup]\hat{s}, \,.\, [_U \ .$$

Similarly

$$(2.16)\quad]\,.\,, \hat{s}]_U = \big(]\,.\,, s_0]_S \cap U\big) \sqcup \big([s_0, \hat{s}]_S \cap U\big) =]\,.\,, s_1[_U \sqcup [s_1, \hat{s}]_U \ .$$

Finally, from (2.13), we have that

$$(2.17)\qquad \hat{s} = \mathrm{Inf}_U]\hat{s}, \,.\, [_U \ .$$

Now it follows from (2.14), (2.15), (2.16) and (2.17) that s_1 is a germ of U, with cogerm $\hat{s}$. Moreover $U_{\leq s} = \{u \in U \mid u < s_1\}$, so this is Case 2 of the theorem.

It remains to see that the two cases of the theorem cannot occur simultaneously. So suppose that there exists $s \in S$, $B \subseteq U$, and $r \in \mathrm{Grm}(U)$ such that for any $u \in U$

$$u \leq s \iff u \leq b, \ \forall b \in B \iff u < r \ .$$

Then for $b \in B$, we have $b \geq u$ for any $u \in]\,.\,, r[_U$, hence $b \geq \mathrm{Sup}_U \,]\,.\,, r[_U = r$. Thus $r \leq b$ for any $b \in B$, hence $r < r$, a contradiction. This completes the proof of Theorem 2.12. □

2.18. Theorem: *Let $(S, \leq)$ be a finite poset, and U be a full subposet of S. Suppose that U detects S, and that for any $s \in S$, one of the following holds:*

1. *there exists a subset B of U such that $U_{\leq s} = U_{\leq B}$.*
2. *there exists a germ r of U such that $U_{\leq s} =]\,.\,, r[_U$.*

Then S is a germ extension of U.

Proof : Let $s \in S - U$.

• Step 1: Let $t \in S$ such that $s' \leq t$ for all $s' \in]\,.\,, s[_S$. If $u \in U_{\leq s}$, that is if $u \in U$ and $u \leq s$, then $u < s$, and then $u \leq t$. In other words $U_{\leq s} \subseteq U_{\leq t}$, thus $s \leq t$, since U detects S. This shows that $s = \mathrm{Sup}_S \,]\,.\,, s[_S$.

• Step 2: Suppose first that there exists $B \subseteq U$ such that $U_{\leq s} = U_{\leq B}$. Let $t \in S$ such that $t \leq s'$, for all $s' \in]s, \,.\, [_S$. If $b \in B$, then $U_{\leq s} \subseteq U_{\leq b}$, hence $s \leq b$, since U detects S, and $s < b$ since $s \notin U$. It follows that $t \leq b$ for all $b \in B$. Hence $U_{\leq t} \subseteq U_{\leq s}$, so $t \leq s$. This shows that $s = \mathrm{Inf}_S]s, \,.\, [_S$. Since $s = \mathrm{Sup}_S \,]\,.\,, s[_S$, it follows (see Example 2.6) that s is a germ of S, equal to its cogerm in S.

• Step 3: Suppose now that there exists a germ r of U such that $U_{\leq s} =]\,.\,, r[_U$. In particular $U_{\leq s} \subset]\,.\,, r]_U = U_{\leq r}$, so $s \leq r$, since U detects S, and $s < r$ since $s \notin U$ and $r \in U$.

Let $t \in]s, .\,[_S$. Let us show that $r \leq t$. Suppose first that there exists $B \subseteq U$ such that $U_{\leq t} = U_{\leq B}$. Let $u \in U$ with $u < r$, and let $b \in B$. Then $u \leq s$, hence $u < t$, thus $u \leq b$. It follows that $b \geq \mathrm{Sup}_U\,]\,.\,, r[_U = r$. This holds for any $b \in B$, so $r \leq t$.

Assume now that there is a germ r' of U such that $U_{\leq t} =]\,.\,, r'[_U$. Then $U_{\leq s}$ is strictly contained in $U_{\leq t}$ since $s < t$ and U detects S. Thus $]\,.\,, r[_U \subset]\,.\,, r'[_U$. In particular $r' \geq u$ for any $u \in]\,.\,, r[_U$, hence $r' \geq \mathrm{Sup}_U\,]\,.\,, r[_U = r$, and $r' > r$ since $]\,.\,, r[_U \subset]\,.\,, r'[_U$. Thus $r \in]\,.\,, r'[_U = U_{\leq t}$, i.e. $r \leq t$ again.

This shows that r is the smallest element of $]s, .\,[_S$. Let $\hat{r}$ be the cogerm of r in U. Then $[r, \hat{r}]_U = \{r = r_0 < r_1 < \ldots < r_n = \hat{r}\}$ (possibly $n = 0$ if $r = \hat{r}$).

• Step 4: Let $t \in S$ with $t > s$, i.e. $t \geq r$, and suppose that $\hat{r} \not\leq t$. Then there exists an integer $m \in \{0, \ldots n-1\}$ such that $r_m \leq t$ but $r_{m+1} \not\leq t$.

Suppose first that there exists $B \subseteq U$ such that $U_{\leq t} = U_{\leq B}$. Thus $r_m \leq b$ for all $b \in B$, but there exists $b_0 \in B$ such that $r_{m+1} \not\leq b_0$. Hence $B \subseteq [r_m, .\,[_U = [r_m, \hat{r}]_U \sqcup]\hat{r}, .\,[_U$. If $r_m \notin B$, then $B \subseteq [r_{m+1}, \hat{r}]_U \sqcup]\hat{r}, .\,[_U$, as r_{m+1} is the smallest element of $]r_m, .\,[_U$. This contradicts $r_{m+1} \not\leq b_0$. Hence $r_m \in B$, and

$$U_{\leq t} \subseteq \{u \in U \mid u \leq r_m\} = U_{\leq r_m} \quad .$$

It follows that $t \leq r_m$, since U detects S. Therefore $t = r_m$ in this case, since we had $r_m \leq t$.

Suppose now that there is a germ r' of U such that $U_{\leq t} =]\,.\,, r'[_U$. Then $r < r'$, since $r \leq t$. Hence $\hat{r} < r'$ by Lemma 2.9. Then $U_{\leq \hat{r}} \subseteq]\,.\,, r'[_U = U_{\leq t}$, and $\hat{r} \leq t$ since U detects S. This contradicts the assumption on t.

This shows that if $t \in S$ and $t \geq s$, then $t \in \{s < r = r_0, \ldots, r_n = \hat{r}\}$ or $t > \hat{r}$.

• Step 5: Let $t \in S$ with $t \leq \hat{r}$. Then $U_{\leq t} \subseteq]\,.\,, \hat{r}]_U =]\,.\,, r[_U \sqcup [r, \hat{r}]_U$. If $U_{\leq t} \subseteq]\,.\,, r[_U = U_{\leq s}$, then $t \leq s$ since U detects S. Otherwise $U_{\leq t}$ has a greatest element $r_m \in \{r = r_0, r_1, \ldots, r_n = \hat{r}\}$, and $U_{\leq t} = U_{\leq r_m}$, thus $t = r_m$.

Hence if $t \in S$ and $t \leq \hat{r}$, then $t < s$ or $t \in \{s < r = r_0, \ldots, r_n = \hat{r}\}$.

• Step 6: Let $t \in S$ such that $t \leq u$ for all $u \in U$ with $u > \hat{r}$. If $u' \in U_{\leq t}$, then $u' \leq u$ for all $u \in U$ with $u > \hat{r}$, hence $u' \leq \mathrm{Inf}_U]\hat{r}, .\,[_U = \hat{r}$. It follows that $U_{\leq t} \subseteq]\,.\,, \hat{r}]_U = U_{\leq \hat{r}}$, hence $t \leq \hat{r}$, as U detects S.

This shows *a fortiori* that if $t \in S$ and $t \leq s'$ for all $s' \in]\hat{r}, .\,[_S$, then $t \leq \hat{r}$, that is $\hat{r} = \mathrm{Inf}_S]\hat{r}, .\,[_S$.

• Step 7: The conclusion of Steps 1, 4, 5 and 6 above show that s is a germ of S under the assumption of Step 3, with cogerm $\hat{r}$ in S. Together with Step 2, this shows that $S - U \subseteq \mathrm{Grm}(S)$, in other words, that S is a germ extension of U. □

2.19. Corollary: *Let $(S, \leq)$ be a finite poset, and $U \subseteq S$. If S is a germ extension of U, so is any full subposet R of S containing U.*

Proof : Let R be a full subposet of S containing U. Then U detects S by Lemma 2.11, so U detects $R \subseteq S$. Moreover, by Theorem 2.12, for any $x \in R$, the set $U_{\leq x}$ is equal to $]\,.\,, r[_U$ for some germ r of the (full) subposet U of S (which is also a full subposet of R), or there exists a subset B of U such that $U_{\leq x} = U_{\leq B}$. By Theorem 2.18, it follows that R is a germ extension of U. □

2.20. Definition and Notation: *Let $(U, \leq)$ be a finite poset. Set*

$$\begin{aligned} \Lambda(U) &= \{s \subseteq U \mid \exists B \subseteq U,\ s = U_{\leq B}\}, \\ \widehat{\mathrm{G}}(U) &= \{s \subseteq U \mid \exists r \in \mathrm{Grm}(U),\ s =]\,.\,, r[_U\} \ . \end{aligned}$$

Let $G(U) = \Lambda(U) \cup \widehat{\mathrm{G}}(U)$, considered as a full subposet of the poset $I_\downarrow(U)$ of lower-subsets of U (ordered by inclusion of subsets of U). The poset $G(U)$ is called the germ closure *of U.*

2.21. Remark: The sets $\Lambda(U)$ and $\widehat{\mathrm{G}}(U)$ are special cases, in the case of the lattice $I_\downarrow(U)$ of lower-subsets of U, of constructions one can define in an arbitrary finite lattice. The set $\Lambda(U)$ is the set of intersections (i.e. meets) of lower intervals of U, i.e. join-irreducible elements of $I_\downarrow(U)$, since $U_{\leq B} = \bigwedge_{b \in B}]\,.\,, b]_U$ (see Section 4 for details).

The terminology *germ closure* is motivated by the following:

2.22. Theorem: *Let $(U, \leq)$ be a finite poset.*

1. *The map $u \in U \mapsto \underline{u} =]\,.\,, u]_U$ is an isomorphism from U onto a full subposet $\underline{U}$ of $G(U)$.*
2. *If S is a full subposet of $G(U)$ containing $\underline{U}$, then S is a germ extension of $\underline{U}$. In particular $G(U)$ is a germ extension of $\underline{U}$.*
3. *Let $(S, \leq)$ be a poset containing U as a full subposet. If S is a germ extension of U, then there exists a unique isomorphism of posets j :*

$S \to S'$ onto a full subposet S' of $G(U)$ such that $j(u) = \underline{u}$ for all $u \in U$.

Proof : For Assertion 1, observe that $]\,.\,,u]_U = \{u' \in U \mid u' \leq u\}$, so $]\,.\,,u]_U$ indeed belongs to $G(U)$. Moreover for $u, v \in U$, the inclusion $]\,.\,,u]_U \subseteq]\,.\,,v]_U$ is equivalent to $u \leq v$.

For Assertion 2, by Corollary 2.19, it suffices to consider the case $S = G(U)$. First it is clear that $\underline{U}$ detects $G(U)$: indeed if $s \in G(U)$ and $u \in U$, then $\underline{u} \subseteq s$ if and only if $u \in s$, because s is a lower subset. In other words $\underline{U}_{\leq s} = \{\underline{u} \mid u \in s\}$.

If there exists a subset B of U such that $s = U_{\leq B}$, then

$$\underline{U}_{\leq s} = \{\underline{u} \mid u \leq b \;\; \forall b \in B\} = \{\underline{u} \in \underline{U} \mid \underline{u} \subseteq \underline{b} \;\; \forall \underline{b} \in \underline{B}\} \;\; ,$$

where $\underline{B} = \{\underline{b} \mid b \in B\} \subseteq \underline{U}$.

Otherwise there exists a germ r of U such that $s =]\,.\,,r[_U$. Clearly in this case

$$\underline{U}_{\leq s} = \{\underline{u} \mid u < r\} = \{\underline{u} \in \underline{U} \mid \underline{u} \subset \underline{r}\} \;\; ,$$

and $\underline{r}$ is a germ of $\underline{U}$ since $u \mapsto \underline{u}$ is an isomorphism of posets from U to $\underline{U}$.

Now the assumptions of Theorem 2.18 are fulfilled. It follows that $G(U)$ is a germ extension of $\underline{U}$, as was to be shown.

For Assertion 3, let $(S, \leq)$ be a finite poset containing U as a full subposet, and assume that S is a germ extension of U. Then U detects S by Lemma 2.11, and moreover, for any $s \in S$, the set $U_{\leq s}$ belongs to $G(U)$, by Theorem 2.12.

For $s \in S$, set $j(s) = U_{\leq s} \in G(U)$, and $S' = j(S)$. Then S' is a full subposet of $G(U)$ (because U detects S), and j is an isomorphism of posets $S \to S'$. Moreover $j(u) = U_{\leq u} = \underline{u}$ for $u \in U$, so S' contains $\underline{U}$. This shows the existence of S' and j in Assertion 3.

For the uniqueness, let $j' : S \to S''$ be an isomorphism of posets from S to a full subposet S'' of $G(U)$, such that $j'(u) = \underline{u}$ for all $u \in U$. Then for $s \in S$ and $u \in U$

$$\underline{u} = j'(u) \subseteq j'(s) \iff u \leq s \iff j(u) = \underline{u} \subseteq j(s) \;\; .$$

In other words $\underline{U}_{\leq j'(s)} = \underline{U}_{\leq j(s)}$. Since $\underline{U}$ detects $G(U)$ by Assertion 2 and Lemma 2.11, it follows that $j(s) = j'(s)$, hence $j = j'$ and $S' = S''$. □

2.23. Proposition: *Let $(U, \leq)$ be a finite poset.*

1. *The poset $G(U)$ is the disjoint union of $\Lambda(U)$ and $\widehat{\mathrm{G}}(U)$.*

2. *If $s, t \in G(U)$, then $s \cap t \in G(U)$. More precisely, if $s \nsubseteq t$ and $t \nsubseteq s$, then $s \cap t \in \Lambda(U)$.*

3. *The subsets $\emptyset$ and U belong to $G(U)$. Thus $G(U)$ is a lattice: for $s, t \in G(U)$, the infimum $s \wedge t$ of $\{s, t\}$ in $G(U)$ is $s \cap t$, and the supremum $s \vee t$ is the intersection of all $x \in G(U)$ such that $x \supseteq s \cup t$.*

Proof : By Definition 2.20, the poset $G(U)$ is the union of $\Lambda(U)$ and $\widehat{\mathrm{G}}(U)$, and by Theorem 2.12, this union is disjoint. Assertion 1 follows.

If A and B are subsets of U, then $U_{\leq A} \cap U_{\leq B} = U_{\leq (A \cup B)}$. Moreover, if r is a germ of U, and if $r \in U_{\leq A}$, then $]\cdot, r[_U \subset U_{\leq A}$. And if $r \notin U_{\leq A}$, then $U_{\leq A} \cap]\cdot, r[_U = U_{\leq A} \cap]\cdot, r]_U = U_{\leq (A \cup \{r\})}$. Finally, if r' is another germ of U, and if $r \leq r'$, then $]\cdot, r[_U \subseteq]\cdot, r'[_U$. And if $r \nleq r'$ and $r' \nleq r$, then $]\cdot, r[_U \cap]\cdot, r'[_U =]\cdot, r]_U \cap]\cdot, r']_U = U_{\leq \{r, r'\}}$. Assertion 2 follows.

The set U is equal to $U_{\leq \emptyset}$, so $U \in G(U)$. If U has no smallest element, then $\emptyset = U_{\leq U}$, so $\emptyset \in G(U)$. And if U has a smallest element u, then u is a germ of U by Proposition 2.8, so $]\cdot, u[_U = \emptyset \in G(U)$ in this case also. The last part of Assertion 3 now follows from Assertion 2, since $G(U)$ has a greatest element U. □

2.24. Theorem: *Let $(S, \leq)$ be a finite poset, let U be a full subposet of S, and assume that S is a germ extension of U. Then:*

1. *If s is a germ of S and if $s \in U$, then s is a germ of U. In other words $U \cap \mathrm{Grm}(S) \subseteq \mathrm{Grm}(U)$.*

2. *Let r be a germ of U, with cogerm $\hat{r} \in U$. Then $[r, \hat{r}]_S = [r, \hat{r}]_U$, and one of the following holds:*

   (a) $r = \mathrm{Sup}_S]\cdot, r[_S$, and then r is a germ of S, with cogerm $\hat{r}$ in S.

   (b) $]\cdot, r[_S$ has a greatest element s. Then r is the smallest element of $]s, \cdot[_S$, and $s \in S - U$ is a germ of S, with cogerm $\hat{r}$ in S. In particular r is not a germ of S.

Proof : For Assertion 1, let $s \in U \cap \mathrm{Grm}(S)$. If $u \in U$ is such that $u \geq v$ for any $v \in]\cdot, s[_U$, let $t \in]\cdot, s[_S$ be minimal such that $t \nleq u$. Then $t \notin U$, hence $t \in \mathrm{Grm}(S)$. In particular $t = \mathrm{Sup}_S]\cdot, t[_S$. Moreover $t' \leq u$ for any $t' \in]\cdot, t[_S$, by minimality of t. Hence $\mathrm{Sup}_S]\cdot, t[_S = t \leq u$, a contradiction, which proves that $t \leq u$ for any $t \in]\cdot, s[_S$. But then $\mathrm{Sup}_S]\cdot, s[_S = s \leq u$. This shows that

$$s = \mathrm{Sup}_U]\cdot, s[_U \quad . \tag{2.25}$$

Now let $\hat{s}$ be the cogerm of s in S. There is an integer $n \in \mathbb{N}$ and elements $s_i \in S$, for $0 \leq i \leq n$, such that

$$[s, \hat{s}]_S = \{s = s_0 < s_1 < \ldots < s_n = \hat{s}\} \ .$$

If $s_i \notin U$ for some $i \in \{1, \ldots, n\}$, then $s_i \in \mathrm{Grm}(S)$, and $s_i > s$. By Lemma 2.9 $s_i > \hat{s} = s_n$, a contradiction. Thus $s_i \in U$ for any $i \in \{0, \ldots, , n\}$, i.e. $[s, \hat{s}]_S = [s, \hat{s}]_U$.

Now if $u \in U$ and $u \geq s$ then $u \in [s, \hat{s}]_S = [s, \hat{s}]_U$, or $u \geq \hat{s}$. Similarly if $u \leq \hat{s}$, then $u \in [s, \hat{s}]_S = [s, \hat{s}]_U$ or $u \leq s$. Thus

$$[s, .\,[_U = [s, \hat{s}]_U \sqcup]\hat{s}, .\,[_U \quad \text{and} \quad]\,.\,, \hat{s}]_U =]\,.\,, s[_U \sqcup [s, \hat{s}]_U \ . \tag{2.26}$$

Now let $u \in U$ such that $u \leq v$ for any $v \in]\hat{s}, .\,[_U$. Suppose that there exists $t \in]\hat{s}, .\,[_S$ such that $u \not\leq t$, and choose a maximal such t. Then $t \notin U$, hence $t \in \mathrm{Grm}(S)$. Let $\hat{t}$ be the cogerm of t in S. If $t = \hat{t}$, then by maximality of t, we have that $u \leq x$ for any $x \in]t, .\,[_S =]\hat{t}, .\,[_S$, so $u \leq \mathrm{Inf}_S]\hat{t}, .\,[_S = \hat{t} = t$, contradicting our assumption on t. Hence $\hat{t} > t$, so there is an integer $m \in \mathbb{N} - \{0\}$ and elements $t_i \in S$, for $0 \leq i \leq m$, such that

$$[\,t, \hat{t}\,]_S = \{\,t = t_0 < t_1 \ldots < t_m = \hat{t}\,\} \ .$$

If $t_i \notin U$ for some $i \in \{1, \ldots, m\}$, then $t_i \in \mathrm{Grm}(S)$, and $t_i > t$. By Lemma 2.9 $t_i > \hat{t} = t_m$, a contradiction. Thus $t_i \in U$ for any $i \in \{1, \ldots, m\}$. In particular $u \leq t_1$. If $u < t_1$, then $u \leq t$, since $t = t_0$ is the greatest element of $]\,.\,, t_1[_S$. This contradiction shows that $u = t_1$ is the smallest element of $]\hat{s}, .\,[_U$. Then t is the greatest element of $]\,.\,, t_1[_S =]\,.\,, u[_S$. It follows that t is unique, that is, there is a unique element t of $]\hat{s}, .\,[_S$ maximal subject to $u \not\leq t$. In other words

$$]\hat{s}, .\,[_S =]\hat{s}, t]_S \sqcup [u, .\,[_S \ ,$$

and $t < u = t_1$.

If $]\hat{s}, t[_S \neq \emptyset$, let t' be a maximal element of this poset. Then $t \notin U$, for otherwise $t' \geq u > t$, since $t' > \hat{s}$. So $t' \in \mathrm{Grm}S$, and $t' < t$. By Lemma 2.9, it follows that $t' \leq \hat{t}' < t$, where $\hat{t}'$ is the cogerm of t' in S. Hence $t' = \hat{t}'$ by maximality of t'. But then

$$]t', .\,[_S = \{t\} \sqcup [u, .\,[_S \ ,$$

hence $\mathrm{Inf}_S]t', .\,[_S = t \neq t'$, contradicting $t' \in \mathrm{Grm}(S)$.

It follows that $]\hat{s}, t[_S = \emptyset$, thus $]\hat{s}, .\,[_S = \{t\} \sqcup [u, .\,[_S$. Then $\mathrm{Inf}_S]\hat{s}, .\,[_S = t \neq s$, contradicting $s \in \mathrm{Grm}(S)$.

This shows finally that $u \leq t$ for any $t \in]\hat{s}, .\,[_S$, thus $u \leq \hat{s} = \mathrm{Inf}_S]\hat{s}, .\,[_S$. Hence

$$(2.27) \qquad \hat{s} = \mathrm{Inf}_U]\hat{s}, .\,[_U \;.$$

Now s is a germ of U, by (2.25), (2.26), and (2.27). This completes the proof of Assertion 1.

For Assertion 2, let r be a germ of U, and let $\hat{r}$ be its cogerm in U. Let $t \in S$ such that $t \leq x$ for any $x \in]\hat{r}, .\,[_S$. Then for any $u \in U$ with $u \leq t$, we have $u \leq v$ for any $v \in]\hat{r}, .\,[_U$. Hence $u \leq \mathrm{Inf}_U]\hat{r}, .\,[_U = \hat{r}$. Thus $U_{\leq t} \subseteq U_{\leq \hat{r}}$, thus $t \leq \hat{r}$ since U detects S by Lemma 2.11. This shows that

$$(2.28) \qquad \hat{r} = \mathrm{Inf}_S]\hat{r}, .\,[_S \;.$$

Now let $t \in S$ with $t \leq \hat{r}$. Then $U_{\leq t} \subseteq]\,.\,, \hat{r}]_U =]\,.\,, r[_U \sqcup [r, \hat{r}]_U$, and there are two cases: either $U_{\leq t} \subseteq]\,.\,, r[_U$, and then $U_{\leq t} \subset U_{\leq r} =]\,.\,, r]_U$, thus $t < r$ as U detects S. Or $U_{\leq t}$ has a greatest element $u \in [r, \hat{r}]_U$. In this case $U_{\leq t} =]\,.\,, u]_U = U_{\leq u}$, thus $t = u$. In other words

$$(2.29) \qquad]\,.\,, \hat{r}]_S =]\,.\,, r[_S \sqcup [r, \hat{r}]_U \;,$$

and in particular $[r, \hat{r}]_S = [r, \hat{r}]_U$.

Now let $t \in S$ with $t \geq r$. By Theorem 2.12, there are two cases: In the first case there exists $B \subseteq U$ such that $U_{\leq t} = U_{\leq B}$. Then in particular $r \in U_{\leq B}$, that is $B \subseteq [r, .\,[_U = [r, \hat{r}]_U \sqcup]\hat{r}, .\,[_U$. If $B \subseteq]\hat{r}, .\,[_U$, then $\hat{r} \in U_{\leq B} = U_{\leq t}$, thus $\hat{r} \leq t$. Otherwise the set B has a smallest element $u \in [r, \hat{r}]_U$, and in this case $U_{\leq B} = U_{\leq u} = U_{\leq t}$, hence $t = u \in [r, \hat{r}]_U$.

The other case is when there exists a germ r' of U such that $U_{\leq t} =]\,.\,, r'[_U$. Then $r < r'$ since $r \in U_{\leq t}$, and $\hat{r} < r'$ by Lemma 2.9. It follows that $U_{\leq \hat{r}} \subseteq U_{\leq t}$, so $\hat{r} \leq t$ in this case also. We get finally that

$$(2.30) \qquad [r, .\,[_S = [r, \hat{r}]_U \sqcup]\hat{r}, .\,[_S \;.$$

It follows from (2.28), (2.29) and (2.30) that r is a germ of S if and only if $r = \mathrm{Sup}_S]\,.\,, r[_S$, and in this case $\hat{r}$ is the cogerm of r in S. This is Case (a) of Assertion 2.

And if $r \neq \mathrm{Sup}_S]\,.\,, r[_S$, there exists $s \in S$ such that $s \geq x$ for any $x \in]\,.\,, r[_S$, but $s \not\geq r$. In particular $]\,.\,, r[_U \subseteq U_{\leq s}$.

If there exists a subset B of U such that $U_{\leq s} = U_{\leq B}$, then $]\,.\,, r[_U \subseteq U_{\leq b}$, for any $b \in B$, thus $b \geq \mathrm{Sup}_U]\,.\,, r[_U = r$. Hence $r \in U_{\leq B}$, that is $r \in U_{\leq s}$, contradicting $s \not\geq r$.

So there is a germ r' of U such that $U_{\leq s} =]\,.\,, r'[_U$. It follows that $]\,.\,, r[_U \subseteq]\,.\,, r'[_U \subseteq]\,.\,, r']_U$, so $r' \geq \mathrm{Sup}_U]\,.\,, r[_U = r$. If $r' > r$, then $r \in]\,.\,, r'[_U = U_{\leq s}$, so

$r \leq s$, a contradiction. Hence $r' = r$, and $U_{\leq s} =]\,.\,, r[_U \subset U_{\leq r}$, so $s < r$. If $t \in]\,.\,, r[_S$, then $U_{\leq t} \subseteq]\,.\,, r[_U = U_{\leq s}$, so $t \leq s$. It follows that s is the greatest element of $]\,.\,, r[_S$, so $\mathrm{Sup}_S]\,.\,, r[_S = s < r$, and r is not a germ of S. Moreover $s \notin U$ since $\mathrm{Sup}_U]\,.\,, r[_U = r$. Finally, the previous discussion shows that s is the only element of S such that $s \geq x$ for any $x \in]\,.\,, r[_S$, but $s \not\geq r$. In particular r is the smallest element of $]s, .\,[_S$. Now (2.28), (2.29) and (2.30) show that s is a germ of S, with cogerm $\hat{r}$ in S. This is case (b) of Assertion 2, and completes the proof of Theorem 2.24. □

2.31. Corollary:

1. *Let T be a finite lattice. Set $U = T - \mathrm{Grm}(T)$, considered as a full subposet of T. Then $T \cong G(U)$.*
2. *Let U be a finite poset, and let $\underline{U}$ denote its isomorphic image in its germ closure $T = G(U)$. Then $\underline{U} = T - \mathrm{Grm}(T)$.*
3. *Let U and V be finite posets. Let $\varphi : G(U) \to G(V)$ be an isomorphism of posets. Then $\varphi(\underline{U}) = \underline{V}$, and in particular U and V are isomorphic.*
4. *In particular, the restriction $\varphi \mapsto \varphi_{|\underline{U}}$ induces a group isomorphism $\mathrm{Aut}\big(G(U)\big) \cong \mathrm{Aut}(U)$.*

Proof : For Assertion 1, the poset T is clearly a germ extension of $U = T - \mathrm{Grm}(T)$. By Theorem 2.22, the map $j : t \in T \mapsto U_{\leq t} \in G(U)$ is a poset isomorphism from T to a full subposet of $G(U)$ containing $\underline{U}$. All we have to show is that this map is surjective. Clearly for $B \subseteq U$, we have $j(\underset{b\in B}{\wedge} b) = U_{\leq B}$, so $\Lambda(U)$ is contained in the image of j. Now if $r \in \mathrm{Grm}(U)$, then $r \notin \mathrm{Grm}(T)$ (since $r \in U$), hence $]\,.\,, r[_T$ has a greatest element $s \in T - U$, by Theorem 2.24. Then $]\,.\,, r[_U = U_{\leq s} = j(s)$. Hence $\widehat{\mathrm{G}}(U)$ is contained in the image of j, completing the proof of Assertion 1.

For Assertion 2, by Theorem 2.22, the poset $T = G(U)$ is a germ extension of $\underline{U}$. Hence $T = \underline{U} \cup \mathrm{Grm}(T)$. So all we have to show is that $\underline{U} \cap \mathrm{Grm}(T) = \emptyset$. Let $s \in \underline{U} \cap \mathrm{Grm}(T)$. Then $s =]\,.\,, u]_U$, for some $u \in U$, and by Theorem 2.24, the element u is a germ of U. This means that $]\,.\,, u[_U \in G(U)$. Since any lower-subset of U properly contained in $]\,.\,, u]_U$ is contained in $]\,.\,, u[_U$, it follows that $]\,.\,, s[_T$ has a greatest element $t =]\,.\,, u[_U$. So $\mathrm{Sup}_T]\,.\,, s[_T = t < s$, so s is not a germ of T, and Assertion 2 follows by contradiction.

Assertion 3 follows as well: if $\varphi : G(U) \to G(V)$ is an isomorphism of posets, then since $\underline{U} = G(U) - \mathrm{Grm}\big(G(U)\big)$ and $\underline{V} = G(V) - \mathrm{Grm}\big(G(V)\big)$, it follows that $\varphi(\underline{U}) = \underline{V}$. Hence U and V are isomorphic.

In the case $U = V$, this shows that $\varphi(\underline{U}) = \underline{U}$ for any automorphism φ of the poset $G(U)$. By Theorem 2.22, the resulting group homomorphism $\mathrm{Aut}\big(G(U)\big) \to \mathrm{Aut}(\underline{U}) \to \mathrm{Aut}(U)$ is injective. It is also surjective, since any automorphism α of U extends to an automorphism of $G(U)$: indeed $\alpha(U_{\leq B}) = U_{\leq \alpha(B)}$, for $B \subseteq U$, and moreover $\alpha\big(\,]\,.\,,r[_U\big) =]\,.\,,\alpha(r)[_U$ for any $r \in U$, and $\alpha\big(\mathrm{Grm}(U)\big) = \mathrm{Grm}(U)$. This completes the proof. □

3. Germ extensible subsets of a lattice

3.1. Definition and Notation: *Let T be a finite lattice. A full subposet U of T is called* germ extensible in T *if the natural map*

$$\nu : s \in G(U) \mapsto \bigvee_{u \in s} u \in T$$

is injective. We denote by $\overline{G}(U)$ the image of ν.

3.2. Theorem: *Let T be a finite lattice, and U be a full subposet of T. Then U is germ extensible in T if and only if*

$$\forall r \in \mathrm{Grm}(U), \;\; r > \bigvee_{\substack{u \in U \\ u < r}} u \;\; \text{in } T \;\; . \tag{3.3}$$

In this case, the map $t \in \overline{G}(U) \mapsto \{u \in U \mid u \leq t\}$ is inverse to the bijection $\nu : G(U) \to \overline{G}(U)$.

Proof : If U is germ extensible in T, let $r \in \mathrm{Grm}(U)$. Then $]\,.\,,r[_U$ and $]\,.\,,r]_U$ are distinct elements of $G(U)$. Thus $\nu\big(\,]\,.\,,r]_U\big) = r > \nu\big(\,]\,.\,,r[_U\big) = \bigvee_{\substack{u \in U \\ u < r}} u$. So Condition 3.3 is necessary.

Conversely, let $B \subseteq U$, and set $s = U_{\leq B} \in G(U)$ and $\wedge B = \bigwedge_{b \in B} b \in T$. Then $\nu(s) = \bigvee_{u \in U_{\leq B}} u \leq \wedge B$, and for any $v \in U$

$$\big(v \leq \wedge B\big) \implies \big(v \in U_{\leq B}\big) \implies \big(v \leq \nu(U_{\leq B}) = \nu(s)\big) \implies \big(v \leq \wedge B\big) \;\; .$$

Thus $s = \{v \in U \mid v \leq \nu(s)\}$.

Now let r be a germ of U, set $s =]\,.\,,r[_U \in G(U)$. If $r > \bigvee_{\substack{u \in U \\ u < r}} u = \nu(s)$, then for any $v \in U$

$$\big(v \in s\big) \implies \big(v < r\big) \implies \big(v \leq \bigvee_{\substack{u \in U \\ u < r}} u = \nu(s)\big) \implies \big(v < r\big) \implies \big(v \in s\big) \;\; ,$$

so $s = \{v \in U \mid v \leq \nu(s)\}$ in this case also. $\square$

3.4. Theorem: *Let T be a finite lattice, and let S be a subset of T. Then there exists a unique germ extensible subset U of T such that $U \subseteq S \subseteq \overline{G}(U)$, namely*

$$U = S - \{s \in \mathrm{Grm}(S) \mid s = \bigvee_{\substack{t \in S \\ t < s}} t\} \quad .$$

In other words, the lattice of subsets of T is the disjoint union of the intervals $[U, \overline{G}(U)]$, when U runs through all germ extensible subsets of T.

Proof : Let $S \subseteq T$, and suppose that U is a germ extensible subset of T such that $U \subseteq S \subseteq \overline{G}(U)$. Since $S \subseteq \overline{G}(U)$, any element t of S is a join of elements of U, that is $t = \bigvee_{\substack{u \in U \\ u \leq t}} u$. It follows that for any $s \in S$

$$\bigvee_{\substack{t \in S \\ t < s}} t = \bigvee_{\substack{u \in U \\ u < s}} u \quad .$$

Let $s \in \mathrm{Grm}(S)$. If $s \in U$, then $s \in \mathrm{Grm}(U)$ by Theorem 2.24, hence $s > \bigvee_{\substack{u \in U \\ u < s}} u$ by Theorem 3.2. In other words if $s = \bigvee_{\substack{t \in S \\ t < s}} t$, then $s \notin U$. Thus

$$U \subseteq S - \{s \in \mathrm{Grm}(S) \mid s = \bigvee_{\substack{t \in S \\ t < s}} t\} \quad .$$

Conversely, let $V = S - \{s \in \mathrm{Grm}(S) \mid s = \bigvee_{\substack{t \in S \\ t < s}} t\}$. Then S is a germ extension of V, so by Theorem 2.22, there is an isomorphism of S onto a full subposet S' of $G(V)$ such that $\underline{V} \subseteq S' \subseteq G(V)$.

If $v \in \mathrm{Grm}(V)$, there are two cases. First if $v \in \mathrm{Grm}(S)$, then $v > \bigvee_{\substack{t \in S \\ t < v}} t$ by definition of V. Then *a fortiori* $v > \bigvee_{\substack{w \in V \\ w < v}} w$. Now if $v \notin \mathrm{Grm}(S)$, then $\underline{v} =]\cdot, v]_V \notin \mathrm{Grm}(S')$, so $]\cdot, \underline{v}[_{S'}$ has a greatest element s' by Theorem 2.24. Thus $]\cdot, v[_S$ has a greatest element s, and then $\bigvee_{\substack{w \in V \\ w < v}} w \leq s < v$ in this case also.

It follows from Theorem 3.2 that V is a germ extensible subposet of T. The inclusions $\underline{V} \subseteq S' \subseteq G(V)$ now read $V \subseteq S \subseteq \overline{G}(V)$ in T, and we have $U \subseteq V \subseteq S \subseteq \overline{G}(U) \cap \overline{G}(V) \subseteq T$.

If $v \in V - U$, then $v \in \mathrm{Grm}(V)$ by Theorem 2.22, so $v > \bigvee_{\substack{w \in V \\ w < v}} w$. But on the other hand $v \in \overline{G}(U)$, so $v = \bigvee_{\substack{u \in U \\ u \leq v}} u = \bigvee_{\substack{u \in U \\ u < v}} u$. Hence $\bigvee_{\substack{u \in U \\ u < v}} u > \bigvee_{\substack{w \in V \\ w < v}} w$, a contradiction, since any element w of V is the join of the elements $u \le w$ of U. It follows that $U = V$, as was to be shown. $\square$

4. Germs and lattices

4.1. Let T be a finite lattice, and let $E = \mathrm{Irr}(T)$ be the set of join-irreducible elements of T. The following constructions are introduced in [3] (Notation 2.3 and 2.6). First, we denote by

$$\Lambda E = \{t \in T \mid t = \mathop{\wedge}_{\substack{e\in E\\ e\ge t}} e\}$$

the set of elements of T which are equal to a meet of join-irreducible elements of T. Moreover, for in T, we set

$$r(t) = \mathop{\vee}_{\substack{e\in E\\ e<t}} e \;\;\text{and}\;\; \sigma(t) = \mathop{\wedge}_{\substack{e\in E\\ e>t}} e \;\;.$$

So $r(t) \le t$, with equality if and only if $t \notin E$. And if $t \in E$, then $r(t)$ is the largest element of $]\cdot, t[_T$. In particular, it follows from Theorem 3.2 that E is a germ extensible subset of T.

On the other hand $\sigma(t) \ge t$ with equality and only if either $t \in \Lambda E - E$, or $t \in E$ and t is equal to the meet of the elements of $E \cap]t, \cdot[_T$.

For any $t \in T$, we denote by $r^\infty(t)$ the limit of the decreasing sequence $t \ge r(t) \ge r^2(t) \ge \ldots$, and by $\sigma^\infty(t)$ the limit of the increasing sequence $t \le \sigma(t) \le \sigma^2(t) \le \ldots$.

Finally ([3] Notation 2.10), we set

$$\begin{aligned} G_T^\sharp &= \{t \in T \mid t = r^\infty\sigma^\infty(t)\},\\ \widehat{G}_T &= G^\sharp - \Lambda E,\\ G_T = E \sqcup G_T^\sharp &= E \sqcup (\Lambda E - E) \sqcup \widehat{G}_T = \Lambda E \sqcup \widehat{G}_T \;\;. \end{aligned} \tag{4.1}$$

We just saw that E is a germ extensible subset of T. The following theorem shows that the corresponding set $\overline{G}(E)$ is equal to the set G_T introduced above. In particular, it only depends on the poset E.

4.2. Theorem: *Let T be a finite lattice, and E the full subposet of join-irreducible elements of T. Then E is germ extensible in T, and moreover $\overline{G}(E) = G_T$.*

Proof : For $t \in T$, we set $\alpha(t) = \{e \in E \mid e \le t\}$.

We have $G_T = \Lambda E \sqcup \widehat{G}_T$. So if $t \in G_T$, then either $t \in \Lambda E$, that is $t = \bigwedge_{\substack{e\in E \\ e\geq t}} e$, and $\alpha(t) = E_{\leq B}$, where $B = \{e \in E \mid e \geq t\}$. Hence $\alpha(t) \in G(E)$ in this case.

Otherwise $t \in \widehat{G}_T$. Then $t \notin \Lambda E$, so we have a sequence

$$t < \sigma(t) < \sigma^2(t) < \ldots < \sigma^n(t) = \sigma^\infty(t) \ \ .$$

All the terms different from t of this sequence are in E: indeed, if $\sigma^i(t) \notin E$ for $i \geq 1$, then $r\sigma^i(t) = \sigma^i(t)$, so

$$t = r^\infty\sigma^\infty(t) \geq r^\infty\sigma^i(t) = \sigma^i(t) \ \ ,$$

contradicting $t < \sigma^i(t)$. So $\sigma^{i-1}(t) \leq r\sigma^i(t) < \sigma^i(t)$ for $i \geq 1$. Moreover if $\sigma^{i-1}(t) < r\sigma^i(t)$, then there are two cases: either $r\sigma^i(t) \notin E$, and then

$$t \leq \sigma^{i-1}(t) < r\sigma^i(t) = r^\infty\sigma^i(t) \leq r^\infty\sigma^\infty(t) = t \ \ ,$$

a contradiction. Or $r\sigma^i(t) \in E$, and then $r\sigma^i(t) \geq \sigma\big(\sigma^{i-1}(t)\big) = \sigma^i(t)$, contradicting $r\sigma^i(t) < \sigma^i(t)$. Hence $r\sigma^i(t) = \sigma^{i-1}(t)$ for any $i \in \{1, \ldots, n\}$.

We set $\gamma = \sigma(t) > t$. Then $\gamma \in E$, and γ is a germ of E. Indeed:

- The element t is the greatest element of $]\cdot,\gamma[_T$, so

  $$]\cdot,\gamma[_E = \{f \in E \mid f < \gamma\} = \{f \in E \mid f \leq t\} = \alpha(t) \ \ .$$

  Hence if $f \in E$ and $f \geq g$ for any $g \in]\cdot,\gamma[_E$, then $f \geq \bigvee_{\substack{e\in E \\ e\leq t}} e = t$, hence $f > t$ and $f \geq \gamma$. Thus $\gamma = \mathrm{Sup}_E\,]\cdot,\gamma[_E$.
- Let $e_i = \sigma^i(t)$, for $i \geq 1$. If $e \in E$ and $e \geq \gamma = e_1$, and either $e > e_n$, or there exists a largest integer $i \in \{1, \ldots, n\}$ such that $e_i \leq e$. If $e_i < e$, then $\sigma(e_i) = e_{i+1} \leq e$, contradicting the definition of i. Hence $e_i = e$ for some $i \in \{1, \ldots, n\}$, or $e > e_n$. Similarly, since $e_i = r(e_{i+1})$ for $i \in \{0, \ldots, n-1\}$, if $e \in E$ and $e \leq e_n$, then e is equal to e_i for some $i \in \{1, \ldots, n\}$, or $e < \gamma$.
- Finally $e_n = \sigma(e_n) = \bigwedge_{\substack{e\in E \\ e>e_n}} e$. Thus if $f \in E$ and $f \leq e$ for all $e \in]e_n,\cdot[_E$, then $f \leq e_n$. In other words $e_n = \mathrm{Inf}_E]e_n,\cdot[_E$.

This shows that γ is a germ of E, with cogerm e_n. Now $\alpha(t) =]\cdot,\gamma[_E$, so $\alpha(t) \in G(E)$ in this case also. It follows that $\alpha(t) \in G(E)$ for any $t \in G_T$.

Conversely, for $s \in G(E)$, set $\nu(s) = \bigvee_{e\in s} e$. Then $\nu(s) \in G_T$: indeed, either $s = E_{\leq B}$ for some $B \subseteq E$, and then $\nu(s) = \bigwedge_{b\in B} b \in \Lambda E$. Or $s =]\cdot,g[_E$, for $g \in \mathrm{Grm}(E)$. In this case $t = \nu(s) = r(g) \in \widehat{G}_T$: indeed, if $\hat{g}$ is the cogerm of g in E, and if $[g, \hat{g}]_E = \{g = e_0 < e_1 < \ldots < e_n = \hat{g}\}$, then clearly $e_i = \sigma^{i+1}(t)$ for $0 \leq i \leq n$. Moreover $\sigma(\hat{g}) = \hat{g}$, since $\sigma(\hat{g}) = \bigwedge\limits_{\substack{f \in E \\ f > \hat{g}}} f$ is equal to the join of all elements $e \in E$ such that $e \leq f$ for all $f \in]\hat{g}, .[_E$, and $\mathrm{Inf}_E]\hat{g}, .[_E = \hat{g}$. Then clearly again $r(e_i) = e_{i-1}$ for $1 \leq i \leq n$, and $t = r(g)$.

So we have the maps α and ν

$$G_T \underset{\nu}{\overset{\alpha}{\rightleftarrows}} G(E) \quad ,$$

which are obviously maps of posets. Moreover $\nu \circ \alpha(t) = t$ for any $t \in G_T$, since $t = \bigvee\limits_{\substack{e \in E \\ e \leq t}} e$ for any $t \in T$. Similarly, if $s \in G(E)$, then $\alpha \circ \nu(s) = s$: indeed either $s = E_{\leq B}$ for some $B \leq U$, and then $\nu(s) = \bigwedge\limits_{b \in B} b$. In this case $e \leq \nu(s)$ for $e \in E$ if and only if $e \in E_{\leq B} = s$, so $\alpha \circ \nu(s) = s$. Or $s =]\,.\,, g[_E$ for some $g \in \mathrm{Grm}(E)$, and then $\nu(s) = r(g)$, so that $e \leq \nu(s)$ for $e \in E$ if and only if $e < g$, that is $\alpha \circ \nu(s) = s$ in this case also. This completes the proof of Theorem 4.2, since by Definition 3.1, the image of ν is equal to $\overline{G}(E)$. □

4.3. Let U be a finite poset, and $I_\downarrow(U)$ be the lattice of lower-subsets of U. The irreducible elements of $I_\downarrow(U)$ are the subsets of the form $]\,.\,, u]_U$, for $u \in U$, so we can identify $\mathrm{Irr}\big(I_\downarrow(U)\big)$ with U. We will now show that $G(U)$ is equal to the subset $G_{I_\downarrow(U)}$ of the lattice $I_\downarrow(U)$ introduced in (4.1).

First let B be a subset of U. Then $\bigcap\limits_{b \in B}]\,.\,, b]_U = U_{\leq B}$. In other words the set ΛU of intersections of irreducible elements of $I_\downarrow(U)$ is equal to the set $\Lambda(U)$ of Definition 2.20.

Now let $S \in \widehat{G}_{I_\downarrow(U)}$. Then S is not a meet of join-irreducible elements of $I_\downarrow(U)$, so $S \neq U_{\leq B}$, for any $B \subseteq U$. Now in the proof of Theorem 4.2, we saw that in the sequence

$$S < \sigma(S) < \sigma^2(S) < \ldots < \sigma^n(S) = \sigma^\infty(S) \quad ,$$

all the terms different from S are join-irreducible in $I_\downarrow(U)$. Moreover $r\sigma^i(S) = \sigma^{i-1}(S)$ for $i \geq 1$.

It follows that there is a sequence $u_1, \ldots, u_n$ of elements of U such that $\sigma^i(U) =]\,.\,, u_i]_U$, for $i \geq 1$. In particular $u_1 < u_2 < \ldots < u_n$. Moreover since $r\big(]\,.\,, u]_U\big) =]\,.\,, u[_U$ for any $u \in U$, we have $S =]\,.\,, u_1[_U$, and $]\,.\,, u_i[_U =]\,.\,, u_{i-1}]_U$ for $i \geq 2$. In other words $S =]\,.\,, u_1[_U$, and u_{i-1} is the largest element of $]\,.\,, u_i[_U$, for $i \geq 2$.

Now $\sigma(S) =]\,.\,,u_1]_U = \bigcap_{\substack{u\in U\\ S\subset]\,.\,,u]_U}}]\,.\,,u]_U$. Since S is not irreducible in $I_\downarrow(U)$, saying that $S\subset]\,.\,,u]_U$ is equivalent to saying that $S\subseteq]\,.\,,u]_U$, that is, u is an upper bound of S. Thus u_1 is the smallest upper bound of S, i.e. $u_1 = \mathrm{Sup}_U]\,.\,,u_1[_U$. Similarly for $1\leq i<n$, saying that $\sigma\big(]\,.\,,u_i]_U\big) =]\,.\,,u_{i+1}]_U$ amounts to saying that u_{i+1} is the smallest element of $]u_i,\,.\,[_U$. Since moreover $r\sigma^i(S) = \sigma^{i-1}(S)$ for $i\geq 1$, it follows that $]\,.\,,u_i[_U =]\,.\,,u_{i-1}]_U$ for $i\geq 2$, i.e. u_{i-1} is the largest element of $]\,.\,,u_i]_U$, for $i\geq 2$. Finally saying that $]\,.\,,u_n]_U = \sigma\big(]\,.\,,u_n]_U\big) = \bigcap_{\substack{u\in U\\ u>u_n}}]\,.\,,u]_U$ amounts to saying that u_n is the greatest lower bound of $]u_n,\,.\,[_U$, i.e. $u_n = \mathrm{Inf}_U]u_n,\,.\,[_U$.

This discussion shows that u_1 is a germ of U, with cogerm u_n. Hence $S =]\,.\,,u_1[_U$ belongs to the set $\widehat{G}(U)$ of Definition 2.20. Conversely, if u_1 is a germ of U, with cogerm u_n, it is straightforward to reverse the above arguments and check that $S =]\,.\,,u_1[_U$ belongs to $\widehat{G}_{I_\downarrow(U)}$, as defined in 4.1, with $\sigma^\infty(S) =]\,.\,,u_n]_U$. This shows that Notation 2.20 is consistent with Notation 4.1.

5. Examples

5.1. The empty subset of a finite (non empty) lattice T is germ extensible (by Theorem 3.2, since the emptyset has no germs at all). Moreover $\overline{G}(\emptyset) = \{0\}$, where 0 is the smallest element of T.

5.2. Let $U = \{u_1 < u_2 < \ldots < u_n\}$ be a totally ordered poset of cardinality $n>0$. Then the only germ of U is u_1, with cogerm u_n. The poset $G(U)$ is equal to $\{V_0\subset V_1\subset\ldots\subset V_n\}$, where $V_i = \{u_1,\ldots,u_i\}$ for $0\leq i\leq n$ (so $V_0=\emptyset$). If U is a subposet of a finite lattice T, then U is germ extensible in T if and only if u_1 is not equal to the smallest element 0 of T. In this case $\overline{G}(U) = \{0\}\sqcup U$.

5.3. Let $U = \{u_1,\ldots,u_n\}$ be a discrete poset of cardinality $n\geq 2$. Then U has no germs (see Proposition 2.8), and $G(U) = \{\emptyset\}\sqcup\big\{\{u\}\mid u\in U\big\}\sqcup\{U\}$. If U is a full subposet of a finite lattice T, then U is germ extensible in T, and $\overline{G}(U) = \{0\}\sqcup U\sqcup\{v\}$, where where 0 is the smallest element of T and $v = u_1\vee u_2\vee\ldots\vee u_n$ in T.

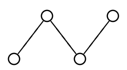
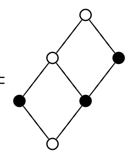
5.4. Let $U = \{a<c>b\}$ be a connected poset of cardinality 3 with two minimal elements. Then $\mathrm{Grm}(U) = \{c\}$, and $G(U) = \big\{\emptyset,\{a\},\{b\},\{a,b\},\{a,b,c\}\big\}$. If U is a full subposet of a finite lattice T, then U is germ extensible in T if and only if $c > a\vee b$ in T. In this case $\overline{G}(U) = \{0,a,b,a\vee b,c\}$.

5.5. Let $V = U^{op} = \{a>c<b\}$ be the opposite poset of the previous

example. Then $\mathrm{Grm}(V) = \{c\}$, and $G(V) = \big\{\emptyset, \{c\}, \{b,c\}, \{a,c\}, \{a,b,c\}\big\}$. If V is a full subposet of a finite lattice T, then V is germ extensible in T if and only if c is not equal to the smallest element 0 of T. In this case $\overline{G}(V) = \{0, c, a, b, a \vee b\}$.

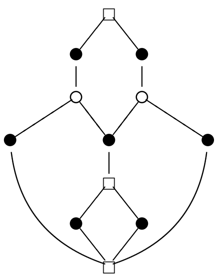
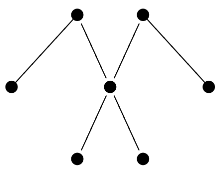
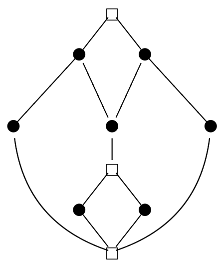
5.6. In the previous two examples, there is an isomorphism of posets $G(U^{op}) \cong G(U)^{op}$. This is a general phenomenon, that will be established in a joint forthcoming paper with Jacques Thévenaz ([6]).

5.7. In all the previous examples, the poset of join-irreducible elements of the lattice $G(U)$ is isomorphic to U, but this need not be true in general. For example

if $U =$ , then $G(U) =$ 

where the irreducible elements of $G(U)$ are the black ones.

5.8. An example of a lattice T, its germs (□), the poset U of its irreducible elements (•), and the germ closure $G(U)$ with its germs (□):

$T =$  $U =$  $G(U) =$ 

5.9. Let T be a finite lattice, and let $s \in \mathrm{Grm}(T)$. Then $s = \mathrm{Sup}_T\,]\cdot, s[_T$, hence $s = \bigvee\limits_{\substack{t \in T \\ t < s}} t$. By Theorem 3.4, it follows that the set $U = T - \mathrm{Grm}(T)$ is germ extensible in T, and such that $\overline{G}(U) = T$. Moreover this set U is the only germ extensible subset of T with this property. This yields in particular another proof of Assertion 1 of Corollary 2.31.

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