K-theory, genotypes, and biset functors

Serge Bouc

Abstract: Let p be an odd prime number. In this paper, we show that the *genome* $\Gamma(P)$ of a finite p-group P, defined as the direct product of the genotypes of all rational irreducible representations of P, can be recovered from the first group of K-theory $K_1(\mathbb{Q}P)$. It follows that the assignment $P \mapsto \Gamma(P)$ is a p-biset functor. We give an explicit formula for the action of bisets on Γ, in terms of generalized transfers associated to left free bisets. Finally, we show that Γ is a rational p-biset functor, i.e. that Γ factors through the Roquette category of finite p-groups.

AMS Subject Classification: 19B28, 20C05, 18A99.

Keywords: K-theory, genotype, Whitehead group, biset functor, Roquette category, transfer.

1. Introduction

Let p be a prime number. This article originates in a joint work with Nadia Romero ([4]), when we started considering the possible applications of *genetic bases* to the computation of Whitehead groups of finite p-groups. Indeed, after the comprehensive book of B. Oliver ([7]), it became clear to N. Romero that these questions have close links to rational representations of p-groups. So the idea emerged that possibly genetic bases would be a natural tool in this context, and a first use of this is made in [8].

In particular, when trying to compute various groups related to the Whitehead group of a finite p-group P (for odd p), a specific product appears, defined in terms of the fields of endomorphisms of the irreducible $\mathbb{Q}P$-modules. After some non trivial reformulation using genetic bases, this product can be viewed as

$$\Gamma(P) = \prod_{S \in \mathcal{B}} \big(N_P(S)/S\big) \;\;,$$

where $\mathcal{B}$ is a genetic basis of P. As the groups $N_P(S)/S$ are called the *types* or *genotypes* of the irreducible $\mathbb{Q}P$-modules, we call $\Gamma(P)$ the *genome* of P. It is the main subject of this paper.

The connection of $\Gamma(P)$ with Whitehead groups and K-theory is established in Theorem 4.3: the genome of P can be recovered as the p-torsion part of $K_1(\mathbb{Q}P)$. This induces a structure of p-biset functor on the correspondence $P \mapsto \Gamma(P)$, which we try to make explicit in Section 5, by giving formulae to compute the action of a (Q,P)-biset on $\Gamma(P)$ (Theorem 5.9). Finally, we show that Γ is a *rational* p-biset functor, hence it factors through the *Roquette category* of finite p-groups introduced in [3].

2. Review of K_1

2.1. Let A be a ring (with 1). Let $GL(A)$ denote the colimit of the linear groups $GL_n(A)$, for $n \in \mathbb{N}_{>0}$, where the inclusion $GL_n(A) \hookrightarrow GL_{n+1}(A)$ is

$$M \in GL_n(A) \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(A) \ \ .$$

The group $K_1(A)$ is defined as the abelianization of $GL(A)$, namely

$$K_1(A) = GL(A)^{ab} = GL(A)/[GL(A), GL(A)] \ \ .$$

2.2. Remark : In particular there is a canonical group homomorphism from the group $A^\times = GL_1(A)$ of invertible elements of A to $K_1(A)$, which factors as

$$A^\times \longrightarrow A^\times/[A^\times, A^\times] \xrightarrow{\alpha_A} K_1(A)$$

2.3. There is an alternative definition of $K_1(A)$: let $\mathcal{P}(A)$ denote the category of pairs (P,a) of a finitely generated projective (left) A-module P, and an automorphism a of P. A morphism $(P,a) \to (Q,b)$ in $\mathcal{P}(A)$ is a morphism of A-modules $f : P \to Q$ such that $b \circ f = f \circ a$.

Let $[P,a]$ denote the isomorphism class of (P,a) in $\mathcal{P}(A)$, and let $K_{det}(A)$ denote the Grothendieck group with generators the set of these equivalence classes, and the relations of the following two forms

- $[P, a \circ a'] = [P,a] + [P,a']$, for any $a, a' \in \mathrm{Aut}_A(P)$,
- $[Q,b] = [P,a] + [R,c]$ whenever there are morphisms $f : [P,a] \to [Q,b]$ and $g : [Q,b] \to [R,c]$ in $\mathcal{P}(A)$ such that the sequence

  $$0 \to P \overset{f}{\to} Q \overset{g}{\to} R \to 0$$

  is an exact sequence of A-modules (in particular, since R is projective, this sequence splits).

If $n \in \mathbb{N}_{>0}$ and $m \in GL_n(A)$, one can view m as an automorphism of the free module A^n. Let $\lambda(m) = [A^n, m] \in K_{det}(A)$.

2.4. Theorem: *The assignment $m \mapsto \lambda(m)$ induces a group isomorphism $K_1(A) \cong K_{det}(A)$.*

Proof : See [5] Theorem 40.6. □

2.5. Let now A and B be two rings, and let L be a (B, A)-bimodule which is finitely generated and projective as a left B-module. If P is a finitely generated projective A-module, then P is a direct summand of some free A-module A^n, and then $L \otimes_A P$ is a direct summand of $L \otimes_A A^n \cong L^n$ as a left B-module. Hence $L \otimes_A P$ is a finitely generated projective left B-module. Then the functor $L \otimes_A - : P \mapsto L \otimes_A P$ induces a functor $T_L : \mathcal{P}(A) \to \mathcal{P}(B)$ such that

$$T_L\big((P, a)\big) = (L \otimes_A P, L \otimes_A a) \quad .$$

One checks easily that the defining relations of $K_{det}(A)$ are preserved by this functor, hence there is a well-defined induced group homomorphism

$$t_L : K_{det}(A) \to K_{det}(B)$$

sending the class $[P, a]$ to the class $[L \otimes_A P, L \otimes_A a]$. This group homomorphism is called the (generalized) *transfer* associated to the bimodule L.

The properties of the tensor product of bimodules now translate to properties of this transfer homomorphism:

2.6. Proposition: *Let A, B, C be rings. In the following assertions, assume that the bimodules involved are finitely generated and projective as left modules. Then:*

1. *if $L \cong L'$ as (B, A)-bimodules, then $t_L = t_{L'}$.*
2. *if L is the (A, A)-bimodule A, then $t_L = \mathrm{Id}_{K_{det}(A)}$.*
3. *if $L \cong L_1 \oplus L_2$ as (B, A)-bimodules, then $t_L = t_{L_1} + t_{L_2}$.*
4. *if L is a (B, A)-bimodule and M is a (C, B)-bimodule, then*

$$t_M \circ t_L = t_{M \otimes_B L} \quad .$$

It follows in particular from (2) and (4) that if L is a (B, A)-bimodule inducing a Morita equivalence from A to B, then t_L is an isomorphism (more precisely, if M is an (A, B)-bimodule such that $M \otimes_B L \cong A$ and $L \otimes_A M \cong B$ as bimodules, then t_L and t_M are inverse to one another).

2.7. The group $K_1(A)$ has been determined for a number of rings A. In particular:

2.8. Theorem:

1. *Let D be a division ring. Then $K_1(D) \cong D^\times/[D^\times, D^\times]$.*
2. *Let F be a field. Then the determinant homomorphism*

$$m \in GL_n(F) \to \det(m) \in F^\times$$

induces an isomorphism $K_1(F) \cong F^\times$.

Proof : See [5] Theorem 38.32. □

2.9. Proposition: *Let $\mathbb{F}$ be a field and G be a finite group of order prime to the characteristic of $\mathbb{F}$. Let $\mathrm{Irr}_{\mathbb{F}}(G)$ denote a set of representatives of isomorphism classes of irreducible $\mathbb{F}G$-modules, and for $V \in \mathrm{Irr}_{\mathbb{F}}(G)$, let $D_V = \mathrm{End}_{\mathbb{F}G}(V)$ denote the skew field of endomorphisms of V.*

Then V is an $(\mathbb{F}G, D_V^{op})$-bimodule, where the action of $g \in G$ and $f \in D_V$ on $v \in V$ is given by $g \cdot v \cdot f = gf(v) = f(gv)$. Let V^* denote the $\mathbb{F}$-dual of V, considered as a $(D_V^{op}, \mathbb{F}G)$-bimodule.

Then the map

$$\tau : K_1(\mathbb{F}G) \xrightarrow{\prod\limits_V t_{V^*}} \prod_{V \in \mathrm{Irr}_{\mathbb{F}}(G)} K_1(D_V^{op})$$

is a well defined isomorphism of abelian groups, with inverse

$$\tau' : \prod_{V \in \mathrm{Irr}_{\mathbb{F}}(G)} K_1(D_V^{op}) \xrightarrow{\prod\limits_V t_V} K_1(\mathbb{F}G) \ .$$

Proof : As $|G|$ is invertible in $\mathbb{F}$, the group algebra $\mathbb{F}G$ is semisimple. Moreover for each $V \in \mathrm{Irr}_{\mathbb{F}}(V)$, the skew field D_V^{op} is also a semisimple $\mathbb{F}$-algebra. This shows that V is projective and finitely generated as an $\mathbb{F}G$-module, and that V^* is projective and finitely generated as a D_V^{op}-module (that is V^* is a finite dimensional D_V-vector space). Hence the generalized transfer maps $t_V : K_1(\mathbb{F}G) \to K_1(D_V^{op})$ and $t_{V^*} : K_1(D_V^{op}) \to K_1(\mathbb{F}G)$ are well defined.

Now for any two finitely generated $\mathbb{F}G$-modules V and W, the map

$$\alpha \otimes w \mapsto \big(v \in V \mapsto \alpha(v)w \in W\big)$$

extends to an isomorphism (see e.g. [6] (2.32))

$$V^* \otimes_{\mathbb{F}G} W \to \mathrm{Hom}_{\mathbb{F}G}(V, W)$$

of $\big((\mathrm{End}_{\mathbb{F}G} V)^{op}, (\mathrm{End}_{\mathbb{F}G} W^{op})\big)$-bimodules, where the bimodule structure on the right hand side is given by

$$\forall h \in (\mathrm{End}_{\mathbb{F}G} V)^{op},\ \forall \psi \in \mathrm{Hom}_{\mathbb{F}G}(V, W),\ \forall k \in (\mathrm{End}_{\mathbb{F}G} W)^{op},\ \ h{\cdot}\psi{\cdot}k = k{\circ}\psi{\circ}h.$$

In case $V, W \in \mathrm{Irr}_{\mathbb{F}}(G)$ and $V \neq W$, this yields $V^* \otimes_{\mathbb{F}G} W = 0$. And if $V = W$, we have an isomorphism $V^* \otimes_{\mathbb{F}G} V \cong D_V^{op}$ of (D_V^{op}, D_V^{op})-bimodules. Then by Assertions (2) and (4) of Proposition 2.6

$$t_{V^*} \circ t_W = \begin{cases} 0 & \text{if } V \neq W \\ \mathrm{Id}_{K_1(D_V^{op})} & \text{if } V = W. \end{cases}$$

In other words $\tau \circ \tau'$ is the identity map of $\prod\limits_{V \in \mathrm{Irr}_{\mathbb{F}}(G)} K_1(D_V^{op})$. Conversely

$$\tau' \circ \tau = \sum_{V \in \mathrm{Irr}_{\mathbb{F}}(G)} t_V \circ t_{V^*} = t_L \ \ ,$$

where L is the $(\mathbb{F}G, \mathbb{F}G)$-bimodule $\mathop{\oplus}\limits_{V \in \mathrm{Irr}_{\mathbb{F}}(G)} (V \otimes_{D_V^{op}} V^*)$. For each $V \in \mathrm{Irr}_{\mathbb{F}}(G)$, the bimodule $V \otimes_{D_V^{op}} V^* \cong \mathrm{End}_{D_V^{op}}(V)$ is isomorphic to the Wedderburn component of $\mathbb{F}G$ corresponding to the simple module V, and the semisimple algebra $\mathbb{F}G$ is equal to the direct sum of its Wedderburn components. Thus $L \cong \mathbb{F}G$, and t_L is equal to the identity map of $K_1(\mathbb{F}G)$. □

2.10. Corollary: *Under the assumptions of Proposition 2.9, there is a group isomorphism*

$$K_1(\mathbb{F}G) \cong \prod_{V \in \mathrm{Irr}_{\mathbb{F}}(G)} D_V^{\times}/[D_V^{\times}, D_V^{\times}] \ \ .$$

Proof : This follows from Proposition 2.9 and Theorem 2.8, since $x \mapsto x^{-1}$ is a group isomorphism $D^{\times} \to (D^{op})^{\times}$, for any skew field D. □

2.11. Recall ([2] Chapter 3) that the *biset category* $\mathcal{C}$ of finite groups has all finite groups as objects, the set of morphisms in $\mathcal{C}$ from a group G to a group H being the Grothendieck group of (finite) (H, G)-bisets, i.e. the Burnside group $B(H, G)$. The composition of morphisms in $\mathcal{C}$ is the linear extension of the product $(V, U) \mapsto V \times_H U$, for a (K, H)-biset V and an (H, G)-biset U.

A *biset functor* is an additive functor from $\mathcal{C}$ to the category $\mathcal{A}b$ of abelian groups.

For a prime number p, a *p-biset functor* is an additive functor from the full subcategory $\mathcal{C}_p$ of $\mathcal{C}$ consisting of p-groups to $\mathcal{A}b$.

Let ${}_1\mathcal{C}$ denote the (non full) subcategory of $\mathcal{C}$ with the same objects, but where the set of morphisms from a group G to a group H is the Grothendieck group ${}_1B(H,G)$ of *left free* (H,G)-bisets. A *deflation biset functor* is an additive functor from ${}_1\mathcal{C}$ to $\mathcal{A}b$.

2.12. Proposition:

1. *Let R be a commutative ring. The assignment $G \mapsto K_1(RG)$ is a deflation functor.*
2. *The assignment $G \mapsto K_1(\mathbb{Q}G)$ is a biset functor.*

Proof : For Assertion 1, if G and H are finite groups, and if U is a finite left free (H,G)-biset, then the corresponding permutation (RH,RG)-bimodule RU is free and finitely generated as a left RH-module. Hence the transfer $t_{RU} : K_1(RG) \to K_1(RH)$ is well defined. If U' is an (H,G)-biset isomorphic to U, then $RU' \cong RU$ as bimodules, hence $t_{RU'} = t_{RU}$. And if U is the disjoint unions of two (H,G)-bisets U_1 and U_2, then $RU \cong RU_1 \oplus RU_2$, thus $t_{RU} = t_{RU_1} + t_{RU_2}$. This shows that one can extend linearly this transfer construction $U \mapsto t_{RU}$ to a group homomorphism

$$u \in {}_1B(H,G) \mapsto K_1(u) \in \mathrm{Hom}_{\mathcal{A}b}\big(K_1(RG), K_1(RH)\big) \enspace .$$

Moreover, if K is a third group, and V is a finite left free (K,H)-biset, then $t_{RV} \circ t_{RU} = t_{RV \otimes_{RH} RU} = t_{R(V \times_H U)}$ since the bimodules $RV \otimes_{RH} RU$ and $R(V \times_H U)$ are isomorphic. Finally, if U is the identity biset at G, namely the set G acted on by left and right multiplication, then $RU \cong RG$ as (RG,RG)-bimodule, thus $t_{RU} = \mathrm{Id}_{K_1(RG)}$. This completes the proof of Assertion (1).

The proof of Assertion (2) is the same, except that the transfer $t_{\mathbb{Q}U} : K_1(\mathbb{Q}G) \to K_1(\mathbb{Q}H)$ is well defined for an arbitrary finite (H,G)-biset U: indeed $\mathbb{Q}U$ is always finitely generated and projective as a $\mathbb{Q}H$-module. □

3. Review of genetic subgroups

3.1. Let p be a prime number. A finite p-group is called a *Roquette* p-group if it has normal rank 1, i.e. if all its normal abelian subgroups are cyclic. The Roquette p-groups (see [9]) are the cyclic groups C_{p^n}, for $n \in \mathbb{N}$, if p is odd. The Roquette 2-groups are the cyclic groups C_{2^n}, for $n \in \mathbb{N}$, the generalized quaternion groups Q_{2^n}, for $n \geq 3$, the dihedral groups D_{2^n}, for $n \geq 4$, and the semidihedral groups SD_{2^n}, for $n \geq 4$.

If P is a Roquette p-group, then P admits a *unique faithful irreducible rational representation* Φ_P ([2] Proposition 9.3.5).

3.2. If S is a subgroup of a finite p-group P, denote by $Z_P(S)$ the subgroup of $N_P(S)$ defined by $Z_P(S)/S = Z\big(N_P(S)/S\big)$. The subgroup S is called *genetic* if it fulfills the following two conditions:

1. if $x \in P$, then $S^x \cap Z_P(S) \leq S$ if and only if $S^x = S$.
2. the group $N_P(S)/S$ is a Roquette p-group.

When S is a genetic subgroup of P, let $V(S) = \mathrm{Indinf}_{N_P(S)/S}^P \Phi_{N_P(S)/S}$ denote the $\mathbb{Q}P$-module obtained by inflation of $\Phi_{N_P(S)/S}$ to $N_P(S)$ followed by induction to P.

Two genetic subgroups S and T of P are said to be *linked modulo* P (notation $S \frown_P T$) if there exists an element $x \in P$ such that $S^x \cap Z_P(T) \leq T$ and ${}^xT \cap Z_P(S) \leq S$ (where as usual $S^x = x^{-1}Sx$ and ${}^xT = xTx^{-1}$).

3.3. Theorem: *Let p be a prime number and P be a finite p-group.*

1. *If V is a simple $\mathbb{Q}P$-module, then there exists a genetic subgroup S of P such that $V \cong V(S)$.*
2. *If S is a genetic subgroup of P, then there is an isomorphism of $\mathbb{Q}$-algebras*
$$\mathrm{End}_{\mathbb{Q}P} V(S) \cong \mathrm{End}_{\mathbb{Q}N_P(S)/S} \Phi_{N_P(S)/S}$$
induced by the induction-inflation functor from $\mathbb{Q}N_P(S)/S$-modules to $\mathbb{Q}P$-modules.
3. *If S and T are genetic subgroups of P, then $V(S) \cong V(T)$ if and only if $S \frown_P T$. In this case, the groups $N_P(S)/S$ and $N_P(T)/T$ are isomorphic.*

Proof : See Theorem 9.4.1, Lemma 9.4.3, Definition 9.4.4, Corollary 9.4.5, Theorem 9.5.6 and Theorem 9.6.1 of [2]. □

It follows in particular that the relation $\frown_P$ is an equivalence relation on the set of genetic subgroups of P. A *genetic basis* of P is by definition a set of representatives of genetic subgroups of P for this equivalence.

It also follows that if V is a simple $\mathbb{Q}P$-module, and if S is a genetic subgroup of P such that $V \cong V(S)$, then the group $N_P(S)/S$ does not depend on the choice of such a genetic subgroup S. This factor group is called the *type* of V ([2] Definition 9.6.8). Laurence Barker ([1]) has introduced the word *genotype* instead of type, and we will follow this terminology.

3.4. Definition: *Let p be a prime number and P be a finite p-group. The* genome $\Gamma(P)$ *of P is the product group*

$$\Gamma(P) = \prod_{S \in \mathcal{B}} \big(N_P(S)/S\big) \ ,$$

where $\mathcal{B}$ is a genetic basis of P. It is well defined up to isomorphism.

More precisely, suppose that $\mathcal{B}$ and $\mathcal{B}'$ are genetic bases of a p-group P. Then for $S \in \mathcal{B}$, there exists a unique $S' \in \mathcal{B}'$ such that there exists some $x \in P$ with

$$(\mathbf{3.5}) \qquad S^x \cap Z_P(S') \leq S' \text{ and } {}^xS' \cap Z_P(S) \leq S \ ,$$

and the correspondence $S \mapsto S'$ is a bijection from $\mathcal{B}$ to $\mathcal{B}'$. Moreover, for each $S \in \mathcal{B}$ corresponding to $S' \in \mathcal{B}'$, the set $\mathcal{D}$ of elements x satisfying (3.5) is a single $\big(N_P(S), N_P(S')\big)$-double coset in P ([2], Proposition 9.6.9).

Let $x \in \mathcal{D}$. Then for each $n \in N_P(S)/S$, there is a unique element $n' \in N_P(S')/S'$ such that $nSx = xS'n'$, and the map $n \mapsto n'$ is a group isomorphism $N_P(S)/S \to N_P(S')/S'$, which only depends on x up to interior automorphism of $N_P(S)/S$. In particular, when p is odd, the group $N_P(S)/S$ is cyclic, so this group isomorphism does not depend on x.

Thus for odd p, this yields a canonical group isomorphism

$$(\mathbf{3.6}) \qquad \prod_{S \in \mathcal{B}} \big(N_P(S)/S\big) \xrightarrow{\gamma_{\mathcal{B}',\mathcal{B}}} \prod_{S' \in \mathcal{B}'} \big(N_P(S')/S'\big) \ .$$

3.7. Remark : Let p be a prime number, and P be a finite p-group. Since the Roquette p-groups are all indecomposable (that is, they cannot be written as a direct product of two non-trivial of their subgroups), the genotypes of the simple $\mathbb{Q}P$-modules are determined by the group $\Gamma(P)$: by the Krull-Remak-Schmidt theorem, the group $\Gamma(P)$ can be written as a direct product of indecomposable groups $\Gamma_1, \ldots, \Gamma_r$, and such a decomposition is unique (up to permutation and isomorphism of the factors). Then $\Gamma_1, \ldots, \Gamma_r$ are the genotypes of the simple $\mathbb{Q}P$-modules.

In terms of the Roquette category $\mathcal{R}_p$ (see Section 7, or [3]), this means that two finite p-groups P and Q become isomorphic in $\mathcal{R}_p$ if and only if their genomes $\Gamma(P)$ and $\Gamma(Q)$ are isomorphic (as groups) (see [3] Proposition 5.14).

4. K-theory and genome

4.1. Lemma: *Let p be a prime, and C be a cyclic p-group. Recall that Φ_C is the unique faithful irreducible rational representation of C, up to isomorphism.*

1. *If $C = \mathbf{1}$, then $\Phi_C = \mathbb{Q}$.*
2. *If $C \neq \mathbf{1}$, let Z be the unique subgroup of order p of C. Then there is an exact sequence*

$$0 \to \Phi_C \to \mathbb{Q}C \to \mathbb{Q}(C/Z) \to 0 \quad , \tag{4.2}$$

   of $(\mathbb{Q}C, \mathbb{Q}C)$-bimodules, where $\mathbb{Q}C \to \mathbb{Q}(C/Z)$ is the canonical surjection.
3. *If C has order p^n, then the algebra $\mathrm{End}_{\mathbb{Q}C}(\Phi_C)$ is isomorphic to the cyclotomic field $\mathbb{Q}(\zeta_{p^n})$, and if $p > 2$, the map sending $c \in C$ to the endomorphism $\varphi \mapsto \varphi c$ of Φ_C is a group isomorphism from C to the p-torsion part ${}_p\mathbb{Q}(\zeta_{p^n})^\times$ of the multiplicative group $\mathbb{Q}(\zeta_{p^n})^\times$.*

Proof : Assertion 1 is trivial. Assertion 2 follows e.g. from [2], Proposition 9.3.5. A different proof consists in observing that if C has order p^n, then the algebra $\mathbb{Q}C$ is isomorphic to $\mathbb{Q}[X]/(X^{p^n} - 1)$, and the projection map $\mathbb{Q}C \to \mathbb{Q}(C/Z)$ becomes the canonical map

$$\mathbb{Q}[X]/(X^{p^n} - 1) \to \mathbb{Q}[X]/(X^{p^{n-1}} - 1) \quad .$$

The kernel of this map is now clearly isomorphic to $\mathbb{Q}[X]/(\gamma_{p^n})$, where γ_{p^n} is the p^n-th cyclotomic polynomial, that is, the p^n-th cyclotomic field, which is clearly a simple faithful module for the cyclic group generated by X in the algebra $\mathbb{Q}[X]/(X^{p^n} - 1)$. Observe moreover that the exact sequence 4.2 is indeed a sequence of $(\mathbb{Q}C, \mathbb{Q}C)$-bimodules.

The first part of Assertion 3 follows easily. For the last part, let ζ_{p^n} be a primitive p^n-th root of unity. Observe that a p-torsion element in $\mathbb{Q}(\zeta_{p^n})^\times$ is a p^n-th root of unity. Hence the p-torsion part of $\mathbb{Q}(\zeta_{p^n})^\times$ is cyclic of order p^n, generated by ζ_{p^n}. □

4.3. Theorem: *Let p be an odd prime, and P be a finite p-group, and $\mathcal{B}$ be a genetic basis of P. If S is a genetic subgroup of P, and $a \in N_P(S)/S$, view a as an automorphism of $\Phi_{N_P(S)/S}$, and let $\tilde{a}$ denote the corresponding automorphism of $V(S) = \mathrm{Indinf}_{N_P(S)}^P \Phi_{N_P(S)/S}$.*

1. *The group homomorphism*

$$\Gamma(P) = \prod_{S\in\mathcal{B}} \big(N_P(S)/S\big) \xrightarrow{\nu_{\mathcal{B}}} K_1(\mathbb{Q}P)$$

sending $a \in N_P(S)/S$*, for* $S \in \mathcal{B}$*, to the class* $[V(S), \tilde{a}]$ *in* $K_1(\mathbb{Q}P)$ *is an isomorphism of the genome* $\Gamma(P)$ *onto the* p*-torsion part* ${}_pK_1(\mathbb{Q}P)$ *of* $K_1(\mathbb{Q}P)$*.*

2. *If* $\mathcal{B}'$ *is another genetic basis of* P*, and* $\gamma_{\mathcal{B}',\mathcal{B}}$ *is the canonical isomorphism defined in 3.6, then*

$$\nu_{\mathcal{B}'} \circ \gamma_{\mathcal{B}',\mathcal{B}} = \nu_{\mathcal{B}} \quad .$$

Proof : Since p is odd, the Roquette p-groups are the cyclic p-groups. Assertion 1 now follows from Proposition 2.9, Theorem 3.3, and Lemma 4.1.

For Assertion 2, let $S \in \mathcal{B}$ and let S' be the unique element of $\mathcal{B}'$ such that $S' \frown_P S$. Let $\varphi : N_P(S)/S \to N_P(S')/S'$ be the restriction of $\gamma_{\mathcal{B}',\mathcal{B}}$ to $N_P(S)/S$. If $a \in N_P(S)/S$, let $a' = \varphi(a)$. Then φ induces an isomorphism of $\mathbb{Q}P$-modules $\tilde{\varphi} : V(S) \to V(S')$ such that the diagram

$$\begin{array}{ccc} V(S) & \xrightarrow{\tilde{a}} & V(S) \\ {\scriptstyle\tilde{\varphi}}\downarrow & & \downarrow{\scriptstyle\tilde{\varphi}} \\ V(S') & \xrightarrow{\tilde{a}'} & V(S') \end{array}$$

is commutative. Hence $\big(V(S), \tilde{a}\big) \cong \big(V(S'), \tilde{a}'\big)$ in $\mathcal{P}(\mathbb{Q}P)$, thus $\big[V(S), \tilde{a}\big] = \big[V(S'), \tilde{a}'\big]$ in $K_1(\mathbb{Q}P)$, as was to be shown. □

4.4. Remark : The elements of odd order of $\mathbb{Q}(\zeta_{p^n})^\times$ are the p^n-th roots of unity. So $\Gamma(P)$ is also the odd-torsion part of $K_1(\mathbb{Q}P)$.

4.5. Corollary: *Let* p *be an odd prime. Then the correspondence sending a finite* p*-group* P *to its genome* $\Gamma(P)$ *is a* p*-biset functor.*

Proof : Indeed by Proposition 2.12, the assignment $P \mapsto K_1(\mathbb{Q}P)$ is a p-biset functor. So its p-torsion part is also a p-biset functor. □

5. Explicit transfer maps

We begin with a slight generalization of the transfer homomorphism, associated to a left-free biset:

5.1. Lemma and Definition: *Let G and H be finite groups, and let Ω be a left free (H,G)-biset. Let $[H\backslash\Omega]$ be a set of representatives of H-orbits on Ω. For $g \in G$, and $x \in \Omega$, let $h_{g,x} \in H$ and $\sigma_g(x) \in [H\backslash\Omega]$ be the elements defined by $xg = h_{g,x}\sigma_g(x)$.*

1. *The map $g \in G \mapsto \prod_{x\in[H\backslash\Omega]} h_{g,x}$ (in any order) induces a well defined group homomorphism*

$$\mathrm{Ver}_\Omega : G/[G,G] \to H/[H,H]$$

called the (generalized) transfer *associated to Ω.*

2. *If $\Omega' \cong \Omega$ as (H,G)-bisets, then* $\mathrm{Ver}_{\Omega'} = \mathrm{Ver}_\Omega$.
3. *If $\Omega = \Omega_1 \sqcup \Omega_2$ as (H,G)-bisets, then* $\mathrm{Ver}_\Omega = \mathrm{Ver}_{\Omega_1} + \mathrm{Ver}_{\Omega_2}$.
4. *If K is another finite group, and Ω' is a finite left free (K,H)-biset, then $\Omega' \times_H \Omega$ is a finite left free K-set, and*

$$\mathrm{Ver}_{\Omega'} \circ \mathrm{Ver}_\Omega = \mathrm{Ver}_{\Omega'\times_H\Omega} \quad .$$

The notation and terminology comes from the classical transfer from $G/[G,G]$ to $H/[H,H]$, when H is a subgroup of G: the corresponding biset Ω is the set G itself, in this case.

Proof : Changing the set of representatives $[H\backslash\Omega]$ amounts to replacing each $x \in [H\backslash\Omega]$ by $\eta_x x$, for some $\eta_x \in H$. This changes the element $h_{g,x}$ to $h'_{g,x} = \eta_x h_{g,x}\eta^{-1}_{\sigma_g(x)}$, so the product over $x \in [H\backslash\Omega]$ of the elements $h'_{g,x}$ is equal to the product of the elements $h_{g,x}$ in the abelianization $H/[H,H]$. Hence Ver_Ω does not depend on the choice of a set of representatives.

It follows moreover from the definition that for $g, g' \in G$ and $x \in [H\backslash\Omega]$, we have $h_{gg',x} = h_{g,x}h_{g',\sigma_g(x)}$. Hence

$$\prod_{x\in[H\backslash\Omega]} h_{gg',x} = \prod_{x\in[H\backslash\Omega]} h_{g,x} \prod_{x\in[H\backslash\Omega]} h_{g',\sigma_g(x)} = \prod_{x\in[H\backslash\Omega]} h_{g,x} \prod_{x\in[H\backslash\Omega]} h_{g',x}$$

in $H/[H,H]$, so Ver_Ω is a group homomorphism. This proves Assertion 1.

For Assertion 2, let $f : \Omega \to \Omega'$ be an isomorphism of (H,G)-bisets. Then the set $f\big([H\backslash\Omega]\big)$ is a set of representatives of the H-orbits on Ω'. Moreover for $x \in [H\backslash\Omega]$ and $g \in G$,

$$f(x)g = f(xg) = f\big(h_{g,x}\sigma_g(x)\big) = h_{g,x}f\big(\sigma_g(x)\big) \quad ,$$

so $\mathrm{Ver}_{\Omega'}(g) = \prod_{x\in\Omega} h_{g,x} = \mathrm{Ver}_{\Omega}(g)$, which proves Assertion 2.

Assertion 3 is clear, since $[H\backslash\Omega] = [H\backslash\Omega_1] \sqcup [H\backslash\Omega_2]$.

For Assertion 4, it is straightforward to check that $\Omega' \times_H \Omega$ is left free. Moreover, the set of pairs $(x', x) \in \Omega' \times_H \Omega$, for $x' \in [K\backslash\Omega']$ and $x \in [H\backslash\Omega]$, is a set of representatives of K-orbits on $\Omega' \times_H \Omega$. Then for $x' \in [K\backslash\Omega']$ and $x \in [H\backslash\Omega]$, and $g \in G$

$$\begin{aligned}(x', x)g &= (x', xg) = \big(x', h_{g,x}\sigma_g(x)\big)\\ &= \big(x'h_{g,x}, \sigma_g(x)\big) = \big(k_{h_{g,x},x'}\tau_{h_{g,x}}(x'), \sigma_g(x)\big)\\ &= k_{h_{g,x},x'}\big(\tau_{h_{g,x}}(x'), \sigma_g(x)\big) \ ,\end{aligned}$$

where $k_{h,x'} \in K$ and $\tau_h(x') \in [K\backslash\Omega']$ are defined by $x'h = k_{h,x'}\tau_h(x')$, for $h \in H$ and $x' \in [K\backslash\Omega']$.

It follows that

$$\mathrm{Ver}_{\Omega'\times_H\Omega}(g) = \prod_{\substack{x\in[H\backslash\Omega]\\ x'\in[K\backslash\Omega']}} k_{h_{g,x},x'} = \mathrm{Ver}_{\Omega'}\big(\prod_{x\in[H\backslash\Omega]} h_{g,x}\big) = \mathrm{Ver}_{\Omega'}\circ\mathrm{Ver}_{\Omega}(g) \ ,$$

which completes the proof. □

5.2. Corollary 4.5 shows that there exists a p-biset functor structure on the assignment $P \mapsto \Gamma(P)$ for p-groups, when p is odd. This raises the following question: suppose that P and Q are finite p-groups, that $\mathcal{B}_P$ is a genetic basis of P, and $\mathcal{B}_Q$ is a genetic basis of Q. When U is a finite (Q, P)-biset, how can we compute the map

$$\Gamma(U) : \Gamma(P) = \prod_{S\in\mathcal{B}_P} \big(N_P(S)/S\big) \to \Gamma(Q) = \prod_{T\in\mathcal{B}_Q} \big(N_Q(T)/T\big)$$

giving the action of the biset U?

This amounts to finding the map

$$\Gamma(U)_{T,S} : \overline{N}_P(S) = N_P(S)/S \to \overline{N}_Q(T) = N_Q(T)/T$$

for each pair (T, S) of a genetic subgroup T of Q and a genetic subgroup S of P, defined as follows: if $a \in \overline{N}_P(S)$, then a can be viewed as an automorphism of the $\mathbb{Q}\overline{N}_P(S)$-module $\Phi_{\overline{N}_P(S)}$, viewed as an ideal of $\mathbb{Q}\overline{N}_P(S)$ as in 4.2. Then $\tilde{a} = \mathrm{Indinf}_{\overline{N}_P(S)}^P a$ is an automorphism of $V(S) = \mathrm{Indinf}_{\overline{N}_P(S)}^P \Phi_{\overline{N}_P(S)}$, hence an element $\hat{a} = [V(S), \tilde{a}]$ of $K_1(\mathbb{Q}P)$. This element is mapped by $t_{\mathbb{Q}U}$ to the element

$$t_{\mathbb{Q}U}(\hat{a}) = [\mathbb{Q}U \otimes_{\mathbb{Q}P} V(S), \mathbb{Q}U \otimes_{\mathbb{Q}P} \tilde{a}]$$

of $K_1(\mathbb{Q}Q)$. This in turn is mapped to the element $t_{V(T)^*} \circ t_{\mathbb{Q}U}(\hat{a})$ of the direct summand $K_1(F_T)$ of $K_1(\mathbb{Q}Q)$ corresponding to the simple $\mathbb{Q}Q$-module $V(T)$ as in Proposition 2.9, where F_T is the field $D_{V(T)} = \mathrm{End}_{\mathbb{Q}Q} V(T)$.

Thus to find $\Gamma(U)_{T,S}(a)$, we have to compute the element

$$[V(T)^* \otimes_{\mathbb{Q}Q} \mathbb{Q}U \otimes_{\mathbb{Q}P} V(S), V(T)^* \otimes_{\mathbb{Q}Q} \mathbb{Q}U \otimes_{\mathbb{Q}P} \tilde{a}]$$

of $K_1(F_T) \cong F_T^\times$, and identify it as an element of $\overline{N}_Q(T)$.

We set $L(U)_{T,S} = V(T)^* \otimes_{\mathbb{Q}Q} \mathbb{Q}U \otimes_{\mathbb{Q}P} V(S)$ for simplicity. First we observe that the induction-inflation functor $\mathrm{Indinf}_{\overline{N}_P(S)}^P$ is isomorphic to the functor $\mathbb{Q}(P/S) \otimes_{\overline{N}_P(S)} (-)$, where $\mathbb{Q}(P/S)$ is endowed with its natural structure of $\Big(\mathbb{Q}P, \mathbb{Q}\big(\overline{N}_P(S)\big)\Big)$-bimodule. Hence

$$\begin{aligned} \mathbb{Q}U \otimes_{\mathbb{Q}P} V(S) &= \mathbb{Q}U \otimes_{\mathbb{Q}P} \mathrm{Indinf}_{\overline{N}_P(S)}^P \Phi_{\overline{N}_P(S)} \\ &\cong \mathbb{Q}U \otimes_{\mathbb{Q}P} \mathbb{Q}(P/S) \otimes_{\mathbb{Q}\overline{N}_P(S)} \Phi_{\overline{N}_P(S)} \\ &\cong \mathbb{Q}(U/S) \otimes_{\mathbb{Q}\overline{N}_P(S)} \Phi_{\overline{N}_P(S)} \ , \end{aligned}$$

where $\mathbb{Q}(U/S)$ is given its natural structure of $\Big(\mathbb{Q}Q, \mathbb{Q}\big(\overline{N}_P(S)\big)\Big)$-bimodule.

Tensoring on the left with $V(T)^*$, and using a similar argument, we get that

$$L(U)_{T,S} \cong \Phi^*_{\overline{N}_Q(T)} \otimes_{\mathbb{Q}\overline{N}_Q(T)} \mathbb{Q}(T\backslash U/S) \otimes_{\mathbb{Q}\overline{N}_P(S)} \Phi_{\overline{N}_P(S)} \ ,$$

where $\mathbb{Q}(T\backslash U/S)$ is the permutation $\big(\mathbb{Q}\overline{N}_Q(T), \mathbb{Q}\overline{N}_P(S)\big)$-bimodule associated to the $\big(\overline{N}_Q(T), \overline{N}_P(S)\big)$-biset $T\backslash U/S$. Moreover $\Phi_{\overline{N}_Q(T)}$ is self dual, since it is the unique faithful rational irreducible representation of $\overline{N}_Q(T)$, so we can replace $\Phi^*_{\overline{N}_Q(T)}$ by $\Phi_{\overline{N}_Q(T)}$ in the right hand side of the previous isomorphism.

Now the biset $T\backslash U/S$ splits as a disjoint union

$$T\backslash U/S = \bigsqcup_{\omega \in N_Q(T)\backslash U/N_P(S)} T\backslash \omega/S$$

of transitive $\big(\overline{N}_Q(T), \overline{N}_P(S)\big)$-bisets, where $N_Q(T)\backslash U/N_P(S)$ is the set of $\big(N_Q(T), N_P(S)\big)$-orbits on U. This yields a decomposition

$$\mathbf{(5.3)} \qquad \mathbb{Q}(T\backslash U/S) \cong \bigoplus_{\omega \in N_Q(T)\backslash U/N_P(S)} \mathbb{Q}\big(T\backslash \omega/S\big)$$

as $\big(\mathbb{Q}\overline{N}_Q(T), \mathbb{Q}\overline{N}_P(S)\big)$-bimodules.

5.4. Lemma: *Let C and D be cyclic p-groups, and let Ω be a transitive (D,C)-biset. Then $\Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}\Omega = 0$ unless Ω is left free, and $\mathbb{Q}\Omega \otimes_{\mathbb{Q}C} \Phi_C = 0$ unless Ω is right free.*

Proof : Suppose that the action of C is not free. This means that C is non-trivial, and that the unique subgroup Z of order p of C acts trivially on Ω: indeed since Ω is a transitive biset, the stabilizers in C of the points of Ω are conjugate in C, hence equal since C is abelian. So these stabilizers all contain Z if one of them is non trivial. Then Ω is inflated from a (C/Z)-set $\overline{\Omega}$, and then $\mathbb{Q}\Omega \cong \mathbb{Q}\overline{\Omega} \otimes_{\mathbb{Q}(C/Z)} \mathbb{Q}(C/Z)$. But $\mathbb{Q}(C/Z) \otimes_{QC} \Phi_C$ is the module of Z-coinvariants on Φ_C, hence it is zero, since Φ_C is faithful. Hence $\mathbb{Q}\Omega \otimes_{\mathbb{Q}C} \Phi_C = 0$ in this case. Similarly, if the action of D is not free, then $\Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}\Omega = 0$. □

5.5. It follows from Lemma 5.4 that to compute

$$\Phi^*_{\overline{N}_Q(T)} \otimes_{\mathbb{Q}\overline{N}_Q(T)} \mathbb{Q}(T\backslash U/S) \otimes_{\mathbb{Q}\overline{N}_P(S)} \Phi_{\overline{N}_P(S)}$$

using decomposition 5.3, we can restrict to orbits $\omega = N_Q(T)uN_P(S)$, where $u \in U$, for which the $\big(\overline{N}_Q(T), \overline{N}_P(S)\big)$-biset $T\backslash\omega/S$ is left and right free. The left stabilizer of the element TuS of this biset is equal to

$$\{xT \in \overline{N}_Q(T) \mid \exists s \in S,\ xu = us\}\ ,$$

hence $T\backslash\omega/S$ is left free if and only if

$$(\mathbf{5.6}) \qquad {}^uS \cap N_Q(T) \leq T\ ,$$

where ${}^uS = \{x \in Q \mid \exists s \in S,\ xu = us\}$ ([2] Notation 2.3.16).

Similarly $T\backslash\omega/S$ is right free if and only if

$$T^u \cap N_P(S) \leq S\ ,$$

where $T^u = \{x \in P \mid \exists t \in T,\ tu = ux\}$.

Finally, the bimodule $L(U)_{T,S}$ is isomorphic to

$$\bigoplus_{\substack{u\in[N_Q(T))\backslash U/N_P(S)] \\ {}^uS\cap N_Q(T)\leq T \\ T^u\cap N_P(S)\leq S}} \Phi_{\overline{N}_Q(T)} \otimes_{\mathbb{Q}\overline{N}_Q(T)} \mathbb{Q}\big(T\backslash N_Q(T)uN_P(S)/S\big) \otimes_{\mathbb{Q}\overline{N}_P(S)} \Phi_{\overline{N}_P(S)},$$

where $[N_Q(T))\backslash U/N_P(S)]$ is a set of representatives of $\big(N_Q(T), N_P(S)\big)$-orbits on U.

5.7. Lemma: *Let p be an odd prime, and let C and D be cyclic p-groups. Let moreover Ω be a left and right free finite (D,C)-biset. Let $a \in C$, viewed as an automorphism of the $\mathbb{Q}C$-module Φ_C. Then the image of $[\Phi_C, a]$ in $K_1(\mathbb{Q}D)$ by the transfer associated to the bimodule $L = \Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}\Omega$ is equal to the image of $\mathrm{Ver}_\Omega(a) \in D = D/[D,D]$ by the map $\alpha_{\mathbb{Q}D}$ of Remark 2.2.*

Proof : By Lemma 5.1 and Proposition 2.6, we can assume that Ω is a transitive biset, of the form $(D\times C)/B$ for some subgroup B of $D\times C$. Then Ω is left and right free if and only if there exists a subgroup E of C and an injective group homomorphism $\varphi : E \to D$ such that $B = \{\big(\varphi(e), e\big) \mid\in E\}$. There are two cases:

- either $E = \mathbf{1}$: in this case $\Omega = D \times C$, so $\mathbb{Q}\Omega \cong \mathbb{Q}D \otimes_{\mathbb{Q}} \mathbb{Q}C$, and $\Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}\Omega \otimes_{\mathbb{Q}C} \Phi_C \cong \Phi_D \otimes_{\mathbb{Q}} \Phi_C$. As a vector space over the cyclotomic field F of endomorphisms of Φ_D, it is isomorphic to $F \otimes_{\mathbb{Q}} \Phi_C$. The action of $a \in C$ on this vector space is given by the matrix of a acting on Φ_C.

  Suppose that a is a generator of C, of order p^n. Then this action is the action by multiplication of a primitive p^n-th root of unity ζ on the field $\mathbb{Q}(\zeta)$. As an element of $K_1(F)$, it is equal to the determinant of the matrix representing this multiplication, i.e. to the norm $N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta)$, which is equal to 1, as the p^n-th cyclotomic polynomial has even degree $p^{n-1}(p-1)$ and value 1 at 0. It follows that $[\Phi_C, a]$ is mapped to the identity element of $K_1(\mathbb{Q}D)$ in this case. Since this holds for a generator a of C, the same is true for any element a of C.

  In this case also, a set of representatives of $[D\backslash\Omega]$ is the set $1 \times C$, which is invariant by right multiplication by C. It follows that the elements $d_{a,x} \in D$ defined for $a \in C$ and $x \in [D\backslash\Omega]$ by $xa = d_{a,x}x'$, for $x' \in [D\backslash\Omega]$, are all equal to 1. So the transfer Ver_Ω is also the trivial homomorphism in this case.

- or $E \neq \mathbf{1}$: let Z denote the unique subgroup of order p of C. Tensoring over $\mathbb{Q}C$ the exact sequence of $(\mathbb{Q}C, \mathbb{Q}C)$-bimodules

  $$0 \to \Phi_C \to \mathbb{Q}C \to \mathbb{Q}(C/Z) \to 0$$

  with $\Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}\Omega$ gives the exact sequence

  $$0 \to \Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}\Omega \otimes_{\mathbb{Q}C} \Phi_C \to \Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}\Omega \to \Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}(\Omega/Z) \to 0 \ .$$

  But Ω/Z is not free as a left D-set, since the unique subgroup $\varphi(Z)$ of order p of D stabilizes $BZ \in \Omega/Z$, as for $z \in Z$ and $e \in E$

  $$\varphi(z)\big(\varphi(e), e\big) = \big(\varphi(ze), e\big) = \big(\varphi(ze), ze\big)z \ .$$

By Lemma 5.4, it follows that $\Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}(\Omega/Z) = 0$, hence

$$\Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}\Omega \otimes_{\mathbb{Q}C} \Phi_C \cong \Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}\Omega \ \ .$$

As a vector space over the cyclotomic field F of endomorphisms of Φ_D, this is isomorphic to $F \otimes_{\mathbb{Q}} \mathbb{Q}[D\backslash\Omega]$. The action of $a \in C$ on this vector space is given for $x \in [D\backslash\Omega]$ and $\lambda \in F$ by

$$(\lambda \otimes x)a = \lambda \otimes xa = \lambda \otimes d_{a,x}\sigma_a(x) = \lambda d_{a,x} \otimes \sigma_a(x) \ \ ,$$

where $d_{a,x} \in D$ and $\sigma_a(x) \in [D\backslash\Omega]$ are defined by $xa = d_{a,x}\sigma_a(x)$. In other words, the matrix of the action of a is the product of the permutation matrix of σ_a with a diagonal matrix of coefficients $d_{a,x}$, for $x \in [D\backslash\Omega]$. In $K_1(F)$, this matrix is equal to its determinant, that is the signature of σ_a, which is equal to 1 as σ_a is a product of cycles of odd length (equal to some power of p), multiplied by the product of the elements $d_{a,x}$, that is the image in $K_1(\mathbb{Q}D)$ of $\mathrm{Ver}_\Omega(a)$, as was to be shown. □

5.8. Remark : Recall that if Q is a central subgroup of finite index n in a group G, then the transfer $G/[G,G] \to Q$ is induced by the map $g \mapsto g^n$ from G to Q (see [10] Theorem 7.47). It follows easily that in the situation of Lemma 5.7, if $\Omega = (D \times C)/B$, where $B = \{(\varphi(e), e) \mid e \in E\}$ for a subgroup E of C and an injective homomorphism $\varphi : E \to D$, the transfer $\mathrm{Ver}_\Omega : C \to D$ is given by $a \mapsto \varphi(a^{|C:E|})$. Moreover $|C : E| = |D\backslash\Omega|$.

5.9. Theorem: *Let p be an odd prime. Let P and Q be finite p-groups, and let U be a finite (Q,P)-biset.*

1. *Let S be a genetic subgroup of P and T be a genetic subgroup of Q. Let $\mathcal{D}(U)_{T,S}$ be the set of orbits $N_Q(T)uN_P(S)$ of those $u \in U$ for which $T^u \cap N_P(S) \leq S$ and ${}^uS \cap N_Q(T) \leq T$ (see 5.6 for notation). Then for $\omega \in \mathcal{D}(U)_{T,S}$, the set $T\backslash\omega/S$ is a left and right free $\big(N_Q(T)/T, N_P(S)/S\big)$-biset, and the map*

$$\Gamma(U)_{T,S} : N_P(S)/S \to N_Q(T)/T$$

sending $a \in N_P(S)/S$ to

$$\prod_{\omega \in \mathcal{D}(U)_{T,S}} \mathrm{Ver}_{T\backslash\omega/S}(a)$$

is a well defined group homomorphism.

2. *Let* $\mathcal{B}_P$ *and* $\mathcal{B}_Q$ *be genetic bases of* P *and* Q*, respectively. Then the map* $\Gamma(U) : \Gamma(P) \to \Gamma(Q)$ *giving the biset functor structure of* Γ *is the map*

$$\Gamma(P) = \prod_{S \in \mathcal{B}_P} \big(N_P(S)/S\big) \to \prod_{T \in \mathcal{B}_Q} \big(N_Q(T)/T\big) = \Gamma(Q)$$

with component (T, S) *equal to* $\Gamma(U)_{T,S}$.

Proof : This results from Paragraph 5.2, Lemma 5.4, Paragraph 5.5, and Lemma 5.7. □

6. Examples

6.1. Proposition: *Let* P *be a finite* p*-group, for* p *odd, and let* $\mathcal{B}$ *be a genetic basis of* P*. Let* $N \trianglelefteq P$*, and* $\overline{P} = P/N$*. Let* $\mathcal{B}_N$ *be the subset of* $\mathcal{B}$ *defined by*

$$\mathcal{B}_N = \{S \in \mathcal{B} \mid S \geq N\} \ .$$

Then:

1. *The set* $\overline{\mathcal{B}} = \{\overline{S} = S/N \mid S \in \mathcal{B}_N\}$ *is a genetic basis of* $\overline{P}$.
2. *Up to the identification of* $N_{\overline{P}}(\overline{S})$ *with* $N_P(S)/S$*, for* $S \in \mathcal{B}_N$*, the inflation morphism*

$$\mathrm{Inf}_{P/N}^P : \Gamma(P/N) = \prod_{\overline{S} \in \overline{\mathcal{B}}} \big(N_{\overline{P}}(\overline{S})/\overline{S}\big) \to \Gamma(P) = \prod_{S \in \mathcal{B}} \big(N_P(S)/S\big)$$

is the embedding in the product of the factors of $\Gamma(P)$ *corresponding to genetic subgroups* S *containing* N.

3. *Similarly, the deflation morphism*

$$\mathrm{Def}_{P/N}^P : \Gamma(P) = \prod_{S \in \mathcal{B}} \big(N_P(S)/S\big) \to \Gamma(P/N) = \prod_{\overline{S} \in \overline{\mathcal{B}}} \big(N_{\overline{P}}(\overline{S})/\overline{S}\big)$$

is the projection onto the product of the factors of $\Gamma(P)$ *corresponding to genetic subgroups* S *containing* N.

Proof : Assertion 1 is clear from the definitions: if $N \leq S \leq P$, then S is genetic in P if and only if S/N is genetic in P/N. Moreover the relation

$\overset{\frown}{-}_{P/N}$ gives the relation $\overset{\frown}{-}_P$ by inflation.

Now the inflation morphism $\mathrm{Inf}_{P/N}^P$ is defined by the $(P,\overline{P})$-biset $U = P/N$, with natural actions of P and $\overline{P}$. Let $u = xN$ be an element of U, for some $x \in P$. Let $T \in \mathcal{B}$ and $\overline{S} \in \overline{\mathcal{B}}$.

$$\begin{aligned} T^u \cap N_{\overline{P}}(\overline{S}) &= \{\bar{y} = yN \in N_{\overline{P}}(\overline{S}) \mid \exists t \in T,\ tu = u\bar{y}\} \\ &= \{\bar{y} = yN \in N_{\overline{P}}(\overline{S}) \mid \exists t \in T,\ txN = xyN\} \\ &= \{\bar{y} = yN \in N_{\overline{P}}(\overline{S}) \mid yN \in T^x\, N\} \\ &= \big(T^x\, N \cap N_P(S)\big)/N = \big(T^x \cap N_P(S)\big)N/N \ \ , \end{aligned}$$

where the last two equalities hold because $S \geq N$. Hence $T^u \cap N_{\overline{P}}(\overline{S}) \leq \overline{S}$ if and only if $T^x \cap N_P(S) \leq S$.

On the other hand

$$\begin{aligned} {}^u\overline{S} \cap N_P(T) &= \{y \in N_P(T) \mid \exists \bar{s} = sN \in \overline{S},\ yu = u\bar{s}\} \\ &= \{y \in N_P(T) \mid \exists s \in S,\ yxN = xNs\} \\ &= \{y \in N_P(T) \mid y \in {}^xS\} \\ &= {}^xS \cap N_P(T) \ \ . \end{aligned}$$

Hence ${}^u\overline{S} \cap N_P(T) \subseteq T$ if and only if ${}^xS \cap N_P(T) \leq T$.

If moreover $T^u \cap N_{\overline{P}}(\overline{S}) \leq \overline{S}$, i.e. $T^x \cap N_P(S) \leq S$, it follows that $T \overset{\frown}{-}_P S$, hence $T = S$ since T and S belong to the same genetic basis $\mathcal{B}$. Moreover $x \in N_P(S)$, and the induced group homomorphism $N_{\overline{P}}(\overline{S})/\overline{S} \to N_P(T)/T$ is the canonical isomorphism $N_{\overline{P}}(\overline{S})/\overline{S} \to N_P(S)/S$. This completes the proof of Assertion 2.

For Assertion 3, we consider the deflation map $\mathrm{Def}_{P/N}^P : \Gamma(P) \to \Gamma(P/N)$. It corresponds to the biset $V = P/N$, with left action of $\overline{P}$ and right action of P. For $v = yN \in V$, for $T \in \mathcal{B}$ and $\overline{S} \in \overline{\mathcal{B}}$, and with the same notation as above, the computation is similar: we have $T^v \cap N_{\overline{P}}(\overline{S}) \leq \overline{S}$ if and only if $T^y \cap N_P(S) \leq S$, and ${}^v\overline{S} \cap N_P(T) \leq T$ if and only if ${}^yS \cap N_P(T) \leq T$. These two conditions are fulfilled if and only if $S = T$ and $y \in N_P(S)$. This completes the proof. □

Recall that the faithful part $\partial F(P)$ of the evaluation of a biset functor F at a group P is the set of faithful elements of $F(P)$, introduced in [2] Definition 6.3.1: it is the set of elements $u \in F(P)$ such that $\mathrm{Def}_{P/N}^P u = 0$ for any non trivial normal subgroup N of P. Equivalently $\mathrm{Def}_{P/Z}^P u = 0$ for any non trivial central subgroup Z of P. The following is now clear:

6.2. Corollary: *Let p be an odd prime and P be a finite p-group. Let $\mathcal{B}$ be a genetic basis of P. Then the faithful part $\partial\Gamma(P)$ of $\Gamma(P)$ is equal to*

$$\partial\Gamma(P) = \prod_{\substack{S\in\mathcal{B}\\ S\cap Z(P)=\mathbf{1}}} \big(N_P(S)/S\big) \enspace .$$

7. Genome and Roquette category

7.1. Let F be a p-biset functor. It is shown in [2] Theorem 10.1.1 that if P is a finite p-group and $\mathcal{B}$ is a genetic basis of P, then the map

$$\mathcal{I}_\mathcal{B} = \bigoplus_{S\in\mathcal{B}} \mathrm{Indinf}_{N_P(S)/S}^P : \bigoplus_{S\in\mathcal{B}} \partial F\big(N_P(S)/S\big) \to F(P)$$

is always split injective. When $\mathcal{I}_\mathcal{B}$ is an isomorphism for one particular genetic basis $\mathcal{B}$ of P, then $\mathcal{I}_{\mathcal{B}'}$ is an isomorphism for any other genetic basis $\mathcal{B}'$ of P.

The functors for which $\mathcal{I}_\mathcal{B}$ is an isomorphism for any finite p-group P and any genetic basis $\mathcal{B}$ of P are called *rational* p-biset functors. It has been shown further ([3]) that these rational p-biset functors are exactly those p-biset functors which factorize through *the Roquette category* $\mathcal{R}_p$ of p-groups: more precisely ([3], Definition 3.3), the category $\mathcal{R}_p$ is defined as the idempotent additive completion of a specific quotient $\mathcal{R}_p^\sharp$ of the category $\mathcal{C}_p$, so there is a canonical additive functor $\pi_p : \mathcal{C}_p \to \mathcal{R}_p$, equal to the composition of the projection functor $\mathcal{C}_p \to \mathcal{R}_p^\sharp$ and the inclusion functor $\mathcal{R}_p^\sharp \to \mathcal{R}_p$. The rational p-biset functors are the additive functors $F : \mathcal{C}_p \to \mathcal{A}b$ for which there exists an additive functor $\overline{F} : \mathcal{R}_p \to \mathcal{A}b$ such that $F = \overline{F} \circ \pi_p$. In this case, the functor $\overline{F}$ is unique.

7.2. Proposition: *Let p be an odd prime. Then the genome p-biset functor Γ is rational.*

Proof : Let P be a p-group, and $\mathcal{B}$ be a genetic basis of P. If $S \in \mathcal{B}$, then $Q = N_P(S)/S$ is cyclic, so the trivial subgroup S/S of Q is the only one intersecting trivially the center of Q, and it is a genetic subgroup of Q. By Corollary 6.2, we have that

$$\partial\Gamma\big(N_P(S)/S\big) = N_P(S)/S \enspace .$$

Now the induction-inflation map $\mathrm{Indinf}_{N_P(S)/S}^P$ is given by the $\big(P, N_P(S)/S\big)$-biset $U = P/S$. Let $T \in \mathcal{B}$, and let $u = xS \in U$ such that

$$T^u \cap N_Q(S/S) \leq S/S \quad \text{and} \quad {}^u(S/S) \cap N_P(T) \leq T \ .$$

The first inclusion means that

$$\{yS \in N_P(S)/S \mid \exists t \in T,\ txS = xSy\} = S/S \ .$$

In other words $T^x \cap N_P(S) \leq S$. The second inclusion means similarly that

$$\{y \in N_P(T) \mid \exists s \in S/S,\ txS = xSs\} \leq T \ ,$$

that is $N_P(T) \cap {}^xS \leq T$. Hence $T \frown_P S$, thus $T = S$ since T and S both belong to a genetic basis of P. Moreover $x \in N_P(S)$, and the morphism we get from $N_P(S)/S$ to $N_P(T)/T$ is the identity map.

In other words, the map

$$\mathrm{Indinf}_{N_P(S)/S}^P : \partial\Gamma\big(N_P(S)/S\big) = N_P(S)/S \to \Gamma(P)$$

is the canonical embedding of $N_P(S)/S$ in $\Gamma(P)$. It clearly follows that the map $\mathcal{I}_B$ is an isomorphism, hence Γ is rational. □

7.3. Corollary: *Let p be an odd prime. Then there exists a unique additive functor $\bar{\Gamma}$ from the Roquette category $\mathcal{R}_p$ to $\mathcal{A}b$ such that $\Gamma = \bar{\Gamma} \circ \pi_p$. Moreover $\bar{\Gamma}(\partial P) = \partial\Gamma(P)$ for any finite p-group P, where ∂P is the edge of P in $\mathcal{R}_p$. In particular $\bar{\Gamma}(\partial C) = C$ for any cyclic p-group C.*

Proof : This follows from the definition and properties of the category $\mathcal{R}_p$ (for the definition of the *edge* ∂P of a p-group P in the Roquette category, see [3] Definition 3.7). □

References

[1] L. Barker. Genotypes of irreducible representations of finite p-groups. *Journal of Algebra*, 306:655–681, 2007.

[2] S. Bouc. *Biset functors for finite groups*, volume 1990 of *Lecture Notes in Mathematics*. Springer, 2010.

[3] S. Bouc. The Roquette category of finite p-groups. *Journal of the European Mathematical Society*, 17:2843–2886, 2015.

[4] S. Bouc and N. Romero. The Whitehead group of (almost) extra-special p-groups with p-odd. *J. Pure Appl. Algebra*, 223 (2019) 86-107.

[5] C. Curtis and I. Reiner. *Methods of representation theory with applications to finite groups and orders*, volume II of *Wiley classics library.* Wiley, 1990.

[6] C. Curtis and I. Reiner. *Methods of representation theory with applications to finite groups and orders*, volume I of *Wiley classics library.* Wiley, 1990.

[7] R. Oliver. *Whitehead groups of finite groups*, volume 132 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1988.

[8] N. Romero. Computing Whitehead groups using genetic bases. *Journal of Algebra*, 450:646–666, 2016.

[9] P. Roquette. Realisierung von Darstellungen endlicher nilpotenter Gruppen. *Arch. Math.*, 9:224–250, 1958.

Serge Bouc
LAMFA-CNRS UMR7352
Université de Picardie Jules Verne
33 rue St Leu, Amiens Cedex 01
France
email: serge.bouc@u-picardie.fr