# Fused Mackey functors

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*A la mémoire de mon père,*
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**Abstract:** Let $G$ be a finite group. In [6], Hambleton, Taylor and Williams have considered the question of comparing Mackey functors for $G$ and biset functors defined on subgroups of $G$ and bifree bisets as morphisms.
This paper proposes a different approach to this problem, from the point of view of various categories of $G$-sets. In particular, the category $G$-$\underline{\mathsf{set}}$ of *fused $G$-sets* is introduced, as well as the category $\underline{\mathbf{S}}(G)$ of spans in $G$-$\underline{\mathsf{set}}$. The *fused Mackey functors* for $G$ over a commutative ring $R$ are defined as $R$-linear functors from $R\,\underline{\mathbf{S}}(G)$ to $R$-modules. They form an abelian subcategory $\mathsf{Mack}_R^f(G)$ of the category of Mackey functors for $G$ over $R$. The category $\mathsf{Mack}_{\mathbb{Z}}^f(G)$ is equivalent to the category of conjugation Mackey functors of [6]. The category $\mathsf{Mack}_R^f(G)$ is also equivalent to the category of modules over the *fused Mackey algebra* $\mu_R^f(G)$, which is a quotient of the usual Mackey algebra $\mu_R(G)$ of $G$ over $R$.

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## 1. Introduction

**1.1.** This note is devoted to the frequently asked question of comparing Mackey functors for a single finite group $G$ (cf. [9]) with biset functors (cf. [3]) defined only on subgroups of $G$ and left-right free bisets as morphisms. More precisely, let $F$ be a biset functor defined on the category $\mathcal{D}$ of all finite groups, where morphisms are given by Grothendieck groups of left-right free bisets. Let moreover $G$ be a fixed finite group. Now:

- when $H$ is a subgroup of $G$, set $M(H) = F(H)$.
- when $H \leq K$ are subgroups of $G$, define a restriction map
$$r_H^K : M(K) \to M(H)$$
by $r_H^K = F({}_HK_K)$, where ${}_HK_K$ is the set $K$, viewed and an $(H, K)$-biset by left and right multiplication, hence also as a morphism from $K$ to $H$ in the category $\mathcal{D}$.
- similarly, define a transfer map
$$t_H^K : M(H) \to M(K)$$
by $t_H^K = F({}_KK_H)$, where ${}_KK_H$ is the set $K$, viewed as a $(K, H)$-biset.

- finally, when $x \in G$, and $H \leq G$, define a conjugation map
$$c_{x,H} : M(H) \to M({}^xH)$$
by $c_{x,H} = F({}_{{}^xH}xH_H)$, where ${}_{{}^xH}xH_H$ is the coset $xH$, wiewed as an $({}^xH, H)$-biset.

One checks easily that this yields a Mackey functor $M$ for $G$.

**1.2.** The question is now to characterize those Mackey functors for $G$ for which the restriction maps $r_H^K$, the transfer maps $t_H^K$, and the conjugation maps $c_{x,H}$ only depend on the above bisets ${}_HK_K$, ${}_KK_H$, and ${}_{{}^xH}xH_H$, respectively. Equivalently, to characterize those Mackey functors for $G$ which can be viewed as additive functors on the (non full) subcategory of $\mathcal{D}$ consisting of subgroups of $G$, where morphisms from $H$ to $K$ are linear combinations of isomorphism classes of $(K, H)$-bisets obtained by composition of the above three types of bisets.

These bisets have been called *conjugation bisets* by Hambleton, Taylor and Williams, who answered first the above question ([6]): the Mackey functors in question are the *conjugation invariant Mackey functors*, namely the Mackey functors $M$ for $G$ such that for any subgroup $H$ of $G$, the centralizer $C_G(H)$ acts trivially on $M(H)$. However, the proof of this characterization given in [6] is rather computational and non canonical (in particular, in Section 7, the definition of the functor $j_\bullet$ requires the choice of sets of representatives of orbits of any finite $G$-set).

The present paper makes a systematic use of Dress definition ([4]) and Lindner definition ([7]) of Mackey functors, to avoid these non canonical choices. This leads to the definition of the category of *fused $G$-sets* (Section 3), and the category of *fused Mackey functors* (Section 4) for a finite group $G$, which is equivalent to the category of "conjugation invariant Mackey functors" of [6]. This category is also equivalent to the category of modules over the *fused Mackey algebra*, introduced in Section 5.

# 2. Conjugation bisets revisited

**2.1.** First a notation : when $G$ is a finite group, and $X$ is a finite $G$-set, let $G\text{-set}{\downarrow_X}$ denote the category of (finite) $G$-sets over $X$: its objects are pairs $(Y, b)$ consisting of a finite $G$-set $Y$, and a morphism of $G$-sets $b : Y \to X$. A morphism $f : (Y, b) \to (Z, c)$ in $G\text{-set}{\downarrow_X}$ is a morphism of $G$-sets $f : Y \to Z$ such that $c \circ f = b$.

There is an obvious notion of disjoint union in $G\text{-set}{\downarrow_X}$, and the corresponding Grothendieck group is called the Burnside group over $X$. It will be denoted by $\mathcal{B}({}_GX)$, or $\mathcal{B}(X)$ when $G$ is clear from the context.

Similarly, when $G$ and $H$ are finite groups, and $U$ is a $(G,H)$-biset, one can define the category $(G,H)\text{-}\mathsf{biset}{\downarrow_U}$ of $(G,H)$-bisets over $U$, and the Burnside group $\mathcal{B}({}_GU_H)$ of $(G,H)$-bisets over $U$.

**2.2.** When $H$ is a subgroup of $G$, and $Y$ is an $H$-set, induction from $H$-sets to $G$-sets is an equivalence of categories from $H\text{-}\mathsf{set}{\downarrow_Y}$ to $G\text{-}\mathsf{set}{\downarrow_{\mathrm{Ind}_H^G Y}}$. A quasi-inverse equivalence is the functor sending the $G$-set $(X,a)$ over $\mathrm{Ind}_H^G Y$ to the $H$-set $a^{-1}(1\times_H Y)$ (cf. [2] Lemma 2.4.1). In particular $\mathcal{B}({}_HY)\cong\mathcal{B}({}_G\mathrm{Ind}_H^G Y)$.

**2.3.** Now an observation: when $H$ and $K$ are subgroups of $G$, the conjugation $(K,H)$-bisets defined in Section 6 of [6] are exactly those over the biset ${}_KG_H$ (the set $G$ on which $K$ and $H$ act by multiplication), i.e. the $(K,H)$-bisets $U$ for which there exists a biset morphism $U\to {}_KG_H$.

Indeed, a conjugation $(K,H)$-biset $U$ is a bifree $(K,H)$-biset isomorphic to a disjoint union of bisets of the form $(K\times H)/S$, where $S$ is a subgroup of $K\times H$ of the form

$$S_{g,A}=\{({}^gx,x)\mid x\in A\}$$

where $A$ is a subgroup of $H$, and $g$ is an element of $G$ such that ${}^gA\leq K$. For such a transitive biset $(K\times H)/S$, the map

$$\forall (k,h)S\in (K\times H)/S,\ (k,h)S\mapsto kgh^{-1}$$

is a morphism of $(K,H)$-bisets.

Conversely, let $U$ be a $(K,H)$-biset for which there exists a biset morphism $\alpha: U\to {}_KG_H$. Then for any $u\in U$, the stabilizer $S_u$ of $u$ in $K\times H$ is the subgroup

$$S_u=\{(k,h)\in K\times H\mid k\cdot u\cdot h^{-1}=u\}$$

of $K\times H$. Then if $(k,h)\in S_u$,

$$\alpha(k\cdot u)=k\alpha(u)=\alpha(u\cdot h)=\alpha(u)h\ \ .$$

Let $A_u$ denote the projection of $S_u$ into $H$, and set $g_u=\alpha(u)$. It follows that $S_u\subseteq S_{g_u,A_u}$.

Conversely, if $(k,h)\in S_{g_u,A_u}$, then $k={}^{g_u}h$, and there exists some $x\in K$ such that $(x,h)\in S_u$, since $h\in A_u$. Thus $x\cdot u\cdot h^{-1}=u$, from which follows that

$$\alpha(x\cdot u)=xg_u=\alpha(u\cdot h)=g_uh\ \ ,$$

hence $x={}^{g_u}h=k$, and $S_u=S_{g_u,A_u}$. Observation 2.3 follows.

**2.4.** In other words, conjugation $(K,H)$-bisets form a category $\mathsf{Conj}_{K,H}^G$, and there is a forgetful functor $\Phi : (K,H)\text{-}\mathsf{biset}\!\downarrow_{{}_KG_H} \to \mathsf{Conj}^G_{K,H}$ sending $(U,a)$ to $U$. This functor preserves disjoint unions, and it induces a surjection on the corresponding sets of isomorphism classes. This means that $\Phi$ induces a surjective group homomorphism (still denoted by $\Phi$) from $\mathcal{B}({}_KG_H)$ to the Grothendieck group $\mathcal{B}^G_{K,H}$ of conjugation $(K,H)$-bisets.

**2.5.** If $H$, $K$ and $L$ are subgroups of $G$, if $(U,a)$ is a $(K,H)$-biset over ${}_KG_H$ and $(V,b)$ is an $(L,K)$-biset over ${}_LG_K$, the composition $(V,b)\circ(U,a)$ is the $(L,H)$-biset over ${}_LG_H$ defined by the following diagram:

$$\begin{array}{ccccc} V & & U & & V\times_K U \\ \big\downarrow{\scriptstyle b} & \circ & \big\downarrow{\scriptstyle a} & = & \big\downarrow{\scriptstyle b\times_K a} \\ {}_LG_K & & {}_KG_H & & G\times_K G \\ & & & & \big\downarrow{\scriptstyle \mu} \\ & & & & {}_LG_H \end{array}$$

where $\mu$ is multiplication in $G$. This composition is associative, and additive with respect to disjoint unions. Hence it induces a composition

$$\widehat{\circ} : \mathcal{B}({}_LG_K)\times\mathcal{B}({}_KG_H)\to\mathcal{B}({}_LG_H)\;\;.$$

Hence, one can define a category $\widehat{\mathbf{B}}(G)$ whose objects are the subgroups of $G$, and such that $\mathrm{Hom}_{\widehat{\mathbf{B}}(G)}(H,K) = \mathcal{B}({}_KG_H)$, for subgroups $H$ and $K$ of $G$. Composition is given by $\widehat{\circ}$, and the identity morphism of the subgroup $H$ of $G$ in the category $\widehat{\mathbf{B}}(G)$ is the class of the biset $({}_HH_H, i_H)$, where $i_H : {}_HH_H \to {}_HG_H$ is the inclusion map from $H$ to $G$.

Since the functor $\Phi$ maps the composition $\widehat{\circ}$ to the composition of bisets, and the identity morphism of $H$ in $\widehat{\mathbf{B}}(G)$ to the identity biset ${}_HH_H$, one can extend $\Phi$ to a functor $\widehat{\mathbf{B}}(G)\to\mathbf{B}(G)$, which is the identity on objects, where $\mathbf{B}(G)$ is the category introduced in Section 3 of [6]: its objects are the subgroups of $G$, and $\mathrm{Hom}_{\mathbf{B}(G)}(H,K) = \mathcal{B}^G_{K,H}$ for any subgroups $H$ and $K$ of $G$, the composition of morphisms being given by linear extension of the composition of bisets.

More precisely, the category $\mathbf{B}(G)$ is the quotient of the category $\widehat{\mathbf{B}}(G)$ obtained by identifying morphisms which have the same image by $\Phi$.

**2.6.** By the above Remark 2.2, when $H$ and $K$ are subgroups of $G$, there is a group isomorphism

$$\mathcal{B}({}_KG_H)\cong\mathcal{B}\big(\mathrm{Ind}_{K\times H}^{G\times G}({}_KG_H)\big)\;\;,$$

(with the usual identification of $(K,H)$-bisets with $(K\times H)$-sets). Now the biset ${}_KG_H$ is actually the restriction to $(K\times H)$ of the $(G,G)$-biset $G$. By

Frobenius reciprocity, it follows that

$$\mathrm{Ind}_{K\times H}^{G\times G}({}_KG_H) \cong \mathrm{Ind}_{K\times H}^{G\times G}\mathrm{Res}_{K\times H}^{G\times G}({}_GG_G) \cong \big(\mathrm{Ind}_{K\times H}^{G\times G}\bullet\big)\times{}_GG_G \ ,$$

where $\bullet$ is a set of cardinality 1. Since $\mathrm{Ind}_{K\times H}^{G\times G}\bullet \cong (G/K)\times(G/H)$, it follows (after switching $G/H$ and $G$) that

$$\mathrm{Ind}_{K\times H}^{G\times G}({}_KG_H) \cong (G/K)\times G\times(G/H) \ ,$$

where the $(G,G)$-biset structure of the right hand side is given by

$$\forall (a,b,x,y,g)\in G^5,\ \ a\cdot(xK,g,yH)\cdot b = (axK, agb, b^{-1}yH) \ .$$

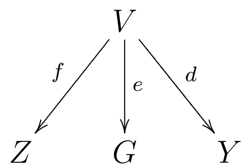
**2.7.** It should now be clear that the additive completion $\widehat{\mathbf{B}}_\bullet(G)$ is equivalent to the category whose objects are finite $G$-sets, where for any two finite $G$-sets $X$ and $Y$

$$\mathrm{Hom}_{\widehat{\mathbf{B}}_\bullet(G)}(X,Y) = \mathcal{B}\big({}_G(Y\times G\times X)_G\big) \ ,$$

the $(G,G)$-biset structure on $(Y\times G\times X)$ being given as above by

$$\forall (a,b,g,x,y)\in G^3\times X\times Y,\ \ a\cdot(y,g,x)\cdot b = (ay, agb, b^{-1}x) \ .$$

Keeping track of the composition $\widehat{\circ}$ along the above isomorphism shows that the composition in the category $\widehat{\mathbf{B}}_\bullet(G)$ can be defined by linearity from the following: if $X$, $Y$, and $Z$ are finite $G$-sets, if



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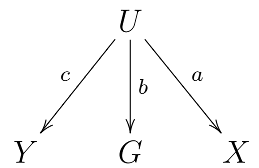
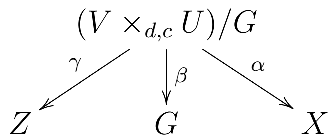


are $(G,G)$-bisets over $(Z\times G\times Y)$ and $(Y\times G\times X)$, respectively, their composition is given by the following $(G,G)$-biset over $(Z\times G\times X)$



where $V\times_{d,c}U$ is the pullback of $V$ and $U$ over $Y$, i.e. the set of pairs $(v,u)\in V\times U$ with $d(v)=c(u)$, and $(V\times_{d,c}U)/G$ the set of orbits of $G$ on it for the action given by $(v,u)\cdot g = (vg, g^{-1}u)$. This makes sense because $d(v\cdot g) = g^{-1}d(v) = g^{-1}c(u) = c(g^{-1}\cdot u)$ if $d(v)=c(u)$. The map $(\gamma,\beta,\alpha)$ is given by

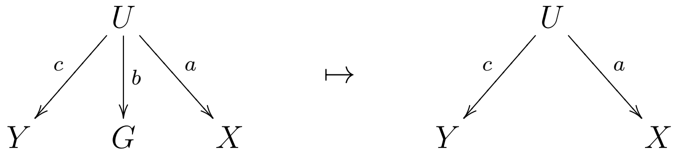
$$(\gamma,\beta,\alpha)\big((v,u)G\big) = \big(f(v), e(v)b(u), a(u)\big) \ .$$

**2.8.** The functor $\Phi:\widehat{\mathbf{B}}(G)\to\mathbf{B}(G)$ extends uniquely to an additive functor

$\Phi_\bullet : \widehat{\mathbf{B}}_\bullet(G) \to \mathbf{B}_\bullet(G)$, and the category $\mathbf{B}_\bullet(G)$ is the quotient of $\widehat{\mathbf{B}}_\bullet(G)$ obtained by identifying morphisms which have the same image by $\Phi_\bullet$. Clearly, two morphisms $f, g \in \mathrm{Hom}_{\widehat{\mathbf{B}}_\bullet(G)}(X, Y)$ are identified if and only if $f - g$ is in the kernel of the group homomorphism

$$\phi : \mathcal{B}\big({}_G(Y \times G \times X)_G\big) \to \mathcal{B}\big({}_G(Y \times X)_G\big)$$

induced by the correspondence



on bisets. In other words, a morphism $f$ in $\widehat{\mathbf{B}}_\bullet(G)$ gives the zero morphism in $\mathbf{B}_\bullet(G)$ if and only if it belongs to $\mathrm{Ker}\,\phi$.

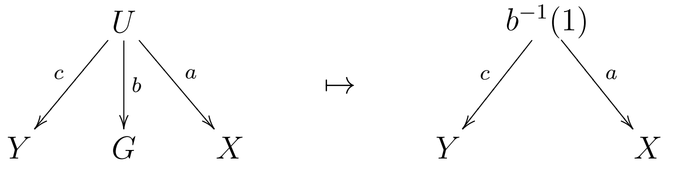
**2.9.** Now the $(G,G)$-biset ${}_GG_G$ is isomorphic to $\mathrm{Ind}_{\Delta(G)}^{G\times G}\bullet$, where $\Delta(G)$ is the diagonal subgroup of $G \times G$. It follows that there is an isomorphism of $(G,G)$-bisets

$$Y \times G \times X \cong \mathrm{Ind}_{\Delta(G)}^{G\times G}(Y \times X) \ \ .$$

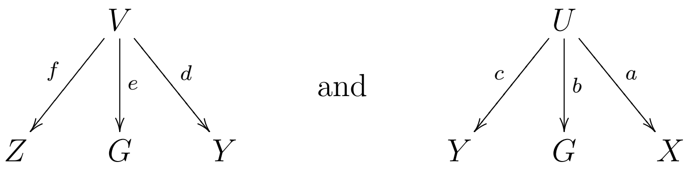
Hence, by Remark 2.2 again, since $\Delta(G) \cong G$,

$$\mathcal{B}\big({}_G(Y \times G \times X)_G\big) \cong \mathcal{B}\big({}_G(Y \times X)\big) \ \ ,$$

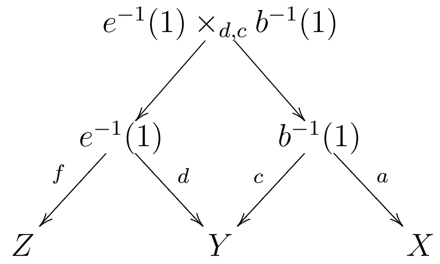
where ${}_G(Y \times X)$ is the usual cartesian product with diagonal $G$-action. More precisely, this isomorphism is induced by the correspondence



It is then easy to check that the composition of



corresponds to the usual pullback diagram




In other words, the category $\widehat{\mathbf{B}}_\bullet(G)$ is equivalent to the category $\mathbf{S}(G)$ whose objects are the finite $G$-sets, where

$$\mathrm{Hom}_{\mathbf{S}(G)}(X, Y) = \mathcal{B}\big({}_G(Y \times X)\big) \ ,$$

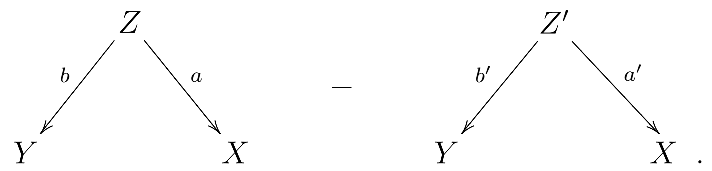
and composition is induced by pullback. It has been shown by Lindner ([7], see also [2]) that the additive functors on this category are precisely the Mackey functors for $G$.

**2.10.** With this equivalence of categories $\widehat{\mathbf{B}}_\bullet(G) \cong \mathbf{S}(G)$, the main result of [6] can be viewed as a characterization of those Mackey functors for $G$, wiewed as additive functors on $\widehat{\mathbf{B}}_\bullet(G)$, which factor through the functor $\Phi_\bullet : \widehat{\mathbf{B}}_\bullet(G) \to \mathbf{B}_\bullet(G)$.

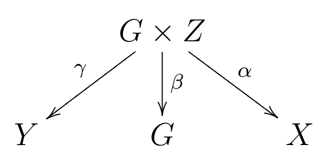
This characterization amounts to a precise description of the identifications effected by $\Phi$ on morphisms: starting with $f \in \mathrm{Hom}_{\mathbf{S}(G)}(X, Y)$, one can lift it to

$$f^+ \in \mathrm{Hom}_{\widehat{\mathbf{B}}_\bullet(G)}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big) \ ,$$

and see when $f^+$ lies in $\mathrm{Ker}\,\phi$. Now $f$ is represented by a difference of two $G$-sets over ${}_G(Y \times X)$ of the form




By induction from $\Delta(G)$ to $G \times G$, the $G$-set on the left hand side lifts to the following $(G \times G)$-set over ${}_{(G\times G)}(Y \times G \times X)$



where the $(G \times G)$-actions on $G \times Z$ and $Y \times G \times X$ are given respectively by $(s,t) \cdot (g,z) = (sgt^{-1}, tz)$ and $(s,t) \cdot (y,g,x) = (sy, sgt^{-1}, tx)$, and where

$$(\gamma, \beta, \alpha)(g,z) = \big(gb(z), g, a(z)\big) \quad .$$

Similarly the $G$-set $\big(Z', (b', a')\big)$ lifts to $\big(G \times Z', (\gamma', \beta', \alpha')\big)$.

Now $f^+$ is in $\mathrm{Ker}\,\phi$ if and only if there is an isomorphism

$$\begin{array}{ccccccc} & G \times Z & & & & G \times Z' & \\ {\scriptstyle\gamma}\swarrow & & \searrow{\scriptstyle\alpha} & \xrightarrow{\theta} & {\scriptstyle\gamma'}\swarrow & & \searrow{\scriptstyle\alpha'} \\ Y & & X & & Y & & X \end{array} \quad .$$

of $(G \times G)$-sets over $Y \times X$. Since $(g,z) = g \cdot (1,z)$ for any $(g,z) \in G \times Z$, it follows that $\theta$ is a map from $G \times Z$ to $G \times Z'$ of the form

$$(g,z) \mapsto \big(gu(z), v(z)\big) \quad ,$$

where $u$ is a map from $Z$ to $G$ and $v$ is a map from $Z$ to $Z'$. Now for any $(s,t) \in G \times G$, the equality

$$\theta\big((s,t) \cdot (g,z)\big) = (s,t) \cdot \theta\big((g,z)\big)$$

gives

$$\big(sgt^{-1}u(tz), v(tz\big) = \big(sgu(z)t^{-1}, tv(z)\big) \quad .$$

This is equivalent to

$$u(tz) = {}^t u(z) \ \text{ and } \ v(tz) = tv(z) \quad .$$

This means that $u$ is a morphism of $G$-sets from $Z$ to $G^c$, which is the set $G$ with $G$-action by conjugation, and $v$ is a morphism of $G$-sets.

Moreover $\theta$ is a bijection if and only if $v$ is.

Finally $\theta$ is a morphism of $(G,G)$-bisets *over* $Y \times X$ if and only if $\alpha' \circ \theta = a$ and $\gamma' \circ \theta = \gamma$, i.e. equivalently if

$$a' \circ v = a \ \text{ and } \ gu(z) \cdot b' \circ v(z) = g \cdot b(z)$$

for any $(g,z) \in G \times Z$. In other words

$$a = a' \circ v \ \text{ and } \ b = u * (b' \circ v) \quad ,$$

where, for any map $w : Z \to Y$, the map $u * w : Z \to Y$ is defined by $(u * w)(z) = u(z) \cdot w(z)$. The map $u * w$ is a map of $G$-sets if $u : Z \to G^c$

and $w : Z \to Y$ are. Note that $w' = u * w$ if and only if $w = \bar{u} * w'$, where $\bar{u} : Z \to G^c$ is defined by $\bar{u}(z) = u(z)^{-1}$.

It follows that $f$ maps to the zero morphism in $\mathbf{B}(G)$ if and only if there exists $u : Z \to G^c$ and an isomorphism $v : Z \to Z'$ such that

$$a' \circ v = a \;\text{ and }\; b' \circ v = u * b \ ,$$

But then $v$ is an isomorphism

$$\left(Y \xleftarrow{\;b'\circ v\;} Z \xrightarrow{\;a'\circ v\;} X\right) \xrightarrow{\;v\;} \left(Y \xleftarrow{\;b'\;} Z' \xrightarrow{\;a'\;} X\right) .$$

of $G$-sets over $Y \times X$, and $f$ is also represented by the difference

$$\left(Y \xleftarrow{\;b\;} Z \xrightarrow{\;a\;} X\right) - \left(Y \xleftarrow{\;u*b\;} Z \xrightarrow{\;a\;} X\right) ,$$

since $a' \circ v = a$ and $b' \circ v = u * b$. These are the morphisms in the category $\mathbf{S}(G)$ that vanish in $\mathbf{B}_\bullet(G)$. In other words:

**2.11. Theorem :** *Let $G$ be a finite group. Let $\underline{\mathbf{S}}(G)$ denote the quotient category of $\mathbf{S}(G)$ defined by setting, for any two finite $G$-sets $Y$ and $Y$*

$$\mathrm{Hom}_{\underline{\mathbf{S}}(G)}(X, Y) = \mathcal{B}\big({}_G(Y \times X)\big)/K(Y, X) \ ,$$

*where $K(Y, X)$ is the subgroup generated by the differences*

**(2.12)**
$$\left(Y \xleftarrow{\;b\;} Z \xrightarrow{\;a\;} X\right) - \left(Y \xleftarrow{\;u*b\;} Z \xrightarrow{\;a\;} X\right) ,$$

*where $a : Z \to X$, $b : Z \to Y$, and $u : Z \to G^c$ are morphisms of $G$-sets.*

*Then the functor $\Phi_\bullet$ induces an equivalence of categories $\underline{\mathbf{S}}(G) \cong \mathbf{B}_\bullet(G)$.*

Since the difference 2.12 factors as

$$\left( Y \xleftarrow{\;b\;} Z \xrightarrow{\;\mathrm{Id}\;} Z \right) \circ \left( \left( Z \xleftarrow{\;\mathrm{Id}\;} Z \xrightarrow{\;\mathrm{Id}\;} Z \right) - \left( Z \xleftarrow{\;u*\mathrm{Id}\;} Z \xrightarrow{\;\mathrm{Id}\;} Z \right) \right) \circ \left( Z \xleftarrow{\;\mathrm{Id}\;} Z \xrightarrow{\;a\;} X \right)$$

the morphisms vanishing in $\underline{\mathbf{S}}(G)$ are generated in the category $\mathbf{S}(G)$ by the morphisms of the form

$$\left( Z \xleftarrow{\;\mathrm{Id}\;} Z \xrightarrow{\;\mathrm{Id}\;} Z \right) - \left( Z \xleftarrow{\;u*\mathrm{Id}\;} Z \xrightarrow{\;\mathrm{Id}\;} Z \right) .$$

**2.13.** It follows that the additive functors from $\underline{\mathbf{S}}(G)$ to the category of abelian groups are exactly those Mackey functors (in the sense of Dress) such that for any $G$-set $Z$ and any $u : Z \to G^c$, the morphism $M_*(u * \mathrm{Id})$ is equal to the identity map of $M(Z)$.

This condition is additive with respect to $Z$, since the map $u * \mathrm{Id}_Z$ maps each $G$-orbit of $Z$ to itself. Hence these functors are exactly the functors for which the map $M_*(u * \mathrm{Id})$ is the identity map of $M(G/H)$, for any subgroup $H$ of $G$ and any $u : G/H \to G^c$. Such a map is of the form $gH \mapsto {}^g c$, where $c \in C_G(H)$. The map $u * \mathrm{Id} : G/H \to G/H$ is the map $gH \mapsto gcH$.

Translated in terms of the usual definition of Mackey functors, this map expresses the action of $c$ on $M(H) = M(G/H)$. This shows that additive functors from $\underline{\mathbf{S}}(G)$ to abelian groups are exactly the Mackey functors for the group $G$ such that, for any $H \leq G$, the centralizer $C_G(H)$ acts trivially on $M(H)$. These are the "conjugation invariant Mackey functors" introduced in [6].

**2.14. Remark :** In view of Paragraph 1.1, one might be tempted to believe that such a conjugation invariant Mackey functor for $G$ can always be obtained from a biset functor defined on all finite groups by the restriction procedure to subgroups of $G$ described in Paragraph 1.1, but this is not true: for example, let $G$ be an elementary abelian group of order 4, and let $A$, $B$, and $C$ denote its subgroups of order 2. The simple Mackey functor $M = S_{A,\mathbb{F}_2}$ for $G$ over the field with 2 elements has value $\mathbb{F}_2$ at $A$, and $\{0\}$ elsewhere (cf. [9] Lemma 15.1). The functor $M$ is obviously a conjugation invariant Mackey functor, but if it were the restriction of a biset functor defined over all finite groups, then in particular its values at $A$, $B$, and $C$ would be isomorphic to one other, as $A$, $B$ and $C$ are all isomorphic to $C_2$.

# 3. Fused $G$-sets

Let $Z$ be any (finite) $G$-set. The multiplication $(u, v) \mapsto u * v$ endows the set $\mathrm{Hom}_{G\text{-}\mathsf{set}}(Z, G^c)$ with a group structure. Moreover, for any finite $G$-set $X$, this group acts on the left on the set $\mathrm{Hom}_{G\text{-}\mathsf{set}}(Z, X)$, via $(u, f) \mapsto u * f$. This action is compatible with the composition of morphisms: if $Y$ is a finite $G$-set, if $u : Z \to G^c$ and $v : Y \to G^c$ are morphisms of $G$-sets, then for any morphisms of $G$-sets $f : Z \to Y$ and $g : Y \to X$, one checks easily that

$$(v * g) \circ (u * f) = \big(u * (v \circ f)\big) * (g \circ f) \quad . \tag{3.1}$$

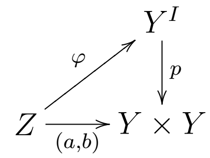
**3.2. Notation :** *Let $G$-$\underline{\mathsf{set}}$ denote the category of* fused $G$-sets*: its objects are finite $G$-sets, and for any finite $G$-sets $Z$ and $Y$*

$$\mathrm{Hom}_{G\text{-}\underline{\mathsf{set}}}(Z, Y) = \mathrm{Hom}_{G\text{-}\mathsf{set}}(Z, G^c)\backslash \mathrm{Hom}_{G\text{-}\mathsf{set}}(Z, Y) \quad .$$

*The composition of morphisms in $G$-$\underline{\mathsf{set}}$ is induced by the composition of morphisms in $G$-$\mathsf{set}$.*

**3.3. Remark :** For any $G$-set $Y$, set $Y^I = Y \times G^c$. This notation is chosen to evoke a path object in homotopy theory (cf. [5] Section 4.12). There is a natural morphism $p : Y^I \to Y \times Y$, defined by $p(y, g) = (y, gy)$, for $y \in Y$ and $g \in G$, and a morphism $i : Y \to Y^I$ defined by $i(y) = (y, 1)$, for $y \in Y$. The composition $p \circ i$ is equal to the diagonal map $Y \to Y \times Y$.

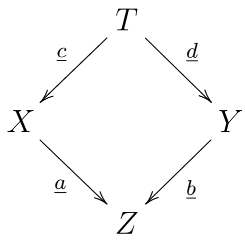
Two morphisms $a, b : Z \to Y$ in $G$-$\mathsf{set}$ are equal in the category $G$-$\underline{\mathsf{set}}$ if and only if the morphism $(a, b) : Z \to Y \times Y$ factors as




for some morphism of $G$-sets $\varphi : Z \to Y^I$.

**3.4. Remark :** It follows from 3.1 that the map $u \mapsto u * \mathrm{Id}_Z$ is a group antihomomorphism from $\mathrm{Hom}_{G\text{-}\mathsf{set}}(Z, G^c)$ to the group of $G$-automorphisms of $Z$. Hence a morphism $\underline{f} : Z \to Y$ in the category $G$-$\underline{\mathsf{set}}$ is an isomorphism if and only if any of its representatives $f : Z \to Y$ in $G$-$\mathsf{set}$ is an isomorphism.

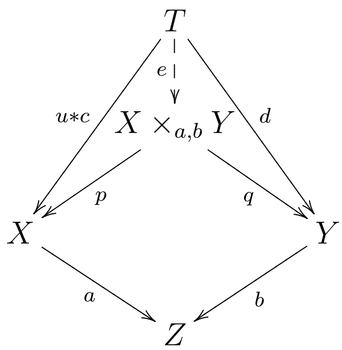
**3.5. Weak pullbacks of fused $G$-sets.** Disjoint union of $G$-sets is a coproduct in $G$-$\underline{\mathsf{set}}$. There is also a weak version of pullback in $G$-$\underline{\mathsf{set}}$ : let




be a commutative diagram in $G$-$\underline{\mathsf{set}}$, where underlines denote the images in $G$-$\underline{\mathsf{set}}$ of morphisms in $G$-$\mathsf{set}$. This means that $\underline{a} \circ \underline{c} = \underline{b} \circ \underline{d}$, i.e. that there exists $u \in \mathrm{Hom}_{G\text{-}\mathsf{set}}(T, G^c)$ such that

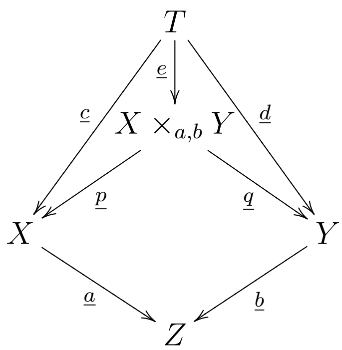
$$b \circ d = u * (a \circ c) \enspace .$$

But $u * (a \circ c) = a \circ (u * c)$. It follows that there is a unique morphism $e \in \mathrm{Hom}_{G\text{-}\mathsf{set}}(T, X \times_{a,b} Y)$ such that the diagram



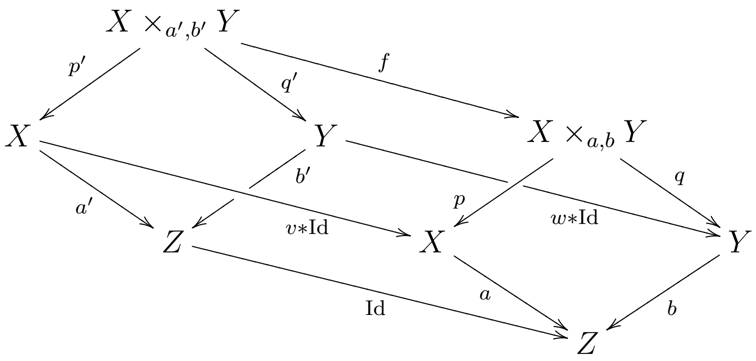

is commutative in $G$-$\mathsf{set}$, where $p : X \times_{a,b} Y \to X$ and $q : X \times_{a,b} Y \to Y$ are the canonical morphisms from the pullback $X \times_{a,b} Y$. In other words, the diagram

**(3.6)**

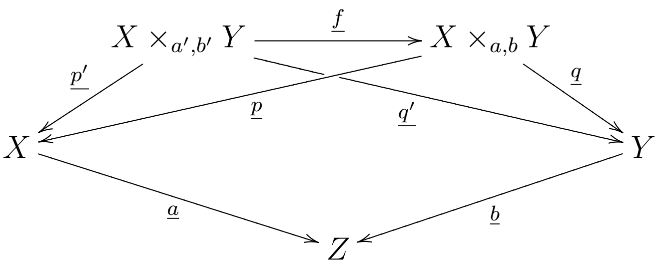



is commutative in $G$-$\underline{\mathsf{set}}$.

But still $(X \times_{a,b} Y, \underline{p}, \underline{q})$ need not be a pullback in $G$-$\underline{\mathsf{set}}$, since the morphism $\underline{e}$ making Diagram 3.6 commutative is generally not unique, as $e$ itself depends on the choice of $u$. Moreover, the lifts $a$ and $b$ of $\underline{a}$ and $\underline{b}$ to $G$-$\mathsf{set}$ are not unique : it should be noted however that if $a' = v * a$ and $b' = w * b$ are other lifts of $a$ and $b$, respectively, where $v \in \mathrm{Hom}_{G\text{-}\mathsf{set}}(X, G^c)$ and $w \in \mathrm{Hom}_{G\text{-}\mathsf{set}}(Y, G^c)$, then the map $f : (x, y) \mapsto \big(v(x)x, w(y)y\big)$ is an isomorphism of $G$-sets from $X \times_{a',b'} Y$ to $X \times_{a,b} Y$, such that the diagram

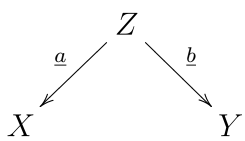



is commutative in $G$-$\mathsf{set}$. Since $\underline{a'} = \underline{a}$, $\underline{b'} = \underline{b}$, $\underline{v * \mathrm{Id}} = \underline{\mathrm{Id}}$, and $\underline{w * \mathrm{Id}} = \underline{\mathrm{Id}}$, this yields a commutative diagram

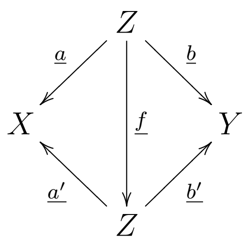



in $G$-$\underline{\mathsf{set}}$, and $\underline{f}$ is an isomorphism. This shows that the weak pullback $X \times_{a,b} Y$ only depends on $\underline{a}$ and $\underline{b}$ in the category $G$-$\underline{\mathsf{set}}$. For this reason, it may be denoted by $X \times_{\underline{a},\underline{b}} Y$.

**3.7. Spans of fused $G$-sets.** Recall (cf. [10], [1] for the general definition) that if $X$ and $Y$ are finite $G$-sets, then a *span* $\Lambda_{Z,\underline{a},\underline{b}}$ over $X$ and $Y$ in the category $G$-$\underline{\mathsf{set}}$ is a diagram of the form




where $Z$ is a finite $G$-set and $\underline{a}, \underline{b}$ are morphisms in the category $G$-$\underline{\mathsf{set}}$. Two spans $\Lambda_{Z,\underline{a},\underline{b}}$ and $\Lambda_{Z',\underline{a'},\underline{b'}}$ over $X$ and $Y$ are equivalent if there exists an isomorphism $\underline{f} : Z \to Z'$ in $G$-$\underline{\mathsf{set}}$ such that the diagram




is commutative. The set of equivalence classes of spans of fused $G$-sets over $X$ and $Y$ is an additive monoid, where the addition is defined by disjoint union (i.e. $\Lambda_{Z_1,\underline{a}_1,\underline{b}_1} + \Lambda_{Z_2,\underline{a}_2,\underline{b}_2} = \Lambda_{Z_1 \sqcup Z_2,\underline{a}_1 \sqcup \underline{a}_2,\underline{b}_1 \sqcup \underline{b}_2}$). The corresponding Grothendieck group is isomorphic to $\mathrm{Hom}_{\underline{\mathbf{S}}(G)}(Y, X)$.

It should be noted that even if there is no pullback construction in the category $G$-$\underline{\mathsf{set}}$, the isomorphism classes of spans in $G$-$\underline{\mathsf{set}}$ can still be composed by *weak pullback*, and this induces the composition of morphisms in $\underline{\mathbf{S}}(G)$.

# 4. Fused Mackey functors

**4.1. Definition :** *Let $R$ be a commutative ring. Let $R\,\mathbf{S}(G)$ (resp. $R\,\underline{\mathbf{S}}(G)$) denote the $R$-linear extension of the category $\mathbf{S}(G)$ (resp. $\underline{\mathbf{S}}(G)$), defined as follows:*

- *The objects of $R\,\mathbf{S}(G)$ and $R\,\underline{\mathbf{S}}(G)$ are finite $G$-sets.*
- *For finite $G$ sets $X$ and $Y$,*

$$\mathrm{Hom}_{R\,\mathbf{S}(G)}(X, Y) = R \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbf{S}(G)}(X, Y) \ ,$$

$$\mathrm{Hom}_{R\,\underline{\mathbf{S}}(G)}(X, Y) = R \otimes_{\mathbb{Z}} \mathrm{Hom}_{\underline{\mathbf{S}}(G)}(X, Y) \ .$$

- *Composition of morphisms is induced by the pullback in $G$-$\mathsf{set}$ (resp. the weak pullback in $G$-$\underline{\mathsf{set}}$).*

*A Mackey functor for $G$ over $R$ in the sense of Lindner ([7]) is an $R$-linear functor from $R\,\mathbf{S}(G)$ to the category $R$-$\mathsf{Mod}$ of $R$-modules.*

*Similarly, a* fused Mackey functor *for $G$ over $R$ is an $R$-linear functor from $R\,\underline{\mathbf{S}}(G)$ to $R$-$\mathsf{Mod}$. A morphism of fused Mackey functors is a natural transformation of functors. Fused Mackey functors for $G$ over $R$ form a category denoted by $\mathsf{Mack}^f_R(G)$.*

The following is an equivalent definition of fused Mackey functors, *à la* Dress:

**4.2. Definition :** *Let $R$ be a commutative ring. A* fused Mackey functor *for the group $G$ over $R$ is a bivariant $R$-linear functor $M = (M^*, M_*)$ from $G$-$\underline{\mathsf{set}}$ to $R$-$\mathsf{Mod}$ such that:*

1. *For any finite $G$-sets $X$ and $Y$, the maps*

$$M(X) \oplus M(Y) \underset{(M^*(\underline{i}_X),M^*(\underline{i}_Y))}{\overset{(M_*(\underline{i}_X),M_*(\underline{i}_Y))}{\rightleftarrows}} M(X \sqcup Y)$$

   *induced by the canonical inclusions $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$ are mutual inverse isomorphisms.*

2. *If*

$$\begin{array}{ccccc} & & X \times_{\underline{a},\underline{b}} Y & & \\ & \swarrow^{\underline{p}} & & \searrow^{\underline{q}} & \\ X & & & & Y \\ & \searrow_{\underline{a}} & & \swarrow_{\underline{b}} & \\ & & Z & & \end{array}$$

   *is a weak pullback diagram in $G$-$\underline{\mathsf{set}}$, then $M^*(\underline{a})M_*(\underline{b}) = M_*(\underline{p})M^*(\underline{q})$.*

*A morphism of fused Mackey functors is a natural transformation of bivariant functors.*

The category $\mathsf{Mack}_R^f(G)$ can be viewed as a full subcategory of the category $\mathsf{Mack}_R(G)$ of Mackey functors for $G$ over $R$. In the case $R = \mathbb{Z}$, this category is equivalent to the category of conjugation invariant Mackey functors introduced in [6].

The inclusion functor $\mathsf{Mack}_R^f(G) \hookrightarrow \mathsf{Mack}_R(G)$ has a left adjoint:

**4.3. Definition :** *Let $M$ be a Mackey functor for $G$ over $R$, in the sense of Lindner, i.e. an $R$-linear functor $R\,\mathbf{S}(G) \to R$-$\mathsf{Mod}$. When $X$ is a finite $G$-set, set*

$$M^f(X) = M(X) / \sum_{Z,a,u} \mathrm{Im}\big(M(\Lambda_{a,\mathrm{Id}_Z}) - M(\Lambda_{u*a,\mathrm{Id}_Z})\big) \ ,$$

*where the summation runs through triples $(Z, a, u)$ consisting of a finite $G$-set $Z$, and morphisms of $G$-sets $a : Z \to X$ and $u : Z \to G^c$, and $\Lambda_{a,\mathrm{Id}_Z}$*

*denotes the span*

$$\begin{array}{ccc} & Z & \\ {}^{a}\swarrow & & \searrow\!\!\searrow{}^{\mathrm{Id}_Z} \\ X & & Z \end{array}$$

*of $G$-sets.*

**4.4. Proposition :** *Let $R$ be a commutative ring, and $G$ be a finite group.*

1. *Let $M$ be a Mackey functor for $G$ over $R$. The correspondence*

$$X \mapsto M^f(X)$$

*is a fused functor $M^f$ for $G$ over $R$.*

2. *The correspondence $\mathcal{F} : M \mapsto M^f$ is a functor from $\mathsf{Mack}_R(G)$ to $\mathsf{Mack}_R^f(G)$, which is left adjoint to the inclusion functor*

$$\mathcal{I} : \mathsf{Mack}_R^f(G) \hookrightarrow \mathsf{Mack}_R(G) \quad .$$

*Moreover $\mathcal{F} \circ \mathcal{I}$ is isomorphic to the identity functor of $\mathsf{Mack}_R^f(G)$.*

**Proof :** For Assertion 1, to prove that $M^f$ is a Mackey functor, observe that if $\Lambda_{Z,a,b}$ is a span of finite $G$-sets of the form

$$\begin{array}{ccc} & Z & \\ {}^{a}\swarrow & & \searrow{}^{b} \\ X & & Y \end{array}$$

and $u : Z \to G^c$ is a morphism of $G$-sets, then

$$\Lambda_{Z,a,b} - \Lambda_{Z,u*a,b} = (\Lambda_{Z,a,\mathrm{Id}_Z} - \Lambda_{Z,u*a,\mathrm{Id}_Z}) \circ \Lambda_{Z,\mathrm{Id}_Z,b} \quad .$$

It follows that the $R$-module

$$\sum_{Z,a,u} \mathrm{Im}\big(M(\Lambda_{a,\mathrm{Id}_Z}) - M(\Lambda_{u*a,\mathrm{Id}_Z})\big)$$

is equal to the sum

$$\sum_{Z,a,b,u} \mathrm{Im}\big(M(\Lambda_{a,b}) - M(\Lambda_{u*a,b})\big) \quad .$$

In other words, it is equal to the image by $M$ of the $R$-submodule $K_R(X,Y)$ of $\mathrm{Hom}_{R\,\mathbf{S}(G)}(Y,X)$ generated by the morphisms $\Lambda_{a,b}-\Lambda_{u*a,b}$, i.e. to the kernel of the quotient morphism

$$\mathrm{Hom}_{R\,\mathbf{S}(G)}(Y,X) \to \mathrm{Hom}_{R\,\underline{\mathbf{S}}(G)}(Y,X) \ .$$

This shows that $K_R$ is an ideal in the category $R\,\mathbf{S}(G)$. So if $M$ is an $R$-linear functor $R\,\mathbf{S}(G)\to R\text{-}\mathsf{Mod}$, the correspondence

$$X \mapsto M^f(X) = M(X)/\sum_{f\in K_R(X,Y)} \mathrm{Im} M(f)$$

is an $R$-linear functor from the quotient category $R\,\underline{\mathbf{S}}(G)$ to $R\text{-}\mathsf{Mod}$.

Assertion 2 is straightforward: first it is clear that $\mathcal{F}\circ\mathcal{I}$ is isomorphic to the identity functor, since $N^f=N$ when $N$ is a fused Mackey functor. This isomorphism $\mathcal{F}\circ\mathcal{I}\cong \mathrm{Id}_{\mathsf{Mack}_R^f(G)}$ provides the counit of the adjunction. Next for any Mackey functor $M$, there is a projection morphism $M\to\mathcal{I}\mathcal{F}(M)$, and this yields the unit of the adjunction. □

**4.5. Remark :** Assertion 2 shows that $\mathsf{Mack}_R^f(G)$ is a *reflective* subcategory of $\mathsf{Mack}_R(G)$ (cf. [8], Chapter IV, Section3).

**4.6. Remark :** If the Mackey functor $M$ is given in the sense of Dress, then for any finite $G$-set $X$

$$M^f(X) = M(X)/\sum_{\substack{a:Z\to X\\ u:Z\to G^c}} \mathrm{Im}\big(M_*(a)-M_*(u*a)\big) \ ,$$

where $Z$ is a finite $G$-set, and $a,u$ are morphisms of $G$-sets.

**4.7. Corollary :**

1. *If $P$ is a projective Mackey functor, then $P^f$ is projective in the category* $\mathsf{Mack}_R^f(G)$.
2. *The category* $\mathsf{Mack}_R^f(G)$ *has enough projective objects. More precisely, if $N$ is a fused Mackey functor, and $\theta: P\to\mathcal{I}(N)$ is an epimorphism in* $\mathsf{Mack}_R(G)$ *from a projective Mackey functor $P$, then $\mathcal{F}(\theta): P^f\to N$ is an epimorphism in* $\mathsf{Mack}_R^f(G)$.

**Proof :** Assertion 1 follows from the fact that $\mathcal{F}$ is left adjoint to the exact functor $\mathcal{I}$. Assertion 2 is then straightforward. □

# 5. The fused Mackey algebra

When $G$ is a finite group, set $\Omega_G = \bigsqcup_{H\leq G} G/H$, and let $RB_{\Omega_G}$ denote the Dress construction for the Burnside functor $RB$ over the ring $R$. Recall that $RB_{\Omega_G}$, as a Mackey functor in the sense of Dress, is obtained by precomposition of $RB$ with the endofunctor $X \mapsto X \times \Omega_G$ of $G$-$\mathsf{set}$.

Also recall (cf. [2] Lemma 7.3.2 and Proposition 4.5.1) that the functor $RB_{\Omega_G}$ is a progenerator of the category $\mathsf{Mack}_R(G)$, and that the algebra $\mathrm{End}_{\mathsf{Mack}_R(G)}(B_{\Omega_G}) \cong B(\Omega_G^2)$ is isomorphic to the Mackey algebra $\mu_R(G)$ of $G$ over $R$, introduced by Thévenaz and Webb ([9]).

It follows from Corollary 4.7 that the functor $(RB_{\Omega_G})^f$ is a progenerator in the category $\mathsf{Mack}_R^f(G)$. Hence this category is equivalent to the category of modules over the algebra $\mathrm{End}_{\mathsf{Mack}_R^f(G)}\big((RB_{\Omega_G})^f\big)$.

**5.1. Definition :** *The* fused Mackey algebra *of $G$ over $R$ is the algebra*

$$\mu_R^f(G) = \mathrm{End}_{\mathsf{Mack}_R^f(G)}\big((RB_{\Omega_G})^f\big) \quad .$$

**5.2. Lemma :** *Let $X$ be a finite $G$-set. Then $(RB_X)^f$ is isomorphic to the Yoneda functor* $\mathrm{Hom}_{R\underline{\mathbf{S}}(G)}(X,-)$.

**Proof :** Denote by $\mathcal{Y}_X$ the Yoneda functor $\mathrm{Hom}_{R\underline{\mathbf{S}}(G)}(X,-)$. For any fused Mackey functor $N$ for $G$ over $R$

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Mack}_R^f(G)}\big((RB_X)^f, N\big) &\cong \mathrm{Hom}_{\mathsf{Mack}_R(G)}\big(RB_X, \mathcal{I}(N)\big)\\ &\cong \mathcal{I}(N)(X) \cong N(X)\\ &\cong \mathrm{Hom}_{\mathsf{Mack}_R^f(G)}(\mathcal{Y}_X, N) \quad .\end{aligned}$$

The lemma follows, since all these isomorphisms are natural. □

**5.3. Theorem :** *The fused Mackey algebra $\mu_R^f(G)$ is isomorphic to the quotient of the algebra $RB(\Omega_G^2) \cong \mu_R(G)$ by the $R$-module generated by dif-*

*ferences of the form*

$$\begin{array}{ccc} & Z & \\ {\scriptstyle b}\swarrow & & \searrow{\scriptstyle a} \\ \Omega_G & & \Omega_G \end{array} \quad - \quad \begin{array}{ccc} & Z & \\ {\scriptstyle u*b}\swarrow & & \searrow{\scriptstyle a} \\ \Omega_G & & \Omega_G \end{array} \;,$$

*where $a, b : Z \to \Omega_G$ and $u : Z \to G^c$ are morphisms of $G$-sets.*

**Proof :** This follows from Lemma 5.2, since the quotient in the theorem is precisely $\mathrm{End}_{R\underline{\mathbf{S}}(G)}(\Omega_G)$. □

**5.4. Remark :** One can deduce from this theorem that the fused Mackey algebra $\mu_R^f(G)$ is always free of finite rank as an $R$-module, and this rank does not depend on the commutative ring $R$. More precisely, Thévenaz and Webb have shown ([9] Proposition 3.2) that the Mackey algebra $\mu_R(G)$ has an $R$-basis consisting of elements of the form

$$t_K^H\, c_{g,K}\, r_{K^g}^L \quad ,$$

where $(H, L, g, K)$ runs through a set of representatives of 4-tuples consisting of two subgroups $H$ and $L$ of $G$, and element $g$ of $G$, and a subgroup $K$ of $H \cap {}^gL$, for the equivalence relation $\equiv$ given by

$$(H, L, g, K) \equiv (H', L', g', K') \Leftrightarrow \left\{ \begin{array}{l} H = H',\ L = L', \\ \text{and} \\ \exists h \in H,\ \exists l \in L,\ g' = hgl,\ K' = {}^hK \quad . \end{array} \right.$$

Similarly, the quotient algebra $\mu_R^f(G)$ of $\mu_R(G)$ has a basis consisting of the images of the elements $t_K^H\, c_{g,K}\, r_{K^g}^L$, where $(H, L, g, K)$ runs through a set of representatives of 4-tuples as above, modulo the relation $\equiv^f$ defined by

$$(H, L, g, K) \equiv^f (H', L', g', K') \Leftrightarrow \left\{ \begin{array}{l} H = H',\ L = L', \\ \text{and} \\ \exists h \in H,\ \exists l \in L,\ \exists x \in C_G(K), \\ \quad g' = hxgl,\ K' = {}^hK \quad . \end{array} \right.$$

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