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The group of endo-permutation modules

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Abstract. The group $D(P)$ of all endo-permutation modules for a finite p-group P is a finitely generated abelian group. We prove that its torsion-free rank is equal to the number of conjugacy classes of non-cyclic subgroups of P. We also obtain partial results on its torsion subgroup. We determine next the structure of $\mathbb{Q}\otimes D(-)$ viewed as a functor, which turns out to be a simple functor $S_{E,\mathbb{Q}}$, indexed by the elementary group E of order p^2 and the trivial $\mathrm{Out}(E)$-module $\mathbb{Q}$. Finally we describe a rather strange exact sequence relating $\mathbb{Q}\otimes D(P)$, $\mathbb{Q}\otimes B(P)$, and $\mathbb{Q}\otimes R(P)$, where $B(P)$ is the Burnside ring and $R(P)$ is the Grothendieck ring of $\mathbb{Q}P$-modules.

Contents

Introduction

Endo-permutation modules for p-groups are ubiquitous in the p-modular representation theory of finite groups. They appear as sources of simple modules for p-soluble groups (see Section 30 in [Th1]), in Puig's description of the source algebra of a nilpotent block (see [Pu1] or Section 50 in [Th1]), and also in the local analysis of Morita or derived equivalences between blocks [Pu3]. So they are fundamental objects which need to be fully understood, the final goal being the complete classification of endo-permutation modules. This paper is a contribution to this classification program.

Recall that if P is a p-group and if k is a field of characteristic p, then a kP-module M is an *endo-permutation module* if $\mathrm{End}_k(M)$ is a permutation kP-module. One is particularly interested in indecomposable endo-permutation modules with vertex P. Twenty years ago, in his seminal papers on endo-permutation modules [Da2, Da3], E.C. Dade introduced most of the basic techniques and defined the group $D(P) = D_k(P)$ of endo-permutation kP-modules, which we call the *Dade group* of P. This can be viewed either as a group of equivalence classes of suitable endo-permutation modules (with multiplication induced by tensor product) or as the group of indecomposable endo-permutation kP-modules with vertex P (because there is one such indecomposable module in each equivalence class, up to isomorphism).

Dade classified all endo-permutation modules when P is abelian and described explicitly the structure of $D(P)$ in that case. Shortly afterwards, L. Puig proved that $D(P)$ is always finitely generated (but his paper [Pu2] only appeared in 1990). So we have

$$D(P) \cong \mathbb{Z}^r \oplus D^t(P)\,, \qquad \text{with} \quad D^t(P) \quad \text{finite}\,.$$

The question of finding the torsion-free rank r and describing the torsion subgroup $D^t(P)$ remained completely open in general. Our first result gives an answer to the first question.

Theorem A. *The torsion-free rank of the Dade group $D(P)$ is equal to the number of conjugacy classes of non-cyclic subgroups of P.*

In fact we define a natural map $\phi : \mathbb{Q} \otimes_{\mathbb{Z}} D(P) \to \mathbb{Q}^r$ which we prove to be an isomorphism, and by means of the inverse of ϕ we produce an explicit basis of $\mathbb{Q} \otimes_{\mathbb{Z}} D(P)$. One of the main ingredients (which follows from deep results of Carlson, Dade, and Puig) is the injectivity of the restriction–deflation map

$$\mathbb{Q} \otimes_{\mathbb{Z}} D(P) \longrightarrow \prod_E \mathbb{Q} \otimes_{\mathbb{Z}} D(E)\,,$$

where E runs over all elementary abelian sections of P of rank 2, together with the fact that $\mathbb{Q} \otimes_{\mathbb{Z}} D(E) \cong \mathbb{Q}$ by Dade's classification in the abelian case. Another main ingredient is tensor induction, a tool which is well adapted because endo-permutation modules behave nicely with respect to tensor products. This tool was first considered by Puig [Pu4] who used it to show that sources of simple modules for p-soluble groups always lie in the torsion subgroup of the Dade group.

One is also particularly interested in the torsion subgroup $D^t(P)$. Indeed the naturally occuring endo-permutation modules mentioned in the first paragraph define interesting elements of $D(P)$ which are actually expected to lie in $D^t(P)$. The cyclic group C_p of order p plays a crucial role here (at least for p odd), because $D(C_p) \cong \mathbb{Z}/2\mathbb{Z}$. Unfortunately, one does not know if the restriction–deflation map

$$\psi_P : D^t(P) \longrightarrow \prod_C D(C)$$

is injective, where C runs over all cyclic sections of P of order p. If it were so, we could determine $D^t(P)$ by methods analogous to those used for Theorem A. Thus we simply bypass the difficulty by taking the quotient $\overline{D^t}(P) = D^t(P)/\mathrm{Ker}(\psi_P)$. This quotient can be explicitly described:

Theorem B. *If p is odd, the group $\overline{D^t}(P)$ is isomorphic to the direct sum of s copies of $\mathbb{Z}/2\mathbb{Z}$, where s is the number of conjugacy classes of non-trivial cyclic subgroups of P.*

We also obtain explicit generators for $\overline{D^t}(P)$. It is our hope that ψ_P is actually injective when p is odd. This question was mentioned to us several years ago by J.F. Carlson and J.L. Alperin. If the answer was positive, Theorem B would completely describe $D^t(P)$ and would also imply that every torsion element of $D(P)$ has order 2 (i.e. is self-dual). This is a non-trivial fact which would be very interesting to obtain.

It is known that the situation is more complicated when $p = 2$. First of all $D(C_2)$ is trivial (so one expects instead that $D(C_4) \cong \mathbb{Z}/2\mathbb{Z}$ should play a role), but there are also unexpected endo-permutation modules for the quaternion groups (see [Da1]). For instance $D(Q_8) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}$. Thus the assumption in Theorem B cannot be removed.

Our methods can also be used to analyse the subgroup $T(P)$ consisting of the *endo-trivial* modules, that is, indecomposable kP-modules M such that $\mathrm{End}_k(M) \cong k \oplus (\text{free})$. Those modules also play an important role in modular representation theory, for instance in self-equivalences of stable module categories (see [Li] and [CR]). We prove a theorem analogous to Theorem A, which gives the torsion-free rank of $T(P)$; this is a recent unpublished theorem of J.L. Alperin but our proof is different (we use tensor induction instead of relative syzygies). In contrast, in analogy with Theorem B, one only has a nearly trivial statement concerning the torsion subgroup of $T(P)$.

Our analysis of $D(P)$ takes place in a functorial framework. We view $D(-)$ as a functor defined on the category of all finite p-groups, the morphisms being compositions of five types of basic morphisms, namely restriction to a subgroup, tensor induction to an overgroup, inflation from a quotient group, deflation to a quotient group (this is Dade's construction of *slash* modules), and isomorphisms. The necessary machinery for dealing with such functors was introduced in [Bo1]. Our original proofs of Theorems A and B used heavily this functorial theory (and in particular some non-trivial results of [Bo1]), but a more elementary approach emerged later from the functorial point of view. For the convenience of the reader, we first give here these non-functorial proofs (Sections 4 and 6). The only result from [Bo1] which we need is a technical lemma on the Möbius function.

However, the functorial approach yields more and we use it in the second part of the paper. The basic functorial constructions are already introduced in Sections 2 and 3, but the more difficult functorial results only appear in the last five sections. For technical reasons, the Dade group $D(P)$ does not in general give rise to a functor in the sense of [Bo1], but $\mathbb{Q} \otimes_{\mathbb{Z}} D(P)$ and $\overline{D^t}(P)$ do. Recall that simple functors $S_{H,V}$ over a field K are parametrized by pairs (H, V) where H is a finite group (a p-group in our situation) and V is a simple $K[\mathrm{Out}(H)]$-module.

Theorem C. *The functor $\mathbb{Q} \otimes_{\mathbb{Z}} D$ is simple. More precisely, it is isomorphic to $S_{E,\mathbb{Q}}$, where E is elementary abelian of rank 2 and $\mathbb{Q}$ is the trivial module.*

Our fourth main result asserts that there is an exact sequence relating the Dade group $D(P)$, the Burnside ring $B(P)$, and the Grothendieck ring $R_{\mathbb{Q}}(P)$ of $\mathbb{Q}P$-modules. The composition factors of the Burnside functor $\mathbb{Q} \otimes_{\mathbb{Z}} B(-)$ were analysed in [Bo1] (for the category of *all* finite groups). For functors defined only on p-groups, we prove a similar result for $K \otimes_{\mathbb{Z}} B$, where K is a field of characteristic different from p (see Section 8). When $K = \mathbb{Q}$, there are just two composition factors $S_{1,\mathbb{Q}}$ and $S_{E,\mathbb{Q}}$, fitting in a non-split exact sequence of functors

$$0 \longrightarrow S_{E,\mathbb{Q}} \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} B \longrightarrow S_{1,\mathbb{Q}} \longrightarrow 0 .$$

But in fact $S_{1,\mathbb{Q}} \cong \mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}$ and $S_{E,\mathbb{Q}} \cong \mathbb{Q} \otimes_{\mathbb{Z}} D$ by Theorem C. Thus we obtain the following result.

Theorem D. *There is an exact sequence of functors*

$$0 \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} D \xrightarrow{\alpha} \mathbb{Q} \otimes_{\mathbb{Z}} B \xrightarrow{\varepsilon} \mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}} \longrightarrow 0 ,$$

where ε is the natural homomorphism mapping a P-set to the corresponding permutation $\mathbb{Q}P$-module.

We also give an explicit description of the map α on a basis of $\mathbb{Q}D(P)$, for every p-group P.

When p is odd, there are analogous results for the functor $\overline{D^t}(-)$ appearing in Theorem B. We prove that it is a simple functor over the field $\mathbb{F}_2$, isomorphic to $S_{C_p,\mathbb{F}_2}$. Moreover, there is an exact sequence of functors

$$0 \longrightarrow \overline{D^t} \longrightarrow \mathbb{F}_2 \otimes_{\mathbb{Z}} R_{\mathbb{Q}} \xrightarrow{\dim} \Gamma_{\mathbb{F}_2} \longrightarrow 0 ,$$

where $\Gamma_{\mathbb{F}_2}$ denotes the constant functor with values $\mathbb{F}_2$.

1. The Dade group

We first recall some standard facts concerning endo-permutation modules and the Dade group. Details can be found in Dade's original paper [Da2] or in Sections 28–29 of [Th]. Let k be a field of characteristic p and let P be a finite p-group. An endo-permutation kP-module M is said to be *capped* if M has at least one indecomposable direct summand with vertex P. By assumption, the P-algebra $A = \mathrm{End}_k(M)$ has a P-invariant basis X and the condition that M be capped is equivalent to the requirement that the Brauer quotient $A[P] = A^P / \sum_{Q<P} A_Q^P$ is non-zero, or equivalently that X has at least one P-fixed point (because $A[P]$ has basis X^P). The endo-permutation modules we shall consider will always be capped and we shall often omit to mention it. The main examples of endo-permutation modules are the kP-modules $\Omega_{P/Q}^n(k)$ where Q is any normal subgroup of P and $\Omega_{P/Q}$ denotes the Heller operator for the quotient group P/Q.

A k-algebra A is called a *Dade P-algebra* if it is a P-algebra (i.e. P acts on A by algebra automorphisms) which is central simple and split (i.e. isomorphic to a matrix algebra over k) and if A has a P-invariant basis with at least one P-fixed point. Thus if M is a capped endo-permutation kP-module, then $A = \mathrm{End}_k(M)$ is a Dade P-algebra. Conversely, if A is a Dade P-algebra, then $A \cong \mathrm{End}_k(M)$ for some k-vector space M and the P-action on A lifts uniquely to a kP-module structure on M (thanks to the fact that P is a p-group), so that M becomes a capped endo-permutation module. Thus the concepts of capped endo-permutation module and Dade P-algebra are equivalent. By the very definition of an endo-permutation module, the emphasis is put on the corresponding Dade P-algebra, so we shall often use the algebra point of view.

The following result is basic for understanding the Dade group.

(1.1) Lemma. *If M is a capped endo-permutation kP-module, then any two indecomposable direct summands of M with vertex P are isomorphic.*

Such an indecomposable direct summand is called a *cap* of M and is uniquely determined up to isomorphism. It is again a capped endo-permutation kP-module. Two capped endo-permutation modules M and N are said to be *equivalent* if they have isomorphic caps, in which case we write $M \sim N$. The set of equivalence classes of capped endo-permutation kP-modules is written $D(P)$, or $D_k(P)$ when we need to emphasize the base field k. The tensor product of kP-modules induces a group structure on $D(P)$. The unit element is the class of the trivial module and turns out to consist of all permutation modules (which are capped, i.e. have a summand isomorphic to k). The inverse of the class of M is the class of the dual module M^*. The group $D(P)$ is abelian and is called the *Dade group* of P.

In any equivalence class, there is a unique indecomposable module up to isomorphism (namely the cap of any module in the class), so it is often convenient to view $D(P)$ as the set of isomorphism classes of indecomposable endo-permutation kP-modules with vertex P. However, the tensor product of two such indecomposable modules need not be indecomposable, so one needs to take its cap to recover the group structure in terms of indecomposable modules.

One can also view the equivalence relation in terms of Dade P-algebras as follows. Two Dade P-algebras A and B are equivalent (written also $A \sim B$) if and only if $iAi \cong jBj$, where i is a primitive idempotent of A^P with $br_P(i) \neq 0$ and j is a primitive idempotent of B^P with $br_P(j) \neq 0$ (both are unique up to conjugation by Lemma 1.1). Here $br_P : A^P \to A[P]$ is the quotient map, called the Brauer homomorphism. Thus $D(P)$ is also the group of equivalence classes of Dade P-algebras and we shall freely use either point of view. The group structure is induced by the tensor product of algebras. Moreover, the inverse of the class of A is the class of the opposite P-algebra A^{op}, because if $\mathrm{End}_k(M) = A$, then $\mathrm{End}_k(M^*) \cong A^{op}$.

There is yet another way of viewing the equivalence relation: $M \sim N$ if and only if there exist modules V and W in the trivial class (i.e. permutation modules with a trivial summand) such that $M \otimes_k V \cong N \otimes_k W$. In terms of Dade P-algebras, $A \sim B$ if and only $A \otimes_k \mathrm{End}_k(V) \cong B \otimes_k \mathrm{End}_k(W)$ for some V and W as above.

Despite the fact that the group structure on $D(P)$ is induced by the tensor product, we shall use an additive notation, which is much more convenient in the formulas and in many computations. Thus, for any endo-permutation modules M and N, we define

$$[M] + [N] = [M \otimes_k N],$$

and similarly in terms of Dade P-algebras. For simplicity, we shall write $\Omega_{P/Q}$ for the class of $\Omega^1_{P/Q}(k)$. For example, the class of $\Omega^n_P(k)$ is equal to $n\Omega_P$, since we have $\Omega^r_P(k) \otimes \Omega^s_P(k) \cong \Omega^{r+s}_P(k) \oplus (\text{free})$.

There is an obvious *restriction map*

$$\mathrm{Res}^P_Q : D(P) \longrightarrow D(Q),$$

where Q is a subgroup of P. If Q is a normal subgroup, there is also an obvious *inflation map*

$$\mathrm{Inf}^P_{P/Q} : D(P/Q) \longrightarrow D(P).$$

For instance $\Omega_{P/Q}$ is the image under $\mathrm{Inf}^P_{P/Q}$ of the class of the Heller translate of the trivial module for the quotient group P/Q. Finally we define the *deflation map*

$$\mathrm{Def}^P_{P/Q} : D(P) \longrightarrow D(P/Q)$$

as follows. If A is a Dade P-algebra, then the Brauer quotient $A[Q] = A^Q / \sum_{R<Q} A^Q_R$ turns out to be a Dade (P/Q)-algebra and we set $\mathrm{Def}^P_{P/Q}([A]) = [A[Q]]$ where $[A]$ denotes the class of A. This is the *slash* construction in Dade's terminology. If X is a P-invariant basis of A, then the Q-fixed points X^Q form a (P/Q)-invariant basis of $A[Q]$. The three above maps are group homomorphisms and are part of the functorial framework which will be introduced in Section 2 and 3 (and fully used in Sections 7–11). The fourth important map is tensor induction and will be introduced in Section 2.

If Q is a subgroup of P and R is a normal subgroup of Q, then Q/R is a section of P and we define

$$\mathrm{Defres}^P_{Q/R} = \mathrm{Def}^Q_{Q/R}\,\mathrm{Res}^P_Q : D(P) \longrightarrow D(Q/R).$$

An important subgroup of $D(P)$ is the kernel of all these maps. We consider

$$\prod_{Q\neq 1} \mathrm{Defres}^P_{N_P(Q)/Q} : D(P) \longrightarrow \prod_{Q\neq 1} D(N_P(Q)/Q),$$

where Q runs over all non-trivial subgroups of P (up to conjugation). We define

$$T(P) = \mathrm{Ker}\big(\prod_{Q\neq 1} \mathrm{Defres}^P_{N_P(Q)/Q}\big) = \bigcap_{Q\neq 1} \mathrm{Ker}(\mathrm{Defres}^P_{N_P(Q)/Q}).$$

Recall that a kP-module M is called *endo-trivial* if $\mathrm{End}_k(M) \cong k \oplus (\text{free})$ as a kP-module. In that case M is a capped endo-permutation module. The connection with $T(P)$ is provided by the following result.

(1.2) Lemma. (Puig [Pu2, 2.1.2]). *Let M be an indecomposable endo-permutation kP-module with vertex P. Then $[M] \in T(P)$ if and only if M is endo-trivial.*

Thus the subgroup $T(P)$ coincides with the group of endo-trivial modules. Another characterization of endo-trivial modules has recently been obtained by J. Carlson [Ca2].

In view of its essential role in the sequel, we recall the following result about Ω_P.

(1.3) Lemma. *Let Q be a subgroup of P.*

(a) $\mathrm{Res}^P_Q(\Omega_P) = \Omega_Q$.

(b) If Q is a non-trivial normal subgroup of P, then $\mathrm{Def}^P_{P/Q}(\Omega_P) = 0$. In other words, Ω_P is endo-trivial.

Proof. (a) We have $\mathrm{Res}^P_Q(\Omega^1_P(k)) = \Omega^1_Q(k) \oplus (\text{free})$.

(b) We have $\mathrm{End}_k(\Omega^1_P(k)) \cong \Omega^1_P(k) \otimes_k \Omega^1_P(k)^* \cong \Omega^1_P(k) \otimes_k \Omega^{-1}_P(k) \cong k \oplus (\text{free})$ as kP-modules. □

We can now recall the two main theorems about the Dade group.

(1.4) Theorem. (Dade [Da3, 10.1 and 12.5]). *Assume that P is abelian.*

(a) $T(P)$ is generated by Ω_P. Thus $T(P)$ is trivial if $|P| = 2$, cyclic of order 2 if P is cyclic of order ≥ 3, and infinite cyclic otherwise.

(b) $D(P)$ is isomorphic to the direct sum of the groups $T(P/Q)$, where Q runs over the subgroups of P. Thus $D(P) \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2\mathbb{Z})^s$, where r is the number of non-cyclic quotients of P and s is the number of cyclic quotients of P of order ≥ 3.

Of course $T(P/Q)$ is identified with a subgroup of $D(P)$ via inflation. Note that (a) is hard to prove, but that (b) follows rather easily from (a) by induction. Another proof of the crucial step for (a) (namely when P has order p^2) is due to Carlson [Ca1].

(1.5) Theorem. (Puig [Pu2, 2.2 and 2.4]).

(a) The kernel of the restriction map $T(P) \longrightarrow \prod_E T(E)$, where E runs over the elementary abelian subgroups of P, is finite.

(b) The group $D(P)$ is finitely generated.

The proof of (a) uses a non-trivial cohomological result of Carlson and some algebraic geometry (though Puig's version is explained in terms of commutative algebra). By induction, (b) follows easily from (a) and Dade's theorem above.

The previous results imply the following injectivity theorem, which is one of our main tools for finding the torsion-free ranks of $D(P)$ and $T(P)$. For simplicity, we write $\mathbb{Q}D(P) = \mathbb{Q} \otimes_{\mathbb{Z}} D(P)$ and $\mathbb{Q}T(P) = \mathbb{Q} \otimes_{\mathbb{Z}} T(P)$.

(1.6) Theorem. *The following maps are injective:*

(a) $\mathrm{Res} : \mathbb{Q}T(P) \longrightarrow \prod_E \mathbb{Q}T(E)$*, where* E *runs over the elementary abelian subgroups of* P *of rank 2.*

(b) $\mathrm{Defres} : \mathbb{Q}D(P) \longrightarrow \prod_{Q/R} \mathbb{Q}D(Q/R)$*, where* Q/R *runs over the elementary abelian sections of* P *of rank 2.*

Proof. (a) By Theorem 1.5 (a), the restriction map to all elementary abelian subgroups is injective after tensoring with $\mathbb{Q}$. But if E_k denotes the elementary abelian group of rank k, we have $\mathbb{Q}T(E_1) = 0$, while the restriction map $\mathrm{Res}^{E_k}_{E_2} : T(E_k) \to T(E_2)$ is an isomorphism if $k \geq 2$, by Theorem 1.4 (a) and the fact that $\mathrm{Res}^{E_k}_{E_2}(\Omega_{E_k}) = \Omega_{E_2}$. The result follows.

(b) We prove the injectivity of the map by induction on $|P|$, starting with the trivial case of a cyclic group. If now P is not cyclic, let $a \in \mathbb{Q}D(P)$ lie in the kernel of all maps $\mathrm{Defres}^P_{Q/R}$ for sections Q/R which are elementary abelian of rank 2. If H is a non-trivial subgroup of P, then $a_H = \mathrm{Defres}^P_{N_P(H)/H}(a)$ also lies in the kernel of all maps $\mathrm{Defres}^{N_P(H)/H}_{Q/R}$ for sections Q/R of rank 2 in $N_P(H)/H$, by transitivity of these maps (which easily follows from the definitions). By induction, we have $a_H = 0$. Since this holds for every non-trivial subgroup H, it follows that $a \in \mathbb{Q}T(P)$, by definition of $T(P)$. Now for every elementary abelian subgroup E of rank 2, $\mathrm{Res}^P_E(a) = 0$ by assumption. Therefore $a = 0$ by (a). □

2. Tensor induction and some functorial constructions

The purpose of this section is to introduce a construction which contains tensor induction as a special case. In fact, it turns out that the five natural operations on Dade P-algebras, namely restriction, inflation, deflation, tensor induction, and isomorphisms, can be recovered by means of a single mechanism. We define this construction here, not only because of its simplicity and its use later in Section 7, but also because the proofs of the results which we need about tensor induction of Dade P-algebras are actually simpler if we adopt this approach. Those results will be proved in the next section.

Throughout this section and the next, it is much more convenient to work with Dade P-algebras rather than endo-permutation modules. However, we first recall the notion of tensor induction of modules, and then of algebras. Let H be a subgroup of a finite group G and let G/H denote the set of right cosets gH where $g \in G$ (i.e. right orbits under the action of H). If M is a kH-module, recall that the *tensor induction* of M is the kG-module $\mathrm{Ten}^G_H(M)$ defined as follows. As a k-vector space, we have

$$\mathrm{Ten}^G_H(M) = \bigotimes_{C \in G/H} (kC \otimes_{kH} M),$$

and the kG-module structure is given by

$$g \cdot [\bigotimes_{C \in G/H} (s_C \otimes m_C)] = \bigotimes_{C \in G/H} (g s_{g^{-1}C} \otimes m_{g^{-1}C})$$

where $g \in G$ and, for each $C \in G/H$, $s_C \in C$ and $m_C \in M$. If $[G/H]$ denotes a set of coset representatives, then we have $\mathrm{Ten}^G_H(M) = \bigotimes_{s \in [G/H]} (s \otimes M)$ and the action of g is given by

$$g \cdot [\bigotimes_{s \in [G/H]} (s \otimes m_s)] = \bigotimes_{s \in [G/H]} (\tau(s) \otimes h_s m_s) = \bigotimes_{s \in [G/H]} (s \otimes h_{\tau^{-1}(s)} m_{\tau^{-1}(s)}),$$

where $gs = \tau(s)h_s$, $h_s \in H$ and τ is a permutation of $[G/H]$. The basic properties of tensor induction (e.g. transitivity, multiplicativity, Mackey formula) can be found in [Be, 3.15], but we shall actually recover them by means of our functorial formalism (at least for permutation modules).

If A is an H-algebra over k , then $kC \otimes_{kH} A$ is an algebra (isomorphic to A), via $(s\otimes a)(s\otimes a') = s\otimes aa'$ where $s \in C$ is fixed (and this is independent of the choice of s). Therefore the tensor product of algebras $\bigotimes_{C\in G/H}(kC \otimes_{kH} A)$ has an algebra structure. It follows that the kG-module $\mathrm{Ten}_H^G(A)$ has a G-algebra structure, since it is easy to check that G acts on $\mathrm{Ten}_H^G(A)$ by algebra automorphisms. Moreover tensor induction of modules and algebras are related in the following way.

(2.1) Lemma. *Let M be a kH-module and consider the corresponding H-algebra $\mathrm{End}_k(M)$. Then there is an isomorphism of G-algebras $\mathrm{Ten}_H^G(\mathrm{End}_k(M)) \cong \mathrm{End}_k(\mathrm{Ten}_H^G(M))$.*

Proof. Let $[G/H]$ be a set of coset representatives. Then there is a canonical isomorphism of k-algebras

$$\phi : \bigotimes_{s\in[G/H]} (s \otimes \mathrm{End}_k(M)) \xrightarrow{\sim} \mathrm{End}_k\Big(\bigotimes_{s\in[G/H]} (s \otimes M)\Big)$$

and therefore $\mathrm{Ten}_H^G(\mathrm{End}_k(M)) \cong \mathrm{End}_k(\mathrm{Ten}_H^G(M))$ as k-algebras. In order to check that ϕ commutes with the action of G , we choose $g \in G$ and write $gs = \tau(s)h_s$ as before. We also have $g^{-1}s = \tau^{-1}(s)h^{-1}_{\tau^{-1}(s)}$. Since H acts on $\mathrm{End}_k(M)$ by conjugation, the action of g on $\mathrm{Ten}_H^G(\mathrm{End}_k(M))$ is given by

$$g \cdot \Big[\bigotimes_{s\in[G/H]} (s \otimes a_s)\Big] = \bigotimes_{s\in[G/H]} \big(\tau(s) \otimes h_s a_s h_s^{-1}\big) ,$$

where $a_s \in \mathrm{End}_k(M)$ for all s . We have to prove that this is the same action as the conjugation action of g on $\mathrm{End}_k(\mathrm{Ten}_H^G(M))$. Given $\bigotimes_{s\in[G/H]}(s \otimes m_s) \in \mathrm{Ten}_H^G(M)$, we have

$$\begin{aligned}
g \Big[\bigotimes_{s\in[G/H]} (s \otimes a_s)\Big] g^{-1}\Big[\bigotimes_{s\in[G/H]} (s \otimes m_s)\Big] &= g \Big[\bigotimes_{s\in[G/H]} (s \otimes a_s)\Big] \Big[\bigotimes_{s\in[G/H]} (s \otimes h_s^{-1} m_{\tau(s)})\Big] \\
&= g \Big[\bigotimes_{s\in[G/H]} (s \otimes a_s h_s^{-1} m_{\tau(s)})\Big] = \bigotimes_{s\in[G/H]} \big(\tau(s) \otimes h_s a_s h_s^{-1} m_{\tau(s)}\big) \\
&= \Big[\bigotimes_{s\in[G/H]} \big(\tau(s) \otimes h_s a_s h_s^{-1}\big)\Big] \Big[\bigotimes_{s\in[G/H]} \big(\tau(s) \otimes m_{\tau(s)}\big)\Big] \\
&= \Big[\bigotimes_{s\in[G/H]} \big(\tau(s) \otimes h_s a_s h_s^{-1}\big)\Big] \Big[\bigotimes_{s\in[G/H]} (s \otimes m_s)\Big] .
\end{aligned}$$

The result follows. □

Now we introduce our functorial formalism. The main idea is to change the point of view about Dade P-algebras. Instead of viewing them as P-algebras in which there exists a P-invariant basis, we consider a permutation kP-module kX with a fixed basis X and we then look at all possible algebra structures on kX . For Dade P-algebras, we actually only need to consider central simple split algebra structures.

We first have to introduce our construction for permutation modules. Let p be a prime number and let k be a field of characteristic p . If P is a finite p-group, we denote by $Perm_k(P)$ the category of finitely generated permutation kP-modules. The objects of $Perm_k(P)$ are the kP-modules M which admit a P-invariant k-basis and the morphisms in $Perm_k(P)$ are the morphisms of kP-modules. Let $\underline{Perm}_k(P)$ denote the following category: the objects of $\underline{Perm}_k(P)$ are the finite P-sets, and a morphism in $\underline{Perm}_k(P)$

from Y to X is a matrix $m(x,y)$, indexed by $X \times Y$, with coefficients in k, which is P-invariant, i.e. such that $m(gx,gy) = m(x,y)$ for all $(x,y) \in X \times Y$ and all $g \in P$. The composition of morphisms is given by usual matrix multiplication.

There is an obvious functor $e_P : \underline{Perm}_k(P) \to Perm_k(P)$, mapping the P-set X to the module kX, and the matrix m indexed by $X \times Y$ to the morphism from kY to kX defined on the k-basis Y of kY by

$$y \in Y \mapsto \sum_{x \in X} m(x,y)x \in kX .$$

The functor e_P is fully faithful by construction, and it is essentially surjective since any permutation module admits a P-invariant k-basis X, hence is isomorphic to kX. Thus e_P is an equivalence of categories, and this allows us to define our constructions on permutation P-algebras only in the category $\underline{Perm}_k(P)$.

It is clear for instance that the direct sum functor in $Perm_k(P)$, mapping a pair of modules M and M' to their direct sum $M \oplus M'$, corresponds to the disjoint union functor, mapping a pair of P-sets (X, X') to their disjoint union $X \sqcup X'$, and a pair of morphisms (m, m') with $m : Y \to X$ and $m' : Y' \to X'$ to the morphism $m \sqcup m' : Y \sqcup Y' \to X \sqcup X'$ defined by

$$(m \sqcup m')(a,b) = \begin{cases} m(a,b) & \text{if } a \in X,\ b \in Y ,\\ m'(a,b) & \text{if } a \in X',\ b \in Y' ,\\ 0 & \text{if } a \in X,\ b \in Y' \text{ or } a \in X',\ b \in Y . \end{cases}$$

In other words, we have an isomorphism

$$e_P(X \sqcup X') \cong e_P(X) \oplus e_P(X')$$

which is natural in X and X'. Similarly, the tensor product of modules corresponds to the cartesian product: define the cartesian product of $m : Y \to X$ and $m' : Y' \to X'$ to be the morphism

$$m \times m' : Y \times Y' \to X \times X' , \qquad (m \times m')(x, x', y, y') = m(x,y)m'(x', y') .$$

Then clearly $e_P(X \times X') \cong e_P(X) \otimes_k e_P(X')$, and this is functorial in X and X'.

Let P and Q be finite p-groups, and let U be a finite biset, which is a left Q-set and a right P-set, such that the actions of P and Q commute. Such a biset will be called a Q-set-P for short. The opposite biset U^{op} is the P-set-Q obtained as usual from the underlying set U by reversing the actions

$$g{\cdot}u{\cdot}h \quad (\text{ in } U^{op}\ ) \ = \ h^{-1}{\cdot}u{\cdot}g^{-1} \quad (\text{ in } U\ ), \qquad \forall g \in P, \forall h \in Q, \forall u \in U .$$

If X is a finite P-set, then the set $\mathrm{Hom}_P(U^{op}, X)$ of all P-equivariant maps $U^{op} \to X$ is a finite Q-set. We define a functor T_U from $\underline{Perm}_k(P)$ to $\underline{Perm}_k(Q)$ on objects by

$$T_U(X) = \mathrm{Hom}_P(U^{op}, X) .$$

If m is a P-invariant matrix indexed by $X \times Y$, let $T_U(m)$ be the matrix indexed by $T_U(X) \times T_U(Y)$, defined by

$$T_U(m)(\phi, \psi) = \prod_{u \in [U/P]} m(\phi(u), \psi(u)) .$$

It is clear that this does not depend on the choice of the set of orbit representatives of U/P, since m is P-invariant.

(2.2) Lemma. *The correspondence* T_U *is a functor from* $\underline{Perm_k}(P)$ *to* $\underline{Perm_k}(Q)$ *.*

Proof. First it is clear that $T_U(m)$ is Q-invariant, since for every $h \in Q$ we have

$$T_U(m)(h\varphi, h\psi) = \prod_{u\in[U/P]} m\big(\varphi(h^{-1}u), \psi(h^{-1}u)\big) = T_U(m)(\varphi, \psi)\,,$$

because the image by h^{-1} of a set of orbit representatives of U/P is another set of representatives.

Now of course, if m represents the identity morphism, then $T_U(m)(\varphi,\psi)$ is non-zero if and only if $\varphi(u) = \psi(u)$ for all $u \in U$, or equivalently if $\varphi = \psi$. So $T_U(m)$ is the identity morphism.

Finally, let Z be another P-set and let n be a matrix representing a morphism in $\underline{Perm_k}(P)$ from Z to Y . The product matrix $m \cdot n$ is defined by

$$(m\cdot n)(x,z) = \sum_{y\in Y} m(x,y)n(y,z)\,.$$

Let $\theta \in \mathrm{Hom}_P(U^{op}, Z)$ and $\varphi \in \mathrm{Hom}_P(U^{op}, X)$. Then

$$T_U(m\cdot n)(\varphi,\theta) = \prod_{u\in[U/P]} \Big(\sum_{y\in Y} m\big(\varphi(u),y\big)n\big(y,\theta(u)\big)\Big)\,. \tag{2.3}$$

Now for a given $u \in [U/P]$, we have

$$\begin{aligned}\sum_{y\in Y} m\big(\varphi(u),y\big)n\big(y,\theta(u)\big) &= \sum_{\substack{y\in[P_u\backslash Y]\\ g\in[P_u/P_{u,y}]}} m\big(\varphi(u),gy\big)n\big(gy,\theta(u)\big)\\ &= \sum_{\substack{y\in[P_u\backslash Y]\\ g\in[P_u/P_{u,y}]}} m\big(\varphi(ug),y\big)n\big(y,\theta(ug)\big)\\ &= \sum_{y\in[P_u\backslash Y]} |P_u : P_{u,y}|\, m\big(\varphi(u),y\big)n\big(y,\theta(u)\big)\,.\end{aligned}$$

As P_u is a p-group and k is of characteristic p , the coefficient $|P_u : P_{u,y}|$ is zero unless $P_{u,y} = P_u$, or equivalently $y \in Y^{P_u}$.

Now expanding the product in Equation (2.3) is equivalent to choosing for each $u \in [U/P]$ an element $y_u \in Y^{P_u}$. This in turn is equivalent to defining a P-morphism ψ from U to Y (by $\psi(u') = gy_u$ if $u \in [U/P]$, $g \in P$, and $u' = ug$). This gives finally

$$\begin{aligned}T_U(m\cdot n)(\varphi,\theta) &= \sum_{\psi\in\mathrm{Hom}_P(U^{op},Y)} \prod_{u\in[U/P]} m\big(\varphi(u),\psi(u)\big)n\big(\psi(u),\theta(u)\big)\\ &= \sum_{\psi\in\mathrm{Hom}_P(U^{op},Y)} T_U(m)(\varphi,\psi)\; T_U(n)(\psi,\theta)\,.\end{aligned}$$

This proves that $m \mapsto T_U(m)$ is multiplicative, and the lemma follows. □

(2.4) Remark. The previous lemma still holds in a more general framework: we only need to suppose that k is a commutative ring of characteristic p and that P is a finite group such that, for any $u \in U$, the stabilizer P_u of u in P is a p-group. The construction of the functor T_U is taken from [Bo2], where it is studied in a much more general framework. In fact, it is possible to define $T_U(M)$ for any kP-module M (and also for Mackey functors) but the definition is more complicated. Our presentation here is easier thanks to our use of invariant bases.

(2.5) Example: Restriction. Let Q be a subgroup of P, and set $U = P$, viewed as a Q-set-P by left and right multiplication. Then if X is a P-set, it is clear that

$$T_U(X) = \mathrm{Hom}_P(U^{op}, X) = \mathrm{Hom}_P(P, X) \cong \mathrm{Res}^P_Q X \, .$$

Similarly, if m is a morphism in $\underline{Perm_k}(P)$ from X to Y, then since $U/P = \{1\}$ we have

$$T_U(m)(\varphi, \psi) = m\big(\varphi(1), \psi(1)\big) \, .$$

In other words, the functor T_U is just the restriction functor in this case.

(2.6) Example: Tensor induction. In the same situation, set $V = P$, viewed as a P-set-Q by left and right multiplication. Then the functor T_V is a functor from $\underline{Perm_k}(Q)$ to $\underline{Perm_k}(P)$. Let S be a set of coset representatives of P/Q. If X is any Q-set, there is a bijection

$$T_V(X) = \mathrm{Hom}_Q(V^{op}, X) = \mathrm{Hom}_Q(P, X) \cong \prod_{s \in S} X \, ,$$

mapping $\varphi \in \mathrm{Hom}_Q(P, X)$ to the sequence $\big(\varphi(s)\big)_{s\in S}$. Conversely, if $(x_s)_{s\in S}$ is a sequence in $\prod_{s\in S} X$, then we can define a morphism of Q-sets φ from V to X by

$$\varphi(sh) = h^{-1}x_s \, , \qquad \forall s \in S, \, \forall h \in Q \, .$$

With this identification, the action of $g \in P$ on $\prod_{s\in S} X$ is given as follows. There is a permutation τ of S and $h_s \in Q$ such that $gs = \tau(s)h_s$ for all $s \in S$. Then we have $g^{-1}s = \tau^{-1}(s)h^{-1}_{\tau^{-1}(s)}$ and it follows that the action of g on the sequence $(x_s)_{s\in S}$ produces the sequence $(h_{\tau^{-1}(s)}x_{\tau^{-1}(s)})_{s\in S}$.

Now the module $k\left(\prod_{s\in S} X\right)$ is isomorphic to the n-th tensor power of kX, where $n = |S|$, and the previous description of the action of P on $\mathrm{Ten}^P_Q(kX)$ shows that

$$k\,T_V(X) \cong k\,\big(\prod_{s\in S} X\big) \cong \mathrm{Ten}^P_Q(kX)$$

as kP-modules. Moreover, if m is a morphism in $\underline{Perm_k}(P)$ from X to Y, then for $\varphi \in \mathrm{Hom}_Q(V^{op}, X)$ and $\psi \in \mathrm{Hom}_Q(V^{op}, Y)$ we have

$$T_V(m)(\psi, \varphi) = \prod_{s\in S} m\big(\psi(s), \varphi(s)\big) \, .$$

Identifying $k\,T_V(X)$ with $\mathrm{Ten}^P_Q(kX)$, this shows that the morphism $T_V(m)$ is given by

$$T_V(m)(x_{s_1}\otimes\ldots\otimes x_{s_n}) = \sum_{t_1,\ldots,t_n\in S} \big(\prod_{i=1}^{n} m(y_{t_i}, x_{s_i})\big)\, t_1\otimes\ldots\otimes t_n = \big(\sum_{t_1\in S} m(t_1, s_1)t_1\big)\otimes\ldots\otimes\big(\sum_{t_n\in S} m(t_n, s_n)t_n\big) \, .$$

In other words $T_V(m)$ is the $|S|$-th tensor power of m. So the functor T_V gives rise to the tensor induction functor (up to some obvious composition with the equivalences $e_P : \underline{Perm_k}(P) \to Perm_k(P)$ and $e_Q : \underline{Perm_k}(Q) \to Perm_k(Q)$).

(2.7) Example: Inflation. Let R be a normal subgroup of P and let $Q = P/R$. Let $U = Q$, viewed as a P-set-Q by projection and multiplication on the left, and by multiplication on the right. If X is any Q-set, then $T_U(X) = \mathrm{Hom}_Q(U^{op}, X)$ is just the set X with its structure of P-set obtained by inflation from Q to P. If m is a morphism of Q-sets from X to Y, as $U/Q = \{1\}$, we have

$$T_U(m)(\varphi, \psi) = m\big(\varphi(1), \psi(1)\big) ,$$

as in Example 2.5 above. So T_U is just the inflation functor in this case.

(2.8) Example: Deflation. In the same situation as in Example 2.7, set $V = Q$, viewed as a Q-set-P. Then if X is a P-set and if X^R denotes the set of R-fixed points, there is an isomorphism of Q-sets

$$T_V(X) = \mathrm{Hom}_P(V^{op}, X) \cong X^R ,$$

mapping $\varphi \in \mathrm{Hom}_P(V^{op}, X)$ to $\varphi(1) \in X^R$. The inverse isomorphism maps $x \in X^R$ to the morphism $\varphi \in \mathrm{Hom}_P(V^{op}, X)$ defined by $\varphi(\overline{g}) = gx$ where $\overline{g} \in Q$ is the image of $g \in P$. It follows that $k\,T_V(X)$ is isomorphic as kQ-module to the Brauer quotient $(kX)[R]$. Here again, as V/P is a single point, it is clear that this isomorphism is functorial in X.

(2.9) Example: Isomorphisms. Let $\theta : P \to Q$ be a group isomorphism. Then we can view $U = Q$ as a Q-set-P, by left multiplication, and right multiplication twisted by θ. It should be clear in that case that the functor T_U is just the functor "change of group via θ". We shall denote it by Iso_P^Q, for it will always be clear in the context which isomorphism is used.

Before introducing algebra structures on our permutation modules, we need to know the basic properties of the functors T_U.

(2.10) Proposition. *Let P and Q be finite p-groups and let U be a finite Q-set-P.*
(a) If X is the trivial P-set, then $T_U(X)$ is the trivial Q-set.
(b) If X and Y are P-sets, then there are isomorphisms

$$T_U(X \times Y) \cong T_U(X) \times T_U(Y)$$

which are natural in X and Y in the category $\underline{Perm}_k(P)$. Moreover if $Y = X$ and σ is the switch endomorphism of $X \times X$ (defined by $\sigma(x_1, x_2, x_1', x_2') = \delta_{x_1, x_2'}\delta_{x_2, x_1'}$, where δ is the Kronecker symbol), then $T_U(\sigma)$ is the switch endomorphism of $T_U(X) \times T_U(X)$.
(c) If U' is another finite Q-set-P, then there is an isomorphism of functors $T_{U \sqcup U'} \cong T_U \times T_{U'}$.

Proof. (a) If X is the trivial P-set, then $T_U(X)$ is the trivial Q-set, since there is a unique morphism of P-sets from any P-set to X.

(b) There is a canonical isomorphism of Q-sets

$$T_U(X \times Y) = \mathrm{Hom}_P(U^{op}, X \times Y) \cong \mathrm{Hom}_P(U^{op}, X) \times \mathrm{Hom}_P(U^{op}, Y) = T_U(X) \times T_U(Y) .$$

To see that it is natural in X and Y in the category $\underline{Perm}_k(P)$, suppose $m : X \to X'$ (resp. $n : Y \to Y'$) are morphisms in $\underline{Perm}_k(P)$. Then m is a P-invariant matrix indexed by $X' \times X$, and n is a P-invariant matrix indexed by $Y' \times Y$. The associated morphism from $X \times Y$ to $X' \times Y'$ is the matrix $m \times n$ indexed by $(X' \times Y') \times (X \times Y)$, given by

$$(m \times n)(x', y', x, y) = m(x', x)\, n(y', y)\,.$$

Now let $\varphi : U^{op} \to (X \times Y)$ and $\varphi' : U^{op} \to (X' \times Y')$ be morphisms of P-sets. Denote by φ_X (respectively φ_Y) the composition of φ with the projection $X \times Y \to X$ (respectively with the projection $X \times Y \to Y$), and define similarly $\varphi'_{X'}$ and $\varphi'_{Y'}$. The morphism $T_U(m \times n)$ is then defined by

$$T_U(m \times n)(\varphi', \varphi) = \prod_{u \in [U/P]} m\big(\varphi'_{X'}(u), \varphi_X(u)\big)\; n\big(\varphi'_{Y'}(u), \varphi_Y(u)\big)\,.$$

This can also be written as

$$T_U(m \times n)(\varphi', \varphi) = \Big(\prod_{u \in [U/P]} m\big(\varphi'_{X'}(u), \varphi_X(u)\big)\Big)\Big(\prod_{u \in [U/P]} n\big(\varphi'_{Y'}(u), \varphi_Y(u)\big)\Big)\,,$$

so that

$$T_U(m \times n)(\varphi', \varphi) = T_U(m)(\varphi'_{X'}, \varphi_X)\; T_U(n)(\varphi'_{Y'}, \varphi_Y)\,,$$

which is equivalent to $T_U(m \times n) = T_U(m) \times T_U(n)$.

Now when $Y = X$, the switch morphism σ is defined by

$$\sigma(x_1, x_2, x'_1, x'_2) = \delta_{x_1, x'_2}\delta_{x'_1, x_2}$$

where δ is the Kronecker symbol. So if $\varphi = (\varphi_1, \varphi_2)$ and $\varphi' = (\varphi'_1, \varphi'_2)$ are morphisms from U^{op} to $X \times X$, then

$$T_U(\sigma)(\varphi, \varphi') = \prod_{u \in [U/P]} \delta_{\varphi_1(u), \varphi'_2(u)}\delta_{\varphi'_1(u), \varphi_2(u)}\,.$$

This is non-zero if and only if $\varphi_1 = \varphi'_2$ and $\varphi'_1 = \varphi_2$. Therefore $T_U(\sigma)$ is the switch endomorphism of $T_U(X) \times T_U(X)$.

(c) If X is a P-set, there is a canonical isomorphism of Q-sets

$$T_{U \sqcup U'}(X) = \mathrm{Hom}_P\big((U \sqcup U')^{op}, X\big) \cong \mathrm{Hom}_P(U^{op}, X) \times \mathrm{Hom}_P(U'^{op}, X) = T_U(X) \times T_{U'}(X)\,,$$

mapping a morphism $\varphi : (U \sqcup U')^{op} \to X$ to the pair $(\varphi_{|U}, \varphi_{|U'})$. Moreover, if m is a morphism from Y to X , then $T_{U \sqcup U'}(m)$ is defined by

$$T_{U \sqcup U'}(m)(\varphi, \psi) = \prod_{u \in \left[(U \sqcup U')/P\right]} m\big(\varphi(u), \psi(u)\big)$$

where $\varphi \in \mathrm{Hom}_P\big((U \sqcup U')^{op}, X\big)$ and $\psi \in \mathrm{Hom}_P\big((U \sqcup U')^{op}, Y\big)$. This product can also be written as

$$\begin{aligned} T_{U \sqcup U'}(m)(\varphi, \psi) &= \left(\prod_{u \in [U/P]} m\big(\varphi_{|U}(u), \psi_{|U}(u)\big)\right) \left(\prod_{u' \in [U'/P]} m\big(\varphi_{|U'}(u'), \psi_{|U'}(u')\big)\right) \\ &= T_U(m)(\varphi_{|U}, \psi_{|U})\; T_{U'}(m)(\varphi_{|U'}, \psi_{|U'})\,. \end{aligned}$$

This proves that $T_{U \sqcup U'}(m) = T_U(m) \times T_{U'}(m)$. □

Now we introduce algebra structures on our permutation modules. Suppose we are given a *permutation P-algebra* A over k, that is, a P-algebra with a P-invariant k-basis X. Equivalently, A is a permutation kP-module together with a multiplication map and a unit map

$$\mu : A \otimes_k A \to A \qquad \text{and} \qquad \varepsilon : k \to A\,,$$

satisfying conditions of associativity and unitarity, which can be expressed in terms of commutative diagrams in the category $Perm_k(P)$ as follows:

$$\begin{array}{ccc} A \otimes_k A \otimes_k A & \xrightarrow{id\otimes\mu} & A \otimes_k A \\ {\scriptstyle \mu\otimes id}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes_k A & \xrightarrow[\mu]{} & A \end{array} \qquad \begin{array}{ccccc} k \otimes_k A & \xrightarrow{\varepsilon\otimes id} & A \otimes_k A & \xleftarrow{id\otimes\varepsilon} & A \otimes_k k \\ & {\scriptstyle \varepsilon}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \varepsilon} & \\ & & A & & \end{array}$$

Using the basis X, we have morphisms $\underline{\mu} : X \times X \to X$ and $\underline{\varepsilon} : \bullet \to X$ in $\underline{Perm_k}(P)$ and the following diagrams are commutative:

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{id\times\underline{\mu}} & X \times X \\ {\scriptstyle \underline{\mu}\times id}\downarrow & & \downarrow{\scriptstyle \underline{\mu}} \\ X \times X & \xrightarrow[\underline{\mu}]{} & X \end{array} \qquad \begin{array}{ccccc} \bullet \times X & \xrightarrow{\underline{\varepsilon}\times id} & X \times X & \xleftarrow{id\times\underline{\varepsilon}} & X \times \bullet \\ & {\scriptstyle \underline{\varepsilon}}\searrow & \downarrow{\scriptstyle \underline{\mu}} & \swarrow{\scriptstyle \underline{\varepsilon}} & \\ & & X & & \end{array}$$

We can take the images of $\underline{\mu}$ and $\underline{\varepsilon}$ by the functor T_U. Since T_U maps the trivial set to the trivial set and since it commutes with direct products, we obtain maps

$$T_U(\underline{\mu}) : T_U(X) \times T_U(X) \to T_U(X) \qquad \text{and} \qquad T_U(\underline{\varepsilon}) : \bullet \to T_U(X)\,,$$

which define a multiplication map and a unit map on $k\,T_U(X)$. Taking the images of the previous diagrams by the functor T_U, we see that $k\,T_U(X)$ is a k-algebra. Moreover, since $k\,T_U(X)$ is a permutation kQ-module, it is a permutation Q-algebra, which we denote by $T_U(A)$.

The algebra $T_U(A)$ only depends on A up to isomorphism. Indeed, if A' is a permutation P-algebra isomorphic to A, with multiplication μ' and unit ε', then A' admits a P-invariant k-basis X', and there exists an isomorphism $\underline{m} : X \to X'$ in $\underline{Perm_k}(P)$ such that the diagrams

$$\begin{array}{ccc} X \times X & \xrightarrow{\underline{\mu}} & X \\ {\scriptstyle \underline{m}\times\underline{m}}\downarrow & & \downarrow{\scriptstyle \underline{m}} \\ X' \times X' & \xrightarrow[\underline{\mu}']{} & X' \end{array} \qquad \begin{array}{ccc} \bullet & \xrightarrow{\underline{\varepsilon}} & X \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \underline{m}} \\ \bullet & \xrightarrow[\underline{\varepsilon}']{} & X' \end{array}$$

are commutative. Taking the images of those diagrams under T_U, we see that $T_U(\underline{m})$ induces an algebra isomorphism $T_U(A) \cong T_U(A')$.

The endomorphism algebra $\mathrm{End}_k(kX)$ of a permutation kP-module kX is a special case of a permutation P-algebra. In terms of the basis X, $\mathrm{End}_k(kX)$ has basis $X \times X$ where (x, y) is the endomorphism of kX mapping $z \in X$ to $\delta_{y,z} \cdot x$. This is the standard basis of a matrix algebra. Moreover the algebra structure of $\mathrm{End}_k(kX)$ is given by $(x, y)(z, t) = \delta_{y,z} \cdot (x, t)$.

(2.11) Lemma. *Let X be a P-set and let U be a finite Q-set-P. Then the Q-algebra $T_U\big(\mathrm{End}_k(kX)\big)$ is isomorphic to $\mathrm{End}_k\big(k\,T_U(X)\big)$.*

Proof. In view of the above observations, $T_U\big(\mathrm{End}_k(kX)\big)$ has basis $T_U(X\times X)$ while $\mathrm{End}_k\big(k\,T_U(X)\big)$ has basis $T_U(X)\times T_U(X)$. So we have to prove that the natural isomorphism $T_U(X\times X)\cong T_U(X)\times T_U(X)$ gives rise to an algebra isomorphism of the corresponding permutation P-algebras. The multiplication in $k(X\times X)$ corresponds to the matrix m indexed by $X^4\times X^2$ given by

$$m(x_1,x_2,x_3,x_4,y_1,y_2)=\begin{cases}1 & \text{if}\quad x_2{=}x_3,\ x_1{=}y_1,\ x_4{=}y_2\,,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore the multiplication in $T_U(X\times X)$ is given by

$$\begin{aligned} m(\phi_1,\phi_2,\phi_3,\phi_4,\psi_1,\psi_2) &= \prod_{u\in[U/P]} m\big(\phi_1(u),\phi_2(u),\phi_3(u),\phi_4(u),\psi_1(u),\psi_2(u)\big)\\ &= \begin{cases}1 & \text{if}\quad \phi_2(u){=}\phi_3(u),\ \phi_1(u){=}\psi_1(u),\ \phi_4(u){=}\psi_2(u)\,,\forall u\in[U/P]\,,\\ 0 & \text{otherwise,}\end{cases}\\ &= \begin{cases}1 & \text{if}\quad \phi_2{=}\phi_3,\ \phi_1{=}\psi_1,\ \phi_4{=}\psi_2\,,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

This corresponds precisely to the multiplication in $k\big(T_U(X)\times T_U(X)\big)$. □

Note that, in the special case of tensor induction (Example 2.6), Lemma 2.11 gives a version of Lemma 2.1 (only for permutation modules).

(2.12) Proposition. *Let A and B be two Dade P-algebras over k and let U be a finite Q-set-P.*
(a) $T_U(A)$ is a Dade Q-algebra.
(b) If A and B are equivalent, then $T_U(A)$ and $T_U(B)$ are equivalent.

Proof. (a) We first observe that if we apply T_U to $\mathrm{End}_k(kX)$ where kX is a permutation kP-module, we obtain a Q-algebra $T_U\big(\mathrm{End}_k(kX)\big)$ which is again isomorphic to the endomorphism algebra of a permutation module, by Lemma 2.11.

Let X be a P-invariant basis of A. By construction, $T_U(A)$ is a permutation Q-algebra, with Q-invariant basis $T_U(X)$. Since X has a P-fixed point x by definition, $T_U(X)=\mathrm{Hom}_P(U^{op},X)$ has a Q-fixed point, namely the constant map on x.

Now A is isomorphic to a matrix algebra, so that we have $A\otimes_k A^{op}\cong\mathrm{End}_k(A)$. As the product for the opposite algebra A^{op} is obtained by composing the multiplication μ with the switch endomorphism of $A\otimes_k A$, it follows from Proposition 2.10 that $T_U(A^{op})\cong T_U(A)^{op}$ as kQ-algebras. Therefore, by Proposition 2.10 again,

$$T_U(A)\otimes_k T_U(A)^{op}\cong T_U(A\otimes_k A^{op})\cong T_U\big(\mathrm{End}_k(A)\big)\,,$$

and by the observation above, $T_U\big(\mathrm{End}_k(A)\big)$ is again isomorphic to the endomorphism algebra of a (permutation) module, hence to a matrix algebra. It follows that $T_U(A)$ is a central simple algebra over k.

Finally we have to see that $T_U(A)$ is split. This is a technical point which we shall prove in full generality in the next section (see Lemma 3.12), but we mention here that it obviously holds if k is either algebraically closed or a finite field, because the Brauer group of such a field is trivial. So $T_U(A)$ is a Dade Q-algebra.

(b) If A and B are equivalent, then

$$A\otimes_k \mathrm{End}_k(kX)\cong B\otimes_k \mathrm{End}_k(kY)$$

for some permutation modules kX and kY. Therefore, by Proposition 2.10 and Lemma 2.11, we obtain

$$T_U(A)\otimes_k \mathrm{End}_k\big(k\,T_U(X)\big)\cong T_U(B)\otimes_k \mathrm{End}_k\big(k\,T_U(Y)\big)\,,$$

and this shows that $T_U(A)$ and $T_U(B)$ are equivalent. □

(2.13) Corollary. *If U is a finite Q-set-P, the functor T_U induces a group homomorphism $D(P) \to D(Q)$, mapping the class of a Dade P-algebra A to the class of the Dade Q-algebra $T_U(A)$.*

This map will be denoted by $D(U) : D(P) \to D(Q)$. In the five examples mentioned before (Examples 2.5–2.9), we obtain the fundamental operations of restriction, tensor induction, inflation, deflation, and isomorphisms.

3. Composition

We need to know how the natural operations introduced in the previous section compose with each other. A particularly important result for the sequel describes the composition of tensor induction and deflation. But our uniform formalism allows us to prove a single result for the composition of the functors T_U, which can be specialized to various situations.

It turns out that the composition of two functors T_U cannot be expressed only in terms of other such functors. There is still another natural operation on Dade algebras and endo-permutation modules which appears, that relies on the action of field endomorphisms (Galois actions for perfect fields).

If a is an endomorphism of k, we denote by

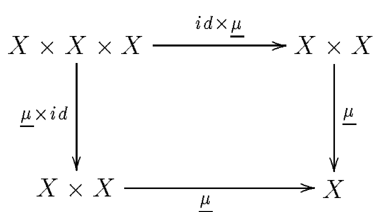
$$\gamma_a : \underline{Perm_k}(P) \longrightarrow \underline{Perm_k}(P)$$

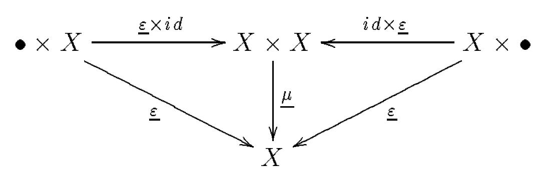
the functor which is the identity on objects and which maps the morphism $m : Y \to X$ to the morphism $\gamma_a(m) : Y \to X$ defined by

$$\gamma_a(m)(x,y) = a\big(m(x,y)\big)\,, \qquad \forall x \in X,\ \forall y \in Y\,.$$

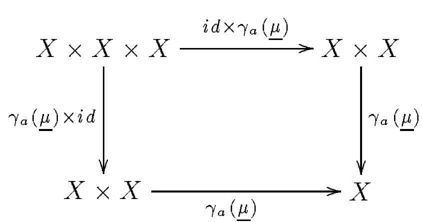
We shall be interested in the special case where $a(\lambda) = \lambda^{p^n}$ for all $\lambda \in k$ (where $n \geq 0$ is an integer), in which case we simply write γ_{p^n} for this functor. Note that this endomorphism is a (Galois) automorphism if and only if k is perfect. So in the general case we have to consider endomorphisms.

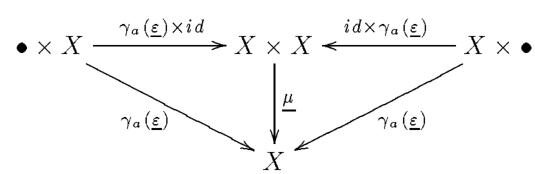
Let A be a permutation P-algebra, with multiplication μ and unit ε. Choosing a P-invariant k-basis X of A, we have morphisms $\underline{\mu} : X \times X \to X$ and $\underline{\varepsilon} : \bullet \to X$ in $\underline{Perm_k}(P)$ and commutative diagrams:





We can take the image of those diagrams by our functor γ_a. It is clear that if m and n are morphisms in $\underline{Perm_k}(P)$, then $\gamma_a(m \times n) = \gamma_a(m) \times \gamma_a(n)$, so the previous diagrams become





and this shows that we have defined another algebra structure on kX , which we will denote by $\gamma_a(A)$. This algebra structure is obtained by twisting by the endomorphism a the multiplication constants of A with respect to the fixed P-invariant k-basis X .

The algebra $\gamma_a(A)$ only depends on A up to isomorphism. Indeed, if A' is a permutation P-algebra isomorphic to A , with multiplication μ' and unit ε' , then A' admits a P-invariant k-basis X' , and there exists an isomorphism $\underline{m} : X \to X'$ in $\underline{Perm_k}(P)$ such that the diagrams

$$\begin{array}{ccc} X \times X & \xrightarrow{\underline{\mu}} & X \\ {\scriptstyle \underline{m}\times\underline{m}}\big\downarrow & & \big\downarrow{\scriptstyle \underline{m}} \\ X' \times X' & \xrightarrow[\underline{\mu}']{} & X' \end{array} \qquad\qquad \begin{array}{ccc} \bullet & \xrightarrow{\underline{\varepsilon}} & X \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle \underline{m}} \\ \bullet & \xrightarrow[\underline{\varepsilon}']{} & X' \end{array}$$

are commutative. The images of those diagrams under γ_a show that $\gamma_a(\underline{m})$ is an isomorphism of P-algebras from $\gamma_a(A)$ to $\gamma_a(A')$. It is also clear that if A and B are permutation P-algebras, then

$$\gamma_a(A \otimes_k B) \cong \gamma_a(A) \otimes \gamma_a(B) \qquad \text{and} \qquad \gamma_a(A^{op}) \cong \gamma_a(A)^{op} \, .$$

(3.1) Lemma. *Let a be an endomorphism of the field k and let A and B be two Dade P-algebras over k .*

(a) $\gamma_a(A)$ is a Dade P-algebra.

(b) If A and B are equivalent, then $\gamma_a(A)$ and $\gamma_a(B)$ are equivalent.

Proof. (a) As $\gamma_a(A)$ has the same P-invariant basis as A , there is a fixed point in this basis. In order to show that $\gamma_a(A)$ is a matrix algebra, it suffices to restrict the group action to the trivial group (in which case we have many more bases available). Now A is isomorphic to $M_n(k)$ for some n and so $\gamma_a(A) \cong \gamma_a\big(M_n(k)\big)$ by the observations above. Choosing the canonical basis of the matrix algebra $M_n(k)$, the multiplication constants have values 0 or 1 and so are fixed under a . It follows that $\gamma_a\big(M_n(k)\big) = M_n(k)$.

(b) If A and B are equivalent, then $A \otimes_k \mathrm{End}_k(kX) \cong B \otimes_k \mathrm{End}_k(kY)$ for some permutation modules kX and kY . Since γ_a preserves tensor products, it suffices to show that $\gamma_a\big(\mathrm{End}_k(kX)\big)$ is again the endomorphism algebra of a permutation module. But since $\mathrm{End}_k(kX)$ has P-invariant basis $X \times X$ with multiplication constants having values 0 or 1, it is fixed under γ_a . □

It follows that γ_a induces a well defined map $\gamma_a : D(P) \longrightarrow D(P)$, mapping the class of a Dade P-algebra A to the class of $\gamma_a(A)$.

(3.2) Lemma. *Let P and Q be p-groups, let U be a Q-set-P, and let a be an endomorphism of k.*
(a) There is an isomorphism of functors $\theta : T_U \circ \gamma_a \to \gamma_a \circ T_U$ from $\underline{Perm}_k(P)$ to $\underline{Perm}_k(Q)$.
(b) We have $D(U) \circ \gamma_a = \gamma_a \circ D(U)$ as maps $D(P) \to D(Q)$.

Proof. Note that we use the same notation $\gamma_a : \underline{Perm}_k(P) \to \underline{Perm}_k(P)$ for every p-group P, without mentioning P. It is clear that $T_U \circ \gamma_a$ and $\gamma_a \circ T_U$ coincide on objects since γ_a is the identity on objects. Thus the map θ_X is equal to the identity for any P-set X. Now if $m : Y \to X$ is a morphism in $\underline{Perm}_k(P)$, we have

$$\begin{aligned}(T_U \circ \gamma_a)(m)(\varphi, \psi) &= \prod_{u \in [U/P]} \gamma_a(m)\big(\varphi(u), \psi(u)\big) = \prod_{u \in [U/P]} a\Big(m\big(\varphi(u), \psi(u)\big)\Big) \\ &= a\left(\prod_{u \in [U/P]} m\big(\varphi(u), \psi(u)\right) = (\gamma_a \circ T_U)(m)(\varphi, \psi)\,,\end{aligned}$$

and (a) follows. Now if X and Y are P-sets, the diagram

$$\begin{array}{ccc} (T_U \circ \gamma_a)(X \times Y) & \xrightarrow{\ \simeq\ } & (T_U \circ \gamma_a)(X) \times (T_U \circ \gamma_a)(Y) \\ {\scriptstyle \theta_{X \times Y}} \Big\downarrow & & \Big\downarrow {\scriptstyle \theta_X \times \theta_Y} \\ (\gamma_a \circ T_U)(X \times Y) & \xrightarrow[\ \simeq\ ]{} & (\gamma_a \circ T_U)(X) \times (\gamma_a \circ T_U)(Y) \end{array}$$

is commutative, since the vertical arrows are both the identity maps, and the horizontal ones are the same map. Applying this to the multiplication map for a permutation P-algebra A, it follows that the Q-algebras $(T_U \circ \gamma_a)(A)$ and $(\gamma_a \circ T_U)(A)$ are isomorphic, proving (b). □

(3.3) Example. For every endomorphism a of k, we have $\gamma_a(\Omega_P) = \Omega_P$. Indeed Ω_P is defined over the prime field $\mathbb{F}_p$ and a is the identity on $\mathbb{F}_p$. It follows that if P is abelian, then $\gamma_a : D(P) \to D(P)$ is the identity because, by Dade's Theorem 1.4, $D(P)$ is generated by elements of the form $\Omega_{P/Q}$. By the injectivity of the map in Theorem 1.6, this implies that γ_a is always the identity on $\mathbb{Q}D(P)$, for every p-group P, because γ_a commutes with restriction and deflation by Lemma 3.2.

(3.4) Example. The only example we know of an endo-permutation module which is not defined over the prime field $\mathbb{F}_p$ occurs for $p = 2$ and the quaternion group Q_8. By a construction of Dade [Da1], there is a 3-dimensional endo-trivial $\mathbb{F}_4 Q_8$-module M, which has order 4 in the Dade group. If a denotes the non-trivial Galois automorphism of $\mathbb{F}_4$, it turns out that $\gamma_a(M) \cong M^*$ and this shows that γ_a is not the identity of the Dade group.

Despite the fact that γ_a is often the identity, we are forced to introduce it in general in order to describe the composition of the functors T_U. Suppose that P, Q, and R are p-groups. If U is a finite Q-set-P and V is a finite R-set-Q, we denote by $V \times_Q U$ the quotient of $V \times U$ by the right Q-action defined by

$$(v, u) \cdot h = (vh, h^{-1}u)\,, \qquad \forall v \in V,\ \forall u \in U,\ \forall h \in Q\,.$$

In other words $V \times_Q U = (V \times U^{op})/Q$. It is an R-set-P for the following double action:

$$l\cdot(v,u)\cdot g = (lv, ug)\,, \qquad \forall v \in V,\ \forall u \in U,\ \forall l \in R,\ \forall g \in P\,.$$

If $(u,v) \in V \times_Q U$, let Q_v denote the right stabilizer of v in Q and $Q_{v,uP}$ the stabilizer of the pair (v, uP), that is,

$$Q_{v,uP} = \{h \in Q \mid vh = v,\ \exists g \in P \text{ such that } hu = ug\}\,.$$

Now we can describe the composition of the functors T_V and T_U.

(3.5) Proposition. *Let U be a finite Q-set-P and let V be a finite R-set-Q, where P, Q, and R are finite p-groups.*

(a) If X is a finite P-set, there is a canonical isomorphism of R-sets

$$(T_V \circ T_U)(X) = \mathrm{Hom}_Q\big(V^{op}, \mathrm{Hom}_P(U^{op}, X)\big) \cong \mathrm{Hom}_P(U^{op} \times_Q V^{op}, X) = T_{V\times_Q U}(X)\,.$$

In other words the functors $T_V \circ T_U$ and $T_{V\times_Q U}$ are isomorphic on objects of $\underline{Perm}_k(P)$.

(b) If m is a morphism in $\underline{Perm}_k(P)$ from X to Y, then

$$T_{V\times_Q U}(m)(\psi,\varphi) = \prod_{(v,u)\in[(V\times_Q U)/P]} m\big(\psi(v,u), \varphi(v,u)\big)\,,$$

$$(T_V \circ T_U)(m)(\psi,\varphi) = \prod_{(v,u)\in[(V\times_Q U)/P]} \Big(m\big(\psi(v,u), \varphi(v,u)\big)\Big)^{|Q_v : Q_{v,uP}|}\,,$$

for all $\psi \in \mathrm{Hom}_P\big((V\times_Q U)^{op}, Y\big)$ and $\varphi \in \mathrm{Hom}_P\big((V\times_Q U)^{op}, X\big)$.

Proof. Statement (a) is straightforward to show, so we prove (b). The expression for $T_{V\times_Q U}(m)(\psi,\varphi)$ is just the definition. On the other hand, using the isomorphism of part (a), the matrix $(T_V \circ T_U)(m)$ can be expressed as

$$(T_V \circ T_U)(m)(\psi,\varphi) = \prod_{v\in[V/Q]} \prod_{u\in[U/P]} m\big(\psi(v,u), \varphi(v,u)\big)\,, \tag{3.6}$$

where we have fixed sets of representatives $[U/P]$ and $[V/Q]$ of U/P and V/Q respectively. In general, the sets $(V\times_Q U)/P$ and $V/Q \times U/P$ are not isomorphic. So the functors $T_{V\times_Q U}$ and $T_V \circ T_U$ are not isomorphic on morphisms. To see how much they differ, consider the map

$$\theta : [V/Q] \times [U/P] \to (V \times_Q U)/P\,, \qquad \theta(v_0, u_0) = (v_0, u_0)P\,.$$

This map is surjective, since if $(v,u) \in V \times_Q U$, there exist $v_0 \in [V/Q]$ (unique) and $h \in Q$ such that $v = v_0 h$. Now in $V \times_Q U$, we have $(v,u) = (v_0, hu)$, and there exist $u_0 \in [U/P]$ (unique) and $g \in P$ such that $hu = u_0 g$. Then we clearly have

$$\theta(v_0,u_0) = (v_0,u_0)P = (v_0, u_0 g)P = (v_0, hu)P = (v_0 h, u)P = (v,u)P\,.$$

Now two pairs (v_0,u_0) and (v_1,u_1) have the same image under θ if and only if there exist $h \in Q$ and $g \in P$ such that

$$v_1 = v_0 h \qquad \text{and} \qquad h^{-1}u_1 = u_0 g\,.$$

The first equality gives $v_1 = v_0 = v_0 h$, since v_1 and v_0 are in the set of representatives $[V/Q]$. The second one gives $u_1 = hu_0 g$. In other words $u_1 P$ is in the left orbit of $u_0 P$ under the right stabilizer Q_{v_0} of v_0. So the inverse image $\theta^{-1}\big((v,u)P\big)$ has cardinality $|Q_v : Q_{v,uP}|$ and we can rewrite Equation (3.6) as

$$(T_V \circ T_U)(m)(\psi,\varphi) = \prod_{(v,u)\in[(V\times_Q U)/P]} \Big(m\big(\psi(v,u), \varphi(v,u)\big)\Big)^{|Q_v : Q_{v,uP}|}\,,$$

as was to be shown. □

(3.7) Corollary. *Let U be a finite Q-set-P and let V be a finite R-set-Q, where P, Q, and R are finite p-groups.*

(a) If the right action of Q on V is free, then the functors $T_V \circ T_U$ and $T_{V\times_H U}$ are isomorphic.

(b) If the right action of P on U is transitive, then the functors $T_V \circ T_U$ and $T_{V\times_H U}$ are isomorphic.

Proof. For (a), we have $Q_v = 1$ for every $v \in V$, while for (b), U/P is a singleton, so that $Q_v = Q_{v,uP}$. So in both cases $[Q_v : Q_{v,uP}] = 1$ and Proposition 3.5 shows that $(T_V \circ T_U)(m) = T_{V\times_Q U}(m)$ for every morphism m. □

The assumption in (a) is satisfied if T_V is the restriction Res^Q_R where $R \le Q$ (because $V = Q$ as an R-set-Q), if T_V is the tensor induction Ten^R_Q where $Q \le R$ (because $V = R$ as an R-set-Q), if T_V is the inflation Inf^R_Q where $Q = R/S$ (because $V = Q$ as an R-set-Q), and also if T_V is an isomorphism Iso^R_Q. Similarly, the assumption in (b) is satisfied if T_U is a restriction, an inflation, a deflation, or an isomorphism.

Many standard results follow from this, including transitivity properties and the Mackey formula. We illustrate the mechanism with the latter formula. Let R and S be subgroups of Q. Let ${}_RQ_Q = Q$ viewed as an R-set-Q (restriction) and similarly ${}_QQ_S = Q$ viewed as a Q-set-S (tensor induction). Making the product over Q, we obtain the R-set-S

$$ {}_RQ_Q \times_Q {}_QQ_S = {}_RQ_S = \coprod_{u\in[R\backslash Q/S]} RuS \cong \coprod_{u\in[R\backslash Q/S]} R \times_{R\cap {}^uS} {}^uS \times_{({}^uS)} S \,, $$

from which it follows that

$$ \mathrm{Res}^Q_R \circ \mathrm{Ten}^Q_S = T_{{}_RQ_Q} \circ T_{{}_QQ_S} = T_{{}_RQ_Q\times_Q {}_QQ_S} = \prod_{u\in[R\backslash Q/S]} \mathrm{Ten}^R_{R\cap {}^uS} \circ \mathrm{Res}^{{}^uS}_{R\cap {}^uS} \circ \mathrm{Iso}^{{}^uS}_S \,, $$

which is the Mackey formula. The product here is the product in $\underline{Perm_k}(R)$, which corresponds to the tensor product of permutation modules.

All the results above have been expressed in terms of the functors T_V and T_U. When these functors are applied to a permutation P-algebra A, we obtain corresponding isomorphisms of R-algebras, as follows.

(3.8) Corollary. *Let U be a finite Q-set-P, let V be a finite R-set-Q, where P, Q, and R are finite p-groups, and let A be a permutation P-algebra. If either the right action of Q on V is free, or the right action of P on U is transitive, then the R-algebras $(T_V \circ T_U)(A)$ and $T_{V\times_H U}(A)$ are isomorphic. In particular, $D(V) \circ D(U) = D(V \times_H U)$ as maps $D(P) \to D(R)$.*

As explained above, this applies whenever $D(V)$ is either a restriction, a tensor induction, an inflation, or an isomorphism, or when $D(U)$ is a restriction, an inflation, a deflation, or an isomorphism. The only remaining case is when $D(V)$ is a deflation and $D(U)$ is a tensor induction. Indeed, if U is an arbitrary transitive Q-set-P, then U decomposes as a product of those five special types of Q-sets-P (see Lemma 7.4).

So finally we come to the formula which describes the last case and which will play a crucial role in the next sections. The result holds when $\mathrm{Def}^P_{P/R} \circ \mathrm{Ten}^P_Q$ is applied to any permutation Q-algebra, but for simplicity we only state the result for the corresponding maps between the Dade groups.

(3.9) Proposition. *Let P be a p-group, let Q be a subgroup of P, and let R be a normal subgroup of P. Then*

$$\mathrm{Def}^P_{P/R} \circ \mathrm{Ten}^P_Q = \gamma_{|R:Q\cap R|} \circ \mathrm{Ten}^{P/R}_{QR/R} \circ \mathrm{Iso}^{QR/R}_{Q/Q\cap R} \circ \mathrm{Def}^Q_{Q/Q\cap R} \,,$$

as maps from $D(Q)$ to $D(P/R)$.

Proof. Let $V = P/R$, viewed as a (P/R)-set-P, let $U = P$, viewed as a P-set-Q, and let A be a Dade Q-algebra with a P-invariant basis X. Then by Examples 2.6 and 2.8, we have

$$\mathrm{Def}^P_{P/R}\,\mathrm{Ten}^P_Q(A) = (T_V \circ T_U)(A) \,.$$

Now $V \times_P U$ is the set P/R, viewed as a (P/R)-set-Q. The stabilizer in P of every $v \in V$ is $P_v = R$ (because R is a normal subgroup), and if $u \in U$, then

$$P_{v,uQ} = \{g \in P_v \mid \exists h \in Q,\ gu = uh\} = {}^uQ \cap R \,.$$

Since R is normal in P, we have $|R : {}^uQ \cap R| = |R : Q \cap R|$ and this shows that the index appearing as an exponent in Proposition 3.5 is constant. Therefore, if we apply Proposition 3.5 to any morphism $m : Z \to Y$ in $\underline{Perm}_k(Q)$, we obtain

$$T_{V\times_P U}(m)(\psi,\varphi) = \prod_{(v,u)\in[(V\times_P U)/Q]} m\big(\psi(v,u),\varphi(v,u)\big) \,,$$

$$\text{and} \qquad (T_V \circ T_U)(m)(\psi,\varphi) = \prod_{(v,u)\in[(V\times_P U)/Q]} \Big(m\big(\psi(v,u),\varphi(v,u)\big)\Big)^{|R:\,{}^uQ\cap R|}$$

$$= \gamma_{|R:Q\cap R|}\left(\prod_{(v,u)\in[(V\times_P U)/Q]} m\big(\psi(v,u),\varphi(v,u)\big)\right) \,,$$

for all $\psi \in \mathrm{Hom}_P\big((V\times_Q U)^{op}, Y\big)$ and $\varphi \in \mathrm{Hom}_P\big((V\times_Q U)^{op}, Z\big)$. Applying this to the morphism $m : X \times X \to X$ which represents the multiplication in A, we see that the multiplication in $(T_V \circ T_U)(A)$ is obtained by applying $\gamma_{|R:Q\cap R|}$ to the multiplication constants in $T_{V\times_P U}(A)$. In other words, this shows that

$$\mathrm{Def}^P_{P/R}\,\mathrm{Ten}^P_Q(A) \cong (T_V \circ T_U)(A) \cong \gamma_{|R:Q\cap R|}\big(T_{V\times_P U}(A)\big) \,.$$

Now let

$Z = P/R$, viewed as a (P/R)-set-(QR/R) (tensor induction),

$I = QR/R$, viewed as a (QR/R)-set-$(Q/Q\cap R)$ (isomorphism),

$W = Q/Q\cap R$, viewed as a $(Q/Q\cap R)$-set-Q (deflation).

Then we have

$$V \times_P U = Z \times_{QR/R} I \times_{Q/Q\cap R} W \,.$$

As the right action of $Q/Q\cap R$ on I and the right action of QR/R on Z are free, Corollary 3.8 implies that this product can be replaced by a composition. So we obtain

$$T_{V\times_P U}(A) \cong (T_Z \circ T_I \circ T_W)(A) = \mathrm{Ten}^{P/R}_{QR/R}\,\mathrm{Iso}^{QR/R}_{Q/Q\cap R}\,\mathrm{Def}^Q_{Q/Q\cap R}(A) \,,$$

and the proposition follows. □

(3.10) Remark. When applied to a permutation P-algebra A, it is possible to express the general result of Proposition 3.5 using the twists γ_{p^n}, thus obtaining a single result including Corollary 3.8 and Proposition 3.9 as special cases. It suffices to notice that the index $|Q_v : Q_{v,uP}|$ only depends on the orbit of $(v,u) \in V \times_Q U$ under the action of $R \times P^{op}$. Therefore $(T_V \circ T_U)(A)$ decomposes as

$$(T_V \circ T_U)(A) = \bigotimes_{(v,u)\in[R\backslash(V\times_Q U)/P]} \gamma_{|Q_v:Q_{v,uP}|}(A_{v,u}) \, ,$$

for suitable R-algebras $A_{v,u}$, while

$$(T_{V\times_Q U})(A) = \bigotimes_{(v,u)\in[R\backslash(V\times_Q U)/P]} A_{v,u} \, ,$$

without any twists.

As mentioned in Example 3.3, the twist $\gamma_{|R:Q\cap R|}$ is the identity if we apply it to the Dade group of an abelian group. Therefore, Proposition 3.9 can be simplified in the following way.

(3.11) Corollary. *Let P be a p-group, let Q be a subgroup of P, and let R be a normal subgroup of P.*
(a) If Q_0 is a normal subgroup of Q such that Q/Q_0 is abelian, then

$$\mathrm{Def}^P_{P/R} \circ \mathrm{Ten}^P_Q \circ \mathrm{Inf}^Q_{Q/Q_0} = \mathrm{Ten}^{P/R}_{QR/R} \circ \mathrm{Iso}^{QR/R}_{Q/Q\cap R} \circ \mathrm{Def}^Q_{Q/Q\cap R} \circ \mathrm{Inf}^Q_{Q/Q_0} \, ,$$

as maps from $D(Q/Q_0)$ to $D(P/R)$.
(b) If Q is abelian, then

$$\mathrm{Def}^P_{P/R} \circ \mathrm{Ten}^P_Q = \mathrm{Ten}^{P/R}_{QR/R} \circ \mathrm{Iso}^{QR/R}_{Q/Q\cap R} \circ \mathrm{Def}^Q_{Q/Q\cap R} \, ,$$

as maps from $D(Q)$ to $D(P/R)$.

Proof. Clearly (b) follows from (a) by taking $Q_0 = 1$. For (a) we apply Proposition 3.9 and compose everything with Inf^Q_{Q/Q_0}. Since $\gamma_{|R:Q\cap R|}$ commutes with tensor induction, isomorphisms, deflation, and inflation (Lemma 3.2) and since it is the identity of $D(Q/Q_0)$ because Q/Q_0 is abelian, the result follows. □

We close this section with the result which was needed in the proof of Proposition 2.12.

(3.12) Lemma. *Let U be a finite Q-set-P, where P and Q are finite p-groups, and let A be a Dade P-algebra. Then the algebra $T_U(A)$ is split (hence is a Dade Q-algebra).*

Proof. It suffices to restrict to the trivial group, so we compose T_U with $\mathrm{Res}^Q_1 = T_V$ where $V = Q$, viewed as a 1-set-Q. Since $(T_V \circ T_U)(A) \cong T_{V\times_Q U}(A)$ by Corollary 3.8, we have reduced to the case of a 1-set-P. We can also assume that this set is transitive, since a disjoint union corresponds to a tensor product and a tensor product of split algebras is split. So U has the form $R\backslash P$ for some subgroup R of P and therefore T_U corresponds to the deflation $\mathrm{Def}^P_{P/R}$ (followed by $\mathrm{Res}^{P/R}_1$ since we only consider the 1-action on the left).

So we have to prove that the Brauer quotient $\mathrm{Def}^P_{P/R}(A) = A[R]$ is split. Recall from the proof of Proposition 2.12 that we already know that $A[R]$ is a central simple k-algebra. Since we only need $\mathrm{Res}^P_R(A)$

in the definition of $A[R]$ (and since the restriction of a split algebra is split!), we can assume that $R = P$. We proceed by induction on $|P|$.

If P is cyclic of order p, then up to equivalence of Dade P-algebras, we have either $A \cong k$ or $A \cong \mathrm{End}_k(\Omega_P(k))$, in which case the result is easy. Moreover the equivalence relation preserves the splitting property. Alternatively, every module is defined over $\mathbb{F}_p$ and indecomposable modules are absolutely indecomposable, so we can work over an algebraically closed field where all central simple algebras are split.

If now $|P| > p$, we choose a normal subgroup S of index p and we recall that $A[P] \cong A[S][P/S]$. By induction, we know that $A[S]$ is split, hence is a Dade (P/S)-algebra. By the cyclic case, we conclude that $A[S][P/S]$ is split. □

4. The torsion-free rank of the Dade group

The purpose of this section is to prove Theorem A of the introduction. More precisely, we shall prove the following result.

(4.1) Theorem. *Let P be a finite p-group. Let $\mathcal{S}$ be the set of non-cyclic subgroups of P and for every $Q \in \mathcal{S}$, choose a normal subgroup $Q_0 \lhd Q$ such that Q/Q_0 is elementary abelian of rank 2. Then the $\mathbb{Q}$-linear map*

$$\psi_P = \prod_{Q \in [\mathcal{S}/P]} \mathrm{Defres}^P_{Q/Q_0} : \mathbb{Q}D(P) \longrightarrow \prod_{Q \in [\mathcal{S}/P]} \mathbb{Q}D(Q/Q_0) \cong \prod_{Q \in [\mathcal{S}/P]} \mathbb{Q}$$

is an isomorphism, where $[\mathcal{S}/P]$ denotes a set of representatives of the conjugacy classes of subgroups in $\mathcal{S}$. In particular, the torsion-free rank of $D(P)$ is equal to the number of conjugacy classes of non-cyclic subgroups of P.

If E is elementary abelian of rank 2, then by Theorem 1.4 there is an isomorphism $\mathbb{Q}D(E) \xrightarrow{\sim} \mathbb{Q}$ mapping Ω_E to 1 (and this explains the right hand side isomorphism in the statement). This isomorphism is canonical since Ω_E is invariant under the group of automorphisms of E. Throughout this section, we shall always identify $\mathbb{Q}D(E)$ with $\mathbb{Q}$ via this isomorphism. This identification is particularly useful for the statement of the following crucial lemma.

(4.2) Lemma. *Let P be an elementary abelian group of rank 3 and let L be a subgroup of P of order p. Then the map*

$$\chi = p\,\mathrm{Def}^P_{P/L} - \sum_{\substack{E \not\geq L \\ \mathrm{rk}(E)=2}} \mathrm{Res}^P_E \,,$$

viewed as a homomorphism $\mathbb{Q}D(P) \to \mathbb{Q}$, is independent of L.

Proof. By Theorem 1.4, $\{\Omega_{P/C} \mid |C| = p\} \cup \{\Omega_P\}$ is a basis of $\mathbb{Q}D(P)$. We compute χ on each basis element. By Lemma 1.3, we have

$$\chi(\Omega_P) = -\sum_{\substack{E \not\geq L \\ \mathrm{rk}(E)=2}} 1 = -p^2 \,,$$

and this is independent of L. If C is a subgroup of order p, we have

$$\mathrm{Def}^P_{P/L}(\Omega_{P/C}) = \begin{cases} 1 & \text{if } C = L\,, \\ 0 & \text{if } C \neq L\,, \end{cases}$$

because in the second case, the Brauer quotient corresponding to L is equal to the Brauer quotient corresponding to CL since C acts trivially, and $\mathrm{Def}^{P/C}_{P/CL}(\Omega_{P/C})=0$ by Lemma 1.3. On the other hand, if E is a subgroup of rank 2,

$$\mathrm{Res}^P_E(\Omega_{P/C})=\begin{cases}1 & \text{if } E\not> C\,,\\ 0 & \text{if } E>C\,,\end{cases}$$

because in the first case, the action of E coincides with the action of $EC/C=P/C$, while in the second we have $\mathrm{Res}^{P/C}_{E/C}(\Omega_{P/C})=\Omega_{E/C}$ which is zero since E/C has order p and $\mathbb{Q}D(E/C)=0$ (in fact $\Omega_{E/C}$ is a torsion element in $D(E/C)$). It follows that

$$\text{if } C=L\,, \text{then}\quad \chi(\Omega_{P/L})=p\cdot 1-\sum_{\substack{E\not>L\\ \mathrm{rk}(E)=2}}1\;=\;p-p^2\,,$$

$$\text{if } C\neq L\,, \text{then}\quad \chi(\Omega_{P/C})=0-\sum_{\substack{E\not>L,\,E\not>C\\ \mathrm{rk}(E)=2}}1\;=\;-(p^2-p)\,.$$

So $\chi(\Omega_{P/C})=p-p^2$ is independent of L. □

(4.3) Remark. Lemma 4.2 is in fact a special case of a result which holds for an arbitrary p-group P: for every normal subgroup N such that P/N is elementary abelian of rank 2, the map

$$\sum_{\substack{X\leq P\\ XN=P}}|X|\,\mu(X,P)\,\mathrm{Defres}^P_{X/(X\cap N)}:\mathbb{Q}D(P)\longrightarrow\mathbb{Q}$$

is independent of N. Here $\mu(X,P)$ denotes the Möbius function of the poset of subgroups of P. This fact will be proved in Remark 10.2. Even for an elementary abelian group of rank 4, it is a rather tedious exercise to give a direct proof using the same method as above.

(4.4) Proposition. *The homomorphism* ψ_P *is injective.*

Proof. By Theorem 1.6, we know that

$$\mathrm{Defres}:\mathbb{Q}D(P)\longrightarrow\prod_{Q/R}\mathbb{Q}D(Q/R)\cong\prod_{Q/R}\mathbb{Q}\tag{4.5}$$

is injective, where Q/R runs over all elementary abelian sections of P of rank 2. But the various maps $\mathrm{Defres}^P_{Q/R}$ are not linearly independent (as Lemma 4.2 shows). We prove the injectivity of the map ψ_P of the statement by induction on $|P|$. The result is clear if $|P|=p$ because $\mathbb{Q}D(P)=0$.

Assume $|P|\geq p^2$ and let $a\in\mathrm{Ker}(\psi_P)$. By the injectivity of the map (4.5), it suffices to show that $\mathrm{Defres}^P_{Q/R}(a)=0$ for every elementary abelian section Q/R of rank 2. For every proper subgroup T of P, we have $\mathrm{Res}^P_T(a)\in\mathrm{Ker}(\psi_T)$, provided we make a consistent choice ot the sections Q/Q_0 for the subgroup T, that is, we keep the same choice of Q_0 for every $Q\leq T$. Note also that the choice of representatives of conjugacy classes does not play any role here since $\mathrm{Defres}^P_{{}^gQ/{}^gR}=\mathrm{Conj}_g\mathrm{Defres}^P_{Q/R}$, where Conj_g is the conjugation map, so that $\mathrm{Ker}(\mathrm{Defres}^P_{{}^gQ/{}^gR})=\mathrm{Ker}(\mathrm{Defres}^P_{Q/R})$.

Since ψ_T is injective by induction, we obtain $\mathrm{Res}^P_T(a)=0$ for every proper subgroup T and consequently $\mathrm{Defres}^P_{Q/R}(a)=0$ whenever Q is a proper subgroup of P. So we are left with the proof that $\mathrm{Def}^P_{P/R}(a)=0$ for every elementary abelian quotient P/R of rank 2. We know by assumption that

$\mathrm{Def}^P_{P/P_0}(a) = 0$. The group $P/(P_0 \cap R)$ is elementary abelian (of rank at most 4) and so it is clearly possible to find a sequence of subgroups S_i ($1 \leq i \leq m$) such that

$$S_1 = P_0 \,, \quad S_m = R \,, \quad S_i \geq P_0 \cap R \,, \quad |P : S_i| = p^2 \,, \quad |P : (S_i \cap S_{i+1})| = p^3 \quad (1 \leq i \leq m-1) \,.$$

So it suffices to prove that if $\mathrm{Def}^P_{P/S_i}(a) = 0$, then $\mathrm{Def}^P_{P/S_{i+1}}(a) = 0$. Let $N = S_i \cap S_{i+1}$ so that P/N has rank 3. Applying $\mathrm{Def}^P_{P/N}$ and using Lemma 4.2 for the group P/N , we obtain

$$p \,\mathrm{Def}^P_{P/S_{i+1}} - \sum_{\substack{E \not\geq S_{i+1} \\ \mathrm{rk}(E/N)=2}} \mathrm{Defres}^P_{E/N} = p \,\mathrm{Def}^P_{P/S_i} - \sum_{\substack{E \not\geq S_i \\ \mathrm{rk}(E/N)=2}} \mathrm{Defres}^P_{E/N} \,.$$

Since we already know that $\mathrm{Defres}^P_{E/N}(a) = 0$ for every such subgroup E (because E is a proper subgroup of P), we are left with

$$p \,\mathrm{Def}^P_{P/S_{i+1}}(a) = p \,\mathrm{Def}^P_{P/S_i}(a) = 0 \,,$$

so that $\mathrm{Def}^P_{P/S_{i+1}}(a) = 0$ as required. □

We shall not prove directly the surjectivity of ψ_P . We shall construct a map α_P in the reverse direction and then show that $\psi_P \,\alpha_P$ is an isomorphism. If Q/R is a section of P , in analogy with the notation $\mathrm{Defres}^P_{Q/R}$, we define

$$\mathrm{Teninf}^P_{Q/R} = \mathrm{Ten}^P_Q \mathrm{Inf}^Q_{Q/R} \,.$$

Now for every elementary abelian section Q/N of P of rank 2, we define

$$\beta^P_{Q,N} = \frac{1}{|Q|} \sum_{\substack{X \leq Q \\ XN=Q}} |X| \,\mu(X,Q) \,\mathrm{Teninf}^P_{X/(X\cap N)} : \mathbb{Q} \longrightarrow \mathbb{Q}D(P) \,.$$

This homomorphism is dual to the one mentioned in Remark 4.3. Note that the conditions on X imply that $X/(X \cap N) \cong Q/N$ is elementary abelian of rank 2, so that $\mathbb{Q}D\big(X/(X \cap N)\big) \cong \mathbb{Q}$ and the domain of the map is indeed $\mathbb{Q}$. Note also that if we needed an integral linear combination, it would suffice to multiply $\beta^P_{Q,N}$ by p , because $p\,|X|\,\mu(X,Q)$ is always a multiple of $|Q|$. There is a close connection between the idempotents of the Burnside ring and the formula for $\beta^P_{Q,N}$ (as well as the formula in Remark 4.3 above). This will be explained in Remark 10.2 and gives a reason why such linear combinations play a crucial role.

Now we define

$$\alpha_P = \sum_{Q \in [\mathcal{S}/P]} \beta^P_{Q,Q_0} : \prod_{Q \in [\mathcal{S}/P]} \mathbb{Q} \longrightarrow \mathbb{Q}D(P) \,,$$

where $\mathcal{S}$ and Q_0 are defined as in the statement of Theorem 4.1. In order to be able to compute $\psi_P \,\alpha_P$, we shall collect properties of the maps $\beta^P_{Q,N}$, but we first need a combinatorial lemma.

We recall that, for p-groups, the Möbius function $\mu(X,P)$ vanishes unless X is a normal subgroup with P/X elementary abelian, and if P/X is elementary abelian of rank r , then $\mu(X,P) = (-1)^r p^{\binom{r}{2}}$ (see [KT, 2.4]). As in [Bo1], whenever N is a normal subgroup of Q , we define

$$m_{Q,N} = \frac{1}{|Q|} \sum_{\substack{X \leq Q \\ XN=Q}} |X| \,\mu(X,Q) \,. \tag{4.6}$$

We need the following technical property about this number.

(4.7) Lemma. *Let N be a normal subgroup of Q with Q/N elementary abelian of rank 2.*

(a) $m_{Q,N}$ is independent of N.

(b) If M is another normal subgroup of Q with Q/M elementary abelian of rank 2, then

$$m_{Q,M} = m_{Q,N} = \frac{1}{|Q|} \sum_{\substack{XM=XN=Q \\ X\cap M=X\cap N}} |X|\,\mu(X,Q) = (1-p)(1-p^2)\dots(1-p^{n-2}),$$

where n is the rank of $Q/\Phi(Q)$ (the largest elementary abelian quotient of Q).

Proof. (a) follows from (b). The expression of $m_{Q,N}$ as a sum is a special case of the formula appearing at the bottom of page 703 of [Bo1]. This uses the fact that Q/M is a b-group, which means that for every non-trivial normal subgroup U of Q/M, we have $m_{Q/M,U} = 0$ (as it is easy to check, or see example 4, page 709 of [Bo1]). It is clear from the formula that $m_{Q,M} = m_{Q,N}$.

For the expression of $m_{Q,N}$ as a product, one notices first that in (4.6) every subgroup X contains the Frattini subgroup $\Phi(Q)$ (otherwise $\mu(X,Q) = 0$), so that one can assume that Q is elementary abelian. Writing E_n for the elementary abelian group of rank n, we prove by induction on k that

$$m_{E_n,E_k} = \prod_{i=1}^{k}(1-p^{n-1-i}), \tag{4.8}$$

for $1 \le k \le n-2$. If $k=1$, we have $m_{E_n,E_1} = (1-p^{n-2})$ by a direct computation (in (4.6), either we have $X = E_n$ and $\mu(E_n,E_n) = 1$ or X has index p and does not contain E_1 in which case $\mu(X,E_n) = -1$ and there are p^{n-1} such subgroups). The induction step is obtained by using the transitivity formula $m_{G,N} = m_{G,M}\, m_{G/M,N/M}$ where $M \le N \le G$. This formula appears on page 703 of [Bo1]. Thus we have

$$m_{E_n,E_k} = m_{E_n,E_{k-1}}\, m_{E_{n-k+1},E_1} = \Big[\prod_{i=1}^{k-1}(1-p^{n-1-i})\Big](1-p^{n-k+1-2}) = \prod_{i=1}^{k}(1-p^{n-1-i}),$$

as required. Now the result follows by taking $k = n-2$. □

We recall that the notation $R \ge_P Q$ means that R contains a P-conjugate of R. We write $T(Q,R)$ for the set of all $g \in P$ such that ${}^gQ \le R$.

(4.9) Proposition. *Let Q be a subgroup of a p-group P and let N be a normal subgroup of Q with Q/N elementary abelian of rank 2.*

(a) If R is a subgroup of P, then

$$\mathrm{Res}^P_R\, \beta^P_{Q,N} = \begin{cases} \displaystyle\sum_{g\in[R\backslash T(Q,R)/Q]} \beta^R_{{}^gQ,\,{}^gN} & \text{if } R \ge_P Q\,, \\ 0 & \text{if } R \not\ge_P Q\,. \end{cases}$$

(b) If M is a normal subgroup of Q with Q/M elementary abelian of rank 2, then $\mathrm{Defres}^P_{Q/M}\, \beta^P_{Q,N}$ is the multiplication by $m_{Q,M}|N_P(Q)/Q|$, where $m_{Q,M}$ is given by (4.6).

(c) $\beta^P_{Q,N}$ is independent of N, that is, $\beta^P_{Q,M} = \beta^P_{Q,N}$, where M is as in (b).

Proof. (a) We first compute $\mathrm{Res}^Q_R\, \beta^Q_{Q,N}$. We apply the Mackey formula, taking into account the fact that each subgroup X is a normal subgroup of Q (since otherwise $\mu(X,Q) = 0$), so that $R\backslash Q/X = RX\backslash Q$.

Since $X/(X\cap N)=E$ has rank 2, $\mathbb{Q}D(E)\cong\mathbb{Q}$ is generated by Ω_E which is invariant under the action of $\mathrm{Aut}(E)$, so that the map $\mathrm{Conj}_g:\mathbb{Q}D(E)\to\mathbb{Q}D(E)$ is the identity. Thus we obtain

$$\begin{aligned}
\mathrm{Res}^Q_R\,\beta^Q_{Q,N} &= \frac{1}{|Q|}\sum_{\substack{X\le Q\\ XN=Q}} |X|\,\mu(X,Q)\,\mathrm{Res}^Q_R\,\mathrm{Ten}^Q_X\,\mathrm{Inf}^X_{X/(X\cap N)}\\
&= \frac{1}{|Q|}\sum_{\substack{X\le Q\\ XN=Q}} |X|\,\mu(X,Q)\sum_{g\in[RX\backslash Q]}\mathrm{Ten}^R_{R\cap X}\,\mathrm{Res}^X_{R\cap X}\,\mathrm{Inf}^X_{X/(X\cap N)}\,\mathrm{Conj}_g\\
&= \frac{1}{|Q|}\sum_{S\le R}\mathrm{Ten}^R_S\sum_{\substack{XN=Q\\ R\cap X=S}} |X|\,\mu(X,Q)\,|Q:RX|\,\mathrm{Res}^X_S\,\mathrm{Inf}^X_{X/(X\cap N)}\,.
\end{aligned}$$

We claim that $\mathrm{Res}^X_S\,\mathrm{Inf}^X_{X/(X\cap N)}=0$ if S does not cover the quotient Q/N (that is, $SN\neq Q$). Indeed S does not cover either the quotient $X/(X\cap N)$, because $XN=Q$. Therefore $\mathrm{Res}^X_S\,\mathrm{Inf}^X_{X/(X\cap N)}$ factorizes through the proper subgroup $S(X\cap N)/(X\cap N)$ of $X/(X\cap N)$. But $X/(X\cap N)=E$ has rank 2 and $\mathbb{Q}D(Y)=0$ for every proper subgroup Y of E, proving the claim.

Now if $SN=Q$ and $S\le X\le Q$, then $XN=Q$ and the inclusion $S\to X$ induces an isomorphism $S/(S\cap N)\cong X/(X\cap N)$. Thus we have $\mathrm{Res}^X_S\,\mathrm{Inf}^X_{X/(X\cap N)}=\mathrm{Inf}^S_{S/(S\cap N)}$, using our usual identification $\mathbb{Q}D(X/(X\cap N))\cong\mathbb{Q}D(S/(S\cap N))\cong\mathbb{Q}$. It follows that

$$\mathrm{Res}^Q_R\,\beta^Q_{Q,N}=\frac{1}{|Q|}\sum_{\substack{S\le R\\ SN=Q}}\mathrm{Ten}^R_S\Big[\sum_{\substack{S\le X\le Q\\ R\cap X=S}} |X|\,\mu(X,Q)\,|Q:RX|\Big]\,\mathrm{Inf}^S_{S/(S\cap N)}\,.$$

Now we have $|X|\cdot|Q:RX|=\dfrac{|Q|}{|RX:X|}=\dfrac{|Q|}{|R:S|}$, and therefore the inner sum is

$$\sum_{\substack{S\le X\le Q\\ R\cap X=S}} |X|\,\mu(X,Q)\,|Q:RX| = \frac{|Q|}{|R:S|}\sum_{\substack{S\le X\le Q\\ R\cap X=S}}\mu(X,Q)\,.$$

But this is zero by a standard combinatorial lemma, provided $R\neq Q$ (see [St, 3.9.3]). It follows that $\mathrm{Res}^Q_R\,\beta^Q_{Q,N}=0$ whenever R is a proper subgroup of Q.

Now we compute $\mathrm{Res}^P_R\,\beta^P_{Q,N}$. From the definition, we have $\beta^P_{Q,N}=\mathrm{Ten}^P_Q\,\beta^Q_{Q,N}$. Therefore, applying again the Mackey formula, we obtain

$$\begin{aligned}
\mathrm{Res}^P_R\,\beta^P_{Q,N} &= \mathrm{Res}^P_R\,\mathrm{Ten}^P_Q\,\beta^Q_{Q,N} = \sum_{g\in[R\backslash P/Q]}\mathrm{Ten}^R_{R\cap {}^gQ}\,\mathrm{Res}^{{}^gQ}_{R\cap {}^gQ}\,\mathrm{Conj}_g\,\beta^Q_{Q,N}\\
&= \sum_{g\in[R\backslash P/Q]}\mathrm{Ten}^R_{R\cap {}^gQ}\,\mathrm{Res}^{{}^gQ}_{R\cap {}^gQ}\,\beta^{{}^gQ}_{{}^gQ,\,{}^gN} = \sum_{g\in[R\backslash T(Q,R)/Q]}\mathrm{Ten}^R_{{}^gQ}\,\beta^{{}^gQ}_{{}^gQ,\,{}^gN}\\
&= \sum_{g\in[R\backslash T(Q,R)/Q]}\beta^R_{{}^gQ,\,{}^gN}\,,
\end{aligned}$$

using the fact that, by the first part of the proof, $\mathrm{Res}^{{}^gQ}_{R\cap {}^gQ}\,\beta^{{}^gQ}_{{}^gQ,\,{}^gN}=0$ if $R\cap {}^gQ<{}^gQ$, that is, if $g\notin T(Q,R)$. In particular, we obtain $\mathrm{Res}^P_R\,\beta^P_{Q,N}=0$ if $T(Q,R)$ is empty, or in other words if $R\not\geq_P Q$.

(b) Taking $R=Q$ in (a), we have $T(Q,R)=N_P(Q)$ and therefore

$$\mathrm{Defres}^P_{Q/M}\,\beta^P_{Q,N}=\mathrm{Def}^Q_{Q/M}\,\mathrm{Res}^P_Q\,\beta^P_{Q,N}=\sum_{g\in[N_P(Q)/Q]}\mathrm{Def}^Q_{Q/M}\,\beta^Q_{Q,\,{}^gN}\,.$$

Thus it suffices to show that each $\mathrm{Def}^Q_{Q/M}\,\beta^Q_{Q,{}^gN}$ is the multiplication by $m_{Q,M}$ and without loss of generality we can assume that ${}^gN = N$. We have

$$\begin{aligned}\mathrm{Def}^Q_{Q/M}\,\beta^Q_{Q,N} &= \frac{1}{|Q|}\sum_{\substack{X\leq Q\\ XN=Q}} |X|\,\mu(X,Q)\,\mathrm{Def}^Q_{Q/M}\,\mathrm{Ten}^Q_X\,\mathrm{Inf}^X_{X/(X\cap N)}\\ &= \frac{1}{|Q|}\sum_{\substack{X\leq Q\\ XN=Q}} |X|\,\mu(X,Q)\,\mathrm{Ten}^{Q/M}_{XM/M}\mathrm{Iso}^{XM/M}_{X/(X\cap M)}\mathrm{Def}^X_{X/(X\cap M)}\,\mathrm{Inf}^X_{X/(X\cap N)}\,,\end{aligned}$$

using Corollary 3.11. Now if $X\cap M \neq X\cap N$, then $\mathrm{Def}^X_{X/(X\cap M)}\,\mathrm{Inf}^X_{X/(X\cap N)}$ factorizes through the proper subgroup $X/(X\cap M)(X\cap N)$ of $X/(X\cap N)$. But $X/(X\cap N) = E$ has rank 2 and $\mathbb{Q}D(Y) = 0$ for every proper subgroup Y of E. Therefore the sum only runs over subgroups X satisfying $X\cap M = X\cap N$ and we have $\mathrm{Def}^X_{X/(X\cap M)}\,\mathrm{Inf}^X_{X/(X\cap N)} = id$ (by definition of deflation in terms of Brauer quotients). Moreover

$$X/(X\cap N) = X/(X\cap M) \cong XM/M \leq Q/M\,,$$

and since $X/(X\cap N)$ and Q/M have order p^2, we also have $XM = Q$, hence $\mathrm{Ten}^{Q/M}_{XM/M} = id$. Finally $\mathrm{Iso}^{Q/M}_{X/(X\cap M)} = id$ by our usual identification $\mathbb{Q}D\big(X/(X\cap M)\big) \cong \mathbb{Q}D(Q/M) \cong \mathbb{Q}$. It follows that

$$\mathrm{Def}^Q_{Q/M}\,\beta^Q_{Q,N} = \frac{1}{|Q|}\sum_{\substack{XM=XN=Q\\ X\cap M = X\cap N}} |X|\,\mu(X,Q)\cdot id = m_{Q,M}\cdot id\,,$$

by Lemma 4.7.

(c) Since $\beta^P_{Q,N} = \mathrm{Ten}^P_Q\,\beta^Q_{Q,N}$, we only have to prove that $\beta^Q_{Q,N}$ is independent of N. By the injectivity of the restriction-deflation map to elementary abelian sections of rank 2 (Theorem 1.6), it suffices to show that $\mathrm{Defres}^Q_{R/S}\,\beta^Q_{Q,N}$ is independent of N for every such section R/S. This is obvious if $R<Q$ because $\mathrm{Res}^Q_R\,\beta^Q_{Q,N} = 0$ by (a). When $R=Q$, we have $\mathrm{Defres}^Q_{Q/S}\,\beta^Q_{Q,N} = m_{Q,S}\cdot id$ by (b) and this is independent of N. □

(4.10) Corollary. *Let $A = (a_{Q,R})$ be the matrix of the linear map $\psi_P\,\alpha_P$ with respect to the canonical basis of $\prod_{Q\in[\mathcal{S}/P]}\mathbb{Q}$, ordered in a way compatible with the relation $\geq_P$.*

(a) A is triangular.

(b) The diagonal entries are $a_{Q,Q} = m_{Q,Q_0}\,|N_P(Q)/Q|$.

(c) ψ_P is surjective.

Proof. (a) If $R \not\geq_P Q$, then $\mathrm{Res}^P_R\,\beta^P_{Q,Q_0} = 0$ by Proposition 4.9 and therefore $\mathrm{Defres}^P_{R/R_0}\,\beta^P_{Q,Q_0} = 0$. This implies that A is triangular.

(b) By Proposition 4.9, $\mathrm{Defres}^P_{Q/Q_0}\,\beta^P_{Q,Q_0} = m_{Q,Q_0}\,|N_P(Q)/Q|\cdot id$.

(c) By Lemma 4.7, we know that the explicit value of m_{Q,Q_0} is non-zero. Therefore the matrix A is invertible, so that $\psi_P\,\alpha_P$ is an isomorphism. In particular ψ_P is surjective. □

Together with Proposition 4.4, part (c) of Corollary 4.10 completes the proof of Theorem 4.1. Moreover, we also deduce that the map α_P is an isomorphism and we now use this for describing two bases of $\mathbb{Q}D(P)$.

(4.11) Proposition. *Let* E *denote an elementary abelian group of rank 2 and for each section* X/Y *isomorphic to* E *, identify* $\mathbb{Q}D(X/Y) \cong \mathbb{Q}$ *with* $\mathbb{Q}D(E)$ *.*

(a) The set $\{\, \beta^P_{Q,Q_0}(\Omega_E) \mid Q \in [\mathcal{S}/P] \,\}$ *is a basis of* $\mathbb{Q}D(P)$ *.*

(b) The set $\{\, \mathrm{Teninf}^P_{Q/Q_0}(\Omega_{Q/Q_0}) \mid Q \in [\mathcal{S}/P] \,\}$ *is a basis of* $\mathbb{Q}D(P)$ *.*

Proof. (a) This set is the image under α_P of the canonical basis of $\prod_{Q \in [\mathcal{S}/P]} \mathbb{Q}$ and α_P is an isomorphism.

(b) For every $Q \leq P$, let J_Q be the subspace of $\mathbb{Q}D(P)$ generated by all elements of the form $\mathrm{Teninf}^P_{R/N}(\Omega_{R/N})$ where $R < Q$ and $R/N \cong E$. We have

$$\beta^P_{Q,Q_0}(\Omega_E) = \frac{1}{|Q|} \sum_{\substack{X \leq Q \\ XQ_0 = Q}} |X|\, \mu(X,Q)\, \mathrm{Teninf}^P_{X/(X\cap Q_0)}(\Omega_{X/(X\cap Q_0)}) \equiv \mathrm{Teninf}^P_{Q/Q_0}(\Omega_{Q/Q_0}) \pmod{J_Q}.$$

Since $\beta^P_{Q,N} = \beta^P_{Q,Q_0}$ whenever $Q/N \cong E$ (Proposition 4.9), we deduce that

$$\mathrm{Teninf}^P_{Q/N}(\Omega_{Q/N}) \equiv \mathrm{Teninf}^P_{Q/Q_0}(\Omega_{Q/Q_0}) \pmod{J_Q}.$$

Therefore, in the expression for $\beta^P_{Q,Q_0}(\Omega_E)$, we can replace each $\mathrm{Teninf}^P_{X/(X\cap Q_0)}(\Omega_{X/(X\cap Q_0)})$ by an element of the form $\mathrm{Teninf}^P_{X/X_0}(\Omega_{X/X_0}) + y$ where $y \in J_X$. It follows by induction that there exist rational numbers λ_R such that

$$\beta^P_{Q,Q_0}(\Omega_E) = \mathrm{Teninf}^P_{Q/Q_0}(\Omega_{Q/Q_0}) + \sum_{\substack{R \in [\mathcal{S}/P] \\ R < Q}} \lambda_R\, \mathrm{Teninf}^P_{R/R_0}(\Omega_{R/R_0}).$$

Thus we obtain a matrix which is triangular with ones along the diagonal and the result follows from (a). □

Since Ω_E is defined over the prime field $\mathbb{F}_p$, the results of this section are independent of the base field k and this gives our next result.

(4.12) Corollary. *For every field* k *of characteristic* p*, there is an isomorphism* $\mathbb{Q}D_{\mathbb{F}_p}(P) \cong \mathbb{Q}D_k(P)$ *induced by scalar extension from* $\mathbb{F}_p$ *to* k *.*

(4.13) Remark. Let $\mathcal{O}$ be a complete discrete valuation ring of characteristic zero with residue field k. One can define the Dade group $D_{\mathcal{O}}(P)$ in a similar fashion using Dade P-algebras over $\mathcal{O}$. The residue map $D_{\mathcal{O}}(P) \to D_k(P)$ is injective (see [Th], Section 29) and it is conjectured that it is an isomorphism. Since Ω_E obviously lifts to $\mathcal{O}$, the results of this section also hold over $\mathcal{O}$ and it follows that the residue map $\mathbb{Q}D_{\mathcal{O}}(P) \to \mathbb{Q}D_k(P)$ is an isomorphism.

(4.14) Remark. There is an expression for Ω_P as a linear combination of the basis $\{\, \mathrm{Teninf}^P_{Q/Q_0}(\Omega_{Q/Q_0}) \,\}$ of $\mathbb{Q}D(P)$. One can show that such a formula only involves the basis elements indexed by the subgroups Q which are elementary abelian. Moreover, even when P itself is elementary abelian, the coefficients have denominators and this shows that this $\mathbb{Q}$-basis is not suitable as an integral basis for the torsion-free part of $D(P)$.

5. The torsion-free rank of the endo-trivial group

The torsion-free rank of the endo-trivial group $T(P)$ has been determined recently by J.L. Alperin, using relative Heller translates of endo-permutation modules (which turn out to be again endo-permutation modules). In this section, we prove Alperin's theorem by another method, using tensor induction.

Let $\mathcal{A}$ be the poset of all elementary abelian subgroups of P of rank ≥ 2. The torsion-free rank of $T(P)$ will turn out to be the number of conjugacy classes of connected components of $\mathcal{A}$. Therefore we first need to understand better those components. Clearly the isolated vertices of $\mathcal{A}$ are precisely the maximal elementary abelian subgroups of rank 2. Let $\mathcal{B}$ be the subposet of $\mathcal{A}$ consisting of all elementary abelian subgroups of P of rank ≥ 3 as well as their subgroups of rank 2. The following group-theoretic result is well-known and not very hard to prove (see Lemma 10.21 in [GLS]).

(5.1) Lemma. *Let $m(P)$ be the maximal rank of an elementary abelian subgroup of P.*

(a) *If $m(P) = 2$, then $\mathcal{A}$ consists of isolated vertices.*

(b) *If $m(P) \geq 3$, then $\mathcal{B}$ is a connected component of $\mathcal{A}$ and all other connected components (if any) are isolated vertices.*

We shall call $\mathcal{B}$ the *big* connected component of $\mathcal{A}$. Let $\mathcal{X}$ be the set of connected components of $\mathcal{A}$ and let $[\mathcal{X}/P]$ be a set of representatives of conjugacy classes in $\mathcal{X}$. Since $\mathcal{B}$ is clearly invariant by conjugation, it must belong to $[\mathcal{X}/P]$ (whenever $m(P) \geq 3$).

(5.2) Theorem. *For every connected component $\mathcal{C} \in [\mathcal{X}/P]$, choose $E_\mathcal{C} \in \mathcal{C}$.*

(a) *The restriction map*

$$\rho = \prod_{\mathcal{C}\in[\mathcal{X}/P]} \mathrm{Res}^P_{E_\mathcal{C}} : \mathbb{Q}T(P) \longrightarrow \prod_{\mathcal{C}\in[\mathcal{X}/P]} \mathbb{Q}T(E_\mathcal{C}) \cong \prod_{\mathcal{C}\in[\mathcal{X}/P]} \mathbb{Q}$$

is an isomorphism.

(b) *If $\mathcal{C}$ is a singleton (so that $E_\mathcal{C}$ is an isolated vertex), let $a_\mathcal{C} = \mathrm{Ten}^P_{E_\mathcal{C}}(\Omega_{E_\mathcal{C}})$. For the big connected component $\mathcal{B}$, let $a_\mathcal{B} = \Omega_P$. Then the set $\{a_\mathcal{C} \mid \mathcal{C} \in [\mathcal{X}/P]\}$ is a $\mathbb{Q}$-basis of $\mathbb{Q}T(P)$.*

Proof. By Theorem 1.5, we know that

$$\mathrm{Res} : \mathbb{Q}T(P) \longrightarrow \prod_E \mathbb{Q}T(E)$$

is injective, where E runs over all elementary abelian subgroups of P. Clearly we only need to choose one subgroup in each conjugacy class. Now if E_k denotes the elementary abelian group of rank k, we have $\mathbb{Q}T(E_1) = 0$, while the restriction map $\mathrm{Res}^{E_k}_{E_2} : T(E_k) \to T(E_2)$ is an isomorphism if $k \geq 2$, by Theorem 1.4 (a) and the fact that $\mathrm{Res}^{E_k}_{E_2}(\Omega_{E_k}) = \Omega_{E_2}$. Therefore all restriction maps within the big component $\mathcal{B}$ are isomorphisms and so it suffices to consider the chosen subgroup $E_\mathcal{B}$. It follows from this that we do not loose injectivity if we only restrict to the subgroups $E_\mathcal{C}$ for $\mathcal{C} \in [\mathcal{X}/P]$. In other words ρ is injective.

We shall show that the set $\{a_\mathcal{C} \mid \mathcal{C} \in [\mathcal{X}/P]\}$ belongs to $\mathbb{Q}T(P)$ and maps via ρ to a basis of $\prod_{\mathcal{C}\in[\mathcal{X}/P]} \mathbb{Q}$. This will prove the surjectivity of ρ as well as statement (b).

First we need to check that $a_\mathcal{C} \in \mathbb{Q}T(P)$. This is clear for the big component since $\Omega_P \in T(P)$ (Lemma 1.3). Now if $E = E_\mathcal{C}$ is an isolated vertex of $\mathcal{A}$, we have $a_\mathcal{C} = \mathrm{Ten}^P_E(\Omega_E) \in \mathbb{Q}D(P)$ and we check that it belongs to $\mathbb{Q}T(P)$ by computing all deflations $\mathrm{Defres}^P_{N/R}$ where $R \neq 1$ and $N = N_P(R)$. By the Mackey formula, we have

$$\mathrm{Def}^N_{N/R}\,\mathrm{Res}^P_N\,\mathrm{Ten}^P_E(\Omega_E) = \mathrm{Def}^N_{N/R}\Big(\sum_{g\in[N\backslash P/E]} \mathrm{Ten}^N_{{}^gE\cap N}\,\mathrm{Res}^{{}^gE}_{{}^gE\cap N}({}^g\Omega_E)\Big)\,.$$

But ${}^g\Omega_E = \Omega_{{}^gE}$ and its restriction to any proper subgroup C is zero since E has rank 2 and $\mathbb{Q}D(C) = 0$ if $|C| = p$. So g only runs over the set $T(E,N)$ of all $g \in P$ such that ${}^gE \leq N$ and we obtain

$$\mathrm{Defres}^P_{N/R}\,\mathrm{Ten}^P_E(\Omega_E) = \sum_{g\in[N\backslash T(E,N)/E]} \mathrm{Def}^N_{N/R}\,\mathrm{Ten}^N_{{}^gE}(\Omega_{{}^gE})\,.$$

We show that each term of the sum is zero. By relabelling ${}^gE$, we can assume that $g = 1$. By Corollary 3.11, we have

$$\mathrm{Def}^N_{N/R}\,\mathrm{Ten}^N_E(\Omega_E) = \mathrm{Ten}^{N/R}_{ER/R}\,\mathrm{Iso}^{ER/R}_{E/E\cap R}\,\mathrm{Def}^E_{E/E\cap R}(\Omega_E)\,.$$

But $\mathrm{Def}^E_{E/E\cap R}(\Omega_E) = 0$ whenever $E\cap R \neq 1$ (Lemma 1.3), so we only need to show that $E \cap R \neq 1$.

Since $E \leq N$, E normalizes R, hence centralizes some non-trivial subgroup Q of R (because E and R are p-groups) and we can assume that Q is elementary abelian. Then EQ is elementary abelian and so $EQ = E$ by maximality of E (since E is an isolated vertex, it is a maximal elementary abelian subgroup). Therefore $Q \leq E$ and $E\cap R \geq Q \neq 1$.

Now we want to prove that $\{\rho(a_{\mathcal{C}}) \mid \mathcal{C} \in [\mathcal{X}/P]\}$ is a $\mathbb{Q}$-basis of $\prod_{\mathcal{C}\in[\mathcal{X}/P]}\mathbb{Q}$. We need to compute the restrictions of $a_{\mathcal{C}} = \mathrm{Ten}^P_E(\Omega_E)$, where $E = E_{\mathcal{C}}$ is an isolated vertex as before. If F is an elementary abelian subgroup, we have

$$\mathrm{Res}^P_F\,\mathrm{Ten}^P_E(\Omega_E) = \sum_{g\in[F\backslash P/E]} \mathrm{Ten}^F_{{}^gE\cap F}\,\mathrm{Res}^{{}^gE}_{{}^gE\cap F}(\Omega_{{}^gE})\,.$$

Since the restriction of $\Omega_{{}^gE}$ to any proper subgroup is zero (because E has rank 2), g only runs over the set $T(E,F)$ of all $g\in P$ such that ${}^gE \leq F$. But since E is isolated, $T(E,F)$ is non empty only if F is conjugate to E. Therefore $\mathrm{Res}^P_{E_{\mathcal{D}}}\,\mathrm{Ten}^P_E(\Omega_E) = 0$ for every component $\mathcal{D}$ not conjugate to $\mathcal{C} = \{E\}$. Moreover, taking now $F = E$, we have $T(E,F) = N_P(E)$ and we obtain

$$\mathrm{Res}^P_E\,\mathrm{Ten}^P_E(\Omega_E) = \sum_{g\in[N_P(E)/E]} \Omega_{{}^gE} = |N_P(E)/E|\cdot\Omega_E\,.$$

This shows that $\rho(a_{\mathcal{C}})$ is a multiple of the basis element indexed by $\mathcal{C}$ in the canonical basis of $\prod_{\mathcal{C}\in[\mathcal{X}/P]}\mathbb{Q}$, whenever $\mathcal{C}$ consists of an isolated vertex. We are left with the big component $\mathcal{B}$ (but only if $m(P) \geq 3$). We have defined $a_{\mathcal{B}} = \Omega_P$ and we have $\mathrm{Res}_F(\Omega_P) = \Omega_F$ for every F. Therefore $\rho(a_{\mathcal{B}})$ is the sum of all basis elements in the canonical basis of $\prod_{\mathcal{C}\in[\mathcal{X}/P]}\mathbb{Q}$. Altogether, it is now clear that the set $\{\rho(a_{\mathcal{C}}) \mid \mathcal{C}\in[\mathcal{X}/P]\}$ is a $\mathbb{Q}$-basis of $\prod_{\mathcal{C}\in[\mathcal{X}/P]}\mathbb{Q}$. □

As in the case of the whole Dade group we deduce the following result.

(5.3) Corollary. *For every field k of characteristic p, there is an isomorphism $\mathbb{Q}T_{\mathbb{F}_p}(P) \cong \mathbb{Q}T_k(P)$ induced by scalar extension from $\mathbb{F}_p$ to k.*

As in Remark 4.13, we also deduce that, if $\mathcal{O}$ is a complete discrete valuation ring of characteristic zero with residue field k, the residue map $\mathbb{Q}T_{\mathcal{O}}(P) \to \mathbb{Q}T_k(P)$ is an isomorphism. However, much more is known, since it has been proved recently by J.L. Alperin that the residue map $T_{\mathcal{O}}(P) \to T_k(P)$ is always an isomorphism.

(5.4) Remark. If E is an isolated vertex of $\mathcal{A}$, we have used the fact that $\mathrm{Res}_C(\Omega_E) = 0$ if $C < E$, because C is cyclic and so $\mathbb{Q}D(C) = 0$. If we want to work over $\mathbb{Z}$, then $D(C) = \mathbb{Z}/2\mathbb{Z}$ if $|C| = p$ and p is odd, so we need to consider $2\Omega_E$ instead to obtain $\mathrm{Res}_C(2\Omega_E) = 0$. It follows that the elements $2a_{\mathcal{C}} = \mathrm{Ten}_{E_{\mathcal{C}}}^P(2\Omega_{E_{\mathcal{C}}})$ (where $\mathcal{C}$ runs over the isolated components up to conjugacy), together with $a_{\mathcal{B}} = \Omega_P$ for the big component (if $m(P) \geq 3$), belong to $T(P)$, are linearly independent over $\mathbb{Z}$, and span a full lattice in $T(P)$.

We also observe that $2a_{\mathcal{C}}$ is the class of the endo-permutation module $M = \mathrm{Ten}_E^P\big(\Omega_E^2(k)\big)$, where $E = E_{\mathcal{C}}$. This class belongs to $T(P)$, but the module M itself may not be endo-trivial (it is only equivalent to an endo-trivial module). The reason is that the endomorphism algebra of $\Omega_E^2(k)$ has the form $k \oplus (\text{free})$ and when we consider its tensor induction, we obtain $k \oplus X$ where X is a permutation module which may not be free. For instance the tensor induction of a free module may not be free.

6. Partial results on the torsion subgroup

The purpose of this section is to prove Theorem B of the introduction. Let p be an odd prime and let P be a finite p-group. Recall that we have defined $\overline{D^t}(P) = D^t(P)/\mathrm{Ker}(\psi_P)$, where $D^t(P)$ is the torsion subgroup of $D(P)$ and where

$$\psi_P : D^t(P) \longrightarrow \prod_A D(A)$$

is the product of the restriction-deflation maps to all cyclic sections A of order p. Since the subgroup $\mathrm{Ker}(\psi_P)$ is quite difficult to handle (and might be trivial), we only deal with the quotient $\overline{D^t}(P)$. Note that if A is cyclic of order p, we have $D(A) = D^t(A) = \overline{D^t}(A) \cong \mathbb{Z}/2\mathbb{Z}$, generated by Ω_A (by direct inspection or by Theorem 1.4). We first need a lemma.

(6.1) Lemma. *Let p be an odd prime and let E be an elementary abelian p-group of rank 2. Then the restriction*

$$\mathrm{Res} : D^t(E) \longrightarrow \prod_{R<E} D^t(R)$$

to all proper subgroups is injective.

Proof. Let $a \in D^t(E)$. By Dade's Theorem 1.4, we have

$$a = \sum_{Q \in \mathcal{C}} n_Q \, \Omega_{E/Q} \, ,$$

where $\mathcal{C}$ is the set of all non-trivial cyclic subgroups of E and where $n_Q \in \mathbb{Z}/2\mathbb{Z}$ for every Q. If $a \in \mathrm{Ker}(\mathrm{Res})$, we have, for every $R \in \mathcal{C}$,

$$0 = \mathrm{Res}^E_R(a) = \sum_{\substack{Q\in\mathcal{C} \\ Q\neq R}} n_Q\, \Omega_R\,,$$

because the action of R on $\Omega^1_{E/Q}(k)$ is obtained using the isomorphism $R \cong E/Q$ whenever $R \neq Q$ (that is, $RQ = E$). It follows that

$$\sum_{\substack{Q\in\mathcal{C} \\ Q\neq R}} n_Q = 0 \in \mathbb{Z}/2\mathbb{Z}\,,$$

for every $R \in \mathcal{C}$. Considering $R, R' \in \mathcal{C}$ with $R \neq R'$, we obtain

$$n_R + \sum_{\substack{Q\in\mathcal{C} \\ Q\neq R,\, Q\neq R'}} n_Q = 0 = n_{R'} + \sum_{\substack{Q\in\mathcal{C} \\ Q\neq R,\, Q\neq R'}} n_Q\,,$$

and therefore $n_R = n_{R'}$. Thus all the numbers n_Q are equal and the sum of p of them is zero in $\mathbb{Z}/2\mathbb{Z}$ (because $|\mathcal{C}| = p+1$). Since p is odd, $n_Q = 0$ for every Q. □

Theorem B of the introduction is contained in the following more precise result.

(6.2) Theorem. *Let P be a finite p-group, where p is an odd prime. Let $\mathcal{C}$ be the set of non-trivial cyclic subgroups of P and for every $C \in \mathcal{C}$, let $\Phi(C)$ be the unique maximal subgroup of C.*

(a) The group homomorphism

$$\phi_P = \prod_{C\in[\mathcal{C}/P]} \mathrm{Defres}^P_{C/\Phi(C)} : \overline{D^t}(P) \longrightarrow \prod_{C\in[\mathcal{C}/P]} D(C/\Phi(C)) \cong \prod_{C\in[\mathcal{C}/P]} \mathbb{Z}/2\mathbb{Z}$$

is an isomorphism, where $[\mathcal{C}/P]$ denotes a set of representatives of the conjugacy classes of subgroups in $\mathcal{C}$.

(b) The set $\{\mathrm{Ten}^P_C(\Omega_{C/\Phi(C)}) \mid C \in [\mathcal{C}/P]\}$ is an $\mathbb{F}_2$-basis of $\overline{D^t}(P)$, mapping via ϕ_P to the canonical basis of $\prod_{C\in[\mathcal{C}/P]} \mathbb{Z}/2\mathbb{Z}$.

Proof. The injectivity of ϕ_P will follow from the injectivity of

$$\psi_P : \overline{D^t}(P) \longrightarrow \prod_A D(A)\,,$$

where A runs over all sections of order p. Let $a \in \mathrm{Ker}(\phi_P)$. We prove that $a = 0$ by induction on $|P|$. For every proper subgroup Q, we clearly have $\mathrm{Res}^P_Q(a) \in \mathrm{Ker}(\phi_Q)$ and therefore $\mathrm{Res}^P_Q(a) = 0$ by induction. By the injectivity of the map ψ_P above, it suffices to prove that $\mathrm{Def}^P_{P/S}(a) = 0$ for every quotient P/S of order p.

If P is cyclic, then necessarily $S = \Phi(P)$ and the condition $\mathrm{Def}^P_{P/S}(a) = 0$ is part of the assumption that $a \in \mathrm{Ker}(\phi_P)$. If P is not cyclic, then there is an elementary abelian quotient P/R of rank 2 with $R < S$ and it suffices to prove that $\mathrm{Def}^P_{P/R}(a) = 0$. But the restriction of $\mathrm{Def}^P_{P/R}(a)$ to every proper subgroup Q/R is zero since $\mathrm{Res}^P_Q(a) = 0$. Therefore, by Lemma 6.1, we have $\mathrm{Def}^P_{P/R}(a) = 0$ as required.

We now prove that, for every $C \in [\mathcal{C}/P]$, $\phi_P\big(\mathrm{Ten}_C^P(\Omega_{C/\Phi(C)})\big)$ is the basis element indexed by C in the canonical basis of $\prod_{C\in[\mathcal{C}/P]} \mathbb{Z}/2\mathbb{Z}$. This will prove the surjectivity of ϕ_P as well as statement (b). For every $D \in \mathcal{C}$, we have by the Mackey formula

$$\mathrm{Def}^D_{D/\Phi(D)} \mathrm{Res}^P_D \mathrm{Ten}^P_C(\Omega_{C/\Phi(C)}) = \sum_{g\in[D\backslash P/C]} \mathrm{Def}^D_{D/\Phi(D)} \mathrm{Ten}^D_{D\cap {}^gC} \mathrm{Res}^{^gC}_{D\cap {}^gC}(\Omega_{{}^gC/\Phi({}^gC)}) ,$$

using the fact that conjugation by g maps $\Omega_{C/\Phi(C)}$ to $\Omega_{{}^gC/\Phi({}^gC)}$. If $D\cap {}^gC < {}^gC$, then $D\cap {}^gC \leq \Phi({}^gC)$ and therefore $\mathrm{Res}^{^gC}_{D\cap {}^gC}(\Omega_{{}^gC/\Phi({}^gC)}) = 0$. So g only runs over the set $T(C,D)$ of all $g\in P$ such that ${}^gC \leq D$. Thus we obtain

$$\mathrm{Defres}^P_{D/\Phi(D)} \mathrm{Ten}^P_C(\Omega_{C/\Phi(C)}) = \sum_{g\in[D\backslash T(C,D)/C]} \mathrm{Def}^D_{D/\Phi(D)} \mathrm{Ten}^D_{^gC}(\Omega_{{}^gC/\Phi({}^gC)}) .$$

By Corollary 3.11, $\mathrm{Def}^D_{D/\Phi(D)} \mathrm{Ten}^D_{^gC}$ factorizes as the composite of $\mathrm{Def}^{^gC}_{{}^gC/{}^gC\cap\Phi(D)}$ followed by an isomorphism and a tensor induction. So if ${}^gC < D$, that is, ${}^gC \leq \Phi(D)$, we have a deflation to the trivial group, hence zero since $D(1) = 0$. This proves that $\mathrm{Defres}^P_{D/\Phi(D)} \mathrm{Ten}^P_C(\Omega_{C/\Phi(C)}) = 0$ whenever D is not conjugate to C. If now $D = C$, we are left with the sum over $g \in [N_P(C)/C]$ and we have $\mathrm{Def}^C_{C/\Phi(C)}(\Omega_{C/\Phi(C)}) = \Omega_{C/\Phi(C)}$. Therefore we obtain

$$\mathrm{Defres}^P_{C/\Phi(C)} \mathrm{Ten}^P_C(\Omega_{C/\Phi(C)}) = |N_P(C)/C| \cdot \Omega_{C/\Phi(C)} = \Omega_{C/\Phi(C)} ,$$

using the fact that $|N_P(C)/C|$ is odd and $\Omega_{C/\Phi(C)}$ has order 2. This completes the proof that the image $\phi_P\big(\mathrm{Ten}^P_C(\Omega_{C/\Phi(C)})\big)$ is the basis element indexed by C. □

It is easy to see by induction (as in the proof of Theorem 1.6) that the injectivity of the map ψ_P would follow if the restriction map

$$\mathrm{Res} : T(P) \longrightarrow \prod_E T(E)$$

was injective, where E runs over elementary abelian subgroups of P. This is one of the important open questions about the Dade group (mentioned to us several years ago by J.F. Carlson and J.L. Alperin). We mention the following consequence.

(6.3) Corollary. *Assume that p is odd and that $\mathrm{Ker}(\psi_P) = 0$. Then $D^t(P)$ is an $\mathbb{F}_2$-vector space of dimension s, where s is the number of conjugacy classes of non-trivial cyclic subgroups of P. In particular every torsion element of $D^t(P)$ is self-dual.*

Proof. The first statement follows from Theorem 6.2 and the second from the fact that the self-dual endo-permutation modules are precisely the elements of order 2 of the Dade group. □

(6.4) Remark. A typical element of order 2 in $\overline{D^t}(P)$ is $\Omega_{P/Q}$ where Q is a proper normal subgroup with P/Q cyclic. Its expression relative to the $\mathbb{F}_2$-basis of $\overline{D^t}(P)$ is

$$\Omega_{P/Q} = \sum_C \mathrm{Ten}^P_C(\Omega_{C/\Phi(C)}) ,$$

where C runs over the cyclic subgroups (up to conjugation) such that Q has index p in QC. This follows easily from the computation of $\mathrm{Defres}^P_{D/\Phi(D)}(\Omega_{P/Q})$ for every cyclic subgroup D.

(6.5) Remark. For the torsion subgroup $T^t(P)$ of the endo-trivial group $T(P)$, one can define similarly $\overline{T^t}(P) = T^t(P)/\mathrm{Ker}(\mathrm{Res})$ where

$$\mathrm{Res} : T(P) \longrightarrow \prod_E T(E)$$

is the restriction to all elementary abelian subgroups (since we have $\mathrm{Ker}(\mathrm{Res}) \subseteq T^t(P)$ by Theorem 1.5). However, there is no interesting result here, because $\overline{T^t}(P)$ is zero if P is not cyclic. Indeed, if P is not a quaternion group, every maximal elementary subgroup E has rank ≥ 2 and $T(E)$ has no torsion, so that $T^t(P)$ is contained in $\mathrm{Ker}(\mathrm{Res})$. If P is a quaternion group, then $T(P) = T^t(P) = \mathrm{Ker}(\mathrm{Res})$ because the target of the restriction map above is $T(C_2) = 0$. Note that if P is cyclic of order ≥ 3, then $T(P) = T^t(P) = \overline{T^t}(P) \cong \mathbb{Z}/2\mathbb{Z}$.

7. Functorial approach

Now we start the second part of this paper, which is concerned with the functorial approach to the Dade group. This approach is the original one we used for the proof of Theorems A and B, from which was then extracted the more elementary version of the previous sections. Our purpose in this section is to show how $\mathbb{Q}D$ and $\overline{D^t}$ can be viewed as functors (in the sense of [Bo1]). We also recall a number of general facts concerning such functors.

We have seen in Sections 2 and 3 that all the natural operations on the Dade group can be described with a single formalism. If P and Q are p-groups, and if U is a Q-set-P, then there is a functor $T_U : \underline{Perm_k}(P) \to \underline{Perm_k}(Q)$ which maps a Dade P-algebra to a Dade Q-algebra. Moreover, this functor induces a group homomorphism

$$D(U) : D(P) \longrightarrow D(Q) .$$

We have seen in Section 3 that when we compose two such functors T_V and T_U, we don't get the functor $T_{V\times_Q U}$, but a kind of twisted version of it, using endomorphisms of the field k. It follows that the map $D(V)\circ D(U)$ is not equal in general to the map $D(V\times_Q U)$. In order to get rid of those twists, we consider a quotient of $D(P)$ defined by

$$D_G(P) = D(P)/G(P) ,$$

where $G(P)$ is the subgroup consisting of the classes of the algebras $\gamma_p(A)\otimes_k A^{op}$, where A is a Dade P-algebra (see Lemma 3.1). Recall that γ_p is the twist induced by the endomorphism of k mapping λ to λ^p. Note that $G(P)$ is indeed a subgroup of $D(P)$ since γ_p commutes with tensor products and with taking opposite algebras. In $D_G(P)$, we have $[\gamma_p(A)] = [A]$, and consequently $[\gamma_{p^n}(A)] = [A]$ for all $n \in \mathbb{N}$.

(7.1) Proposition. *Let P be a p-group.*

(a) The subgroup $G(P)$ is finite, that is, $G(P) \subseteq D^t(P)$. In particular $\mathbb{Q}D(P) \cong \mathbb{Q}D_G(P)$.

(b) More precisely, we have $G(P) \subseteq \mathrm{Ker}(\psi_P)$, where $\psi_P : D^t(P) \to \prod_A D(A)$ is the product of the restriction-deflation maps to all cyclic sections A of order p.

(c) For every $n \in \mathbb{N}$, the functor γ_{p^n} induces the identity map on $D_G(P)$ and also on both $\mathbb{Q}D(P)$ and $\overline{D^t}(P) = D^t(P)/\mathrm{Ker}(\psi_P)$.

Proof. (a) First recall that, by Example 3.3, if A is an abelian group, then the map $\gamma_p : D(A) \to D(A)$ is the identity. By Theorem 1.5, $D(P)$ is a finitely generated abelian group. Therefore, by Theorem 1.6, the restriction-deflation map

$$\mathrm{Defres} : D(P) \longrightarrow \prod_E D(E)$$

has finite kernel, where E runs over all elementary abelian sections of P of rank 2. Since γ_p commutes with Defres (by Lemma 3.2) and is the identity on $D(E)$ because E is abelian, the class of the algebra $\gamma_p(A) \otimes_k A^{op}$ lies in $\mathrm{Ker}(\mathrm{Defres})$ for every Dade P-algebra A. It follows that $G(P)$ is contained in the finite group $\mathrm{Ker}(\mathrm{Defres})$ and so $G(P) \subseteq D^t(P)$.

(b) Exactly the same argument applies to the restriction-deflation map

$$\psi_P : D^t(P) \to \prod_A D(A)\,,$$

where A runs over all cyclic sections of order p. Therefore $G(P) \subseteq \mathrm{Ker}(\psi_P)$.

(c) Clearly γ_p induces the identity map on $D_G(P)$, by definition of $D_G(P)$, hence also on the subquotient $\overline{D^t}(P)$. Moreover, since $G(P)$ is finite, $D(P) \to D_G(P)$ induces an isomorphism $\mathbb{Q}D(P) \cong \mathbb{Q}D_G(P)$ and the result follows. □

It is precisely part (c) of the proposition which will allow us to put on $\mathbb{Q}D$ and $\overline{D^t}$ a structure of functor in the sense of [Bo1]. However, D itself need not be a functor in the sense of [Bo1], because of the presence of the twists γ_{p^n} (see Example 3.4).

(7.2) Remark. Though we shall not adopt this point of view in this paper, we would like to mention that it is possible to define a similar formalism which involves the twists by endomorphisms of k and which allows us to view D as a functor, without taking the quotient D_G. We define an $\mathrm{End}(k)$-graded Q-set-P to be a pair (U, a) where U is a Q-set-P and a is a function from $Q\backslash U/P$ to $\mathrm{End}(k)$. The disjoint union of (U, a) and (U', a') is defined in the obvious way. Now if R is another p-group and if (V, b) is an $\mathrm{End}(k)$-graded R-set-Q, then the product $(V, b) \times_Q (U, a)$ is the $\mathrm{End}(k)$-graded R-set-P

$$(V, b) \times_Q (U, a) = (V \times_Q U, b \times_Q a)\,,$$

where $b \times_Q a$ is the function from $R\backslash(V \times_Q U)/P$ to $\mathrm{End}(k)$ defined by

$$(b \times_Q a)(v, u) = b(v) \circ a(u) \circ \pi_{v,u}\,,$$

where $\pi_{v,u}$ is raising to the power $|Q_v : Q_{v,uP}|$ (see Proposition 3.5). If (U, a) is an $\mathrm{End}(k)$-graded Q-set-P, we define a functor $T_{(U,a)} : \underline{Perm_k}(P) \to \underline{Perm_k}(Q)$ by

$$T_{(U,a)}(X) = \mathrm{Hom}_P(U^{op}, X)\,.$$

If m is a morphism in $\underline{Perm}_k(P)$ from Y to X, we set

$$T_{(U,a)}(m)(\varphi,\psi) = \prod_{u\in U/P} a(u)\Big(m\big(\varphi(u),\psi(u)\big)\Big) .$$

Clearly, if we take for a the constant function equal to the identity endomorphism of k, then the functor $T_{(U,a)}$ is just our previous functor T_U. With those definitions, is is easy to see that we get a map

$$D(U,a) : D(P) \to D(Q) ,$$

which is additive in (U,a). Moreover $D(V,b)\circ D(U,a) = D\big((V,b)\times_Q (U,a)\big)$ by Proposition 3.5.

Now we introduce the functorial framework of [Bo1]. If P and Q are p-groups, we denote by $\Gamma(Q,P)$ the Grothendieck group of finite Q-sets-P, with addition induced by the decomposition into disjoint union. Thus $\Gamma(Q,P)$ is the free abelian group on transitive bisets up to conjugation. An element of $\Gamma(Q,P)$ is called a virtual biset. The product of bisets $(V,U)\mapsto V\times_Q P$ can be extended to a bilinear product

$$\circ : \Gamma(R,Q)\times\Gamma(Q,P)\to\Gamma(R,P) .$$

Let $\mathcal{C}$ be the following category:

- The objects of $\mathcal{C}$ are the finite p-groups.
- If P and Q are finite p-groups, then $\mathrm{Hom}_{\mathcal{C}}(P,Q) = \Gamma(P,Q)$.
- The composition of morphisms is given by the product $\circ$ defined above.

The category $\mathcal{C}$ is a preadditive category (in the sense of [Mc]). Similarly, when K is a commutative ring, we consider the category $\mathcal{C}_K$ obtained by tensoring everything with K: the objects of $\mathcal{C}_K$ are the same as the objects of $\mathcal{C}$, but

$$\mathrm{Hom}_{\mathcal{C}_K}(P,Q) = K\otimes_{\mathbb{Z}}\Gamma(P,Q) .$$

The composition of morphisms is defined by K-linearity.

Let us fix some notation about subgroups of products and bisets. If P and Q are finite groups, then any P-set-Q is a disjoint union of transitive ones, and the isomorphism classes of transitive P-sets-Q are in bijection with the conjugacy classes of subgroups of $Q\times P$: if L is such a subgroup, we denote by $(P\times Q)/L$ the set of right cosets of L in $P\times Q$, viewed as a P-set-Q by

$$g\cdot(a,b)L\cdot h = (ga,h^{-1}b)L , \qquad \forall g,a\in P,\ \forall b,h\in Q .$$

Define the following subgroups of P and Q:

(7.3)
$$\begin{aligned} {}_pL &= \{g\in P \mid \exists h\in Q,\ (g,h)\in L\} , & {}_kL &= \{g\in P\mid (g,1)\in L\} ,\\ L_p &= \{h\in Q\mid \exists g\in P,\ (g,h)\in L\} , & L_k &= \{h\in Q\mid (1,h)\in L\} , \end{aligned}$$

where the letter p stands for projection and k for kernel. Then ${}_kL\trianglelefteq {}_pL$, $L_k\trianglelefteq L_p$, and the quotients ${}_pL/{}_kL$ and L_p/L_k are naturally isomorphic. Let us recall the following basic decomposition lemma. We use the notation ${}_{L_p}Q_Q$ for Q viewed as a L_p-set-Q in the obvious way, and similarly in other situations.

(7.4) Lemma. *([Bo1], Lemma 3). With the notation above, the morphism* $(P\times Q)/L$ *from* Q *to* P *is equal to the product of the morphisms* ${}_{L_p}Q_Q$ *(restriction),* ${}_{L_p/L_k}(L_p/L_k)_{L_p}$ *(deflation),* ${}_{pL/_kL}(L_p/L_k)_{L_p/L_k}$ *(isomorphism),* ${}_{pL}\big({}_pL/_kL\big)_{{}_pL/_kL}$ *(inflation), and* ${}_PP_{{}_pL}$ *(induction).*

If R is another p-group and M is a subgroup of $Q\times R$, let

$$L*M=\{\,(x,y)\in P\times R\;\big|\;\exists z\in Q,\;(x,z)\in L,\;(z,y)\in M\,\}.$$

This is a subgroup of $P\times R$. With this notation, recall the following "Mackey formula".

(7.5) Lemma. *([Bo1], Proposition 1). Let* P, Q, *and* R *be finite* p*-groups. If* L *is a subgroup of* $P\times Q$ *and* M *is a subgroup of* $Q\times R$, *then*

$$\big[(P\times Q)/L\big]\times_Q\big[(Q\times R)/M\big]=\sum_{x\in[L_p\backslash Q/{}_pM]}(P\times R)/(L*{}^{(x,1)}M)\,,$$

where L_p *and* ${}_pM$ *are defined by (7.3).*

Now we come to the main fact for our purposes.

(7.6) Proposition. *With the notation of the beginning of this section, the following hold.*

(a) The correspondence D_G *which maps the* p*-group* P *to* $D_G(P)$ *and the virtual biset* $U-U'$ *to the group homomorphism* $D_G(U)-D_G(U')$ *is an additive functor from* $\mathcal{C}$ *to the category of abelian groups.*

(b) $\mathbb{Q}D$ *is a* $\mathbb{Q}$*-linear functor from* $\mathcal{C}_{\mathbb{Q}}$ *to the category of* $\mathbb{Q}$*-vector spaces.*

(c) $\overline{D^t}$ *is an* $\mathbb{F}_2$*-linear functor from* $\mathcal{C}_{\mathbb{F}_2}$ *to the category of* $\mathbb{F}_2$*-vector spaces.*

Proof. (a) Let U be a Q-set-P and let V be an R-set-Q, where P, Q and R are p-groups. First note that, by Lemma 3.2, $D(U):D(P)\to D(Q)$ induces a group homomorphism

$$D_G(U):D_G(P)\to D_G(Q)\,.$$

Now we have to prove that $D(V)\circ D(U)$ and $D(V\times_Q U)$ induce the same map

$$D_G(V)\circ D_G(U)=D_G(V\times_Q U):D_G(P)\longrightarrow D_G(R)\,.$$

This is a direct consequence of Remark 3.10, since all the twists γ_{p^n} act as the identity of $D_G(R)$ by Proposition 7.1. Alternatively, one can decompose U and V, reducing first to the case where U and V are transitive bisets (using Proposition 2.10), and then to the case where U and V are among the special types of bisets appearing in Lemma 7.4 above. By Corollary 3.8, we know that $D(V)\circ D(U)=D(V\times_Q U)$ in all cases except one, namely when $D(U)$ corresponds to a tensor induction and $D(V)$ to a deflation. Now in this case, by Proposition 3.9 and its proof, for every Dade P-algebra A, we have

$$(T_V\circ T_U)(A)\cong\gamma_{p^n}\big(T_{V\times_Q U}(A)\big)$$

for some $n\in\mathbb{N}$, so that $D_G(V)\circ D_G(U)=D_G(V\times_Q U)$ since γ_{p^n} acts as the identity on $D_G(R)$.

(b) This follows from (a) by tensoring with $\mathbb{Q}$, since $\mathbb{Q}D(P)\cong\mathbb{Q}D_G(P)$.

(c) Clearly the torsion subgroup $D^t_G(P)$ of $D_G(P)$ defines a subfunctor of D_G. With the notation of Proposition 7.1, we have to prove that $\mathrm{Ker}(\psi_P)/G(P)$ defines a subfunctor of $D^t_G(P)$, for the quotient $\overline{D^t}$

will then be a functor in the category $\mathcal{F}_{\mathbb{Z}}$. Note that $\overline{D^t}(P)$ is, by construction, an abelian group of exponent 2, hence an $\mathbb{F}_2$-vector space, so $\overline{D^t}$ will in fact be an $\mathbb{F}_2$-linear functor from $\mathcal{C}_{\mathbb{F}_2}$ to the category of $\mathbb{F}_2$-vector spaces.

In order to prove that $\mathrm{Ker}(\psi_P)/G(P)$ defines a subfunctor of $D^t_G(P)$, it is easy to verify that it is invariant under restriction, tensor induction, inflation, and deflation. For a change, we use a more functorial argument. We prove that $\mathrm{Ker}(\psi_P)/G(P)$ is equal to the kernel of all maps $D^t_G(\varphi)$, for all morphisms $\varphi : P \to C$ in $\mathcal{C}_{\mathbb{Z}}$. Since this clearly defines a subfunctor of $D^t_G(P)$, the result follows.

By linearity, it suffices to consider a morphism $\varphi : P \to C$ in $\mathcal{C}_{\mathbb{Z}}$ defined by a transitive biset, that is, $\varphi = (C \times P)/L$ for some subgroup L of $C \times P$. If there is a subgroup A of C and a subgroup B of P such that $L = A \times B$, then the morphism φ factors through the trivial group (by Lemma 7.4). Therefore

$$D^t_G(\varphi) : D^t_G(P) \longrightarrow D^t_G(C)$$

is zero since $D^t_G(1) = 0$. Otherwise there is a subgroup B of P and a surjection $s : B \to C$ such that $L = \{(s(b), b) \mid b \in B\}$. Then the morphism φ is a restriction-deflation map to a section of order p of P, so that $D^t_G(\varphi)$ vanishes on $\mathrm{Ker}(\psi_P)/G(P)$. □

We denote by $\mathcal{F}_K$ the category of K-linear functors from $\mathcal{C}_K$ to the category of K-modules. Thus the functor $K \otimes_{\mathbb{Z}} D_G$ is an object of $\mathcal{F}_K$, but our goal in this paper is to study $\mathbb{Q}D$ as an object of $\mathcal{F}_{\mathbb{Q}}$ and $\overline{D^t}$ as an object of $\mathcal{F}_{\mathbb{F}_2}$. Recall that $\mathcal{F}_K$ is an abelian category.

We end this section with generalities about the category $\mathcal{F}_K$, which are proved in [Bo1]. The emphasis in [Bo1] is put on functors defined on a similar category whose objects are *all* finite groups. Here instead we consider only finite p-groups, but this does not change anything to the general facts which we now recall. If F is an object of $\mathcal{F}_K$, and if P is a p-group, then $F(P)$ is a module for the endomorphism algebra $\mathcal{E}nd_{\mathcal{C}_K}(P)$ of P in $\mathcal{C}_K$. In fact the correspondence

$$F \mapsto F(P)$$

is a functor from $\mathcal{F}_K$ to $\mathcal{E}nd_{\mathcal{C}_K}(P)$-**Mod**, called the evaluation functor (at P). This evaluation functor has a left adjoint, defined as follows: if V is an $\mathcal{E}nd_{\mathcal{C}_K}(P)$-module and Q is a p-group, set

$$L_{P,V}(Q) = \mathrm{Hom}_{\mathcal{C}_K}(P, Q) \otimes_{\mathcal{E}nd_{\mathcal{C}_K}(P)} V\,. \tag{7.7}$$

This makes sense because $\mathrm{Hom}_{\mathcal{C}_K}(P, Q)$ has a natural structure of right $\mathcal{E}nd_{\mathcal{C}_K}(P)$-module, given by right composition of morphisms in $\mathcal{C}_K(P, Q)$. If φ is a morphism in $\mathcal{C}_K$ from Q to Q', then composition of morphisms (on the left) defines a map

$$L_{P,V}(\varphi) : L_{P,V}(Q) \to L_{P,V}(Q')\,.$$

So $L_{P,V}$ is an object of $\mathcal{F}_K$. Moreover, the correspondence $V \mapsto L_{P,V}$ is clearly a functor from the category of $\mathcal{E}nd_{\mathcal{C}_K}(P)$-modules to $\mathcal{F}_K$, and this functor is left adjoint to the evaluation functor at P.

If the module V is a simple $\mathcal{E}nd_{\mathcal{C}_K}(P)$-module, then $L_{P,V}$ is not simple in general, but it admits a unique maximal (proper) subfunctor $J_{P,V}$ defined by

$$J_{P,V}(Q) = \Big\{ \sum_i \psi_i \otimes v_i \;\Big|\; \forall \psi \in \mathrm{Hom}_{\mathcal{C}_K}(Q, P),\quad \textstyle\sum_i (\psi\psi_i) v_i = 0 \Big\} \tag{7.8}$$

for every p-group Q. The quotient $L_{P,V}/J_{P,V}$ is a simple object of $\mathcal{F}_K$ and is denoted by $S_{P,V}$.

Now in the algebra $\mathcal{E}nd_{\mathcal{C}_K}(P)$, let I_P denote the K-submodule generated by the endomorphisms of P which factor through a group Q of order strictly smaller than the order of P. It is clear that I_P is a two-sided ideal of $\mathcal{E}nd_{\mathcal{C}_K}(P)$. Moreover, if A is the K-submodule generated by the bisets which are both free and transitive on the right and on the left, then A is a subalgebra isomorphic to the algebra over K of the group $\mathrm{Out}(P)$ of outer automorphisms of P. The isomorphism is obtained by mapping the automorphism φ of P to the set P_φ, which is the set P, with double action given by

$$g\cdot x\cdot g' = g\,x\,\varphi(g')\,, \qquad \forall x\in P,\ \forall g,g'\in P\,,$$

where the product on the right hand side is the product inside the group P. Finally, there is a decomposition

$$\mathcal{E}nd_{\mathcal{C}_K}(P) = A\oplus I_P\,, \qquad \text{with } A\cong K\mathrm{Out}(P)\,. \tag{7.9}$$

With this decomposition, the quotient algebra $\mathcal{E}nd_{\mathcal{C}_K}(P)/I_P$ is isomorphic to $K\mathrm{Out}(P)$ and we can view any $K\mathrm{Out}(P)$-module V as an $\mathcal{E}nd_{\mathcal{C}_K}(P)$-module, still denoted by V.

(7.10) Proposition. *([Bo1], Proposition 2). Let $S_{P,V} = L_{P,V}/J_{P,V}$.*

(a) *The correspondence $(P,V)\mapsto S_{P,V}$ is a bijection between the set of isomorphism classes of pairs (P,V) consisting of a p-group P and a simple $K\mathrm{Out}(P)$-module V, and the set of isomorphism classes of simple objects of $\mathcal{F}_K$.*

(b) *The inverse bijection maps a simple object S to the pair $\big(P,S(P)\big)$, where P is a group of minimal order such that $S(P)\neq 0$. This condition forces $S(P)$ to be a $K\mathrm{Out}(P)$-module, which turns out to be simple.*

In particular, when K is a field, we denote by $S_{P,K}$ the simple functor associated to the pair (P,K), where K is the trivial $K\mathrm{Out}(P)$-module.

Recall finally ([Bo1] section 7.1) that there is a morphism $X\mapsto\widetilde{X}$ of K-algebras (with unity) from the Burnside algebra $KB(P)$ of P over K to the algebra $\mathcal{E}nd_{\mathcal{C}_K}(P)$. If X is a P-set, the set

$$\widetilde{X} = X\times P \tag{7.11}$$

is a P-set-P with double action given by

$$g_1\cdot(x,g)\cdot g_2 = (g_1x, g_1gg_2)\,, \qquad \forall x\in X,\ \forall g,g_1,g_2\in P\,.$$

If X is the transitive P-set P/Q, then it is easy to check that $\widetilde{X}$ is a transitive P-set-P isomorphic to $(P\times P)/\Delta(Q)$, where $\Delta(Q)$ is the diagonal inclusion of Q in $P\times P$.

We have the following idempotents of $\mathbb{Z}[\frac{1}{p}]B(P)$, indexed by subgroups Q of P:

$$e_Q^P = \frac{1}{|N_P(Q)|}\sum_{R\leq Q}|R|\,\mu(R,Q)\,(P/R)\,,$$

where μ denotes the Möbius function of the poset of subgroups of P. These idempotents are characterized by the fact that if X is a P-set, then

$$X\cdot e_Q^P = |X^Q|e_Q^P\,.$$

It follows from this characterization that the restriction of e_Q^P to a subgroup R of P is the sum of the idempotents e_S^R associated to the subgroups S of R which are conjugate to Q in P, up to conjugation by R. In particular, the restriction of e_P^P to any proper subgroup of P is zero.

If p is invertible in K, the idempotents e_Q^P have natural images in $KB(P)$, still denoted by e_Q^P. When Q runs over a set of representatives of the conjugacy classes of subgroups of P, the idempotents e_Q^P are mutually orthogonal, and their sum is the identity of $KB(P)$. So we obtain the following result.

(7.12) Proposition. *If p is invertible in K, the elements $\widetilde{e_Q^P}$ are mutually orthogonal idempotents of sum 1 in $\mathcal{E}nd_{\mathcal{C}_K}(P)$.*

For any functor F in the category $\mathcal{F}_K$, one deduces a decomposition of $F(P)$ as a direct sum

$$F(P) = \bigoplus_Q F(\widetilde{e_Q^P})\, F(P)\,,$$

and this will play a crucial role in Section 9.

8. The Burnside functor

One of the main examples of functors in the category $\mathcal{F}_K$ is the Burnside functor KB, where $KB(P)$ is the Burnside algebra of the p-group P over the commutative ring K. When K is a field, the simple functors which appear as composition factors of KB are easier to describe than arbitrary simple functors. It turns out that the functors $\mathbb{Q}D$ and $\overline{D^t}$, which we are interested in, are isomorphic to composition factors of $\mathbb{Q}B$ and $\mathbb{F}_2B$ respectively. In view of this, we need to understand better the composition factors of KB and this is the purpose of this section.

When $K = \mathbb{Q}$, this question was studied in [Bo1] (for all finite groups instead of p-groups). The methods can be modified to find all subfunctors of the functor KB defined on the category $\mathcal{C}_K$ of p-groups, when K is a field of characteristic q different from p. In particular when $K = \mathbb{Q}$, we recover the results of [Bo1], specialized to the category of p-groups. The description of all subfunctors of KB requires some combinatorial computations, involving the constants $m_{G,N}$ defined in (4.6).

It is clear that KB is a functor in the category $\mathcal{F}_K$, since if U is a Q-set-P, we have a K-linear homomorphism

$$KB(U) : KB(P) \to KB(Q)$$

mapping a P-set Y to the Q-set $U \times_P Y$. If such a biset U is transitive, then it decomposes as a product of bisets corresponding to a restriction, a deflation, an isomorphism, an inflation, and an induction (Lemma 7.4).

Since the characteristic q of K is distinct from p, the K-algebra $KB(P)$ is semi-simple for any p-group P and its primitive idempotents are the idempotents e_Q^P defined in the previous section. It is easy to compute the value of the five natural operations above on the primitive idempotents e_Q^P, with the single exception of deflation which requires a little work. We first recall this. Details can be found in [Bo1], page 701.

(8.1) Lemma. *Let P be a p-group, let Q be a subgroup of P, and let N be a normal subgroup of P. Then*

$$\mathrm{Def}^P_{P/N}(e_Q^P) = \frac{|N_P(QN)/QN|}{|N_P(Q)/Q|}\, m_{Q,Q\cap N} \cdot e^{P/N}_{QN/N}\,,$$

where $m_{Q,Q\cap N}$ is the rational number defined in (4.6).

Given a positive integer r, we define the ideal

$$I_r(P) = < e_Q^P \;\mid\; |Q/\Phi(Q)| \geq p^r >,$$

where $\Phi(Q)$ denotes the Frattini subgroup of Q. The condition $|Q/\Phi(Q)| \geq p^r$ means that Q cannot be generated by fewer than r elements, or in other words that the elementary abelian group $Q/\Phi(Q)$ has rank $\geq r$. For suitable values of r, we want to prove that I_r is a subfunctor of KB. These values will be those of the set

$$\mathcal{I} = \{0\} \cup \{r \in \mathbb{N} \mid r > 0,\ p^{r-2} \equiv 1 \ (\mathrm{mod}\ q)\} \cup \{\infty\},$$

where $I_\infty(P)$ denotes the zero ideal of $KB(P)$. If s denotes the order of p modulo q, then $\mathcal{I}$ consists of all positive integers congruent to 2 modulo s (together with 0 and ∞). If $q = 0$, then s has to be understood as ∞ and we have $\mathcal{I} = \{0, 2, \infty\}$ in that case. If q divides $p-1$, then $s = 1$ and $\mathcal{I}$ consists of all positive integers (together with 0 and ∞).

For every $r \in \mathcal{I}$, $r \neq \infty$, we write $s(r)$ for the successor of r for the natural ordering of $\mathcal{I}$. Thus $s(0) = 2$ (unless q divides $p-1$ in which case $s(0) = 1$), and $s(r) = r + s$ if $r > 0$. Finally we denote by E_r an elementary abelian group of rank r.

(8.2) Theorem. *Let K be a field of characteristic $q \neq p$.*

(a) For every $r \in \mathcal{I}$, the family of ideals $I_r(P)$, where P runs over all finite p-groups, forms a subfunctor of KB.

(b) The functors I_r (for $r \in \mathcal{I}$) are the only subfunctors of KB. In other words, the functor KB has a unique filtration

$$\begin{aligned}
&KB = I_0 \supset I_2 \supset I_\infty = \{0\} && \text{if } q = 0,\\
&KB = I_0 \supset I_1 \supset I_2 \supset I_3 \supset \dots && \text{if } q \neq 0 \text{ and } q \mid p-1,\\
&KB = I_0 \supset I_2 \supset I_{2+s} \supset I_{2+2s} \supset \dots && \text{if } q \neq 0 \text{ and } q \nmid p-1.
\end{aligned}$$

(c) If $r \in \mathcal{I}$, $r \neq \infty$, the quotient $I_r/I_{s(r)}$ is isomorphic to the simple functor $S_{E_r,K}$, where K denotes the trivial module.

(d) If P is a p-group, the dimension of $S_{E_r,K}(P)$ is equal to the number of conjugacy classes of subgroups Q of P such that $p^r \leq |Q/\Phi(Q)| < p^{s(r)}$.

Proof. We first prove (b). If F is a non-zero subfunctor of KB, then $F(P)$ is an ideal of $KB(P)$ for any p-group P. This follows from the isomorphism $\widetilde{X} \times_P Y \simeq X \times Y$ whenever X and Y are P-sets. Here $\widetilde{X} \in \mathcal{E}nd_{\mathcal{C}_K}(P)$ is defined by (7.11). It follows that there is a family $\mathcal{A}(P)$ such that

$$F(P) = < e_Q^P \mid Q \in \mathcal{A}(P) > .$$

If Q is a subgroup of P, we know that $\mathrm{Ind}_Q^P\big(F(Q)\big) \subseteq F(P)$ and $\mathrm{Res}_Q^P\big(F(P)\big) \subseteq F(Q)$, since F is a subfunctor of KB. Since $e_Q^P = |N_P(Q) : Q|^{-1}\mathrm{Ind}_Q^P(e_Q^Q)$, and since $\mathrm{Res}_Q^P(e_Q^P) = e_Q^Q$, we conclude that $Q \in \mathcal{A}(P)$ if and only if $Q \in \mathcal{A}(Q)$. In other words, there is a family $\mathcal{A}$ of p-groups such that, for any p-group P,

$$F(P) = < e_Q^P \mid Q \in \mathcal{A}, Q \leq P > .$$

If $N \trianglelefteq P$, we must have $\mathrm{Inf}_{P/N}^P\big(F(P/N)\big) \subseteq F(P)$. Since

$$\mathrm{Inf}_{P/N}^P e_{P/N}^{P/N} = \sum_{\substack{XN=P \\ X \bmod P}} e_X^P ,$$

it follows that if $P/N \in \mathcal{A}$, then $P \in \mathcal{A}$. Finally, let $\mathcal{B}$ be the family of elements of $\mathcal{A}$ having no proper quotient in $\mathcal{A}$. Then of course $\mathcal{A}$ is the family of groups having a quotient in $\mathcal{B}$.

If $P \in \mathcal{B}$, we must have $m_{P,N} = 0$ for every non-trivial normal subgroup N of P. Indeed $\mathrm{Def}^P_{P/N}(e^P_P) = m_{P,N}\, e^{P/N}_{P/N}$ by Lemma 8.1, and if $m_{P,N} \neq 0$, we have $P/N \in \mathcal{A}$, which is impossible since $P \in \mathcal{B}$. Since $m_{P,\Phi(P)}$ is always equal to 1, this proves that P is elementary abelian. If $P \neq \{1\}$ is elementary abelian of order p^r, and if N is a subgroup of order p of P, we have

$$m_{P,N} = \frac{1}{|P|} \sum_{XN=P} |X|\,\mu(X,P) = 1 - p^{r-2}$$

since a proper subgroup X of P such that $XN = P$ is a complement of N, hence is a maximal subgroup of P, so that $\mu(X,P) = -1$. Moreover, there are p^{r-1} such complements.

In other words, the family $\mathcal{B}$ is contained in the family of elementary abelian p-groups of order p^r with either $r = 0$ or $p^{r-2} = 1$ in K. If $\mathcal{B}$ is non-empty, and if r is minimal such that the elementary abelian p-group E_r of order p^r is in $\mathcal{B}$, it follows that $\mathcal{A}$ is the family of p-groups having E_r as a quotient, that is, the family of p-groups P such that $|P/\Phi(P)| \geq p^r$. This proves that any non-zero subfunctor of KB is equal to I_r with either $r = 0$ or $p^{r-2} \equiv 1 \pmod{q}$.

(a) We have to prove that any such I_r is indeed a subfunctor of KB. This is trivial if $r = 0$ since $I_0 = KB$. If now $r > 0$, it should be clear from the previous remarks that I_r is stable under induction and restriction. It is invariant under inflation, because

$$\mathrm{Inf}^P_{P/N} e^{P/N}_{Q/N} = \sum_{\substack{XN=Q \\ X \bmod N_P(U)}} e^P_X\,,$$

and moreover, if $XN = Q$ and if Q/N has an elementary abelian quotient of rank $\geq r$, then so does X since $X/X \cap N \cong Q/N$. In order to check that I_r is stable under deflation, let Q be a subgroup of P such that $|Q/\Phi(Q)| \geq p^r$ and let N be a normal subgroup of P. By Lemma 8.1, we have

$$\mathrm{Def}^P_{P/N}(e^P_Q) = \frac{|N_P(QN)/QN|}{|N_P(Q)/Q|}\, m_{Q,Q\cap N}\, e^{P/N}_{QN/N}\,,$$

and this is non-zero if and only if $m_{Q,Q\cap N}$ is non-zero. Let $M = Q \cap N$ and $R = Q/M \cong QN/N$. Finally, all we have to check is that if Q is a p-group such that $|Q/\Phi(Q)| \geq p^r$, if M is a normal subgroup of Q such that $m_{Q,M} \neq 0$, and if $R = Q/M$, then we have $|R/\Phi(R)| \geq p^r$. As in the proof of Lemma 4.7, we have

$$m_{Q,M} = m_{Q,M\Phi(Q)} = m_{Q/\Phi(Q),M\Phi(Q)/\Phi(Q)}$$

so it is enough to suppose that Q is elementary abelian. If $|Q| = p^s$ and $|M| = p^t$, then we have $s \geq r$ by hypothesis, and we can suppose $t > 0$. Moreover, by (4.8), we have

$$m_{Q,M} = (1 - p^{s-2})(1 - p^{s-3}) \ldots (1 - p^{s-t-1})\,.$$

If $s - t - 1 \leq r - 2$, this is zero in K. Hence $s - t \geq r$, which means that $|R/\Phi(R)| \geq p^r$.

(c) It follows from (b) that each non-zero subfunctor I_r has a unique maximal subfunctor $I_{s(r)}$ and so the quotient $I_r/I_{s(r)}$ is a simple functor S_r. Clearly E_r is a minimal group for S_r, and $S_r(E_r)$ is one-dimensional, generated by $e^{E_r}_{E_r}$, which is invariant by any automorphism of E_r. So S_r is the simple functor $S_{E_r,K}$.

(d) If P is a p-group, it is clear that the dimension of $I_r(P)$ is equal to the number of conjugacy classes of subgroups Q of P such that $p^r \leq |Q/\Phi(Q)|$, and similarly for $I_{s(r)}(P)$. Therefore the dimension of

$$S_{E_r,K}(P) = I_r(P)/I_{s(r)}(P)$$

is equal to the number of conjugacy classes of subgroups Q of P such that $p^r \leq |Q/\Phi(Q)| < p^{s(r)}$. □

(8.3) Corollary. *Suppose K is a field of characteristic 0.*

(a) There is a non-split exact sequence of K-linear functors

$$0 \longrightarrow S_{E_2,K} \longrightarrow KB \longrightarrow S_{1,K} \longrightarrow 0 .$$

(b) If P is a p-group, the dimension of $S_{1,K}(P)$ is the number of conjugacy classes of cyclic subgroups of P, and the dimension of $S_{E_2,K}(P)$ is the number of conjugacy classes of non-cyclic subgroups of P.

(c) The functor $S_{1,K}$ is isomorphic to $KR_{\mathbb{Q}}$ and the right hand side morphism in (a) can be chosen to be the natural morphism $KB \to KR_{\mathbb{Q}}$ (mapping a P-set to its permutation $\mathbb{Q}P$-module).

Proof. (a) The set $\mathcal{I}$ in Theorem 8.2 is equal to $\{0, 2, \infty\}$. This means that the functor KB has a unique proper non-zero subfunctor $S_{E_2,K}$, with quotient isomorphic to $S_{1,K}$.

(b) By Theorem 8.2, the dimension of $S_{1,K}(P)$ is the number of conjugacy classes of subgroups Q such that $1 \leq |Q/\Phi(Q)| < p^2$, that is, the cyclic subgroups of P. On the other hand the dimension of $S_{E_2,K}(P)$ is the number of conjugacy classes of subgroups of P having a quotient isomorphic to E_2. Those subgroups are precisely the non-cyclic subgroups of P.

(c) There is a natural morphism $\varepsilon : KB \to KR_{\mathbb{Q}}$ such that $\varepsilon(P) : KB(P) \to KR_{\mathbb{Q}}(P)$ maps a P-set X to the corresponding permutation $\mathbb{Q}P$-module $\mathbb{Q}X$. Now it is well-known that the dimension of $KR_{\mathbb{Q}}(P)$ is equal to the number of conjugacy classes of cyclic subgroups of P, so that

$$\dim_K \big(KR_{\mathbb{Q}}(P)\big) = \dim_K \big(S_{1,K}(P)\big)$$

by the theorem. It follows that ε must be surjective (actually a well-known easy fact!) and the functor $S_{1,K}$ must be isomorphic to $KR_{\mathbb{Q}}$. $\square$

Note that Corollary 8.3 is a special case of the analysis of the composition factors of KB made in [Bo1] for *all* finite groups (see [Bo1], Propositions 10 and 12, and Example 4 page 709).

(8.4) Corollary. *Suppose K is a field of characteristic q dividing $p-1$.*

(a) The functor KB has a unique filtration

$$KB = I_0 \supset I_1 \supset I_2 \supset I_3 \supset \ldots$$

and for every $r \geq 0$, the simple functor $S_{E_r,K}$ is a composition factor of KB.

(b) If P is a p-group, the dimension of $S_{E_r,K}(P)$ is the number of conjugacy classes of subgroups Q of P such that $Q/\Phi(Q)$ has order p^r. In particular, the dimension of $S_{E_1,K}(P)$ is the number of conjugacy classes of non-trivial cyclic subgroups of P.

(c) $S_{1,K}$ is isomorphic to the constant functor Γ_K with $\Gamma_K(P) = K$ for every P.

(d) $KR_{\mathbb{Q}}$ is isomorphic to a quotient functor of KB and there is a non-split exact sequence of K-linear functors

$$0 \longrightarrow S_{E_1,K} \longrightarrow KR_{\mathbb{Q}} \longrightarrow \Gamma_K \longrightarrow 0 .$$

Proof. (a) This follows directly from Theorem 8.2 since $\mathcal{I} = \mathbb{N} \cup \{\infty\}$ when q divides $p-1$.

(b) This follows immediately from Theorem 8.2.

(c) This follows from (b) by taking $r = 0$.

(d) There is again a morphism $\varepsilon : KB \to KR_{\mathbb{Q}}$, mapping a P-set to the corresponding permutation $\mathbb{Q}P$-module. The kernel of this morphism must be one of the subfunctors I_r of KB. It cannot be $I_0 = KB$ since ε is non-zero (the trivial P-set is not mapped to zero). It cannot be I_1, since otherwise the image of ε would be the simple functor $S_{1,K} = \Gamma_K$, which is one dimensional in every evaluation. But for the cyclic group C of order p, the image of $\varepsilon(C)$ is $KR_{\mathbb{Q}}(C)$ and has dimension 2 (since the characters $\mathrm{Ind}_1^C 1$ and 1 are not proportional modulo q). Finally, the kernel of ε cannot be one of the subfunctors I_r for $r \geq 3$, for reasons of dimension. Indeed the dimension of $KR_{\mathbb{Q}}(P)$ is the number of conjugacy classes of cyclic subgroups of P, and this is also the dimension of $KB(P)/I_2(P)$.

It follows that the kernel of ε is I_2, and that ε is surjective. Thus $KR_{\mathbb{Q}}$ is a quotient of KB, having a unique non-zero proper subfunctor, isomorphic to $I_1/I_2 \cong S_{E_1,K}$. The quotient functor is the constant functor Γ_K (and the morphism $KR_{\mathbb{Q}} \to \Gamma_K$ is actually the reduction modulo q of the dimension). □

We note that the case $q \mid p-1$ is the only one where a constant functor Γ_K exists.

9. A characterization of some simple functors

We prove in this section that the simple functors $S_{E,K}$, where E is an elementary abelian p-group and K is a field of characteristic different from p, can be characterized in some precise way by their values on elementary abelian groups. This is important for our purposes since both $\mathbb{Q}D$ and $\overline{D^t}$ turn out to be simple functors of this type.

Recall that, for every p-group A and every $K\mathrm{Out}(A)$-module V, we have

$$S_{A,V} = L_{A,V}/J_{A,V} ,$$

where $L_{A,V}$ and $J_{A,V}$ are defined by (7.7) and (7.8). We first need two lemmas.

(9.1) Lemma. *Let A be an abelian p-group and let V be a $K\mathrm{Out}(A)$-module. If P is a p-group such that $L_{A,V}(P) \neq 0$, then the group A is isomorphic to a section of P.*

Proof. If $L_{A,V}(P) \neq 0$, then there exists elements $\varphi_i \in \mathrm{Hom}_{\mathcal{C}_K}(A,P)$ and elements $v_i \in V$, for $1 \leq i \leq n$, such that

$$\sum_{i=1}^{n} \varphi_i \otimes v_i \neq 0 \qquad \text{in} \quad L_{A,V}(P) = \mathrm{Hom}_{\mathcal{C}_K}(A,P) \otimes_{\mathrm{End}_{\mathcal{C}_K}(P)} V .$$

Moreover, by linearity, we can suppose that the elements φ_i are transitive P-sets-A. So there exists such a transitive biset φ and an element $v \in V$ such that $\varphi \otimes v$ is non-zero in $L_{A,V}(P)$. There is a subgroup L of $P \times A$ such that $\varphi = (P \times A)/L$. Using the notation 7.3, we let

$$B = L_p/L_k \cong {}_pL/{}_kL ,$$

so that there are surjective morphisms $s : {}_pL \to B$ with kernel ${}_kL$ and $t : L_p \to B$ with kernel L_k such that

$$L = \{(g,a) \in P \times A \mid g \in {}_pL,\ a \in L_p,\ s(g) = t(a)\} .$$

It is easy to see from Lemma 7.5 that

$$\varphi = (P \times A)/L = (P \times B)/M \times_B (B \times A)/N , \tag{9.2}$$

where

$$M=\{(g,s(g))\mid g\in {}_pL\}\subseteq P\times B \qquad \text{and} \qquad N=\{(t(a),a)\mid a\in L_p\}\subseteq B\times A\,.$$

Now the group $B=A'/A''$ is a section of the abelian group A, where we have set $A'=L_p$ and $A''=L_k$. The groups A, B, A', and A'' are all abelian, hence isomorphic to their dual (or character group). As A' is a subgroup of A, its dual $\hat{A}'\cong A'$ is a quotient of $\hat{A}\cong A$. Now B is a quotient of A', hence also a quotient of A. Thus there exists a surjective morphism $u:A\to B$.

Now set

$$R=\{(a,u(a))\mid a\in A\}\subseteq A\times B \qquad \text{and} \qquad S=\{(u(a),a)\mid a\in A\}\subseteq B\times A\,.$$

It follows easily from Lemma 7.5 that

$$(B\times A)/S\times_A(A\times B)/R=(B\times B)/\{(b,b)\mid b\in B\}\,,$$

which is the identity endomorphism of B in the category $\mathcal{C}_K$ (this is inflation followed by deflation). We can insert this relation in (9.2), and this gives

$$\varphi=(P\times B)/M\times_B(B\times A)/S\times_A(A\times B)/R\times_B(B\times A)/N\,.$$

Setting

$$\psi=(P\times B)/M\times_B(B\times A)/S \qquad \text{and} \qquad \theta=(A\times B)/R\times_B(B\times A)/N\,,$$

we have $\varphi=\psi\circ\theta$. Now if $|B|<|A|$, the endomorphism θ of A is in the ideal I_A defined in (7.9), since it factors through B. Therefore its action on the $K\mathrm{Out}(A)$-module V is zero, and it follows that in $L_{A,V}(P)$, we have

$$\varphi\otimes v=(\psi\circ\theta)\otimes v=\psi\otimes\theta\cdot v=0\,.$$

This shows that $|B|=|A|$, hence $B\cong A$. Now B is also a section of P, and the lemma follows. □

(9.3) Lemma. *Let K be a field of characteristic different from p, let A be an elementary abelian p-group, and let V be a $K\mathrm{Out}(A)$-module. Then any subfunctor F of $L_{A,V}$ is generated by its values on elementary abelian p-groups, that is,*

$$F(P)=\sum_{\substack{E\ \ p-\text{elementary abelian}\\ \varphi:E\to P}}F(\varphi)\big(F(E)\big)\,,$$

for every p-group P.

Proof. We prove the lemma by induction on the order of P, the result being trivial if P itself is elementary abelian. Let P be an arbitrary p-group, and suppose the result holds for all groups of order strictly smaller than $|P|$. Let $f\in F(P)$. By Proposition 7.12, we can decompose f as

$$f=\sum_{\substack{Q\leq P\\ Q \bmod P}}F(\widetilde{e_Q^P})(f)\,.$$

Recall that

$$\widetilde{e_Q^P}=\frac{1}{|N_P(Q)|}\sum_{R\leq Q}|R|\,\mu(R,Q)\,(P\times P)/\Delta(R)\,.$$

This expression shows that if Q is a proper subgroup of P, then $\widetilde{e_Q^P}$ is a linear combination of elements of type $(P\times P)/\Delta(R)$, where R is a subgroup of Q, hence a proper subgroup of P. By Lemma 7.5, the biset $(P\times P)/\Delta(R)$ factorizes as

$$(P\times P)/\Delta(R)=(P\times R)/\Delta(R)\times_R (R\times P)/\Delta(R)\,,$$

and it follows that

$$F\big((P\times P)/\Delta(R)\big)(f)\in F\big((P\times R)/\Delta(R)\big)\big(F(R)\big)\,.$$

By induction hypothesis, this gives

$$F\big((P\times P)/\Delta(R)\big)(f)\in F\big((P\times R)/\Delta(R)\big)\left(\sum_{\varphi:E\to R}F(\varphi)\big(F(E)\big)\right)\subseteq\sum_{\psi:E\to P}F(\psi)\big(F(E)\big)\,.$$

It follows that we only have to deal with the term $F(\widetilde{e_P^P})(f)$, so we can suppose that $f=F(\widetilde{e_P^P})(f)$. On the other hand, the element f is in $F(P)\subseteq L_{A,V}(P)$, so it can be written

$$f=\sum_{i=1}^n \varphi_i\otimes v_i\,,$$

for suitable elements $v_i\in V$ and $\varphi_i\in \mathrm{Hom}_{\mathcal{C}_K}(A,P)$, that we can suppose to be transitive P-sets-A. Now by definition of the functor $L_{A,V}$, we have

$$f=F(\widetilde{e_P^P})(\sum_{i=1}^n\varphi_i\otimes v_i)=\sum_{i=1}^n(\widetilde{e_P^P}\times_P\varphi_i)\otimes v_i\,.$$

There exist subgroups L_i of $P\times A$ such that $\varphi_i=(P\times A)/L_i$. Using the notation 7.3 and setting $B_i=(L_i)_p/(L_i)_k$, there are subgroups M_i of $P\times B_i$ and N_i of $B_i\times A$ such that

$$\varphi_i=(P\times A)/L_i=(P\times B_i)/M_i\times_{B_i}(B_i\times A)/N_i\,,$$

and it follows that

$$(\widetilde{e_P^P}\times_P\varphi_i)\otimes v_i=L_{A,V}\big(\widetilde{e_P^P}\circ(P\times B_i)/M_i\big)\Big(\big((B_i\times A)/N_i\big)\otimes v_i\Big)\,.$$

Now the element $\big((B_i\times A)/N_i\big)\otimes v_i$ is in $L_{A,V}(B_i)$, which is zero if A is not a section of B_i by Lemma 9.1. Moreover, if A is a section of B_i, as B_i is also a section of A, we have $A\cong B_i$.

In this case there is a subgroup P_i of P and a surjective homomorphism $s_i:P_i\to A$ such that

$$L_i=\{\big(g,s_i(g)\big)\mid g\in P_i\}\subseteq P\times A\,,$$

and we have the factorization

$$\varphi_i=(P\times A)/L_i=(P\times P_i)/\Delta(P_i)\times_{P_i}(P_i\times A)/L_i'$$

where L_i' is the group L_i, viewed as a subgroup of $P_i\times A$. Now Proposition 14 of [Bo1] shows that

$$\widetilde{e_P^P}\times_P(P\times P_i)/\Delta(P_i)=(P\times P_i)/\Delta(P_i)\times_{P_i}\widetilde{\mathrm{Res}_{P_i}^P e_P^P}\,,$$

and this is zero if $P_i \neq P$. So we have $P_i = P$ for all i and

$$\varphi_i = (P \times A)/\Delta_{s_i}(P)\,, \qquad \text{where } \Delta_{s_i}(P) = \{\big(g, s_i(g)\big) \mid g \in P\}\ .$$

Therefore we obtain

$$\begin{aligned}
\widetilde{e_P^P} \times_P \varphi_i &= \frac{1}{|P|} \sum_{R \leq P} |R|\, \mu(R,P)\, (P \times P)/\Delta(R) \times_P (P \times A)/\Delta_{s_i}(P) \\
&= \frac{1}{|P|} \sum_{R \leq P} |R|\, \mu(R,P)\, (P \times A)/\Delta_{s_i}(R)\,,
\end{aligned}$$

where $\Delta_{s_i}(R) = \{\big(r, s_i(r)\big) \mid g \in R\}$. The sum only runs over subgroups R such that $\mu(R,P) \neq 0$, so that R contains the Frattini subgroup Φ of P. Setting $\overline{P} = P/\Phi$ and $\overline{R} = R/\Phi$, the surjection $s_i : P \to A$ factorizes as the projection $\pi : P \to \overline{P}$ followed by a surjection $\overline{s}_i : \overline{P} \to A$, because A is elementary abelian. It is easy to check that

$$(P \times A)/\Delta_{s_i}(R) = (P \times \overline{P})/\Delta_\pi(P) \times_{\overline{P}} (\overline{P} \times A)/\Delta_{\overline{s}_i}(\overline{R})\,,$$

where $\Delta_\pi(P) = \{\big(g, \pi(g)\big) \mid g \in P\}$ and $\Delta_{\overline{s}_i}(\overline{R}) = \{\big(\overline{r}, \overline{s}_i(\overline{r})\big) \mid \overline{r} \in \overline{R}\}$. Therefore we obtain

$$\begin{aligned}
f &= \sum_{i=1}^{n} (\widetilde{e_P^P} \times_P \varphi_i) \otimes v_i \\
&= \sum_{i=1}^{n} \frac{1}{|P|} \sum_{R \leq P} |R|\, \mu(R,P)\, \big((P \times A)/\Delta_{s_i}(R)\big) \otimes v_i \\
&= \sum_{i=1}^{n} \Big[(P \times \overline{P})/\Delta_\pi(P) \times_{\overline{P}} \big(\frac{1}{|P|} \sum_{R \leq P} |R|\, \mu(R,P)\, (\overline{P} \times A)/\Delta_{\overline{s}_i}(\overline{R})\big)\Big] \otimes v_i \\
&= L_{A,V}\big((P \times \overline{P})/\Delta_\pi(P)\big)\,(\overline{f}) \\
&= \mathrm{Inf}_{\overline{P}}^{P}(\overline{f})\,,
\end{aligned}$$

where $\overline{f} \in L_{A,V}(\overline{P})$ denotes the expression

$$\overline{f} = \frac{1}{|P|} \sum_{R \leq P} |R|\, \mu(R,P)\, \big((\overline{P} \times A)/\Delta_{\overline{s}_i}(\overline{R})\big) \otimes v_i\ .$$

But we have already noticed that inflation followed by deflation is the identity, that is,

$$(\overline{P} \times P)/\{\big(\pi(g), g\big) \mid g \in P\} \times_P (P \times \overline{P})/\{\big(g, \pi(g)\big) \mid g \in P\} = (\overline{P} \times \overline{P})/\{(\overline{g}, \overline{g}) \mid \overline{g} \in \overline{P}\} = id_{\overline{P}}\,,$$

as morphisms in $\mathcal{C}_K$. Therefore $\overline{f} = \mathrm{Def}_{\overline{P}}^{P} \mathrm{Inf}_{\overline{P}}^{P}(\overline{f}) = \mathrm{Def}_{\overline{P}}^{P}(f)$, and so $\overline{f} \in F(\overline{P})$ since $f \in F(P)$ and F is a subfunctor of $L_{A,V}$. Hence

$$f = \mathrm{Inf}_{\overline{P}}^{P}(\overline{f}) \in F(\varphi)(\overline{P})\,,$$

for some morphism $\varphi : \overline{P} \to P$, as was to be shown. □

Now we can state our characterization of the simple functors $S_{E,K}$.

(9.4) Theorem. *Let K be a field of characteristic different from p, let E be an elementary abelian p-group, and let F be an object of $\mathcal{F}_K$.*

(a) Suppose F has the following two properties:

(1) $F(E) \cong K$ is the trivial $K\mathrm{Out}(E)$-module.

(2) If P is elementary abelian, $\dim_K\big(F(P)\big) = \dim_K\big(S_{E,K}(P)\big)$.

Then the functor F admits a direct summand isomorphic to $S_{E,K}$.

(b) Suppose moreover that the following condition holds:

(3) For any p-group P, the map

$$\prod_{\varphi} F(\varphi) : F(P) \to \prod_{\substack{A \quad p-\text{elementary abelian} \\ \varphi:P\to A}} F(A)$$

is injective.

Then the functor F is isomorphic to the simple functor $S_{E,K}$.

Proof. Since $S_{E,K}$ vanishes on proper sections of E, hypothesis (2) implies that $F(P) = 0$ if P is a proper section of E. Therefore the ideal I_P of $\mathcal{E}nd_{\mathcal{C}_K}(P)$ acts by zero (see (7.9)) and so $F(E)$ is not only isomorphic to K as $K\mathrm{Out}(E)$-module (hypothesis (1)), but also as $\mathcal{E}nd_{\mathcal{C}_K}(P)$-module. Therefore, by adjunction, there is a morphism $\theta : L_{E,K} \to F$ which is the identity of K when evaluated at E. Hence the kernel of θ is contained in the unique maximal proper subfunctor $J = J_{E,K}$ of $L_{E,K}$. Now the image F' of θ contains a submodule $F'' = \theta(J)$ such that $F'/F'' \cong S_{E,K}$. Hypothesis (2) implies that for any elementary abelian p-group A, we have $F'(A) = F(A)$ and $F''(A) = 0$.

But for any p-group P, we know from Lemma 9.3 that

$$J(P) = \sum_{\varphi:A\to P} J(\varphi)\big(J(A)\big),$$

where the sum runs over the elementary abelian groups A, and morphisms $\varphi : A \to P$ in $\mathcal{C}_K$. Taking images by θ, we have

$$F''(P) = \theta_P\big(J(P)\big) = \sum_{\varphi:A\to P} \theta_P J(\varphi)\big(J(A)\big) = \sum_{\varphi:A\to P} F''(\varphi)\theta_A\big(J(A)\big) = \sum_{\varphi:A\to P} F''(\varphi)F''(A) = 0.$$

Hence $F'' = 0$, and F' is isomorphic to $S_{E,K}$, so we have an injection $i : S_{E,K} \to F$.

The next observation is that there is a natural notion of duality in $\mathcal{F}_K$. If F is any K-linear functor from $\mathcal{C}_K$ to K-vector spaces, the dual F^* of F is defined on each object P by

$$F^*(P) = \mathrm{Hom}_K\big(F(P), K\big).$$

If Q is another p-group, and $\varphi : P \to Q$ is a Q-set-P, then one defines $F^*(\varphi) = F(\varphi^{op})^*$, where $F(\varphi^{op})^*$ is the transpose map of $F(\varphi^{op})$. This definition extends by linearity. There is a canonical morphism of functors δ from F to its bidual F^{**}, defined for a p-group P by

$$\delta_P(x)(\varphi) = \varphi(x), \qquad \forall x \in F(P),\ \forall \varphi \in F(P)^*.$$

The map δ_P is an isomorphism if $F(P)$ is finite dimensional over K.

It is clear that the dual of a simple functor is simple, and the classification of simple functors shows that the functor $S_{E,K}$ is self dual. Now hypotheses (1) and (2) hold clearly for F^* if they do for F. Our previous argument shows therefore that $S_{E,K}$ is a subfunctor of F^*. By duality, we obtain a surjective morphism $s : F^{**} \to S_{E,K}$. Now $s \circ \delta \circ i$ is an endomorphism of $S_{E,K}$, which is non-zero since its evaluation at E is the identity of K. As $S_{E,K}$ is simple, it follows that $s \circ \delta \circ i$ is an isomorphism, so that $S_{E,K}$ is a direct summand of F, proving (a).

Now we can write $F \cong S_{E,K} \oplus X$. Hypothesis (2) shows that $X(P) = 0$ if P is elementary abelian. Since hypothesis (3) holds for any subfunctor of F if it does for F, it follows that for any p-group P, the map

$$\prod_{\varphi} X(\varphi) : X(P) \to \prod_{\substack{A \;\; p-\text{elementary abelian} \\ \varphi: P \to A}} X(A)$$

is injective. As the right hand side is zero, we have $X(P) = 0$, hence $X = 0$ and F is isomorphic to $S_{E,K}$, proving (b). $\square$

10. The torsion-free part of the Dade functor

The purpose of this section is to prove Theorems C and D of the introduction. For convenience, we first restate Theorem C.

(10.1) Theorem. *The functor $\mathbb{Q}D$ is simple and is isomorphic to $S_{E,\mathbb{Q}}$, where E is an elementary abelian group of order p^2.*

We shall give two independent proofs of this theorem. The first uses the main theorem of Section 4, which gives the dimension of $\mathbb{Q}D(P)$. Our second proof is more conceptual and involves more of the functorial formalism. It is based on the characterization of $S_{E,\mathbb{Q}}$ obtained in Section 9 and it is independent of the results of Section 4. In both cases we need to apply the results of Section 8 giving the dimension of $S_{E,\mathbb{Q}}(P)$, though in the second proof this is only used when P is elementary abelian.

First proof of Theorem 10.1. Clearly if we evaluate $\mathbb{Q}D$ at E, we get $\mathbb{Q}D(E) \cong \mathbb{Q}$ and this isomorphism is an isomorphism of $\mathbb{Q}\mathrm{Out}(E)$-modules, since $\mathbb{Q}D(E)$ is generated over $\mathbb{Q}$ by Ω_E, which is invariant under the action of any automorphism of E. Moreover, the group E is minimal for $\mathbb{Q}D$, so in fact the ideal I_E of $\mathcal{E}nd_{\mathcal{C}_\mathbb{Q}}(E)$ acts by zero and $\mathbb{Q}D(E)$ is isomorphic to $\mathbb{Q}$ as $\mathcal{E}nd_{\mathcal{C}_\mathbb{Q}}(E)$-module, see (7.9).

By adjunction, we now have a morphism of functors $\theta : L_{E,\mathbb{Q}} \to \mathbb{Q}D$, see (7.7). This morphism is non-zero, so its kernel is contained in the unique maximal proper subfunctor $J_{E,\mathbb{Q}}$ of $L_{E,\mathbb{Q}}$, see (7.8). This shows that the image of θ has a quotient isomorphic to $S_{E,\mathbb{Q}} = L_{E,\mathbb{Q}}/J_{E,\mathbb{Q}}$.

But we know from Theorem 4.1 that for any p-group P, the dimension of $\mathbb{Q}D(P)$ is equal to the number $nc(P)$ of conjugacy classes of non-cyclic subgroups of P. On the other hand, we have shown in Corollary 8.3 that the dimension of $S_{E,\mathbb{Q}}(P)$ is also $nc(P)$. Since the image of θ_P has a quotient isomorphic to $S_{E,\mathbb{Q}}(P)$, it follows that θ_P is surjective, and that $\mathbb{Q}D$ is isomorphic to $S_{E,\mathbb{Q}}$. $\square$

Second proof of Theorem 10.1. We want to apply Theorem 9.4 for $K = \mathbb{Q}$, for the functor $F = \mathbb{Q}D$ and for the elementary abelian p-group E of order p^2. So we need to check the three hypotheses of Theorem 9.4. We first note that $\mathbb{Q}D(E) \cong \mathbb{Q}$ is the trivial $\mathbb{Q}\mathrm{Out}(E)$-module since it is generated by Ω_E, which is invariant under automorphisms of E. Thus hypothesis (1) holds.

If P is elementary abelian, then the dimension of $\mathbb{Q}D(P)$ is given by Dade's Theorem 1.4, and is equal to the number of non-cyclic quotients of P, which is equal to the number of non-cyclic subgroups of P, by duality. But by Corollary 8.3, the dimension of $S_{E,\mathbb{Q}}(P)$ is equal to the number of non-cyclic subgroups of P. Hence hypothesis (2) holds for $\mathbb{Q}D$.

Finally hypothesis (3) also holds for $\mathbb{Q}D$, by Theorem 1.6. We conclude that $\mathbb{Q}D$ is isomorphic to $S_{E,\mathbb{Q}}$, by Theorem 9.4. □

It is now straightforward to explain our original proof of Theorem A (giving the dimension of $\mathbb{Q}D(P)$), which was based on Theorem 10.1 (and its second proof above). It suffices to notice that the dimension of $\mathbb{Q}D(P)$ is that of $S_{E,\mathbb{Q}}(P)$, which is known to be the number of conjugacy classes of non-cyclic subgroups of P, by Corollary 8.3.

(10.2) Remark. In Section 4, we have encountered the homomorphism

$$\beta^P_{Q,N} = \frac{1}{|Q|} \sum_{\substack{X \leq Q \\ XN=Q}} |X|\,\mu(X,Q)\,\mathrm{Teninf}^P_{X/(X\cap N)} : \mathbb{Q} \to \mathbb{Q}D(P)\,,$$

where P is a p-group, $Q \leq P$, and $N \trianglelefteq Q$ is such that Q/N is elementary abelian of order p^2. Clearly, this is defined by a formula analogous to the expression for the idempotents of the Burnside ring. The explicit connection is made by means of the action of the Burnside algebra $KB(Q)$ on $\mathbb{Q}D(Q)$ (see Proposition 7.12). Indeed we have

$$\beta^P_{Q,N} = \mathrm{Ten}^P_Q \circ \widetilde{e^Q_Q} \circ \mathrm{Inf}^Q_{Q/N}\,,$$

by a direct computation (using the bisets corresponding to Ten^P_Q and $\mathrm{Inf}^Q_{Q/N}$, together with the fact that $\mathbb{Q}D$ vanishes on proper sections of Q/N). However, the fact that $\beta^P_{Q,N}$ is independent of N (Proposition 4.9) still requires arguments involving the numbers $m_{Q,N}$ (see the proof of Proposition 4.9 or [Bo1]). On the other hand we also mentioned in Remark 4.3 the homomorphism

$$\delta^P_{Q,N} = \frac{1}{|Q|} \sum_{\substack{X \leq Q \\ XN=Q}} |X|\,\mu(X,Q)\,\mathrm{Defres}^P_{X/(X\cap N)} : \mathbb{Q}D(P) \to \mathbb{Q}\,,$$

which is dual to $\beta^P_{Q,N}$, and we claimed that it is also independent of N. The proof of this is now straightforward. It simply follows from the fact that $\mathbb{Q}D$ is self-dual, because $S_{E,\mathbb{Q}}$ is self-dual, and the duality transforms $\delta^P_{Q,N}$ into $\beta^P_{Q,N}$ and conversely.

(10.3) Remark. The self-dual property of $\mathbb{Q}D$ means that there exist non-degenerate bilinear forms

$$<-,->_P \;:\; \mathbb{Q}D(P) \times \mathbb{Q}D(P) \longrightarrow \mathbb{Q}\,,$$

which satisfy suitable functorial relations (i.e. Ten^P_Q is adjoint of Res^P_Q and $\mathrm{Def}^P_{P/N}$ is adjoint of $\mathrm{Inf}^P_{P/N}$). The forms are symmetric because they are determined by the form $<-,->_E$ (by simplicity of $\mathbb{Q}D \cong S_{E,\mathbb{Q}}$) and $\mathbb{Q}D(E)$ is one-dimensional. It would be useful to have a natural description of the form $<-,->_P$ on endo-permutation modules, but we can only describe it on the basis of $\mathbb{Q}D(P)$ given in Proposition 4.11.

Now we come to Theorem D of the introduction.

(10.4) Theorem. *There is an exact sequence of functors*

$$0 \longrightarrow \mathbb{Q}D \xrightarrow{\alpha} \mathbb{Q}B \xrightarrow{\varepsilon} \mathbb{Q}R_{\mathbb{Q}} \longrightarrow 0$$

where for every p-group P, $\varepsilon(P) : \mathbb{Q}B(P) \to \mathbb{Q}R_{\mathbb{Q}}(P)$ is the morphism mapping a P-set to the corresponding permutation module over $\mathbb{Q}$.

Proof. Corollary 8.3 shows that $\mathbb{Q}B$ has a unique non-zero proper subfunctor I_2, isomorphic to $S_{E,\mathbb{Q}}$, and that the quotient functor is isomorphic to $\mathbb{Q}R_{\mathbb{Q}}$ via the morphism ε. The result follows from the isomorphism $\mathbb{Q}D \cong S_{E,\mathbb{Q}}$. □

(10.5) Remark. We have no natural description of the map α in terms of arbitrary endo-permutation modules, but we can give an explicit description of the map on a basis of $\mathbb{Q}D(P)$. We first note that α is unique up to a scalar multiple, because $\mathbb{Q}D \cong S_{E,\mathbb{Q}}$ is simple and its endomorphism ring is isomorphic to the endomorphism ring of $\mathbb{Q}$ as $\mathbb{Q}\mathrm{Out}(E)$-module, by adjunction. In other words, α is completely determined by its evaluation at E

$$\alpha_E : \mathbb{Q}D(E) \cong \mathbb{Q} \longrightarrow \mathbb{Q}B(E) .$$

Recall that the image of α is the subfunctor I_2 defined in Section 8 and that $I_2(E)$ is one-dimensional, generated by the primitive idempotent e_E^E of $\mathbb{Q}B(E)$. Thus we specify α_E by mapping Ω_E to e_E^E.

Now for an arbitrary p-group P, we know that $\mathbb{Q}D(P)$ has a basis consisting of the elements $\mathrm{Ten}_Q^P\mathrm{Inf}_{Q/Q_0}^Q(\Omega_{Q/Q_0})$, where Q runs over the non-cyclic subgroups of P up to conjugation, and Q_0 denotes a fixed normal subgroup of Q such that $Q/Q_0 \cong E$. Since α is functorial, we immediately deduce that

$$\alpha_P\big(\mathrm{Ten}_Q^P\mathrm{Inf}_{Q/Q_0}^Q(\Omega_{Q/Q_0})\big) = \mathrm{Ind}_Q^P\,\mathrm{Inf}_{Q/Q_0}^Q(e_{Q/Q_0}^{Q/Q_0}) = \mathrm{Ind}_Q^P\Big(\sum_{\substack{XQ_0=Q\\ X \bmod Q}} e_X^Q\Big)$$

$$= \mathrm{Ind}_Q^P\Big(\sum_{\substack{X\leq Q\\ XQ_0=Q}} \frac{1}{|Q:N_Q(X)|}\, e_X^Q\Big) = \sum_{\substack{X\leq Q\\ XQ_0=Q}} \frac{|N_P(X)|}{|Q|}\, e_X^P .$$

This can also be written in terms of the standard basis of $\mathbb{Q}B(P)$ by expanding e_X^P.

11. Partial results on the torsion functor

In this final section, we consider the functor $\overline{D^t}$ and prove results analogous to those proved for $\mathbb{Q}D$ in the previous section. We suppose that p is an odd prime.

(11.1) Theorem. *If p is odd, the functor $\overline{D^t}$ is simple and is isomorphic to $S_{C,\mathbb{F}_2}$, where C is a cyclic group of order p.*

We shall again give two independent proofs of this theorem. The first uses the main theorem of Section 6, which gives the dimension of $\overline{D^t}(P)$, while the second uses more functorial formalism, since it is based on the characterization of $S_{C,\mathbb{F}_2}$ obtained in Section 9.

First proof of Theorem 11.1. Clearly if we evaluate $\overline{D^t}$ at C, we get $\overline{D^t}(C) \cong \mathbb{F}_2$ and this has to be an isomorphism of $\mathbb{Q}\mathrm{Out}(C)$-modules. Moreover, the group C is minimal for $\overline{D^t}$, so in fact the ideal I_C of $\mathcal{E}nd_{\mathcal{C}_{\mathbb{F}_2}}(C)$ acts by zero and $\overline{D^t}(C)$ is isomorphic to $\mathbb{F}_2$ as $\mathcal{E}nd_{\mathcal{C}_{\mathbb{F}_2}}(C)$-module, see (7.9).

By adjunction, we now have a morphism of functors $\theta : L_{C,\mathbb{F}_2} \to \overline{D^t}$, see (7.7). This morphism is non-zero, so its kernel is contained in the unique maximal proper subfunctor $J_{C,\mathbb{F}_2}$ of $L_{C,\mathbb{F}_2}$, see (7.8). This shows that the image of θ has a quotient isomorphic to $S_{C,\mathbb{F}_2} = L_{C,\mathbb{F}_2}/J_{C,\mathbb{F}_2}$.

But we know from Theorem 6.1 that for any p-group P, the dimension of $\overline{D^t}(P)$ is equal to the number $c(P)$ of conjugacy classes of non-trivial cyclic subgroups of P. On the other hand, we have shown in Corollary 8.4 that the dimension of $S_{C,\mathbb{F}_2}(P)$ is also $c(P)$ (note that 2 divides $p-1$ since p is odd!). Since the image of θ_P has a quotient isomorphic to $S_{C,\mathbb{F}_2}(P)$, it follows that θ_P is surjective, and that $\overline{D^t}$ is isomorphic to $S_{C,\mathbb{F}_2}$. □

Second proof of Theorem 11.1. We want to apply Theorem 9.4 for $K = \mathbb{F}_2$, for the functor $F = \overline{D^t}$ and for the cyclic group C of order p. So we need to check the three hypotheses of Theorem 9.4. We first note that $\overline{D^t}(C) \cong \mathbb{F}_2$ is the trivial $\mathbb{Q}\mathrm{Out}(C)$-module. Thus hypothesis (1) holds.

If P is elementary abelian, then the dimension of $\overline{D^t}(P)$ is given by Dade's Theorem 1.4, and is equal to the number of non-trivial cyclic quotients of P, which is equal to the number of non-trivial cyclic subgroups of P, by duality. But by Corollary 8.4, the dimension of $S_{C,\mathbb{F}_2}(P)$ is equal to the number of non-trivial cyclic subgroups of P (again note that 2 divides $p-1$ since p is odd). Hence hypothesis (2) holds for $\overline{D^t}$.

Finally hypothesis (3) also holds for $\overline{D^t}$, by definition, since $\overline{D^t}(P) = D^t(P)/\mathrm{Ker}(\psi_P)$ and the injectivity of the map induced by ψ_P guarantees hypothesis (3). We conclude that $\overline{D^t}$ is isomorphic to $S_{C,\mathbb{F}_2}$, by Theorem 9.4. □

There is a short exact sequence for $\overline{D^t}$ which is very similar to the one we obtained for $\mathbb{Q}D$.

(11.2) Theorem. *If p is odd, there is an exact sequence of functors*

$$0 \longrightarrow \overline{D^t} \stackrel{\beta}{\longrightarrow} \mathbb{F}_2 R_{\mathbb{Q}} \stackrel{\dim}{\longrightarrow} \Gamma_{\mathbb{F}_2} \longrightarrow 0 ,$$

where $\Gamma_{\mathbb{F}_2}$ denotes the constant functor.

Proof. Corollary 8.4 (which can be applied since $2 \mid p-1$) shows that $\mathbb{F}_2 R_{\mathbb{Q}}$ has a unique non-zero proper subfunctor, isomorphic to $S_{C,\mathbb{F}_2}$, and that the quotient functor is isomorphic to $\Gamma_{\mathbb{F}_2}$ via the dimension morphism. The result follows from the isomorphism $\overline{D^t} \cong S_{C,\mathbb{F}_2}$. □

(11.3) Remark. The situation is the same as for $\mathbb{Q}D$. We have no natural description of the map β in terms of arbitrary endo-permutation modules of finite order, but we can give an explicit description of the map on a basis of $\overline{D^t}(P)$.

The representation ring $\mathbb{F}_2 R_{\mathbb{Q}}(C)$ has dimension 2 with basis $\{1, \mathrm{Ind}_1^C(1)\}$, and the kernel of the dimension map modulo 2 is one-dimensional over $\mathbb{F}_2$, generated by $\mathrm{Ind}_1^C(1) - 1$. So we necessarily have $\beta_C(\Omega_C) = \mathrm{Ind}_1^C(1) - 1$. Now for an arbitrary p-group P, we know that $\overline{D^t}(P)$ has an $\mathbb{F}_2$-basis consisting of the elements $\mathrm{Ten}_Z^P \mathrm{Inf}_{Z/\Phi(Z)}^Z(\Omega_{Z/\Phi(Z)})$, where Z runs over the non-trivial cyclic subgroups of P up to conjugation, and $\Phi(Z)$ denotes the unique subgroup of Z of index p. Since β is functorial, we immediately deduce that

$$\beta_P\big(\mathrm{Ten}_Z^P \mathrm{Inf}_{Z/\Phi(Z)}^Z(\Omega_{Z/\Phi(Z)})\big) = \mathrm{Ind}_Z^P \mathrm{Inf}_{Z/\Phi(Z)}^Z\big(\mathrm{Ind}_1^{Z/\Phi(Z)}(1) - 1\big) = \mathrm{Ind}_{\Phi(Z)}^P(1) - \mathrm{Ind}_Z^P(1) .$$

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