The Dade group of a p-group

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Abstract

Let p be a prime number. This paper solves the question of the structure of the group $D(P)$ of endo-permutation modules over an arbitrary finite p-group P, that was open after Dade's original papers in 1978 ([19], [20]), and it gives a proof of the conjectures proposed in [4] and [10]. This leads to a presentation of $D(P)$ by explicit generators and relations, generalizing the presentation obtained by Dade when P is abelian.

A key result of independent interest is the explicit description of the kernel of the natural map from the Burnside group to the group of rational characters, in terms of the extraspecial group of order p^3 and exponent p if $p \neq 2$, or of all dihedral groups of order at least 8 if $p = 2$.

Keywords : Endo-permutation, Dade group, Burnside biset functor rational representation, p-group

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1. Introduction

This paper describes the structure of the Dade group $D(P)$ of a finite p-group P, that was defined by E. C. Dade in 1978 ([19], [20]), in order to classify endo-permutation kP-modules, where k is a field of characteristic $p > 0$, or more generally, a "p-local" ring.

Endo-permutation modules for p-groups appear as a crucial tool in many aspects of the p-modular representation theory of finite groups, e.g. as source modules of simple modules, or in the description of source algebras of blocks (L. Puig [22]), or the description of derived or stable equivalences between blocks (L. Puig [24]). They are a generalization of the notion of endo-trivial module or invertible module, studied in particular by J. L. Alperin ([1], [3]) and J. F. Carlson ([15], [16]).

In his original papers, Dade determined the structure of $D(P)$ when P is abelian. Then Puig proved ([23]) that for an arbitrary finite p-group P, the group $D(P)$ is a finitely generated (abelian) group. Next J. Thévenaz and I showed ([12]) that the free rank of $D(P)$ is equal to the number of conjugacy classes of non cyclic subgroups of P.

Recently, in a series of remarkable papers ([13], [14], [17]), J. F. Carlson and J. Thévenaz completed the classification of endo-trivial modules, and proved detection theorems for the Dade group. Their results are the first essential ingredient for the present work. Their third paper ([17]) also gave methods to start an induction procedure, by which N. Mazza and I determined ([11]) the structure of $D(P)$, when P is any (almost) extraspecial p-group.

The second ingredient of the present paper is the notion of *biset functor* for finite groups, introduced in [5], and specialized to p-groups in [12]. Biset functors over p-groups seem specially well suited to study the subgroup $D^\Omega(P)$ of $D(P)$ generated by the relative syzygies Ω_X associated to (non empty) finite P-sets X, where the module Ω_X is defined as the kernel of the augmentation map $kX \to k$ (see [2] or [6] for details).

In [6], a formula for tensor induction of relative syzygies in the Dade group was stated, showing that the correspondence sending a p-group P to $D^\Omega(P)$ is a biset functor. In [9], it was shown that this functor D^Ω is a quotient of the $\mathbb{Z}$-dual B^* of the Burnside biset functor B, and that there is an exact sequence of biset functors over p-groups

$$0 \to R_\mathbb{Q}^* \to B^* \to D^\Omega/D^\Omega_{tors} \to 0 \quad ,$$

where $R_\mathbb{Q}^*$ is the $\mathbb{Z}$-dual of the functor of rational representations, and D^Ω_{tors} is the torsion subfunctor of D^Ω.

In this sequence, the embedding $R_\mathbb{Q}^* \to B^*$ is the transpose of the natural transformation $\chi : B \to R_\mathbb{Q}$, whose evaluation at P maps the P-set X to the corresponding $\mathbb{Q}P$-module $\mathbb{Q}X$. A key result in this paper is Theorem 6.12, which may be of independent interest : it shows that the functor $K = \mathrm{Ker}\ \chi$ is generated by its values at the extraspecial group of order p^3 and exponent p for $p \neq 2$, or at the dihedral groups of order at least 8 for $p = 2$. In other words, it gives an explicit way to build all the "virtual P-sets with zero character" from specific ones for these extraspecial or dihedral p-groups.

On the other hand, the surjectivity of χ is known since 1972 by the Ritter-Segal theorem, which was stated in a more explicit form in [7]. This was the starting point of the study of the functor $R_\mathbb{Q}$ of rational representations of p-groups ([8]), which is the third ingredient of this paper. In particular, the notion of *genetic section* of a p-group was defined there, and was developed in [10], where the notion of *rational biset functor* was also introduced.

This led to precise conjectures on the torsion part of the Dade group (Conjectures 6.2 and 6.3 of [10]). In my talk at Oberwolfach in March 2003 ([4]), I formulated another conjecture on the Dade group, saying that $D = D^\Omega$ if $p \neq 2$, and that D/D^Ω is isomorphic to a specific subfunctor of $\mathbb{F}_2 R_\mathbb{Q}$ if $p = 2$, associated to quaternion 2-groups. In the present paper, I will give a proof of all these conjectures, in Theorem 7.7, Theorem 8.2, and Theorem 10.2.

The main consequence will be presentation of $D(P)$ by explicit generators and relations (Theorem 9.5), generalizing the presentation obtained by Dade when P is abelian.

The paper is organized as follows : Section 2 recalls some definitions and results from rational representation theory of p-groups. Section 3 is devoted to definitions and results on biset functors. Section 4 is a brief presentation of the Dade group, its functorial properties, and theorems of Dade and Carlson-Thévenaz. In Section 5, I will state some results on genetic bases, in particular Lemma 5.2, that is the key to both Theorem 6.12 and Theorem 8.2. The key result on the functor K is stated in Section 6. Section 7 is an application to the Dade group, showing that $D = D^\Omega + D_{tors}$, and that $D = D^\Omega$ if $p \neq 2$. Section 8 is devoted to the structure of the torsion part of the Dade group. Section 9 gives a presentation of $D(P)$ by generators and relations. Finally Section 10 describes the functorial structure of D/D^Ω for $p = 2$.

2. Rational representations

Throughout this paper, the symbol p will denote a prime number. All p-groups will be finite ones. If P is a p-group, then $\Phi(P)$ denotes its Frattini subgroup, and $Z(P)$ its center. The largest elementary abelian subgroup of $Z(P)$ is denoted by $\Omega_1 Z(P)$.

If n is a positive integer, the symbol C_n will denote a cyclic group of order n. If $n \geq 2$, then D_{2^n} denotes a dihedral group of order 2^n, with the convention that D_4 is the Klein four group. If $n \geq 3$, then Q_{2^n} denotes a generalized quaternion group of order 2^n. If $n \geq 4$, then SD_{2^n} denotes a semi dihedral group of order 2^n.

2.1. Basic subgroups and associated simple modules : Recall some notation and definitions from [8] :

2.2. Notation : *if P is a group, denote by $R_\mathbb{Q}(P)$ the Grothendieck group of finitely generated $\mathbb{Q}P$-modules, and by $\mathrm{Irr}_\mathbb{Q}(P)$ a set of representatives of isomorphism classes of irreducible $\mathbb{Q}P$-modules. There is a natural bilinear form on $R_\mathbb{Q}(P)$, with values in $\mathbb{Z}$, defined by*

$$\langle V, W\rangle_P = \dim_\mathbb{Q} \mathrm{Hom}_{\mathbb{Q}P}(V, W)$$

for $\mathbb{Q}P$-modules V and W. If V is a simple $\mathbb{Q}P$-module, there is a unique linear form $m(V, -)$ on $R_\mathbb{Q}(P)$, with values in $\mathbb{Z}$, sending the $\mathbb{Q}P$-module W to the multiplicity $m(V, W)$ of V as a summand of W.

2.3. Definition : ([8] 2.3) *Let P be a finite p-group. A subgroup Q of P is called basic if the following two conditions hold :*

1. *The quotient $N_P(Q)/Q$ is cyclic or generalized quaternion.*
2. *If R is any subgroup of P and if $R \cap N_P(Q) \subseteq Q$, then $|R| \leq |Q|$.*

If Q is a proper basic subgroup of P, then there is a unique subgroup $\tilde{Q} \supset Q$ of P with $|\tilde{Q} : Q| = p$, and the kernel of the projection map

$$\mathbb{Q}P/Q \to \mathbb{Q}P/\tilde{Q}$$

is an irreducible $\mathbb{Q}P$-module, denoted by V_Q.

The group P itself is a basic subgroup of P, and by convention V_P is the trivial $\mathbb{Q}P$-module $\mathbb{Q}$. With this notation, any irreducible $\mathbb{Q}P$-module is isomorphic to V_Q, for some basic subgroup Q of P.

If Q and Q' are basic subgroups of P, then the $\mathbb{Q}P$-modules V_Q and $V_{Q'}$ are isomorphic if and only if $Q \div_P Q'$, where $\div_P$ is the relation defined in [8] 2.7 by

$$Q \div_P Q' \Leftrightarrow \begin{cases} |Q| = |Q'| \\ \exists x \in P,\ Q^x \cap N_P(Q') \subseteq Q' \end{cases} .$$

2.4. Remark : In particular, a normal subgroup N of P is basic if and only if P/N is cyclic or generalized quaternion. Moreover, if Q is a basic subgroup such that $Q \div_P N$, then $Q = N$.

2.5. Definition and Notation : ([8] Proposition 3.7) *A p-group P has normal p-rank 1 if it does not have any normal subgroup isomorphic to $(C_p)^2$. Up to isomorphism, such a group has a unique rational faithful irreducible representation, denoted by Φ_P.*

Recall (Theorem 5.4.10 of [21]) that if P is a p-group of normal p-rank 1 and order p^n, then $P \cong C_{p^n}$ if $p \neq 2$, and if $p = 2$, then $P \cong C_{2^n}$, or $P \cong Q_{2^n}$ ($n \geq 3$), or $P \cong D_{2^n}$ ($n \geq 4$), or $P \cong SD_{2^n}$ ($n \geq 4$).

2.6. Definition : ([8] Definition 3.5) *A section (T, S) of the group P is a pair of subgroups of P such that $S \trianglelefteq T \subseteq P$. A section (T, S) of P is proper if $(T, S) \neq (P, \mathbf{1})$. If R is a group, the section (T, S) will be said of type R if the factor group T/S is isomorphic to R.*

The section (T, S) of the p-group P will be called genetic if the following three conditions hold :

1. *The group T/S has normal p-rank 1.*
2. *The $\mathbb{Q}P$-module $V(T, S) = \mathrm{Ind}_T^P \mathrm{Inf}_{T/S}^T \Phi_{T/S}$ is simple.*
3. *$\langle V(T, S), V(T, S) \rangle_P = \langle \Phi_{T/S}, \Phi_{T/S} \rangle_{T/S}$.*

The simple $\mathbb{Q}P$-module $V(T, S)$ will be called the simple module associated to (T, S). It was shown in Theorem 3.4 of [8] that for any simple $\mathbb{Q}P$-module V, there exists a genetic section (T, S) of P such that $V \cong V(T, S)$.

Moreover ([8] Theorem 7.11 and Proposition 7.1), if (T,S) and (T',S') are genetic sections of P, the corresponding modules $V(T,S)$ and $V(T',S')$ are isomorphic if and only if $(T,S) \text{---}_P (T',S')$, i.e. if there exists $x \in P$ such that

$$T^x{\cdot}S' = S^x{\cdot}T' \qquad \text{and} \qquad T^x \cap S' = S^x \cap T' \quad .$$

In particular, the relation ---_P ("linked modulo P") is an equivalence relation on the set of genetic sections of P. Observe that if $(T,S) \text{---}_P (T',S')$, then the groups T/S and T'/S' are isomorphic. In other words (T,S) and (T',S') have the same type.

2.7. Definition : *If V is a simple $\mathbb{Q}P$-module, and (T,S) is a genetic section of P such that $V \cong V(T,S)$, then T/S is called the type of V.*

The genetic sections of P can be characterized as follows :

2.8. Proposition : ([10] Proposition 4.4) *Let P be a finite p-group, and let (T,S) be a section of P. Let $Z_P(S)$ denote the subgroup of P defined by*

$$Z_P(S)/S = Z\big(N_P(S)/S\big) \quad .$$

Then the following conditions are equivalent :

1. *The section (T,S) is an genetic section of P.*
2. *The group $N_P(S)/S$ has normal p-rank 1, the group T is equal to $N_P(S)$, and if $x \in P$ is such that $S^x \cap Z_P(S) \subseteq S$, then $x \in N_P(S)$.*

In particular, if (T,S) is a genetic section of P, then $T = N_P(S)$. Hence (T,S) is actually determined by S. This leads to the following :

2.9. Definition and Notation : *A subgroup S of the p-group P will be called genetic if the section $(N_P(S),S)$ is a genetic section of P. The simple module $V(N_P(S),S)$ will be denoted by $V(S)$.*

Similarly, if S and S' are genetic subgroups of P, I will say that $S \text{---}_P S'$ if $(N_P(S),S) \text{---}_P (N_P(S'),S')$.

The relation ---_P is an equivalence relation on the set of genetic subgroups of P. By Theorem 3.11 of [8], the correspondence $S \mapsto V(S)$ is a one to one correspondence between the set of equivalence classes of genetic subgroups modulo ---_P, and the set of isomorphism classes of rational irreducible representations of P. Moreover by Theorem 7.11 of [8], if S and S' are genetic subgroups of P such that $S \text{---}_P S'$, the set of elements $x \in P$ such that $S \text{---} {}^xS'$ is a single double coset $N_P(S)yN_P(S')$ in P. In particular if $S \text{---} {}^xS$, then $x \in N_P(S)$.

2.10. Definition : *A subset $\mathcal{S}$ of the set of genetic subgroups of P will be called a genetic basis of P if it is a set of representatives of the set of equivalence classes of genetic subgroups of P for the relation ---_P .*

2.11. Remark : A normal subgroup N of P is genetic if and only if P/N has normal p-rank 1. In that case, if S is any genetic subgroup of P such that $S \,\text{---}_P\, N$, then $S = N$. In particular, the group N belongs to every genetic basis of P.

2.12. Remark : (see Proposition 7.4 in [8]) If S is a genetic subgroup of P, and if R_S/S is a basic subgroup of $N_P(S)/S$, intersecting trivially the center of $N_P(S)/S$, then R_S is a basic subgroup of P, corresponding to the same simple QP-module (i.e. with the above notation $V_{R_S} \cong V(S)$).

If $N_P(S)/S$ is cyclic or generalized quaternion, then $R_S = S$. If $N_P(S))/S$ is dihedral or semi dihedral, then R_S/S is any non central subgroup of order 2 of $N_P(S)/S$.

Basic subgroups obtained from genetic ones by this operation were called *origins* in [8]. In general, not all basic subgroups are origins, but there is at least an origin in each equivalence class of basic subgroups for the relation $\doteqdot_P$ (see Corollary 7.5 in [8] for details).

3. Biset functors

3.1. Notation and Definition : *Denote by $\mathcal{C}_p$ the following category :*

- *The objects of $\mathcal{C}_p$ are the finite p-groups.*
- *If P and Q are finite p-groups, then $\mathrm{Hom}_{\mathcal{C}_p}(P,Q) = B(Q \times P^{op})$ is the Burnside group of finite (Q,P)-bisets. An element of this group is called a virtual (Q,P)-biset.*
- *The composition of morphisms is $\mathbb{Z}$-bilinear, and if P, Q, R are finite p-groups, if U is a finite (Q,P)-biset, and V is a finite (R,Q)-biset, then the composition of (the isomorphism classes of) V and U is the (isomorphism class) of $V \times_Q U$. The identity morphism Id_P of the p-group P is the class of the set P, with left and right action by multiplication.*

Let $\mathcal{F}_p$ denote the category of additive functors from $\mathcal{C}_p$ to the category $\mathbb{Z}$-Mod of abelian groups. An object of $\mathcal{F}_p$ is called a biset functor (defined over p-groups, with values in $\mathbb{Z}$-Mod).

If F is an object of $\mathcal{F}_p$, if P and Q are finite p-groups, and if $\varphi \in \mathrm{Hom}_{\mathcal{C}_p}(P,Q)$, then the image of $w \in F(P)$ by the map $F(\varphi)$ will generally be denoted by $\varphi(w)$. The composition $\psi \circ \varphi$ of morphisms $\varphi \in \mathrm{Hom}_{\mathcal{C}_p}(P,Q)$ and $\psi \in \mathrm{Hom}_{\mathcal{C}_p}(Q,R)$ will also be denoted by $\psi \times_Q \varphi$.

3.2. Notation : *The Burnside biset functor (defined e.g. as the Yoneda functor $\mathrm{Hom}_{\mathcal{C}_p}(\mathbf{1},-)$), will be denoted by B. The functor of rational representations (see Section 1 of [8]) will be denoted by $R_{\mathbb{Q}}$.*

3.3. Examples : Recall that this formalism of bisets gives a single framework for the usual operations of induction, restriction, inflation, deflation, and transport by isomorphism via the following correspondences :

- If Q is a subgroup of P, then let $\mathrm{Ind}_Q^P \in \mathrm{Hom}_{\mathcal{C}_p}(Q, P)$ denote the set P, with left action of P and right action of Q by multiplication.
- If Q is a subgroup of P, then let $\mathrm{Res}_Q^P \in \mathrm{Hom}_{\mathcal{C}_p}(P, Q)$ denote the set P, with left action of Q and right action of P by multiplication.
- If $N \trianglelefteq P$, and $Q = P/N$, then let $\mathrm{Inf}_Q^P \in \mathrm{Hom}_{\mathcal{C}_p}(Q, P)$ denote the set Q, with left action of P by projection and multiplication, and right action of Q by multiplication.
- If $N \trianglelefteq P$, and $Q = P/N$, then let $\mathrm{Def}_Q^P \in \mathrm{Hom}_{\mathcal{C}_p}(P, Q)$ denote the set Q, with left action of Q by multiplication, and right action of P by projection and multiplication.
- If $\varphi : P \to Q$ is a group isomorphism, then let $\mathrm{Iso}_P^Q = \mathrm{Iso}_P^Q(\varphi) \in \mathrm{Hom}_{\mathcal{C}_p}(P, Q)$ denote the set Q, with left action of Q by multiplication, and right action of P by taking image by φ, and then multiplying in Q.

3.4. Remark : If P and Q are p-groups, then any element $\varphi \in \mathrm{Hom}_{\mathcal{C}_p}(P, Q)$ is a $\mathbb{Z}$-linear combination of transitive (Q, P)-bisets, and by Lemma 7.4 of [12], every transitive (Q, P)-biset can be factored in the category $\mathcal{C}_p$ as a composition

$$\mathrm{Ind}_V^Q \circ \mathrm{Inf}_{V/U}^V \circ \mathrm{Iso}_{T/S}^{V/U}(\varphi) \circ \mathrm{Def}_{T/S}^T \circ \mathrm{Res}_T^P$$

where (T, S) is a section of P, and (V, U) is a section of Q, such that there exists a group isomorphism $\varphi : T/S \to V/U$.

3.5. Notation : *If (T, S) is a section of P, set*

$$\mathrm{Indinf}_{T/S}^P = \mathrm{Ind}_T^P \mathrm{Inf}_{T/S}^T \qquad \textit{and} \qquad \mathrm{Defres}_{T/S}^P = \mathrm{Def}_{T/S}^T \mathrm{Res}_T^P \quad .$$

Then $\mathrm{Indinf}_{T/S}^P \cong P/S$ as $(P, T/S)$-biset, and $\mathrm{Defres}_{T/S}^P \cong S\backslash P$ as $(T/S, P)$-biset.

3.6. Opposite bisets : If P and Q are finite p-groups, and if U is a finite (Q, P)-biset, then let U^{op} denote the opposite biset : as a set, it is equal to U, and it is a (P, Q)-biset for the following action

$$\forall h \in Q, \forall u \in U, \forall g \in P, \; g.u.h \text{ (in } U^{op}) = h^{-1}ug^{-1} \text{ (in } U) \quad .$$

This definition can be extended by linearity, to give an isomorphism

$$\varphi \mapsto \varphi^{op} : \mathrm{Hom}_{\mathcal{C}_p}(P, Q) \to \mathrm{Hom}_{\mathcal{C}_p}(Q, P) \quad .$$

It is easy to check that $(\varphi \circ \psi)^{op} = \psi^{op} \circ \varphi^{op}$, with obvious notation, and the functor

$$\left\{ \begin{array}{l} P \mapsto P \\ \varphi \mapsto \varphi^{op} \end{array} \right.$$

is an equivalence of categories from $\mathcal{C}_p$ to the dual category.

3.7. Example : if P is a p-group, and (T, S) is a section of P, then

$$(\mathrm{Indinf}_{T/S}^P)^{op} \cong \mathrm{Defres}_{T/S}^P$$

as $(T/S, P)$-bisets.

3.8. Definition and Notation : *If F is a biset functor, the* <u>*dual*</u> *biset functor F^* is defined by*

$$F^*(P) = \mathrm{Hom}_{\mathbb{Z}}(F(P), \mathbb{Z}) \quad ,$$

for a p-group P, and by

$$F^*(\varphi)(\alpha) = \alpha \circ F(\varphi^{op}) \quad ,$$

for any $\alpha \in F^*(P)$, any p-group Q, and any $\varphi \in \mathrm{Hom}_{\mathcal{C}_p}(P, Q)$.

3.9. Some idempotents in $\mathrm{End}_{\mathcal{C}_p}(P)$ **:** Let P be a finite p-group, and let $N \trianglelefteq P$. Then it is clear from the definitions that

$$\mathrm{Def}_{P/N}^P \circ \mathrm{Inf}_{P/N}^P = (P/N) \times_P (P/N) = \mathrm{Id}_{P/N} \quad .$$

It follows that the composition $e_N^P = \mathrm{Inf}_{P/N}^P \circ \mathrm{Def}_{P/N}^P$ is an idempotent in $\mathrm{End}_{\mathcal{C}_p}(P)$. Moreover $e_1^P = \mathrm{Id}_P$, and if M and N are normal subgroups of P, then $e_N^P \circ e_M^P = e_{NM}^P$.

3.10. Lemma : ([10] Lemma 2.5) *If $N \trianglelefteq P$, define $f_N^P \in \mathrm{End}_{\mathcal{C}_p}(P)$ by*

$$f_N^P = \sum_{\substack{M \trianglelefteq P \\ N \subseteq M}} \mu_{\trianglelefteq P}(N, M) e_M^P \quad ,$$

where $\mu_{\trianglelefteq P}$ denotes the Möbius function of the poset of normal subgroups of P. Then the elements f_N^P, for $N \trianglelefteq P$, are orthogonal idempotents of $\mathrm{End}_{\mathcal{C}_p}(P)$, and their sum is equal to Id_P.

Moreover, it is easy to check from the definition that for $N \trianglelefteq P$,

$$f_N^P = \mathrm{Inf}_{P/N}^P \circ f_{\mathbf{1}}^{P/N} \circ \mathrm{Def}_{P/N}^P \quad , \qquad (\mathbf{3.11})$$

and

$$e_N^P = \mathrm{Inf}_{P/N}^P \circ \mathrm{Def}_{P/N}^P = \sum_{\substack{M \trianglelefteq P \\ M \supseteq N}} f_M^P \quad .$$

3.12. Lemma : *If N is a non trivial normal subgroup of P, then*

$$f_1^P \circ \mathrm{Inf}_{P/N}^P = 0 \quad \textit{and} \quad \mathrm{Def}_{P/N}^P \circ f_{\mathbf{1}}^P = 0 \quad .$$

Proof: Indeed by 3.11

$$\begin{aligned} f_1^P \circ \mathrm{Inf}_{P/N}^P &= f_1^P \circ \mathrm{Inf}_{P/N}^P \circ \mathrm{Def}_{P/N}^P \circ \mathrm{Inf}_{P/N}^P \\ &= \sum_{\substack{M \trianglelefteq N \\ M \supseteq N}} f_1^P f_M^P \mathrm{Inf}_{P/N}^P = 0 \quad , \end{aligned}$$

since $M \neq \mathbf{1}$ when $M \supseteq N$. The other equality of the lemma follows by taking opposite bisets. $\square$

3.13. Remark : It was also shown in Section 2.7 of [10] that

$$f_{\mathbf{1}}^P = \sum_{N \subseteq \Omega_1 Z(P)} \mu(\mathbf{1}, N) P/N \quad ,$$

where μ is the Möbius function of the poset of subgroups of N.

3.14. Notation and Definition : *If F is a a biset functor, and if P is a p-group, then the idempotent $f_{\mathbf{1}}^P$ of $\mathrm{End}_{\mathcal{C}_p}(P)$ acts on $F(P)$. Its image*

$$\partial F(P) = f_{\mathbf{1}}^P F(P)$$

is a direct summand of $F(P)$ as $\mathbb{Z}$-module : it will be called the set of <u>*faithful*</u> *elements of $F(P)$.*

The reason for this name is that any element $u \in F(P)$ which is inflated from a proper quotient of P is such that $F(f_{\mathbf{1}}^P)u = 0$. From Lemma 3.12, it is also clear that

$$\partial F(P) = \bigcap_{\mathbf{1} \neq N \trianglelefteq P} \mathrm{Ker}\ \mathrm{Def}_{P/N}^P \quad .$$

3.15. Notation : *If F is a biset functor, and P is a p-group, set*

$$\overline{F}(P) = F(P) / \sum_{\substack{(T,S) \\ |T/S| < |P|}} \mathrm{Indinf}_{T/S}^P F(T/S) \quad ,$$

and dually

$$\underline{F}(P) = \bigcap_{\substack{(T,S) \\ |T/S| < |P|}} \mathrm{Ker}\ \mathrm{Defres}_{T/S}^P \quad .$$

where in both expressions (T, S) runs through proper sections of P.

4. The Dade group

In this section, I will briefly recall some basic definitions and constructions on the Dade group of endo-permutation modules. Most of them go back to Dade's paper ([19]), and are also exposed in Sections 28-29 of Thévenaz's book ([25]). The "functorial approach" to the Dade group was introduced in Sections 2 and 3 of [12].

4.1. Let k be a field of characteristic p. If P is a p-group, the Dade group $D(P) = D_k(P)$ can be defined as the group of equivalence classes of capped endo-permutation modules, or as the group of equivalence classes of Dade P-algebras over k, for a suitable equivalence relation in each case. It is an abelian group, and Puig ([23]) has shown that it is finitely generated.

The (equivalence classes of) endo-trivial modules form a subgroup $T(P)$ of $D(P)$, called the group of endo-trivial modules. Another crucial subgroup of $D(P)$ is the subgroup $D^{\Omega}(P)$ generated by relative syzygies (see [2] or [6] for details).

4.2. Functorial properties : The Dade group has some important functorial properties : if Q is a subgroup of P, and if N is a normal subgroup of P, or if $\varphi : P \to P'$ is a group isomorphism, then there are maps of restriction, induction, deflation, inflation, and transport by isomorphism

$$\textbf{(4.3)} \qquad \left\{ \begin{array}{c} D(P) \xrightarrow{\mathrm{Res}^P_Q} D(Q) \xrightarrow{\mathrm{Ten}^P_Q} D(P) \\ D(P) \xrightarrow{\mathrm{Def}^P_{P/N}} D(P/N) \xrightarrow{\mathrm{Inf}^P_{P/N}} D(P) \\ D(P) \xrightarrow{\mathrm{Iso}^{P'}_P} D(P') \end{array} \right.$$

coming respectively from the restriction, tensor induction, Brauer quotient, inflation, and transport by group isomorphism of Dade algebras (note the change of notation for induction).

These five operations can be unified in a single formalism using bisets : if P and Q are p-groups, if U is a finite (Q, P)-biset, then in Corollary 2.3 of [12], we introduced a map

$$D(U) : D(P) \to D(Q) \quad ,$$

such that the maps in 4.3 are equal to $D(U)$, where U is the corresponding biset Res^P_Q, Ind^P_Q, $\mathrm{Def}^P_{P/N}$, $\mathrm{Inf}^P_{P/N}$, and Iso^P_Q defined in Section 3.3. The definition of this map $D(U)$ associated to an arbitrary finite (Q, P) biset U followed from the existence of a corresponding functor $T_U : \underline{Perm}_k(P) \to \underline{Perm}_k(Q)$, which is a sort of generalized tensor induction, where $\underline{Perm_k}(P)$ is a category equivalent to the category of finitely generated permutation kP-modules (see Section 2 of [12], in particular Lemma (2.2) and Corollary (2.13), for details).

Moreover the correspondence $U \mapsto D(U)$ is additive (by Proposition 2.10 of [12]), so it can be extended to a correspondence

$$\varphi \in \mathrm{Hom}_{\mathcal{C}_p}(P,Q) \mapsto D(\varphi) \in \mathrm{Hom}_{\mathbb{Z}}(D(P), D(Q)) \quad .$$

However, these constructions *do not* define a biset functor structure : the reason is that in general, with obvious notation

$$D(\psi) \circ D(\varphi) \neq D(\psi \circ \varphi) \quad ,$$

as can be read from Proposition 3.5 and Proposition 3.10 of [12] : in general, the right hand side is equal to a linear combination $\sum_i r_i D(U_i)$, where the U_i's are transitive bisets, and the r_i's are integers, whereas the left hand side is equal to $\sum_i r_i \gamma_{a_i} D(U_i)$, for the same U_i' and r_i's, but with a new kind of operation inserted, namely the γ_{a_i}'s. These operations are called for short "Galois torsions", and they are precisely defined in Section 3 of [12] : if a is any endomorphism of the ground field k, then for any p-group P, there is a linear map $\gamma_a = \gamma_{a,P} : D(P) \to D(P)$. They only appear when one composes a tensor induction followed by a deflation (see Proposition 3.10 of [12] for details).

This remark and Lemma 3.10 show in particular that the maps $D(f_N^P)$, for $N \trianglelefteq P$, which only involve deflation and inflation maps, are orthogonal idempotents of $\mathrm{End}_{\mathbb{Z}}(D(P))$, and their sum is the identity.

4.4. Notation : *If P and Q are p-groups, and if $\varphi \in \mathrm{Hom}_{\mathcal{C}_p}(P,Q)$, the map $D(\varphi) : D(P) \to D(Q)$ will simply be denoted by φ, and the image of $u \in D(P)$ by this map will be denoted by $\varphi(u)$ or φu. The faithful part $f_1^P D(P)$ of the Dade group will be denoted by $\partial D(P)$.*

It is easy to see that the maps γ_a restrict to the identity on $D^\Omega(P)$ (see Section 1.6 of [9]). Moreover, if $\varphi \in \mathrm{Hom}_{\mathcal{C}_p}(P,Q)$, then $D(\varphi)(D^\Omega(P)) \subseteq D^\Omega(Q)$, by Section 4 and 5 of [6]. The above discussion now shows that the correspondence $P \mapsto D^\Omega(P)$ is a biset functor.

4.5. Some known Dade groups : In his original papers, Dade determined $D(P)$, when P is abelian :

4.6. Theorem : (Dade [19], [20]) *Let P be an abelian p-group. Then $D(P)$ is generated by the elements $\Omega_{P/Q}$, for $Q \subseteq P$, subject to the relations $\tau_Q \Omega_{P/Q} = 0$, if P/Q is cyclic, where $\tau_Q = 1$ if $|P/Q| \leq 2$, and $\tau_Q = 2$ otherwise.*

In particular $D(P) = D^\Omega(P)$ if P is abelian.

The structure of $D(P)$ is also known for any 2-group P of normal 2-rank 1 : when P is generalized quaternion, the result is due to Dade, and the other cases have been solved by Carlson and Thévenaz :

4.7. Theorem : (Dade [18], Carlson-Thévenaz [13] Theorem 10.3)

1. *$D(D_{2^n}) \cong \mathbb{Z}^{2n-3}$.*
2. *$D(SD_{2^n}) \cong \mathbb{Z}^{2n-4} \oplus \mathbb{Z}/2\mathbb{Z}$.*
3. *$D(Q_{2^n}) \cong \mathbb{Z}^{2n-5} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, for $n \geq 4$.*
4. *$D(Q_8) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, if the ground field contains all cubic roots of unity, and $D(Q_8) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ otherwise.*

This result is actually more precise : Lemma 2.10 of [13] shows that if P is one of these 2-groups, then $\partial D(P) = T(P) = \mathrm{Ker}\ \mathrm{Def}^P_{P/Z}$, and $D(P) = T(P) + \mathrm{Ind}^P_{P/Z} D(P/Z)$, where Z is the center of P. This allows induction, since P/Z is dihedral. Then Theorem 5.4 and Theorem 7.1 of [13] show that $D(P) = D^\Omega(P)$ when P is dihedral, or semi dihedral.

If P is generalized quaternion of order 2^n, then Theorem 6.3 and Theorem 6.5 of [13] show that $T(P)$ is equal to the torsion part $D_{tors}(P)$ of $D(P)$, and that $T(P) \cong \mathbb{Z}/4\mathbb{Z} \oplus Z/2\mathbb{Z}$ if $n \geq 4$, or if $n \geq 3$ and the ground field has all cubic roots of unity. If $n = 3$ and the ground field has no non trivial cubic roots of unity, then $T(P) = D_{tors}(P) = D^\Omega_{tors}(P) \cong \mathbb{Z}/4\mathbb{Z}$.

4.8. Lemma : *Let Q be a generalized quaternion 2-group of order 2^n. Suppose that $n \geq 4$, or that $n = 3$, and the ground field has all cubic roots of unity. Then*

1. *There are exactly two elements η_Q and η'_Q of order 2 in the set $D(Q) - D^\Omega(Q)$, and*
$$\eta_Q + \eta'_Q = 2\Omega_{Q/\mathbf{1}} \quad .$$
2. *If R is a proper subgroup of Q, then $\mathrm{Res}^Q_R \eta_Q \in D^\Omega(R)$.*
3. *If R is a generalized quaternion group containing Q, then*
$$\mathrm{Ten}^R_Q \eta_Q - \eta_R \in D^\Omega(R) \quad .$$
4. *If a is any endomorphism of the ground field, then $\gamma_a(\eta_Q) = \eta_Q$ if $n \geq 4$.*

Proof: Assertion 1 follows from the fact that $\Omega_{Q/\mathbf{1}}$ generates a cyclic summand of order 4 in $D_{tors}(P) \cong \mathbb{Z}/4\mathbb{Z} \oplus Z/2\mathbb{Z}$. And in the group $\mathbb{Z}/4\mathbb{Z} \oplus Z/2\mathbb{Z}$, the elements of order 2 are $(0,0)$, $(2,0)$, $(0,1)$, and $(2,1)$.

Assertion 2 is trivial if $n = 3$, because by construction in that case all proper restrictions of η_Q are 0. If $n \geq 4$ and R is cyclic, then $\mathrm{Res}^Q_R \eta_Q = 2\Omega_{R/\mathbf{1}} = 0$ by construction (see Lemma 6.4 and Theorem 6.5 of [13]). Suppose now that R is quaternion, and $|Q : R| = 2$. There are two conjugacy classes of quaternion subgroups of order 8 in Q, and one of them is contained in R. Let H and H' denote representatives for these classes, and suppose $H \subseteq R$. Then $\mathrm{Res}^Q_H \eta_Q$ is equal to 0 or $2\Omega_H$, by the proof of Theorem 6.5 of [13], and then $\mathrm{Res}^Q_H \eta'_Q$ is respectively equal to $2\Omega_H$ or 0. Replacing η_Q by η'_Q if necessary, I can assume $\mathrm{Res}_H^Q \eta_Q = 0$. Now the element $u = \mathrm{Res}_R^Q \eta_Q$ is such that $\mathrm{Res}_L^R u = 0$, for any quaternion subgroups L of order 8 of R. Then $u = 0$ by Theorem 2.8 of [13] (see also the proof of Theorem 6.5 there). This shows Assertion 2, since $\eta_Q + \eta'_Q = 2\Omega_{Q/\mathbf{1}}$.

Assertion 3 is equivalent to $\mathrm{Ten}_Q^R \eta_Q \notin D^\Omega(R)$. Proceed by induction on $|R : Q|$, and suppose that $|R : Q| = 2$. If $u = \mathrm{Ten}_Q^R \eta_Q \in D^\Omega(R)$, then u is equal to 0 or $2\Omega_{R/\mathbf{1}}$. By Mackey formula, the restriction of u to the generalized quaternion subgroup $Q' \neq Q$ of R is equal to $\mathrm{Ten}_{Q\cap Q'}^{Q'} \mathrm{Res}_{Q\cap Q'}^Q \eta_Q$. This is equal to 0 by the above argument, since $Q \cap Q'$ is cyclic. It follows that $u = 0$, since $\mathrm{Res}_{Q'}^R 2\Omega_{R/\mathbf{1}} = 2\Omega_{Q'/\mathbf{1}} \neq 0$. Now $\mathrm{Res}_Q^R u = \eta_Q + {}^r\eta_Q$, where $r \in R - Q$. Moreover ${}^r\eta_Q \neq \eta_Q$: if $n \geq 4$, this is because r exchanges the conjugacy classes of quaternion subgroups of order 8 of Q. And if $n = 3$, it can be checked directly from the construction : with the notation of Theorem 6.3 of [13], the action of the automorphism of Q_8 which exchanges the generators x and y is equivalent to replacing the cubic root ω by its conjugate ω^2. So the elements of the outer automorphism group $\mathrm{Out}(Q_8) \cong S_3$ with odd signature exchange η_Q and η'_Q. In both cases $\mathrm{Res}_Q^R u \neq 0$, which is a contradiction.

Finally Assertion 4 is clear from the construction Theorem 6.5 of [13]. □

5. Genetic bases

5.1. Lemma : *Let P be a p-group, and Z be a central subgroup of order p in P. If S is a genetic subgroup of P, then*

$$\mathrm{Def}_{P/Z}^P V(S) = \{0\} \Leftrightarrow S \not\supseteq Z \quad .$$

Proof: If $S = P$, the result is trivial. Suppose that $S \neq P$, and let R/S be a basic subgroup of $N_P(S)/S$, intersecting trivially the center of $N_P(S)/S$. Then R is a basic subgroup of P for $V(S)$, and $V(S)$ is isomorphic to the kernel of the projection map $\mathbb{Q}P/R \to \mathbb{Q}P/\tilde{R}$, where $\tilde{R}$ is the unique subgroup of P such that $|\tilde{R} : R| = p$. Hence $V(S) = \mathbb{Q}P/R - \mathbb{Q}P/\tilde{R}$ in $R_\mathbb{Q}(P)$.

Then either $S \supseteq Z$, and then $\mathrm{Def}_{P/Z}^P V(S) = \mathbb{Q}P/R - \mathbb{Q}P/\tilde{R} \neq 0$ in $R_\mathbb{Q}(P/Z)$, or $S \not\supseteq Z$, and then $R \not\supseteq Z$ because $ZS/S = \Omega_1 Z(N_P(S)/S)$, thus $\tilde{R} = RZ$ and $\mathrm{Def}_{P/Z}^P V(S) = \mathbb{Q}P/RZ - \mathbb{Q}P/RZ = 0$ in $R_\mathbb{Q}(P/Z)$. □

5.2. Lemma : *Let P be a p-group, and let E be a non-central normal elementary abelian subgroup of order p^2 of P. Set $Z = E \cap Z(P)$, where $Z(P)$ is the center of P.*

1. *If R is a genetic subgroup of $C_P(E)$ such that $Z \not\subseteq R$, then R is a genetic subgroup of P, and $N_P(R) \subseteq C_P(E)$. Moreover if V is the simple $\mathbb{Q}P$-module associated to R, and W is the simple $\mathbb{Q}C_P(E)$-module associated to R, then $V \cong \mathrm{Ind}_{C_P(E)}^P W$, and $\langle V, V\rangle_P = \langle W, W\rangle_{C_P(E)}$.*
2. *If Q is a genetic subgroup of P such that $Z \not\subseteq Q$, then there exists a genetic subgroup R of $C_P(E)$ such that $Z \not\subseteq R$ and $R \,\text{—}_P\, Q$.*

Proof: Set $H = C_P(E)$. Then $|P : H| = p$, and $|Z| = p$, since E is not central in P, and since the p-Sylow subgroups of $\mathrm{Aut}(E)$ have order p.

Let R be a genetic subgroup of H such that $Z \not\subseteq R$, and let W be the corresponding rational irreducible representation of H. Then $\mathrm{Def}_{H/Z}^H W = \{0\}$ by Lemma 5.1.

Now set $V = \mathrm{Ind}_H^P W$. Then since $Z \subseteq H$

$$\mathrm{Def}_{P/Z}^P V = \mathrm{Ind}_{H/Z}^{P/Z} \mathrm{Def}_{H/Z}^H W = \{0\} \quad .$$

Restriction to H gives

$$\mathrm{Res}_H^P V \cong \mathop{\oplus}_{x \in P/H} {}^x W \quad .$$

Let I denote the stabilizer of W in P, i.e. the group of elements $x \in P$ such that the representations W and ${}^xW$ of H are isomorphic. Then $I \supseteq H$, so $I = H$ or $I = P$.

If $I = H$, then the representations ${}^xW$, for $x \in P/H$, are non isomorphic to each other. In this case V is an irreducible representation of P : indeed if V_1 is any simple summand of V, then some conjugate of W is a direct summand of $W_1 = \mathrm{Res}_H^P V_1$. Since $W_1 \cong {}^xW_1$ for any $x \in P$, is follows that all the conjugates of W in P are direct summands of W_1. Thus $W_1 = \mathrm{Res}_H^P V$, and $V_1 = V$. Moreover

$$\langle V, V\rangle_P = \langle W, \mathrm{Res}_H^P V\rangle_H = \sum_{x \in P/H} \langle W, {}^xW\rangle_H = \langle W, W\rangle_H \quad .$$

This means in particular that the section $(N_H(R), R)$ is a genetic section of P for the irreducible representation V. Thus $N_H(R) = N_P(R)$ by Proposition 2.8, and R is a genetic subgroup of P.

If $I = P$, then ${}^xW \cong W$ for any $x \in P$, and $\mathrm{Res}_H^P V \cong pW$. Let

$$V = V_1 \oplus \ldots \oplus V_n$$

be a decomposition of V as a direct sum of irreducible rational representations of P. Then $\mathrm{Def}_{P/Z}^P V_i = \{0\}$, for $i = 1, \ldots n$. Fix i in $\{1, \ldots, n\}$, and let S_i denote any simple summand of $\mathrm{Res}_E^P V_i$. Then either $S_i \cong \mathbb{Q}$, or there exists a subgroup F_i of index p in E, such that S_i is isomorphic to the kernel of the projection map $\mathbb{Q}E/F_i \to \mathbb{Q}$. Now $\mathrm{Def}^E_{E/Z} S_i = \{0\}$, thus $S_i \not\cong \mathbb{Q}$, and the subgroup F_i is not equal to Z. It follows that $N_P(F_i) = C_P(F_i) = C_P(E)$, since F_i has order p and $E = F_iZ$. Hence the stabilizer of S_i in P is equal to H, and by Clifford theory

$$V_i \cong \mathrm{Ind}_H^P \tilde{S}_i \quad ,$$

where $\tilde{S}_i$ is the S_i-isotypic component of $\mathrm{Res}^P_E V_i$. By restriction to H, this gives

$$\mathrm{Res}^P_H V_i \cong \bigoplus_{x \in P/H} {}^x\tilde{S}_i \quad ,$$

and the representations ${}^x\tilde{S}_i$ appearing in the right hand side are mutually non isomorphic irreducible representations of H, since the stabilizer of $\tilde{S}_i$ in P is equal to the stabilizer of S_i, i.e. the normalizer of F_i in P. It follows that

$$\mathrm{Res}^P_H V \cong pW \cong \bigoplus_{i=1}^{n} \bigoplus_{x \in P/H} {}^x\tilde{S}_i \quad .$$

This gives a contradiction, since W has to be isomorphic to some conjugate of some $\tilde{S}_i$, but W is invariant by P, whereas the stabilizer of $\tilde{S}_i$ in P is equal to H. This proves Assertion 1 of the lemma.

For Assertion 2, consider a genetic subgroup Q of P, such that $Z \not\subseteq Q$, and let $V = V(S)$ denote the corresponding rational irreducible representation of P. Then $\mathrm{Def}^P_{P/Z} V = \{0\}$ by Lemma 5.1.

Let S be a simple summand of $\mathrm{Res}^P_E V$. Then $\mathrm{Def}^E_{E/Z} S = \{0\}$, thus as above S is isomorphic to the kernel of the projection map $\mathbb{Q}E/F \to \mathbb{Q}$, for some subgroup F of index p in E, and $F \neq Z$. Hence the stabilizer of S in P is equal to $N_P(F) = C_P(F) = C_P(E) = H$, and

$$V \cong \mathrm{Ind}^P_H W \quad ,$$

where W is equal to the S-isotypic component of $\mathrm{Res}^P_E V$.

Moreover $\langle V, V \rangle_P = \langle W, W \rangle_H$, since the stabilizer of W in P is equal to H, and $\mathrm{Def}^H_{H/Z} W = \{0\}$. Let R be any genetic subgroup of H associated to the representation W. Then $R \not\supseteq Z$, and $(N_H(R), R)$ is a genetic section of P for V. Thus $N_H(R) = N_P(R)$, and $(N_P(R), R) \,\frac{\quad}{}_P\, (N_P(Q), Q)$, as was to be shown. □

5.3. Corollary : *In the situation of Lemma 5.2, there exists a genetic basis $\mathcal{S}$ of P and a decomposition*

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2$$

in disjoint union such that

1. *if $Q \in \mathcal{S}_2$, then $Q \supseteq Z$, and*
2. *the set $\{{}^xQ \mid Q \in \mathcal{S}_1,\ x \in [P/C_P(E)]\}$ is a set of representatives of genetic subgroups R of $C_P(E)$ such that $R \not\supseteq Z$, for the relation $-\!\!-_{C_P(E)}$, where $[P/C_P(E)]$ is any chosen set of representatives of $C_P(E)$-cosets in P.*

Proof: Let $\mathcal{S}$ be a genetic basis of P. Set $\mathcal{S}_2 = \{Q \in \mathcal{S} \mid Q \supseteq Z\}$, and $\mathcal{S}_1 = \mathcal{S} - \mathcal{S}_2$. By Lemma 5.2, I can assume that if $Q \in \mathcal{S}_1$, then $N_P(Q) \subseteq C_P(E)$, and Q is a genetic subgroup of $C_P(E)$. Then the set

$$\Gamma = \{{}^xQ \mid Q \in \mathcal{S}_1,\ x \in [P/C_P(E)]\}$$

is a set of genetic subgroups of $C_P(E)$, not containing Z, and they are not equivalent to each other for the relation $-\!\!-_{C_P(E)}$: indeed, if there exist $Q, Q' \in \mathcal{S}_1$ and elements $x, x' \in [P/C_P(E)]$ such that ${}^{x'}Q' -\!\!- {}^{hx}Q$ for some $h \in C_P(E)$, then in particular $Q' -\!\!- {}^{x'^{-1}hx}Q$, thus $Q' = Q$ since $\mathcal{S}$ is a genetic basis of P. Moreover $Q' = Q = {}^{x'^{-1}hx}Q$ since Q is genetic in P. Thus $x'^{-1}hx \in N_P(Q) \subseteq C_P(E)$, thus $x \in h^{-1}x'C_P(E) \subseteq C_P(E)x'C_P(E) = x'C_P(E)$, since $C_P(E) \trianglelefteq P$, and $x = x'$.

Conversely, if R is a genetic subgroup of $C_P(E)$ such that $R \not\supseteq Z$, then R is a genetic subgroup of P, and there exists an element $y \in P$ and an element Q of $\mathcal{S}$ such that $R -\!\!- {}^yQ$. This implies in particular that

$$Q \cap Z = Q \cap Z \cap N_P(R^y) \subseteq Z \cap R^y = (Z \cap R)^y = \mathbf{1} \quad ,$$

hence $Q \in \mathcal{S}_1$. Now y is equal to ux, for some unique element x in $[P/C_P(E)]$, and some $u \in C_P(E)$, because $C_P(E) \trianglelefteq P$. Thus $R -\!\!-_{C_P(E)} {}^xQ$, and ${}^xQ$ is an element of Γ. □

6. The kernel of $B \to R_{\mathbb{Q}}$

6.1. Notation : *Let B denote the Burnside biset functor (over $\mathbb{Z}$), and $R_{\mathbb{Q}}$ denote the functor of rational representations. Let K denote the kernel of the natural morphism from B to R_Q.*

By the Ritter-Segal Theorem, there is an exact sequence of biset functors

$$0 \to K \to B \to R_{\mathbb{Q}} \to 0 \quad .$$

6.2. Remark : For any p-group P, the corresponding sequence

$$0 \to K(P) \to B(P) \to R_{\mathbb{Q}}(P) \to 0 \quad .$$

is a split exact sequence, since the group $R_{\mathbb{Q}}(P)$ is a free group with basis $\mathrm{Irr}_{\mathbb{Q}}(P)$ and rank equal to the number of conjugacy classes of cyclic subgroups of P. It follows that $K(P)$ is a free group, of rank equal to the number of conjugacy classes of non cyclic subgroups of P. In particular $K(P) = \{0\}$ if P is cyclic.

6.3. Notation : *If P is a p-group, and $\mathcal{B}$ is a set of representatives of proper basic subgroups of P, for the relation $\div_P$, set $\overline{\mathcal{B}} = \mathcal{B} \sqcup \{P\}$.*

Denote by $[s_P]$ a set of representatives of conjugacy classes of subgroups of P. The elements P/Q, for $Q \in [s_P]$, form a $\mathbb{Z}$-basis of $B(P)$. If Q is a subgroup of P, set

$$S_Q^{\mathcal{B}} = P/Q - P/P - \sum_{R\in\mathcal{B}} m(V_R, \mathbb{Q}P/Q)(P/R - P/\tilde{R}) \quad .$$

6.4. Lemma : *Let $\kappa_{\mathcal{B}}$ the endomorphism of $B(P)$ defined by*

$$\kappa_{\mathcal{B}}(P/Q) = S_Q^{\mathcal{B}} \quad ,$$

for $Q \in [s_P]$. Then $\kappa_{\mathcal{B}}$ is an idempotent endomorphism of $B(P)$, whose image is equal to $K(P)$.

In other words, the elements $S_Q^{\mathcal{B}}$, for $Q \in [s_P]$, generate $K(P)$. Moreover the element $u = \sum_{Q\in[s_P]} u_Q P/Q$ of $B(P)$, where $u_Q \in \mathbb{Z}$, is in $K(P)$ if and only if $u = \sum_{Q\in[s_P]} u_Q S_Q^{\mathcal{B}}$.

Proof: Let $u = \sum_{Q\in[s_P]} u_Q P/Q$ be an element of $B(P)$, where $u_Q \in \mathbb{Z}$. Its image in $R_{\mathbb{Q}}(P)$ is equal to

$$v = \sum_{Q\in[s_P]} u_Q \mathbb{Q}P/Q \quad .$$

Now $m(\mathbb{Q}, \mathbb{Q}P/Q) = 1$ for any subgroup Q of P, thus $m(\mathbb{Q}, v) = \sum_{Q\in[s_P]} u_Q$. Moreover $m(V_R, v) = \sum_{Q\in[s_P]} u_Q m(V_R, \mathbb{Q}P/Q)$, for $R \in \mathcal{B}$. Since V_R is isomorphic to the kernel of the projection map $\mathbb{Q}P/R \to \mathbb{Q}P/\tilde{R}$, it follows that V_R is equal to the image in $R_{\mathbb{Q}}(P)$ of the element $P/R - P/\tilde{R}$ of $B(P)$. Since $\mathbb{Q}$ is the image of P/P in $R_{\mathbb{Q}}(P)$, it follows that v is also equal to the image in $R_{\mathbb{Q}}(P)$ of the element

$$u' = (\sum_{Q\in[s_P]} u_Q)P/P + \sum_{R\in\mathcal{B}} \sum_{Q\in[s_P]} u_Q m(V_R, \mathbb{Q}P/Q)(P/R - P/\tilde{R}) \quad .$$

Thus $u - u' = \sum_{Q\in[s_P]} u_Q S_Q^{\mathcal{B}} = \kappa_{\mathcal{B}}(u)$ is in $K(P)$. Moreover u is in $K(P)$ if and only if $u' = 0$, i.e. if $u = \kappa_{\mathcal{B}}(u)$. The lemma follows. □

6.5. Corollary : *Let E be an elementary abelian p-group of order p^2. Then $K(E) = \partial K(E)$ is free of rank one, generated by the element*

$$\varepsilon_E = E/\mathbf{1} - \sum_{\substack{F \subseteq E \\ |F|=p}} E/F + pE/E \tag{6.6}$$

of $B(E)$.

Proof: Indeed $K(E)$ is free of rank 1 by Remark 6.2. In this case, by Remark 2.4, there is a unique set $\mathcal{B}$ of representatives of proper basic subgroups of E, consisting of all subgroups of E of order p, and its easy to check that $\kappa_{\mathcal{B}}(E/Q) = 0$ if $Q \neq \mathbf{1}$, and $\kappa_{\mathcal{B}}(E/\mathbf{1}) = \varepsilon_E$. □

6.7. Notation : *Fix an elementary abelian p-group E_{p^2} of order p^2,and set $\varepsilon = \varepsilon_{E_{p^2}}$, defined in 6.6. Let K_ε denote the subfunctor of K generated by ε, i.e. the intersection of all subfunctors L of K such that $L(E_{p^2}) \ni \varepsilon$.*

Thus for any p-group P

$$K_\varepsilon(P) = \mathrm{Hom}_{\mathcal{C}_p}(E, P) \times_E \varepsilon \quad .$$

6.8. Lemma : *Let P be a p-group.*

1. *If Q is a subgroup of P such that $Q \cap Z(P) \neq \mathbf{1}$, then $f_{\mathbf{1}}^P P/Q = 0$.*
2. *If the center of P is not cyclic, then $\partial B(P) \subseteq K_\varepsilon(P)$.*

Proof: If Q is a subgroup of P and $Z = Q \cap Z(P) \neq \mathbf{1}$, then $P/Q = \mathrm{Inf}_{P/Z}^P (P/Z)\big/(Q/Z)$, thus $f_1^P P/Q = 0$ by Lemma 3.12. This proves Assertion 1.

Now proving Assertion 2 amounts to showing that $f_1^P P/Q \in K_\varepsilon(P)$, for any subgroup Q of P. Consider first the case $Q = \mathbf{1}$. Since the center $Z(P)$ of P is not cyclic, there exists a subgroup $E \subseteq Z(P)$ which is elementary abelian of order p^2. In particular the element

$$\varepsilon_E = E/\mathbf{1} - \sum_{\substack{F \in E \\ |F|=p}} E/F + pE/E$$

of $B(E)$ is the image of ε under any isomorphism $E_{p^2} \to E$. Thus $\varepsilon_E \in K_\varepsilon(E)$. Inducing up to P gives the element

$$e = P/\mathbf{1} - \sum_{\substack{F \in E \\ |F|=p}} P/F + pP/E$$

of $K_\varepsilon(P)$. But $f_1^P e = f_1^P P/1$ by Assertion 1, since $E \subseteq Z(P)$. Hence $f_1^P P/\mathbf{1} \in K_\varepsilon(P)$.

I will now show that $f_1^P P/Q \in K_\varepsilon(P)$ for any subgroup Q of P by induction on the index $|P:Q|$. If $Q=P$, then $f_1^P P/P = 0 \in K_\varepsilon(P)$, since the center of P is non trivial. Now let Q be any subgroup of P, and suppose that $f_1^P P/R \in K_\varepsilon(P)$ for any subgroup R of P with $|R|>|Q|$. If $Q\cap Z(P)\neq \mathbf{1}$, then $f_1^P P/Q = 0 \in K_\varepsilon(P)$. If $Q\cap Z(P)=\mathbf{1}$, then $Z(P)$ embeds in the center of $\overline{N}=N_P(Q)/Q$, so this center is not cyclic. The special case above shows that the element

$$f_1^{\overline{N}}\overline{N}/\mathbf{1} = \sum_{\overline{Z}\subseteq\Omega_1 Z(\overline{N})} \mu(\mathbf{1},\overline{Z})\overline{N}/\overline{Z}$$

belongs to $K_\varepsilon(\overline{N})$. Taking inflation from $\overline{N}$ to N, and then induction from N to P gives the element

$$w = \mathrm{Indinf}_{N_P(Q)/Q}^P f_1^{\overline{N}}\overline{N}/\mathbf{1} = \sum_{Q'/Q\subseteq\Omega_1 Z(\overline{N})} \mu(\mathbf{1},Q'/Q)P/Q'$$

of $K_\varepsilon(P)$. It follows that

$$f_1^P w = f_1^P P/Q + \sum_{\substack{Q'/Q\subseteq\Omega_1 Z(\overline{N})\\ Q'/Q\neq\mathbf{1}}} \mu(\mathbf{1},Q'/Q) f_1^P P/Q' \in K_\varepsilon(P) \quad .$$

By induction hypothesis, all terms in the summation are in $K_\varepsilon(P)$, and by difference $f_1^P P/Q \in K_\varepsilon(P)$, as was to be shown. □

6.9. Notation :

1. *If $p\neq 2$, denote by X_{p^3} an extraspecial group of order p^3 and exponent p, and by Z its center. Choose two non conjugate non central subgroups I and J of order p in X_{p^3}. Let δ be the element of $B(X_{p^3})$ defined by*

$$\delta = (X_{p^3}/I - X_{p^3}/IZ) - (X_{p^3}/J - X_{p^3}/JZ) \quad .$$

2. *If $p=2$, and if $n\geq 3$ is an integer, denote by D_{2^n} a dihedral group of order 2^n, and by Z its center. Choose two non conjugate non central subgroups I_n and J_n of order 2 in D_{2^n}. Let δ_n be the element of $B(D_{2^n})$ defined by*

$$\delta_n = (D_{2^n}/I_n - D_{2^n}/I_nZ) - (D_{2^n}/J_n - D_{2^n}/J_nZ) \quad .$$

6.10. Remark : Let P be one of the groups X_{p^3} or D_{2^n}, for $n\geq 3$. Then the center Z of P is cyclic of prime order, and P has a unique faithful rational representation : if P is dihedral of order at least 16, this follows from Proposition 3.7 in [8]. If $P=D_8$, this follows from the remark preceding Notation 3.8, in the same paper. And if $P=X_{p^3}$, this follows from the fact that P has

$p+3$ conjugacy classes of cyclic subgroups, and that $P/Z\cong(C_p)^2$ has $p+2$ such subgroups.

Now in each case, it is easy to see that any non central subgroup Q of prime order of P is basic, and the corresponding simple module is isomorphic to the kernel of the projection map $\mathbb{Q}P/Q\to\mathbb{Q}P/QZ$. This shows that the elements $(P/Q-P/QZ)-(P/R-P/RZ)$, for any non-central subgroups Q and R of prime order of P, are in $K(P)$. In particular $\delta\in K(X_{p^3})$, and $\delta_n\in K(D_{2^n})$, for $n\geq 3$.

6.11. Lemma :

1. *If $p\neq 2$, then $JZ\cong E_{p^2}$, and* $\mathrm{Res}^{X_{p^3}}_{JZ}\,\delta=\varepsilon_{JZ}$.
2. *If $p=2$, then $J_3Z\cong E_{p^2}$, and* $\mathrm{Res}^{D_8}_{J_3Z}\delta_3=\varepsilon_{J_3Z}$

Proof: This is a straightforward consequence of the Mackey formula. □

6.12. Theorem :

1. *If $p\neq 2$, then the functor K is generated by δ : in other words for any p-group P*
$$K(P)=\mathrm{Hom}_{\mathcal{C}_p}(X_{p^3},P)\times_{X_{p^3}}\delta\quad .$$
In particular $\overline{K}(P)=\{0\}$ except if P is isomorphic to X_{p^3}, or if P is elementary abelian of order p^2.
2. *If $p=2$, then the functor K is generated by the elements δ_n, for $n\geq 3$: in other words for any 2-group P*
$$K(P)=\sum_{n\geq 3}\mathrm{Hom}_{\mathcal{C}_2}(D_{2^n},P)\times_{D_{2^n}}\delta_n\quad .$$
In particular $\overline{K}(P)=\{0\}$, except if P is dihedral of order at least 4.

Proof: If $p\neq 2$, denote by L the subfunctor of K generated by δ, and if $p=2$, denote by L the subfunctor of K generated by the elements δ_n, for $n\geq 3$. I will show that $L(P)=K(P)$ for any p-group P by induction on the order of $|P|$.

Step 1 : The induction starts with the case where P is cyclic : since $K(P)=\{0\}$ in this case, the result is trivial.

Suppose that P is a p-group such that $K(P')=L(P')$ for any p-group with $|P'|<|P|$. By Lemma 3.10 there is a decomposition
$$K(P)\cong\mathop{\oplus}_{N\trianglelefteq P}f_N^PK(P)\quad ,$$
and $f_N^P=\mathrm{Inf}_{P/N}^P\circ f_{\mathbf{1}}^{P/N}\circ\mathrm{Def}_{P/N}^P$. Now by induction hypothesis, it follows that $f_N^PK(P)\subseteq L(P)$ if $N\neq 1$, since $K(P/N)=L(P/N)$. Hence in order to show that $K(P)=L(P)$, it suffices to show that $f_1^PK(P)=\partial K(P)\subseteq L(P)$.

Now if the center of P is not cyclic, then $\partial K(P) \subseteq K_\varepsilon(P)$ by Lemma 6.8, and $K_\varepsilon(P) \subseteq L(P)$ by Lemma 6.11. Thus $\partial K(P) \subseteq L(P)$, and I can suppose that the center of P is cyclic.

Step 2 : Suppose that P admits a normal elementary abelian subgroup E of rank 2. Set $H = C_P(E)$ and $Z = E \cap Z(P)$. Then $|P : H| = |Z| = p$, since E is not central in P. According to Corollary 5.3, there is a genetic basis $\mathcal{S}$ of P and a decomposition $\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2$ with the following properties :

1. if $S \in \mathcal{S}_2$, then $S \supseteq Z$, and
2. the set $\{{}^xS \mid S \in \mathcal{S}_1,\ x \in [P/H]\}$ is a set of representatives of genetic subgroups R of H such that $R \not\supseteq Z$, for the relation $\,—_H$, where $[P/H]$ is any chosen set of representatives of H-cosets in P.

For each $S \in \mathcal{S}$, choose a basic subgroup R_S/S of $N_P(S)/S$, intersecting trivially the center of $N_P(S)/S$, and denote by $\overline{\mathcal{B}}$ the set of subgroups R_S, for $S \in \mathcal{S}$. Then $\overline{\mathcal{B}}$ is a set of representatives of basic subgroups of P for the relation $\doteqdot_P$, and in particular it contains P. Set moreover $\mathcal{B} = \overline{\mathcal{B}} - \{P\}$.

Observe now that $R_S \supseteq Z$ if $S \supseteq Z$. Conversely, if $S \not\supseteq Z$, then ZS/S is the only subgroup of the center of $N_P(S)/S$ of order p, thus $R_S \cap ZS = S$, and $R_S \not\supseteq Z$. This gives a decomposition of $\overline{\mathcal{B}}$ as

$$\overline{\mathcal{B}} = \overline{\mathcal{B}}_1 \sqcup \overline{\mathcal{B}}_2$$

where

$$\begin{aligned} \overline{\mathcal{B}}_1 &= \{R \in \overline{\mathcal{B}} \mid R \not\supseteq Z\} = \{R_S \mid S \in \mathcal{S}_1\} \quad . \\ \overline{\mathcal{B}}_2 &= \{R \in \overline{\mathcal{B}} \mid R \supseteq Z\} = \{R_S \mid S \in \mathcal{S}_2\} \end{aligned}$$

By Lemma 6.4, the group $K(P)$ is generated by the elements

$$S_Q^{\mathcal{B}} = P/Q - P/P - \sum_{R \in \mathcal{B}} m(V_R, \mathbb{Q}P/Q)(P/R - P/\tilde{R}) \quad ,$$

for $Q \in [s_P]$. Hence in order to show that $f_1^P K(P) = \partial K(P) \subseteq L(P)$, it is enough to show that $f_1^P S_Q^{\mathcal{B}} \in L(P)$ for any subgroup Q of P. Moreover $f_1^P P/Q = P/Q - P/QZ$ for any $Q \subseteq P$, since $Z = \Omega_1 Z(P)$ is the only central subgroup of order p of P. Thus if $R \in \mathcal{B}$ and $R \supseteq Z$, then $\tilde{R} \supseteq Z$, and $f_1^P P/R = f_1^P P/\tilde{R} = 0$. And if $R \not\supseteq Z$, then $\tilde{R} = RZ$, and $f_1^P (P/R - P/\tilde{R}) = P/R - P/RZ$ in this case. Since $f_1^P P/P = 0$, one has that

$$f_1^P S_Q^{\mathcal{B}} = P/Q - P/QZ - \sum_{R \in \overline{\mathcal{B}}_1} m(V_R, \mathbb{Q}P/Q)(P/R - P/RZ) \quad .$$

6.13. Remark : This shows in particular $f_1^P S_Q^{\mathcal{B}} = 0$ if $Q \supseteq Z$. Indeed in that case if $R \in \mathcal{B}$ is such that $m(V_R, \mathbb{Q}P/Q) \neq 0$, then there exists $x \in P$ such that $Q^x \cap N_P(R) \subseteq R$ (see Proposition 2.5 in [8]). Thus

$$Z = Q^x \cap Z \subseteq Q^x \cap N_P(R) \subseteq R \quad .$$

Step 3 : Suppose first that $Q \subseteq H$. Since $\overline{\mathcal{B}}_1 = \{R_S \mid S \in \mathcal{S}_1\}$, if $R = R_S \in \overline{\mathcal{B}}_1$, then $S \subseteq H$ by choice of $\mathcal{S}$, and then by Lemma 5.2, the corresponding simple module $V_R = V(S)$ is isomorphic to $\mathrm{Ind}_H^P W_R$, where $W_R = W(S)$ is the simple $\mathbb{Q}H$-module corresponding to the genetic subgroup S of H. Moreover in that case $\langle V_R, V_R \rangle_P = \langle W_R, W_R \rangle_H$.

Let V be any $\mathbb{Q}P$-module. Then

$$\begin{aligned}
\langle V_R, V \rangle_P &= m(V_R, V)\langle V_R, V_R \rangle_P \\
&= \langle \mathrm{Ind}_H^P W_R, V \rangle_P = \langle W_R, \mathrm{Res}_H^P V \rangle_H \\
&= m(W_R, \mathrm{Res}_H^P V)\langle W_R, W_R \rangle_H \quad ,
\end{aligned}$$

thus $m(V_R, V) = m(W_R, \mathrm{Res}_H^P V)$. This gives

$$f_1^P S_Q^{\mathcal{B}} = P/Q - P/QZ - \sum_{R \in \overline{\mathcal{B}}_1} m(W_R, \mathrm{Res}_H^P \mathbb{Q}P/Q)(P/R - P/RZ) \quad ,$$

and since $\mathrm{Res}_H^P \mathbb{Q}P/Q \cong \oplus_{x \in [P/H]} \mathbb{Q}H/Q^x$, where $[P/H]$ is some set of representatives of H-cosets in P, this leads to

$$\textbf{(6.14)} \qquad f_1^P S_Q^{\mathcal{B}} = P/Q - P/QZ - \sum_{\substack{R \in \overline{\mathcal{B}}_1 \\ x \in [P/H]}} m(W_R, \mathbb{Q}H/Q^x)(P/R - P/RZ)$$

By Corollary 5.3 again, it is possible to choose a genetic basis $\mathcal{T}$ of H such that

$$\{{}^x S \mid S \in \mathcal{S}_1,\ x \in [P/H]\} = \{S \in \mathcal{T} \mid S \not\supseteq Z\} \quad .$$

This also means that the set

$$\mathcal{D}_1 = \{{}^x R_S \mid S \in \mathcal{S}_1,\ x \in [P/H]\} = \{{}^x R \mid R \in \overline{\mathcal{B}}_1,\ x \in [P/H]\}$$

can be completed to a set $\overline{\mathcal{D}}$ of representatives of basic subgroups of H for the relation $\doteqdot_H$. Set $\mathcal{D} = \overline{\mathcal{D}} - \{H\}$, and for $R \in \overline{\mathcal{D}}$, denote by W_R the corresponding simple $\mathbb{Q}H$-module (so in particular, this notation is consistent for $R \in \overline{\mathcal{B}}_1$). Then the elements

$$S_T^{\mathcal{D}} = H/T - H/H - \sum_{R \in \mathcal{D}} m(W_R, \mathbb{Q}H/T)(H/R - H/\tilde{R}) \quad ,$$

for $T \subseteq H$, generate $K(H)$, hence by induction hypothesis, they belong to $L(H)$. This shows that the element

$$S_Q^{\mathcal{D}} - S_{QZ}^{\mathcal{D}} = H/Q - H/QZ - \sum_{R\in\mathcal{D}} m(W_R, \mathbb{Q}H/Q - \mathbb{Q}H/QZ)(H/R - H/\tilde{R})$$

belongs to $L(H)$. Now if $R \in \mathcal{D}$ and $R \supseteq Z$, then $W_R = \mathrm{Inf}_{H/Z}^H W'_R$, where $W'_R = \mathrm{Def}_{H/Z}^H W_R$. Moreover $\langle W_R, W_R\rangle_H = \langle W'_R, W'_R\rangle_{H/Z}$, thus

$$m(W_R, \mathbb{Q}H/Q) = m(W'_R, \mathbb{Q}H/QZ) = m(W_R, \mathbb{Q}H/QZ) \quad .$$

This shows that $m(W_R, \mathbb{Q}H/Q - \mathbb{Q}H/QZ) = 0$ if $R \supseteq Z$, i.e. if $R \notin \mathcal{D}_1$. And if $R \in \mathcal{D}_1$, then $\tilde{R} = RZ$, and

$$\langle W_R, \mathbb{Q}H/QZ\rangle_H = \langle \mathrm{Def}_{H/Z}^H W_R, \mathbb{Q}(H/Z)/(QZ/Z)\rangle_{H/Z} = 0 \quad ,$$

since $\mathrm{Def}_{H/Z}^H W_R = \{0\}$. Hence $m(W_R, \mathbb{Q}H/QZ) = 0$, and this gives finally

$$S_Q^{\mathcal{D}} - S_{QZ}^{\mathcal{D}} = H/Q - H/QZ - \sum_{R\in\mathcal{D}_1} m(W_R, \mathbb{Q}H/Q)(H/R - H/RZ) \quad .$$

Comparing with expression 6.14, since obviously

$$m(W_R, \mathbb{Q}H/Q^x) = m(W_{{}^xR}, \mathbb{Q}H/Q) \quad ,$$

it follows that

$$f_1^P S_Q^{\mathcal{B}} = \mathrm{Ind}_H^P (S_Q^{\mathcal{D}} - S_{QZ}^{\mathcal{D}}) \quad .$$

showing that $f_1^P S_Q^{\mathcal{B}} \in L(P)$ if $Q \subseteq H$.

Step 4 : Suppose now that $Q \not\subseteq H$. This case will be handled by the following lemma :

6.15. Lemma : *Let P be a p-group, and suppose that E is a normal subgroup of P, which is elementary abelian of rank 2, and not contained in the center of P. Set $Z = E \cap Z(P)$. Let S be a subgroup of P, such that $S \not\supseteq Z$, and $S \not\subseteq C_P(E)$.*

1. *The group $C_S(E) = S \cap C_P(E)$ is a normal subgroup of SE, and the quotient $SE/C_S(E)$ is isomorphic to X_{p^3} if $p \neq 2$, and to D_8 if $p = 2$.*
2. *There exist a subgroup T of $C_P(E)$ and a morphism φ in $\mathcal{C}_p$ from X_{p^3} to P if $p \neq 2$ (resp. from D_8 to P if $p = 2$), such that the element*

$$(P/S - P/SZ) - (P/T - P/TZ)$$

is the image by φ of the element δ of $K(X_{p^3})$ (resp. of the element δ_3 of $K(D_8)$).

Proof: Since E is normal in P, and not central in P, the group $C_P(E)$ has index p in P. Now $C_P(E) \trianglelefteq P$, hence S normalizes $C_P(E)$, hence also $S \cap C_P(E)$. Since E centralizes this latter group, it follows that $C_S(E) \trianglelefteq SE$. Moreover $E \cap C_S(E) = E \cap S$. If this is not trivial, then it has order p, otherwise $E \subseteq S$, hence $Z \subseteq S$. Now S normalizes $S \cap E$, hence it centralizes it. Thus S centralizes $E = (S \cap E)Z$, and this contradicts the hypothesis.

Hence $E \cap C_S(E) = \mathbf{1}$, thus E embeds as a normal elementary abelian subgroup $\overline{E}$ of order p^2 in the quotient $X = SE/C_S(E)$. Now X is equal to the semi-direct product of $\overline{E}$ by its subgroup $\overline{S} = S/C_S(E)$, which has order $|S : S \cap C_P(E)| = p$. Moreover X is not abelian, since otherwise $[S, E] \subseteq E \cap C_S(E) = \mathbf{1}$, thus $S \subseteq C_P(E)$. Assertion 1 of the lemma follows.

The group X has center of order p, equal to $\overline{Z} = ZC_S(E)/C_S(E)$. Let θ be any group isomorphism from X_{p^3} to X if $p \neq 2$ (resp. from D_8 to X if $p = 2$) sending the subgroup IZ of X_{p^3} (resp. the subgroup I_3Z of D_8) to $\overline{S}\,\overline{Z}$, and the subgroup JZ of X_{p^3} (resp. the subgroup J_3Z of D_8) to $\overline{E}$. Such an isomorphism exists since the automorphism group of X_{p^3} (resp. of D_8) acts 2-transitively on its elementary abelian subgroups of order p^2. Then $\theta(I)$ (resp. $\theta(I_3)$) is some conjugate of $\overline{S}$ in X, and $\theta(J)$ (resp. $\theta(J_3)$) is some subgroup $\overline{T}$ of $\overline{E}$. Let $w = \delta$ if $p \neq 2$ (resp. $w = \delta_3$ if $p = 2$). Then

$$\theta(w) = (X/\overline{S} - X/\overline{S}\,\overline{Z}) - (X/\overline{T} - X/\overline{T}\,\overline{Z}) \quad .$$

Taking inflation to SE, and then induction to P gives the element

$$\mathrm{Indinf}_{SE/C_S(E)}^{P} \theta(w) = (P/S - P/SZ) - (P/T - P/TZ) \quad ,$$

where T is the preimage in SE of the subgroup $\overline{T}$ of $\overline{E} = EC_S(E)/C_S(E)$ under the projection $SE \to SE/C_S(E)$.

In particular $T \subseteq E(S \cap C_P(E)) = C_P(E)$ (since $E \subseteq C_P(E) \subseteq SC_P(E) = P$), proving Assertion 2 of the lemma. □

Step 5 : It remains to show that $f_1^P S_Q^{\mathcal{B}} \in L(P)$ if Q is a subgroup of P, such that $Q \not\supseteq Z$ and $Q \not\subseteq H = C_P(E)$. By Lemma 6.15, there exists a subgroup T of H such that the element

$$u = (P/Q - P/QZ) - (P/T - P/TZ)$$

is the image of δ (resp. δ_3) by some morphism in the category $\mathcal{C}_p$. Since $\delta \in L(X_{p^3})$ (resp. $\delta_3 \in L(D_8)$), it follows that $u \in L(P)$. Since moreover $L(P) \subseteq K(P)$, the element u is equal to

$$(S_Q^{\mathcal{B}} - S_{QZ}^{\mathcal{B}}) - (S_T^{\mathcal{B}} - S_{TZ}^{\mathcal{B}})$$

by Lemma 6.4. It follows that

$$(f_1^P S_Q^{\mathcal{B}} - f_1^P S_{QZ}^{\mathcal{B}}) - (f_1^P S_T^{\mathcal{B}} - f_1^P S_{TZ}^{\mathcal{B}}) \in L(P) \quad .$$

Since $T \subseteq H$, the previous discussion shows that $f_1^P S_T^{\mathcal{B}} \in L(P)$. Moreover $f_1^P S_{QZ}^{\mathcal{B}} = f_1^P S_{TZ}^{\mathcal{B}} = 0$ by Remark 6.13. Hence $f_1^P S_Q^{\mathcal{B}} \in L(P)$, as announced.

Step 6 : To complete the proof of Theorem 6.12, it remains to consider the case where P has no elementary abelian normal subgroup of order p^2, i.e. the case where P has normal p-rank 1. If p is odd, there is nothing more to do, since then P is cyclic, and $K(P) = \{0\}$. If $p = 2$, then :

- if P is cyclic, again there is nothing more to do, since $K(P) = \{0\}$.
- if P is generalized quaternion, then $\partial K(P) = \{0\}$: indeed the only subgroup of P intersecting trivially the center is the trivial group, and the trivial group is a normal basic subgroup of P, hence it belongs to any set $\mathcal{B}$ of representatives of basic subgroups of P, by Remark 2.4. Clearly then $S_{\mathbf{1}}^{\mathcal{B}} = 0$.
- if P is semi-dihedral, then $\partial K(P)$ is free of rank 1 : up to conjugation, there are only two subgroups of P which intersect trivially the center of P, namely the trivial group and the non central subgroup Q of order 2, which is a basic subgroup of P. Then any set $\mathcal{B}$ of representatives of basic subgroups of P modulo the relation $\doteqdot_P$ must contain Q (up to conjugation). Then clearly $f_1^P S_Q^{\mathcal{B}} = 0$. On the other hand

  $$\begin{aligned} f_1^P S_{\mathbf{1}}^{\mathcal{B}} &= P/\mathbf{1} - P/Z - m(V_Q, \mathbb{Q}P/\mathbf{1})(P/Q - P/QZ) \\ &= P/\mathbf{1} - P/Z - 2P/Q + 2P/QZ \\ &= \mathrm{Ind}_{QZ}^P \varepsilon_{QZ} \end{aligned}$$

  Thus in this case also $\partial K(P) \subseteq K_\varepsilon(P) \subseteq L(P)$.
- The only remaining case is when P is dihedral, say $P \cong D_{2^n}$, with $n \geq 4$. In this case up to conjugation, there are 3 subgroups which intersect trivially the center of P, namely the trivial group, and the non central subgroups I_n and J_n of order 2. Since $I_n \doteqdot_P J_n$, I can suppose that $\mathcal{B}$ contains J_n (and not I_n). In this case

  $$f_1^P S_Q^{\mathcal{B}} = \begin{cases} P/\mathbf{1} - P/Z - 2(P/J_n - P/J_nZ) & \text{if } Q = \mathbf{1} \\ (P/I_n - P/I_nZ) - (P/J_n - P/J_nZ) & \text{if } Q =_P I_n \\ 0 & \text{otherwise} \end{cases}$$

  Thus $f_{\mathbf{1}}^P S_{\mathbf{1}}^{\mathcal{B}} = \mathrm{Ind}_{J_nZ}^P \varepsilon_{J_nZ}$ is in $L(P)$, by Lemma 6.11, and $f_{\mathbf{1}}^P S_{I_n}^{\mathcal{B}} = \delta_n \in L(P)$. Thus $\partial K(P) \subseteq L(P)$ also in this case.

The only thing to prove now is that $\overline{K}(P) = \{0\}$ except if $P \cong X_{p^3}$ or $P \cong E_{p^2}$ if $p \neq 2$, or if P is dihedral of order at least 4 if $p = 2$. But by Remark 3.4, if Q is any p-group, then any transitive (P, Q)-biset factors as a composition of a deflation-restriction to some section of Q, followed by a group isomorphism, followed by an induction-inflation from a section of P.

Since any section of X_{p^3} is cyclic, isomorphic to E_{p^2} or to X_{p^3}, since any section of a dihedral group is cyclic or dihedral of order at least 4 (with the usual convention that a dihedral group of order 4 is elementary abelian), it follows that for any $\varphi \in \mathrm{Hom}_{\mathcal{C}_p}(X_{p^3}, P)$ (resp. for any $\varphi \in \mathrm{Hom}_{\mathcal{C}_2}(D_{2^n}, P)$), the element $\varphi(\delta)$ (resp. the element $\varphi(\delta_n)$) is a linear combination of elements of the form $\mathrm{Indinf}_{T/S}^P v$, for sections T/S of P which are also sections of X_{p^3} (resp. of D_{2^n}), and elements $v \in K(T/S)$. Here of course I can assume that T/S is not cyclic, otherwise $K(T/S) = \{0\}$.

Hence $\overline{K}(P) = \{0\}$ if P is not isomorphic to E_{p^2} or X_{p^3} for $p \neq 2$ (resp. if P is not dihedral of order at least 4 if $p = 2$). This completes the proof of Theorem 6.12. □

6.16. Corollary : *Let P be a p-group. Then $K(P)$ is equal to the set of linear combinations of elements of the form $\mathrm{Indinf}_{T/S}^P \theta(\kappa)$, where (T, S) is a section of P, where θ is a group isomorphism from one of the groups X_{p^3}, E_{p^2}, or D_{2^n} $(n \geq 3)$ to T/S, and κ is respectively equal to δ, ε, δ_n.*

6.17. Remark : One can check easily that if P is isomorphic to one of the groups E_{p^2}, X_{p^3}, D_{2^n} for $n \geq 3$, then $\overline{K}(P)$ is actually non zero, respectively isomorphic to $\mathbb{Z}$, $\mathbb{Z}/p\mathbb{Z}$, or $\mathbb{Z}/2\mathbb{Z}$, and generated by the image of ε, δ, or δ_n.

7. The Dade group modulo torsion

In this section I will show that $D(P) = D^\Omega(P) + D_{tors}(P)$, for any p-group P. In the case $p \neq 2$, this will be enough to conclude that $D = D^\Omega$.

7.1. Theorem : *Let n be a positive integer. Then*

$$(nD \cap D^\Omega) + D^\Omega_{tors} = nD^\Omega + D^\Omega_{tors} \quad .$$

Proof: If $n = 1$, there is nothing to prove, so assume $n \geq 2$. By Theorem 1.8 of [9], there is an exact sequence of bisets functors

$$0 \to R^*_{\mathbb{Q}} \to B^* \to D^\Omega/D^\Omega_{tors} \to 0 \quad ,$$

where R^*_Q and B^* are the respective $\mathbb{Z}$-dual of $R_{\mathbb{Q}}$ and B, and the inclusion $R^*_{\mathbb{Q}} \to B^*$ is the transpose of the natural morphism $B \to R_{\mathbb{Q}}$. If P is a p-group, evaluation at P gives the exact sequence of abelian groups

$$0 \to R^*_{\mathbb{Q}}(P) \to B^*(P) \to (D^\Omega/D^\Omega_{tors})(P) \to 0 \quad ,$$

and this sequence is actually split exact, since $(D^\Omega/D^\Omega_{tors})(P)$ is a free abelian group. In particular, this sequence remains exact after tensoring (over $\mathbb{Z}$) with $\Gamma_n = \mathbb{Z}/n\mathbb{Z}$, and this shows that the sequence

$$0 \to \Gamma_n R^*_\mathbb{Q} \to \Gamma_n B^* \to \Gamma_n(D^\Omega/D^\Omega_{tors}) \to 0$$

is an exact sequence of biset functors, where the $\otimes_\mathbb{Z}$ symbols have been dropped (so e.g. $\Gamma_n R_\mathbb{Q}$ denotes $\Gamma_n \otimes_\mathbb{Z} R_\mathbb{Q}$). Now there are canonical isomorphisms

$$\begin{aligned} \Gamma_n R^*_\mathbb{Q} &\cong \mathrm{Hom}_{\Gamma_n}(\Gamma_n R_\mathbb{Q}, \Gamma_n) \\ \Gamma_n B^* &\cong \mathrm{Hom}_{\Gamma_n}(\Gamma_n B, \Gamma_n) \quad . \end{aligned}$$

Moreover

$$\begin{aligned} \Gamma_n(D^\Omega/D^\Omega_{tors}) &\cong (D^\Omega/D^\Omega_{tors})\Big/n(D^\Omega/D^\Omega_{tors}) \\ &\cong (D^\Omega/D^\Omega_{tors})\Big/\Big((nD^\Omega + D^\Omega_{tors})/D^\Omega_{tors}\Big) \\ &\cong D^\Omega/(nD^\Omega + D^\Omega_{tors}) \quad . \end{aligned}$$

Setting $T_n = D^\Omega/(nD^\Omega + D^\Omega_{tors})$, this gives the exact sequence

(7.2) $$0 \to \mathrm{Hom}_{\Gamma_n}(\Gamma_n R_\mathbb{Q}, \Gamma_n) \to \mathrm{Hom}_{\Gamma_n}(\Gamma_n B, \Gamma_n) \to T_n \to 0 \quad .$$

Note that $T_n(P)$ is a free Γ_n-module, for any p-group P.

Now on the other hand the exact sequence of biset functors

$$0 \to K \to B \to R_\mathbb{Q} \to 0$$

remains exact after tensoring with Γ_n, since every evaluation of this sequence is a split exact sequence of abelian groups. Hence there is an exact sequence

$$0 \to \Gamma_n K \to \Gamma_n B \to \Gamma_n R_\mathbb{Q} \to 0 \quad ,$$

and every evaluation of this sequence is a split exact sequence of Γ_n-modules. Now taking Γ_n-duals leads to the exact sequence

(7.3)
$$0 \to \mathrm{Hom}_{\Gamma_n}(\Gamma_n R_\mathbb{Q}, \Gamma_n) \to \mathrm{Hom}_{\Gamma_n}(\Gamma_n B, \Gamma_n) \to \mathrm{Hom}_{\Gamma_n}(\Gamma_n K, \Gamma_n) \to 0 \quad .$$

Comparing the sequences 7.2 and 7.3 gives the following natural isomorphism of biset functors

(7.4) $$T_n \cong \mathrm{Hom}_{\Gamma_n}(\Gamma_n K, \Gamma_n) \quad .$$

This means that for any p-group P, there is a non-degenerate scalar product

$$(\ ,\ )_P : T_n(P) \times \Gamma_n K(P) \to \Gamma_n$$

with the property that for any p-group Q and any $\varphi \in \mathrm{Hom}_{\mathcal{C}_p}(Q,P)$, one has that

$$\textbf{(7.5)}\qquad \forall u\in T_n(P),\ \forall v\in \Gamma_nK(Q),\ \Big(u,\varphi(v)\Big)_P=\Big(\varphi^{op}(u),v\Big)_Q \quad .$$

Now set

$$F=\Big((nD\cap D^\Omega)+D^\Omega_{tors}\Big)\Big/(nD^\Omega+D^\Omega_{tors})\quad .$$

Then F is a biset subfunctor of T_n. Suppose that $F\neq\{0\}$, and let P be a p-group of minimal order such that $F(P)\neq\{0\}$. Then $F(P)$ is a subset of

$$\underline{T}_n(P)\subseteq\{u\in T_n(P)\mid \mathrm{Defres}^P_{T/S}(u)=0,\ \forall S\trianglelefteq T\subseteq P,\ |T/S|<|P|\}\quad .$$

By the above duality 7.5, it follows that if $u\in\underline{T_n}(P)$, then for any proper section T/S of P, and any $v\in\Gamma_nK(P)$

$$\Big(u,\mathrm{Indinf}^P_{T/S}(v)\Big)_P=0\quad .$$

Hence $(u,w)_P=0$ for any $w\in\sum\limits_{\substack{S\trianglelefteq T\subseteq P\\|T/S|<|P|}}\mathrm{Indinf}^P_{T/S}\Gamma_nK(T/S)$. By Theorem 6.12, this is the whole of $\Gamma_nK(P)$, unless P is isomorphic to X_{p^3} or E_{p^2} if $p\neq 2$, or if P is dihedral of order at least 4 if $p=2$. Thus if P is not isomorphic to one of the groups in this list, then $(u,v)_P=0$ for any $v\in\Gamma_nK(P)$, hence $u=0$ since the scalar product is non degenerate. Thus $\underline{T}_n(P)=\{0\}$.

Hence the minimal group P is one of the groups X_{p^3}, E_{p^2} or D_{2^n} (note that $E_{2^2}\cong D_{2^2}$). If $F(P)=\{0\}$ in each of these cases, this gives a contradiction, proving that $F=\{0\}$. But if P is one of the groups X_{p^3}, E_{p^2} or D_{2^n}, then $D(P)=D^\Omega(P)$: for $P=X_{p^3}$, this follows from Theorem 10.2 of [11], for $P=E_{p^2}$, this follows from Theorem 4.6, and for $P=D_{2^n}$, this follows from Theorem 4.7. Hence $nD(P)\cap D^\Omega(P)=nD^\Omega(P)$, and $F(P)=\{0\}$. This completes the proof of the theorem. □

7.6. Remark : Using the fact that Γ_n is a self injective ring, one can show that the above isomorphism 7.4 actually restricts to an isomorphism

$$\underline{T}_n(P)\cong \mathrm{Hom}_{\Gamma_n}(\overline{\Gamma_nK}(P),\Gamma_n)\quad ,$$

for any p-group P. Moreover

$$\begin{aligned}
\overline{\Gamma_nK}(P) &= \Gamma_nK(P)/\sum_{\substack{S\trianglelefteq T\subseteq P\\|T/S|<|P|}}\mathrm{Indinf}^P_{T/S}\Gamma_nK(T/S)\\
&\cong \Big(K(P)/nK(P)\Big)\Big/\sum_{\substack{S\trianglelefteq T\subseteq P\\|T/S|<|P|}}\mathrm{Indinf}^P_{T/S}\Big(K(T/S)/nK(T/S)\Big)\\
&\cong K(P)/\Big(\sum_{\substack{S\trianglelefteq T\subseteq P\\|T/S|<|P|}}\mathrm{Indinf}^P_{T/S}K(T/S)+nK(P)\Big)\\
&\cong \overline{K}(P)/n\overline{K}(P)\cong\Gamma_n\overline{K}(P)\quad ,
\end{aligned}$$

so in fact there is always a group isomorphism $\underline{T}_n(P) \cong \mathrm{Hom}_{\Gamma_n}(\Gamma_n\overline{K}(P), \Gamma_n)$.

7.7. Theorem :

1. *$D = D^\Omega + D_{tors}$.*
2. *If $p \neq 2$, then $D = D^\Omega$.*

(Assertion 2 was the first part of Conjecture B in [4]).

Proof: Indeed, by Proposition 7.4.9 of [6], for any p-group P, the quotient $D(P)/(D^\Omega(P) + L(P))$ is a finite p-group, where $L(P)$ is the intersection of the kernels of the maps $\mathrm{Defres}^P_{T/S}$, where T/S is an elementary abelian section of P. Thus if $u \in D(P)$, then there is $m \in \mathbb{N}$ such that $p^m u \in D^\Omega(P) + L(P)$.

Now the group $L(P)$ is always a subgroup of $D_{tors}(P)$, by Theorem 1.6 of [12]. Let e denote its exponent. Then $ep^m u \in D^\Omega(P)$, hence

$$ep^m u \in ep^m D(P) \cap D^\Omega(P) \subseteq ep^m D^\Omega(P) + D^\Omega_{tors} \quad ,$$

by Theorem 8.2. Hence there exists $v \in D^\Omega(P)$ such that the difference $ep^m u - ep^m v = ep^m(u - v)$ is a torsion element of $D^\Omega(P)$. Hence $u - v$ is a torsion element of $D(P)$, proving Assertion 1.

If $p \neq 2$, it has been shown by Carlson and Thévenaz (see Theorem 13.1 of [14]) that $L(P) = \{0\}$. Hence with the same notation $e = 1$, and the difference $w = p^m(u-v)$ is in $D^\Omega_{tors}(P)$. This group is a 2-group of exponent 2 (by Corollary 7.6 of [10], or by Corollary 13.2 of [14]). Thus $w = p^m w$, and $p^m(u - v - w) = 0$. But by Corollary 13.2 of [14], there is no p-torsion in $D(P)$. Thus $u = v + w \in D^\Omega(P)$, and $D(P) = D^\Omega(P)$ in this case. □

8. The torsion part of the Dade group

8.1. Proposition : *Let P be a p-group, and suppose that E is a normal subgroup of P, which is elementary abelian of rank 2. Let Z be a subgroup of order p of $E \cap Z(P)$. Then*

$$D_{tors}(P) \cap \mathrm{Ker\ Res}^P_{C_P(E)} \cap \mathrm{Ker\ Def}^P_{P/Z} = \{0\} \quad .$$

Proof: by induction on the order of P : suppose that the result holds for all p-groups Q with $|Q| < |P|$.

Step 1 : If E is central in P, the result is trivial, for in that case $C_P(E) = P$, and $\mathrm{Res}^P_{C_P(E)}$ is the identity map.

If E is not central in P, then $|P : C_P(E)| = p$, and $E \cap Z(P)$ has order p, hence it is equal to Z. Let $u \in D_{tors}(P)$, such that $\mathrm{Res}^P_{C_P(E)} u = 0$ and $\mathrm{Def}^P_{P/Z} u = 0$.

Let H be a proper subgroup of P, containing E, and consider $v = \mathrm{Res}^P_H u$. Then the hypothesis of the proposition holds for H, its normal subgroup E, and its subgroup Z contained in $E \cap Z(H)$. Moreover v is in $D_{tors}(H)$, and

$$\begin{aligned} \mathrm{Res}^H_{C_H(E)} v &= \mathrm{Res}^P_{C_H(E)} u \\ &= \mathrm{Res}^{C_P(E)}_{C_H(E)} \mathrm{Res}^P_{C_P(E)} u = 0 \quad . \end{aligned}$$

Similarly

$$\begin{aligned} \mathrm{Def}^H_{H/Z} v &= \mathrm{Def}^H_{H/Z} \mathrm{Res}^P_H u \\ &= \mathrm{Res}^{P/Z}_{H/Z} \mathrm{Def}^P_{P/Z} u = 0 \quad . \end{aligned}$$

The induction hypothesis now shows that $v = 0$. Thus $\mathrm{Res}^P_H u = 0$ for any proper subgroup of P containing E.

Consider now a subgroup Z' of order p of $Z(P)$, not contained in E (equivalently $Z' \neq Z$, since E is not central in P). The group E embeds in the group $\overline{P} = P/Z'$, since $E \cap Z' = \mathbf{1}$, and its image $\overline{E} = EZ'/Z'$ is a normal subgroup of $\overline{P}$.

If $u \in P$ is such that $uZ' \in C_{\overline{P}}(\overline{E})$, then $[u, E] \subseteq Z'$, hence $[u, E] \subseteq E \cap Z' = \mathbf{1}$. This shows that $C_{\overline{P}}(\overline{E}) = C_P(E)/Z'$. Moreover $\overline{Z} = Z.Z'/Z'$ is a central subgroup of order p of $\overline{P}$, contained in $\overline{E}$.

Set $w = \mathrm{Def}^P_{P/Z'} u$. Then $w \in D_{tors}(\overline{P})$, and

$$\begin{aligned} \mathrm{Res}^{\overline{P}}_{C_{\overline{P}}(\overline{E})} w &= \mathrm{Res}^{P/Z'}_{C_P(E)/Z'} \mathrm{Def}^P_{P/Z'} u \\ &= \mathrm{Def}^{C_P(E)}_{C_P(E)/Z'} \mathrm{Res}^P_{C_P(E)} u = 0 \quad . \end{aligned}$$

Similarly

$$\begin{aligned} \mathrm{Def}^{\overline{P}}_{\overline{P}/\overline{Z}} w &= \mathrm{Def}^P_{P/ZZ'} u \\ &= \mathrm{Def}^{P/Z}_{P/ZZ'} \mathrm{Def}^P_{P/Z} u = 0 \quad . \end{aligned}$$

Then $w = 0$ by induction hypothesis. It follows that $\mathrm{Def}^P_{P/X} u = 0$ for any central subgroup X of order p of P, and then $\mathrm{Def}^P_{P/N} u = 0$ for any non trivial normal subgroup N of P.

On the other hand, Carlson and Thévenaz have shown that the element u of $D_{tors}(P)$ is equal to 0 if and only if $\mathrm{Defres}^P_{N_P(Q)/Q} u = 0$ for any subgroup Q of P such that $N_P(Q)/Q$ is cyclic, if $p \neq 2$, or cyclic, generalized quaternion, or semi-dihedral, if $p=2$ (see [14] Theorem 13.4). Let Q be such a subgroup. If $M=Q\cap Z(P)$ is non trivial, then setting $\overline{P}=P/M$ and $\overline{Q}=Q/M$

$$\mathrm{Defres}^P_{N_P(Q)/Q}u=\mathrm{Defres}^{\overline{P}}_{N_{\overline{P}}(\overline{Q})/\overline{Q}}\mathrm{Def}^P_{\overline{P}}u=0 \quad .$$

And if $M=\mathbf{1}$, then $Z(P)$ embeds in the center of the group $N_P(Q)/Q$, which is always cyclic when $N_P(Q)/Q$ is cyclic, generalized quaternion, or semi-dihedral. This cannot happen if $Z(P)$ is not cyclic, and $u=0$ in this case.

Step 2 : Hence I can assume that $Z(P)$ is cyclic. Then by Assertion (b) of Theorem 13.1 of [14], the element u of $D_{tors}(P)$ is equal to 0 if and only if $\mathrm{Defres}^P_{T/S}u=0$, for any section (T,S) of P such that T/S is cyclic of order p if $p\neq 2$, or cyclic of order 4 or quaternion of order 8 if $p=2$.

Let (T,S) be such a section. If $M=S\cap Z(P)\neq\mathbf{1}$, then setting $\overline{P}=P/M$, $\overline{T}=T/M$ and $\overline{S}=S/M$

$$\mathrm{Defres}^P_{T/S}u=\mathrm{Defres}^{\overline{P}}_{\overline{T}/\overline{S}}\mathrm{Def}^P_{\overline{P}}u=0 \quad .$$

Similarly, if $H=TE\neq P$, then

$$\mathrm{Defres}^P_{T/S}u=\mathrm{Defres}^H_{T/S}\mathrm{Res}^P_H u=0 \quad ,$$

since H is a proper subgroup of P containing E.

Step 3 : So I can suppose that $TE=P$, and $S\cap Z(P)=\mathbf{1}$. Set $M=S\cap C_P(E)$. Then M is normalized by T and centralized by E, thus $M\trianglelefteq P$, and $M\cap Z(P)=\mathbf{1}$ since M is a subgroup of S. Thus $M=\mathbf{1}$. Since $C_P(E)$ has index p in P, this implies that $|S|\leq p$, and there are two cases :

- if $|S|=p$, then $S\subseteq T\subseteq SC_P(E)=P$, thus $T=S(T\cap C_P(E))$, and in particular $T/S\cong T\cap C_P(E)$ is isomorphic to C_p for $p\neq 2$, or to C_4 or Q_8 for $p=2$. Moreover S in central in T, thus $T=S\times(T\cap C_P(E))$. The group SE is isomorphic to the (non trivial) semi-direct product of C_p by E_{p^2}, hence it is isomorphic to X_{p^3} if $p\neq 2$, and to D_8 if $p=2$. The group $E\cap T$ has order at most p (otherwise $E\subseteq T$, thus $E\subseteq T\cap C_P(E)$, and this cannot happen since $T\cap C_P(E)$ is one of the groups C_p, C_4 or Q_8).

  If $E\cap T=\mathbf{1}$, then $P=SE\times(T\cap C_P(E))$, since

  $$SE\cap(T\cap C_P(E))=S(E\cap T)\cap C_P(E)=S\cap C_P(E)=\mathbf{1} \quad .$$

  In that case the center of P cannot be cyclic, since both groups SE and $T\cap C_P(E)$ are non trivial.

It follows that $E \cap T$ has order p. If $E \cap T \neq Z$, then $E = (E \cap T)Z$ is centralized by T, hence contained in the center of P. Thus $E \cap T = Z$. In that case

$$SE \cap (T \cap C_P(E)) = SZ \cap C_P(E) = Z(S \cap C_P(E)) = Z \quad ,$$

and P is isomorphic to the central product of SE and $T \cap C_P(E)$. Thus if $p \neq 2$, then $T \cap C_P(E) = T \cap E = Z$, for $T \cap E \subseteq T \cap C_P(E)$, and both have order p, and $P = SE \cong X_{p^3}$ in this case. And if $p = 2$, the group P is isomorphic to $D_8 * C_4$ or $D_8 * Q_8$.

- if $S = \mathbf{1}$, then T is isomorphic to C_p if $p \neq 2$, or to C_4 or Q_8 if $p = 2$. Again if $E \cap T = \mathbf{1}$ then $P = E \rtimes T \cong E_{p^2} \rtimes C_p \cong X_{p^3}$ if $p \neq 2$. And if $p = 2$, then $P \cong E_4 \rtimes C_4$ or $P \cong E_4 \rtimes Q_8$, and in each case $Z(P) = Z \times Z(T) \cong C_2 \times C_2$, which is not cyclic. And if $E \cap T \neq \mathbf{1}$, then $E \cap T = Z$ as above.

  If $p \neq 2$, then T has order p, hence $T \subseteq E$, hence $P = E$, a contradiction. And if $p = 2$, then $P \cong E \rtimes T/\Delta$, where Δ is the unique subgroup of order 2 of $Z \times Z(T)$ which is neither contained in Z nor in $Z(T)$. If $T \cong C_4$, then $P \cong D_8$, and $u = 0$ since $D(D_8)$ is torsion free by Theorem 10.3 of [13]. And if $T \cong Q_8$, then clearly $[P, P] = (Z \times [T, T])/\Delta$, which has order 2, and is also equal to the Frattini subgroup of P, since $P/[P, P]$ is elementary abelian of order 8. In this case also $Z(P)$ is cyclic of order 4, generated by the element et, where $e \in E - Z$, and t is a generator of $C_T(E)$. Hence P is almost extraspecial of order 16, isomorphic to $P \cong D_8 * C_4$ again.

Step 4 : Finally, the only cases left to consider are

- Case 1 : $p \neq 2$ and $P \cong X_{p^3}$,
- Case 2 : $p = 2$ and $P \cong D_8 * C_4$,
- Case 3 : $p = 2$ and $P \cong D_8 * Q_8$.

By Theorem 9.1 of [11], the group $\partial D_{tors}(P)$ is cyclic of order 2 in Case 1 and Case 2, or cyclic of order 4 in Case 3 and the ground field does not contain primitive cubic roots of unity. In Case 3, when the ground field contains primitive cubic roots of unity, then $\partial D_{tors}(P) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, by Theorem 10.1 of [11].

Note that by Theorem 4.6 and Theorem 4.7, this can be expressed as

$$\partial D_{tors}(P) \cong \begin{cases} T_{tors}(C_p) & \text{in Case 1} \\ T_{tors}(C_4) & \text{in Case 2} \\ T_{tors}(Q_8) & \text{in Case 3} \end{cases} \quad .$$

Using this observation, these results can be interpreted as follows : the genetic subgroups S of P which do not contain $Z = \Omega_1 Z(P) = \Phi(P)$ are precisely the non central subgroups of order p. For such a subgroup set $R = N_P(S)$. Then

$$R/S \cong \begin{cases} C_p & \text{in Case 1} \\ C_4 & \text{in Case 2} \\ Q_8 & \text{in Case 3} \end{cases} .$$

By Theorem 6.1 of [10], and with the same notation, the map

$$\mathrm{Teninf}_{R/S}^{P} : T_{tors}(R/S) \to D_{tors}(P)$$

is injective, and its image is contained in $\partial D_{tors}(P) = \mathrm{Ker}\ \mathrm{Def}_{P/Z}^{P}$: indeed for $v \in T_{tors}(R/S) = f_1^{R/S} D_{tors}(R/S)$, since $\mathrm{Def}_{R/SZ}^{R/S} v = 0$, it follows that

$$\mathrm{Teninf}_{R/S}^{P} v = a_{R,S}(v) = (P/S - P/SZ)v \quad ,$$

where $P/S - P/SZ$ is a virtual $(P, R/S)$-biset. Then clearly

$$\mathrm{Def}_{P/Z}^{P} \mathrm{Teninf}_{R/S}^{P} v = (P/SZ - P/SZ)v = 0 \quad ,$$

since the Galois torsions have no effect here : the group P/Z is elementary abelian, hence $D(P/Z) = D^{\Omega}(P/Z)$.

Hence in each case above, the map $\mathrm{Teninf}_{R/S}^{P}$ induces an isomorphism from $T_{tors}(R/S)$ to $\partial D_{tors}(P)$. It follows that the (restriction of the) map $b_{R,S} = S\backslash P - SZ\backslash P$ is the inverse isomorphism $\partial D_{tors}(P) \to T_{tors}(R/S)$.

Now the elementary abelian subgroups of P of rank 2 are all maximal elementary abelian subgroups of P. If E is one of them, then $E = SZ$, where S is any non central subgroup of order p of E. In particular $C_P(E)$ is equal to $R = N_P(S)$.

Now if $u \in \partial D_{tors}(P)$ is such that $\mathrm{Res}_{C_P(E)}^{P} u = 0$, then

$$\begin{aligned} b_{R,S}(u) &= (S\backslash P - SZ\backslash P)u \\ &= f_{\mathbf{1}}^{R/S} \mathrm{Def}_{R/S}^{R} \mathrm{Res}_{R}^{P} u = 0 \quad , \end{aligned}$$

hence $u = 0$ since $b_{R,S}$ is an isomorphism. This completes the proof of Proposition 8.1. □

8.2. Theorem : *Let P be a p-group, and let $\mathcal{S}$ be a genetic basis of P. Then the map*

$$\mathcal{I}_{\mathcal{S}} = \bigoplus_{S \in \mathcal{S}} \mathrm{Teninf}_{N_P(S)/S}^{P} : \bigoplus_{S \in \mathcal{S}} T_{tors}(N_P(S)/S) \to D_{tors}(P)$$

is an isomorphism.

(This was Conjecture A in [4], or Conjecture 6.2 in [10]).

Proof: Let

$$\mathcal{D}_{\mathcal{S}} : D_{tors}(P) \to \bigoplus_{S \in \mathcal{S}} T_{tors}(N_P(S)/S)$$

be the map in the other direction, defined as in Section 6 of [10] by $\mathcal{D}_\mathcal{S} = \oplus_{S\in\mathcal{S}}\, b_{N_P(S),S}$. Since $\mathcal{D}_\mathcal{S}\circ\mathcal{I}_\mathcal{S} = \mathrm{Id}$ by Theorem 6.1 of [10], proving Theorem 8.2 is equivalent to proving that the map $\mathcal{D}_\mathcal{S}$ is injective, and I will proceed by induction on the order of P.

Before that, suppose that the map $\mathcal{I}_\mathcal{S}$ is an isomorphism, for a particular genetic basis $\mathcal{S}$ of P. It follows that

$$D_{tors}(P) \cong \mathop{\oplus}_{S\in\mathcal{S}} T_{tors}(N_P(S)/S) \quad ,$$

and up to isomorphism, the right hand side is independent of the choice of $\mathcal{S}$, since the groups $N_P(S)/S$ are the types of the irreducible rational representations of P. Now if $\mathcal{S}'$ is another genetic basis of P, then the map $\mathcal{I}_{\mathcal{S}'}$ is always injective, by Theorem 6.1 of [10]. Hence it is an isomorphism, since it is an injection from a finite group to an isomorphic finite group. In other words, it is enough to prove Theorem 8.2 for any particular genetic basis $\mathcal{S}$ of P.

Suppose now that Theorem 8.2 holds for any p-group Q with $|Q| < |P|$. Let $\mathcal{S}$ be a genetic basis of P, and let Z be any central subgroup of order p in P. Set

$$\mathcal{S}' = \{S \in \mathcal{S} \mid S \supseteq Z\} \quad .$$

Then the set $\overline{\mathcal{S}}' = \{S/Z \mid S \in \mathcal{S}'\}$ is a genetic basis for $\overline{P} = P/Z$. Let $u \in \mathrm{Ker}\ \mathcal{D}_\mathcal{S}$, and consider $v = \mathrm{Def}^P_{P/Z}u$. If $\overline{S} = S/Z$ is an element of $\overline{\mathcal{S}}'$, for $S \in \mathcal{S}$, then $N_{\overline{P}}(\overline{S}) = \overline{R} = R/S$, where $R = N_P(S)$ and

$$\begin{aligned} b_{\overline{R},\overline{S}}(v) &= b_{\overline{R},\overline{S}}\mathrm{Def}^P_{P/Z}u \\ &= \mathrm{Def}^P_{P/Z}b_{R,S}(u) = 0 \quad . \end{aligned}$$

Now the induction hypothesis shows that $v = 0$. Since this holds for any central subgroup Z of order p of P, it follows that $u \in \partial D_{tors}(P)$. Thus

$$\mathrm{Ker}\ \mathcal{D}_\mathcal{S} \subseteq \partial D_{tors}(P) \quad ,$$

for any genetic basis $\mathcal{S}$ of P.

Now let $u \in \partial D_{tors}(P)$. Theorem 13.4 of [14] shows that u is equal to 0 if and only if $\mathrm{Defres}^P_{N_P(Q)/Q}u = 0$, for any subgroup Q of P such that $N_P(Q)/Q$ is cyclic, semi dihedral or generalized quaternion. Since $\mathrm{Def}^P_{P/Z}u = 0$ for any non-trivial central subgroup of P, it follows that $\mathrm{Defres}^P_{N_P(Q)/Q}u = 0$ if $M = Q \cap Z(P) \neq \mathbf{1}$, for

$$\mathrm{Defres}^P_{N_P(Q)/Q}u = \mathrm{Defres}^{\overline{P}}_{N_{\overline{P}}(\overline{Q})/\overline{Q}}\mathrm{Def}^P_{\overline{P}}u \quad ,$$

where $\overline{P} = P/M$ and $\overline{Q} = Q/M$.

And if $M = \mathbf{1}$, then $Z(P)$ embeds in the center of $N_P(Q)/Q$, which is always cyclic if Q cyclic, semi dihedral or generalized quaternion. This shows that $u = 0$ if $Z(P)$ is not cyclic, hence $\partial D_{tors(P)} = \{0\}$ in this case, and Ker $\mathcal{D}_\mathcal{S} = \{0\}$ also.

Now if $Z(P)$ is cyclic, let $Z = \Omega_1 Z(P)$ be its subgroup of order p. If P has normal p-rank 1, then $\mathbf{1}$ is a normal genetic subgroup of P, thus $\mathbf{1} \in \mathcal{S}$. Moreover $\partial D_{tors}(P)$ is equal to $T_{tors}(P)$. Thus for $u \in \partial D_{tors}(P)$

$$b_{P,\mathbf{1}}(u) = f_{\mathbf{1}}^P u = u \quad .$$

It follows that Ker $\mathcal{D}_\mathcal{S} = \{0\}$ in this case.

The remaining case is when P admits a normal subgroup E which is elementary abelian of rank 2. Set $H = C_P(E)$. Since $Z(P)$ is cyclic, it follows that $|P : H| = p$, and that $Z = E \cap Z(P)$. By Corollary 5.3, there exists a genetic basis $\mathcal{S}$ of P and a decomposition

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2$$

in disjoint union such that

1. if $S \in \mathcal{S}_2$, then $S \supseteq Z$, and
2. the set $\Gamma = \{{}^xS \mid S \in \mathcal{S}_1,\ x \in [P/H]\}$ is a set of representatives of genetic subgroups R of H such that $R \not\supseteq Z$, for the relation $\frac{\quad}{}_H$, where $[P/H]$ is any chosen set of representatives of H-cosets in P.

Let $u \in$ Ker $\mathcal{D}_\mathcal{S}$. Then $u \in$ Ker $\mathrm{Def}_{P/Z}^P = \partial D_{tors}(P)$. Set $v = \mathrm{Res}_H^P u$. Then there is a genetic basis $\mathcal{T}$ of H such that $\Gamma = \{S \in \mathcal{T} \mid S \not\supseteq Z\}$. Let $S \in \mathcal{S}_1$, and $x \in [P/H]$. Then ${}^xS \in \mathcal{T}$, and

$$N_H({}^xS) = {}^xN_H(S) = {}^xN_P(S) \quad .$$

Moreover

$$\begin{aligned} b_{{}^xR,{}^xS}(v) &= ({}^xS\backslash H - {}^xSZ\backslash H)\mathrm{Res}_H^P u \\ &= ({}^xS\backslash P - {}^xSZ\backslash P)u = xb_{R,S}(x^{-1}u) = xb_{R,S}(u) = 0 \quad . \end{aligned}$$

Moreover if $S \in \mathcal{T} - \Gamma$, then $S \supseteq Z$, so setting $R = N_P(S)$ again, and denoting by $\hat{S}$ the subgroup of R such that $\hat{S}/S$ is central of order p in R/S,

$$\begin{aligned} b_{R,S}(v) &= (S\backslash H - \hat{S}\backslash H)\mathrm{Res}_H^P u \\ &= (\overline{S}\backslash\overline{H} - \overline{\hat{S}}\backslash\overline{H})\mathrm{Res}_{\overline{H}}^{\overline{P}}\mathrm{Def}_{\overline{P}}^P u = 0 \quad , \end{aligned}$$

where $\overline{P} = P/Z$, $\overline{S} = S/Z$, $\overline{H} = H/Z$, $\overline{\hat{S}} = \hat{S}/Z$.

It follows that $b_{N_H(S),S}(v) = 0$ for any $S \in \mathcal{T}$, thus $v = 0$ by induction hypothesis. Now $u \in$ Ker $\mathrm{Res}_{C_P(E)}^P \cap$ Ker $\mathrm{Def}_{P/Z}^P$. Hence $u = 0$ by Proposition 8.1. This completes the proof of Theorem 8.2. □

8.3. Corollary : *Let P be a p-group. Then*

$$D_{tors}(P) \cong (\mathbb{Z}/2\mathbb{Z})^{n_P} \oplus (\mathbb{Z}/4\mathbb{Z})^{m_P} \quad ,$$

where m_P is equal to the number of isomorphism classes of rational irreducible representations of P of generalized quaternion type, and n_P is the number of isomorphism classes of rational irreducible representations of P whose type is

- *cyclic of order at least 3, or semi dihedral, or generalized quaternion, if the ground field contains primitive cubic roots of unity.*
- *cyclic of order at least 3, or semi dihedral, or generalized quaternion of order at least 16 otherwise.*

(This was Conjecture 6.3 in [10]).

8.4. Corollary : *If P is a 2-group, then $D(P) = D^{\Omega}(P) + {}_2D(P)$, where ${}_2D(P) = \{u \in D(P) \mid 2u = 0\}$.*

Proof: Indeed $D(P) = D^{\Omega}(P) + D_{tors}(P)$ by Theorem 7.7. Moreover Corollary 8.3 shows that $2D_{tors}(P) = 2D^{\Omega}_{tors}(P)$: indeed, it is enough to check this equality when P is cyclic, semi dihedral, or generalized quaternion, since for an arbitrary 2-group P, any element in $D_{tors}(P)$ is a sum of elements obtained by inflation and tensor induction from sections of P of this type. Now by Theorem 4.7, if P is cyclic or semi dihedral, then $D_{tors}(P) = D^{\Omega}_{tors}(P)$. And if P is generalized quaternion, then $2D_{tors}(P)$ is a group of order 2, generated by $2\Omega_{P/\mathbf{1}}$.

Now if P is an arbitrary 2-group, and if $u \in D(P)$, there is an element $v \in D^{\Omega}(P)$ such that $w = u - v$ is a torsion element of $D(P)$. So there exists an element $t \in D^{\Omega}_{tors}(P)$ such that $2w = 2t$, i.e. $w - t \in {}_2D(P)$. Now $u = v + t + (w - t)$, and Corollary 8.4 follows. □

8.5. Corollary : *Let $\mathcal{O}$ be a commutative noetherian local ring with maximal ideal $\mathfrak{p}$, complete for the $\mathfrak{p}$-adic topology, such that the residue field $k = \mathcal{O}/\mathfrak{p}$ has characteristic p. If P is a p-group, then reduction mod $\mathfrak{p}$ induces a group isomorphism $D_{\mathcal{O}}(P) \to D_k(P)$ from the group of Dade P -algebras over $\mathcal{O}$ to the corresponding group over k. In other words, every endo-permutation kP-module can be lifted to an endo-permutation $\mathcal{O}P$-lattice.*

Proof: The assumption is Assumption 2.1 of [25], except that k is not assumed to be algebraically closed. Now with the Notation of Section 29 of [25], reduction mod $\mathfrak{p}$ induces an injection $D_{\mathcal{O}}(P) \to D_k(P)$, by Proposition 29.4 of [25]. This injection obviously restricts to an isomorphism $D^{\Omega}_{\mathcal{O}}(P) \to D^{\Omega}_k(P)$. Also, this injection is an isomorphism if P has normal p-rank 1, by Dade's Theorem 4.6, and by Carlson-Thévenaz explicit description of $D_k(P)$, when P is a 2-group of normal 2-rank 1.

Now it follows from Theorem 8.2 and Theorem 7.7, that the reduction map $D_{\mathcal{O}}(P) \to D_k(P)$ is always an isomorphism. □

9. Generators and relations

9.1. Notation : *If P is a p-group, denote by $[s_P]$ a set of representatives of conjugacy classes of subgroups of P, ordered by the relation*

$$U \leq_P V \Leftrightarrow \exists x \in P,\ U \subseteq {}^xV \quad .$$

Let μ_P denote the Möbius function of the poset $([s_P], \leq_P)$. If $V \in [s_P]$, let $\Delta_{P/V}$ denote the element of $D^\Omega(P)$ defined by

$$\Delta_{P/V} = \sum_{\substack{U \in [s_P] \\ U \leq_P V}} \mu_P(U, V)\Omega_{P/U} \quad .$$

If Q is any subgroup of P, set $\Delta_{P/Q} = \Delta_{P/V}$, if $V \in [s_P]$ is a conjugate of Q in P.

If Q and S are subgroups of P, recall from Notation 2.4 of [8] that

$$\mathcal{I}_P(Q, S) = \{x \in Q\backslash P/N_P(S) \mid Q^x \cap N_P(S) \subseteq S\} \quad ,$$

and set $i_P(Q, S) = |\mathcal{I}_P(Q, S)|$.

Define similarly

$$\mathcal{J}_P(Q, S) = \{x \in Q\backslash P/N_P(S) \mid |J_x(Q, S)| = p,\ J_x(Q, S) \not\subseteq Z(N_P(S)/S)\} \quad ,$$

where $J_x(Q, S) = (Q^x \cap N_P(S))S/S$, and set $j_P(Q, S) = |\mathcal{J}_P(Q, S)|$.

9.2. Remark : If $N_P(S)/S$ is cyclic or generalized quaternion, then $j_P(Q, S)$ is equal to 0 for any Q.

If Q is a generalized quaternion group, recall (Lemma 4.8) that η_Q denotes any element of order 2 in $D(Q) - D^\Omega(Q)$.

9.3. Notation : *If P is a 2-group, and S is a genetic subgroup of P such that $N_P(S)/S$ is generalized quaternion, choose such an element $\eta_{N_P(S)/S} \in D(N_P(S)/S)$, and set*

$$\Lambda_S = \mathrm{Teninf}_{N_P(S)/S}^P \eta_{N_P(S)/S} \quad .$$

9.4. Remark : By Lemma 4.8, the element Λ_Q does not exist if $Q \cong Q_8$ and the ground field does not have primitive roots of unity. In all other cases, there are exactly two elements η_Q and η'_Q of order 2 in $D(Q) - D^\Omega(Q)$, and $\eta_Q + \eta'_Q = 2\Omega_{Q/\mathbf{1}}$ in $D(Q)$.

9.5. Theorem : *Let P be a p-group, and let $\mathcal{S}$ be a genetic basis of P. Denote by $\mathcal{Q}$ the subset of $\mathcal{S}$ consisting of elements S such that $N_P(S)/S$ is generalized quaternion, if the ground field contains all cubic roots of unity, or generalized quaternion of order at least 16 otherwise.*

Then the Dade group $D(P)$ is generated by

- *the elements $\Delta_{P/Q}$, for $Q \in [s_P]$, and*
- *if $p = 2$, the elements Λ_S, for $S \in \mathcal{Q}$.*

These generators are subject to the following relations 9.6 and 9.7 :

$$(\mathbf{9.6}) \qquad \forall S \in \mathcal{S}, \qquad \tau_S \sum_{Q \in [s_P]} (a_S i_P(Q,S) + j_P(Q,S))\Delta_{P/Q} = 0 \quad ,$$

$$\text{where} \quad \tau_S \;=\; \begin{cases} 1 & \text{if } |N_P(S)/S| \leq 2 \\ & \text{or if } N_P(S)/S \text{ is dihedral} \\ 2 & \text{if } N_P(S)/S \text{ is cyclic of order at least 3} \\ & \text{or semi dihedral} \\ 4 & \text{if } N_P(S)/S \text{ is generalized quaternion} \end{cases}$$

$$\text{and} \quad a_S \;=\; \begin{cases} 1 & \text{if } N_P(S)/S \text{ is cyclic or generalized quaternion} \\ 2 & \text{if } N_P(S)/S \text{ is dihedral or semi dihedral} \end{cases} \quad .$$

$$(\mathbf{9.7}) \qquad \forall S \in \mathcal{Q}, \qquad 2\Lambda_S = 0 \quad .$$

Moreover, these generators and relations form a presentation of $D(P)$ as an abelian group : more precisely, the generators $\Delta_{P/Q}$, for $Q \in [s_P]$, subject to the relations 9.6, form a presentation of $D^\Omega(P)$ as an abelian group. The elements Λ_S, for $S \in \mathcal{Q}$ generate a subgroup $D_\mathcal{Q}(P)$, isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{|\mathcal{Q}|}$, and

$$D(P) = D^\Omega(P) \oplus D_\mathcal{Q}(P) \quad .$$

Proof: Recall from Theorem 1.7 of [9] that there exists a unique natural transformation of biset functors $\Theta : B^* \to D^\Omega$ such that $\Theta_P(\omega_X) = \Omega_X$, for any p-group P and any finite P-set X, and Θ is surjective. Let $L = \mathrm{Ker}\,\Theta$.

Then L is a biset functor, and since composition of Θ with the projection $D^\Omega \to D^\Omega/D^\Omega_{tors}$ leads to the exact sequence

$$0 \to R^*_\mathbb{Q} \to B^* \to D^\Omega/D^\Omega_{tors} \to 0 \quad ,$$

it follows that L is a subfunctor of $R^*_\mathbb{Q}$, hence L is a subfunctor of the dual of a rational biset functor. By Proposition 7.4 of [10], the functor L is rational, and thus for any genetic basis $\mathcal{S}$ of P, the map

$$\mathcal{I}_\mathcal{S} = \mathop{\oplus}_{S \in \mathcal{S}} \mathrm{Teninf}_{N_P(S)/S}^P : \mathop{\oplus}_{S \in \mathcal{S}} \partial L(N_P(S)/S) \to L(P)$$

is an isomorphism.

Now the element $\Delta_{P/Q}$ of $D^\Omega(P)$ is the image by Θ_P of the element $\delta_{P/Q}$ of the canonical basis of $B^*(P)$ (see Remark 2.3 of [10]). In particular the elements $\Delta_{P/Q}$, for $Q \in [s_P]$, generate $D^\Omega(P)$.

Let P be any p-group. Evaluating at P the exact sequence

$$0 \to L \to B^* \to D^\Omega \to 0 \quad ,$$

and taking images by the idempotent f_1^P of $\mathrm{End}_{\mathcal{C}_p}(P)$, leads to the exact sequence of abelian groups

$$0 \to \partial L(P) \to \partial B^*(P) \to \partial D^\Omega(P) \to 0 \quad .$$

But $R_\mathbb{Q}^*$ has a $\mathbb{Z}$-basis consisting of the elements V^*, where V runs through a set of representatives of isomorphism classes of rational irreducible representation of P, and $V^*(W) = m(V, W)$ for any $W \in R_\mathbb{Q}(P)$ (see Lemma 3.2 of [9]). It follows that $\partial R_\mathbb{Q}(P)$ has a basis consisting of the elements V^*, where V runs through a set of representatives of isomorphism classes of *faithful* rational irreducible representation of P.

Now suppose that P has normal p-rank 1, and set $Z = \Omega_1 Z(P)$. Then there is a unique such representation Φ_P, by Proposition 3.7 of [8]. Thus $\partial R_\mathbb{Q}^*(P)$ is free of rank 1, generated by the element Φ_P^*. Viewed as an element of $B^*(P)$, it is equal to

$$\Phi_P^* = \sum_{Q \in [s_P]} m(\Phi_P, \mathbb{Q}P/Q)\delta_{P/Q} \quad .$$

Then by Lemma 4.1 of [10], if P is cyclic or generalized quaternion, then

$$m(\Phi_P, \mathbb{Q}P/Q) = \begin{cases} 0 & \text{if } Q \neq \mathbf{1} \\ 1 & \text{if } Q = \mathbf{1} \end{cases} \quad ,$$

and if P is dihedral or semi dihedral, then

$$m(\Phi_P, \mathbb{Q}P/Q) = \begin{cases} 0 & \text{if } Q \supseteq Z \\ 1 & \text{if } Q \not\supseteq Z,\ |Q| = 2 \\ 2 & \text{if } Q = \mathbf{1} \end{cases} \quad .$$

If P is cyclic or generalized quaternion, then $\partial R_\mathbb{Q}(P)$ is generated by $\Phi_P^* = \delta_{P/\mathbf{1}} = \omega_{P/\mathbf{1}}$. The order τ_P of $\Theta_P(\Phi_P^*) = \Omega_{P/\mathbf{1}}$ is equal to 1 if $|P| \leq 2$, to 2 if P is cyclic of order at least 3, and to 4 if P is generalized quaternion. Hence $\partial L(P)$ is generated by $\rho_P = \tau_P \omega_{P/\mathbf{1}}$ in this case.

If P is semi dihedral, then $\partial R_\mathbb{Q}^*(P)$ is generated by

$$\Phi_P^* = 2\delta_{P/\mathbf{1}} + \delta_{P/R} = \omega_{P/\mathbf{1}} + \omega_{P/R} \quad ,$$

where R is a non central subgroup of P of order 2. Then $\Theta_P(\Phi_P^*)$ is equal to $\Omega_{P/\mathbf{1}} + \Omega_{P/R}$, which has order 2 in $D(P)$, by Theorem 7.1 of [13]. Hence $\partial L(P)$ is generated by $\rho_P = 2(\omega_{P/\mathbf{1}} + \omega_{P/R})$ in this case.

Finally if P is dihedral, then $\partial R_{\mathbb{Q}}(P)$ is generated by

$$\Phi_P^* = 2\delta_{P/\mathbf{1}} + \delta_{P/R} + \delta_{P/R'} = \omega_{P/R} + \omega_{P/R'} \quad ,$$

where R and R' are non central subgroups of order 2 of P, not conjugate in P. Then $u = \Theta_P(\Phi_P^*) = \Omega_{P/R} + \Omega_{P/R'}$ is a torsion element of $D(P)$, which is torsion-free by Theorem 4.7. Hence $u = 0$. So $\partial L(P)$ is generated by $\rho_P = \omega_{P/R} + \omega_{P/R'}$ in this case.

Let $S \in \mathcal{S}$. If Q is a subgroup of P, then

$$\begin{aligned}
\mathrm{Indinf}_{N_P(S)/S}^P \Phi_{N_P(S)/S}^*(P/Q) &= \Phi_{N_P(S)/S}^*(\mathrm{Defres}_{N_P(S)/S}^P(P/Q)) \\
&= m(\Phi_{N_P(S)/S}, \mathrm{Defres}_{N_P(S)/S}^P(\mathbb{Q}P/Q)) \\
&= m(\mathrm{Indinf}_{N_P(S)/S}^P \Phi_{N_P(S)/S}, \mathbb{Q}P/Q) \\
&= a_S i_P(Q,S) + j_P(Q,S) \quad .
\end{aligned}$$

Here the equality in the second line follows from the definition. The equality in the third line follows by Frobenius reciprocity, from the fact that $(N_P(S), S)$ is a genetic section of P, and the last equality follows from Lemma 4.1 of [10].

Then

$$\mathrm{Indinf}_{N_P(S)/S}^P \Phi_{N_P(S)/S}^* = \sum_{Q \in [s_P]} (a_S i_P(Q,S) + j_P(Q,S))\delta_{P/Q} \quad ,$$

and it follows that $D^\Omega(P)$ has a presentation as an abelian group generated by the element $\Delta_{P/Q}$, for $Q \in [s_P]$, subject to the relations 9.6.

Now the elements Λ_S, for $S \in \mathcal{Q}$, are of order 2, since the element $\eta_{N_P(S)/S}$ has order 2, and since the map

$$\mathrm{Teninf}_{N_P(S)/S}^P : T_{tors}(N_P(Q)/Q) \to D(P)$$

is injective, by Theorem 6.1 of [10]. Let $D_{\mathcal{Q}}(P)$ denote the subgroup generated by the elements Λ_S, for $S \in \mathcal{Q}$.

Then $D_{tors}(P) = D_{tors}^\Omega(P) + D_{\mathcal{Q}}(P)$: to prove this, by Theorem 8.2, it suffices to check that if P is a quaternion group, then $D_{tors}(P)$ is generated by $D^\Omega(P)$ and one chosen element η_Q. But this is obvious, from Lemma 4.8.

Now by Theorem 7.7, it follows that $D(P) = D^\Omega(P) + D_{\mathcal{Q}}(P)$, and

$$D(P)/D^\Omega(P) \cong D_{tors}(P)/D_{tors}^\Omega(P) \cong D_{\mathcal{Q}}(P)/(D^\Omega(P) \cap D_{\mathcal{Q}}(P)) \quad .$$

By Corollary 8.3 and by Corollary 7.6 of [10], the group $D_{tors}(P)/D_{tors}^\Omega(P)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{q_P}$, where q_P is the number of isomorphism classes of rational irreducible representations of P which are of generalized quaternion type if the ground field contains all cubic roots of unity, and of generalized quaternion type of order at least 16 otherwise. In other words $q_P = |\mathcal{Q}|$.

Now $D_{\mathcal{Q}}(P)$ is an elementary abelian 2-group generated by $|\mathcal{Q}|$ elements, so its rank is at most equal to $|\mathcal{Q}|$. Since its factor group

$$D_{\mathcal{Q}}(P)/(D^{\Omega}(P) \cap D_{\mathcal{Q}}(P))$$

has rank equal to $|\mathcal{Q}|$, it follows that $D^{\Omega}(P) \cap D_{\mathcal{Q}}(P) = \{0\}$, that the elements Λ_S, for $S \in \mathcal{Q}$, are linearly independent over $\mathbb{F}_2$, and that

$$D(P) = D^{\Omega}(P) \oplus D_{\mathcal{Q}}(P) \quad .$$

This completes the proof of the theorem. □

9.8. Remark : If the group P is abelian, then by Remark 2.11, there is a unique genetic basis $\mathcal{S}$ of P, consisting of subgroups S of P such that P/S is cyclic. If Q is any subgroup of P, and if $S \in \mathcal{S}$, then $j_P(Q,S) = 0$ by Remark 9.2, and $i_P(Q,S)$ is equal to 1 if $Q \subseteq S$, and to 0 otherwise. Hence the relations 9.6 become

$$\forall S \in \mathcal{S}, \quad \tau_S \sum_{Q \subseteq S} \Delta_{P/Q} = 0 \quad .$$

Moreover

$$\begin{aligned} \sum_{Q \subseteq S} \Delta_{P/Q} &= \sum_{Q \subseteq S} \sum_{\substack{U \in [s_P] \\ U \leq_P Q}} \mu_P(U,Q) \Omega_{P/U} \\ &= \sum_{U \subseteq P} \Big(\sum_{U \subseteq Q \subseteq S} \mu(U,Q) \Big) \Omega_{P/U} \\ &= \Omega_{P/S} \end{aligned}$$

by the defining property of the Möbius function, since the poset s_P is equal to the poset of all subgroups of P if P is abelian. Thus Theorem 9.5 is another form of Dade's Theorem (Theorem 4.6) if P is abelian.

10. The functor D/D^{Ω}

In this section, I will show that D/D^{Ω} is a biset functor. Of course, by Theorem 7.7, this is non trivial only if $p = 2$, which I shall assume throughout this section.

10.1. Notation : *(see Notation 4.1 of [8]) If Q is a 2-group of normal 2-rank 1, let H_Q denote the subfunctor of $\mathbb{F}_2R_\mathbb{Q}$ generated by the image $\overline{\Phi_Q} \in \mathbb{F}_2R_\mathbb{Q}(Q)$ of the unique (up to isomorphism) rational irreducible representation Φ_Q of Q.*

If P is any 2-group, then $\mathbb{F}_2R_\mathbb{Q}(P)$ has a canonical basis consisting of the images $\overline{V}$ of the rational irreducible $\mathbb{Q}P$-modules V, up to isomorphism. If $u \in \mathbb{F}_2R_\mathbb{Q}(P)$, denote by $\gamma(V,u)$ the coefficient of $\overline{V}$ in the decomposition of u in this basis.

The next theorem is a precise form of the second part of Conjecture B in [4] :

10.2. Theorem : *If P is a 2-group, there is an exact sequence of abelian groups*

$$0 \longrightarrow D^\Omega(P) \xrightarrow{i_P} D(P) \xrightarrow{\sigma_P} H_Q(P) \longrightarrow 0 \quad ,$$

where i_P is the inclusion, and $Q \cong Q_8$ if the ground field contains all cubic root of unity, and $Q \cong Q_{16}$ otherwise.

Moreover this sequence is functorial in the following sense : if P' is a 2-group, and if $\psi \in \mathrm{Hom}_{\mathcal{C}_2}(P,P')$, then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^\Omega(P) & \xrightarrow{i_P} & D(P) & \xrightarrow{\sigma_P} & H_Q(P) & \longrightarrow & 0 \\ & & {\scriptstyle D^\Omega(\psi)}\big\downarrow & & {\scriptstyle D(\psi)}\big\downarrow & & {\scriptstyle H_Q(\psi)}\big\downarrow & & \\ 0 & \longrightarrow & D^\Omega(P') & \xrightarrow{i_{P'}} & D(P') & \xrightarrow{\sigma_{P'}} & H_Q(P') & \longrightarrow & 0 \end{array}$$

is commutative.

In other words, there is a natural structure of biset functor on D/D^Ω, and $D/D^\Omega \cong H_Q$.

In general however, because of "Galois torsion" (see Section 3 in [12]), there is no natural biset functor structure on D, so there is no genuine "exact sequence of biset functors"

$$0 \to D^\Omega \to D \to H_Q \to 0 \quad .$$

If the ground field does not contain non trivial cubic root of unity, then D is a genuine biset functor, by Lemma 4.8, and this sequence is an exact sequence of biset functors.

Proof: If the ground field contains all cubic roots of unity, fix a group $Q \cong Q_8$, otherwise fix a group $Q \cong Q_{16}$. In any case, choose an element η_Q of order 2, in $D(Q) - D^\Omega(Q)$. If P is any 2-group, set

$$F(P) = \mathrm{Hom}_{\mathcal{C}_2}(Q,P)(\eta_Q) \quad .$$

Then $F(P)$ is a subgroup of $D(P)$, and $2F(P) = 0$. Define a map $\pi_P : F(P) \to H_Q(P)$ by

(10.3) $$\pi_P(\varphi(\eta_Q)) = \varphi(\overline{\Phi_Q}) \quad ,$$

for $\varphi \in \mathrm{Hom}_{\mathcal{C}_2}(Q, P)$.

10.4. Lemma : *The map π_P is well defined.*

Proof: This amounts to showing that if $\varphi \in \mathrm{Hom}_{\mathcal{C}_2}(Q, P)$ is such that $\varphi(\eta_Q) = 0$, then $\varphi(\overline{\Phi_Q}) = 0$. Now if $\varphi(\eta_Q) = 0$, then $\psi(\varphi(\eta_Q)) = 0$ for any morphism $\psi \in \mathrm{Hom}_{\mathcal{C}_p}(P, R)$, where R is a generalized quaternion group, such that $|R| \geq |Q|$.

I have to be careful here, since in general, due to "Galois torsion", it is *not* true that

$$\psi(\varphi(\eta_Q)) = (\psi\varphi)(\eta_Q) \quad .$$

However the quotient $D(R)/D^\Omega(R)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$: indeed, by the choice of Q, the hypothesis $|R| > |Q|$, and the fact that generalized quaternion groups have a unique rational irreducible representation of quaternion type, up to isomorphism, the set $\mathcal{Q}$ of Theorem 9.5 for the group R is of cardinality 1 (more precisely $\mathcal{Q} = \{\mathbf{1}\}$).

Hence there is a linear form

$$\lambda_R : D(R) \to \mathbb{Z}/2\mathbb{Z}$$

defined by $\lambda_R(u) = 1$ if $u \notin D^\Omega(R)$, and $\lambda_R(u) = 0$ otherwise.

The point now is that $\psi(\varphi(\eta_Q))$ differs from $(\psi\varphi)(\eta_Q)$ by an element of $D^\Omega(R)$ (even $2D^\Omega(R)$) : this follows from the case where ψ and φ are bisets, using Proposition 3.10 of [12], and from the fact that for any endomorphism a of the ground field the difference $\gamma_a(\eta_Q) - \eta_Q$ is a multiple of $2\Omega_Q$. In particular, this shows that

$$\lambda_R\Big(\psi(\varphi(\eta_Q))\Big) = \lambda_R\Big((\psi\varphi)(\eta_Q)\Big) = 0 \quad .$$

10.5. Lemma : *Let Q and R be generalized quaternion groups, and let $f \in \mathrm{Hom}_{\mathcal{C}_2}(Q, R)$. Then*

$$\lambda_R(f(\eta_Q)) = \gamma(\Phi_R, f(\overline{\Phi_Q})) \quad .$$

Proof: Since $\lambda_R(f(\eta_Q))$ and $\gamma(\Phi_R, f(\overline{\Phi_Q}))$ are linear in f, it suffices to consider the case where f is a transitive biset, hence (by Remark 3.4) of the form

$$\mathrm{Indinf}_{T/S}^R \mathrm{Iso}_{Y/X}^{T/S} \mathrm{Defres}_{Y/X}^Q \quad ,$$

where T/S is a section of R and Y/X is a section of Q, such that there exists a group isomorphism $Y/X \to T/S$.

If Y is proper subgroup of Q, then $\mathrm{Res}_Y^Q \eta_Q$ is in $D^\Omega(Y)$, by Lemma 4.8. Thus if $Y \neq Q$, then $f(\eta_Q) \in D^\Omega(R)$, hence $\lambda_R(f(\eta_Q)) = 0$. But the restriction of Φ_Q to any maximal subgroup H of Q is equal to $2\Phi_H$, by Lemma 3.14 of [8]. Hence $f(\Phi_Q) \in 2R_\mathbb{Q}(R)$, thus $\gamma(\Phi_R, f(\overline{\Phi_Q})) = 0$ in this case.

Suppose now that $Y = Q$. If $X \neq \mathbf{1}$, then $\mathrm{Def}^Q_{Q/X}\eta_Q = 0$ (by construction, or because if Z is the center of Q, then $\mathrm{Def}^Q_{Q/Z}\eta_Q$ is in $D_{tors}(Q/Z) = 0$, since Q/Z is dihedral). Thus if $X \neq 1$, then $\lambda_R(f(\eta_Q)) = 0$. On the other hand, any proper deflation of Φ_Q is equal to 0, by Lemma 3.12 of [8], hence $\gamma(\Phi_R, f(\overline{\Phi_Q})) = 0$ in this case also.

Now assume $X = \mathbf{1}$. Then T/S is isomorphic to Q. But any section of the generalized quaternion group R which is itself a generalized quaternion group is actually a subgroup of R. This shows that $S = \mathbf{1}$, and I can suppose that Q is a subgroup of R. In this case $\mathrm{Ten}^R_Q\eta_Q$ is equal to η_R or $\eta_R + 2\Omega_R$, by Lemma 4.8. But on the other hand $\mathrm{Ind}^R_Q\Phi_Q = \Phi_R$, thus $\gamma(\Phi_R, f(\overline{\Phi_Q})) = 1$, proving the lemma. □

10.6. End of proof of Lemma 10.4 : consider the element $u = \varphi(\overline{\Phi_Q})$ of $\mathbb{F}_2 R_{\mathbb{Q}}(P)$. The previous discussion and Lemma 10.5 show that $\gamma(\Phi_R, \psi(u))$ is equal to 0 for any $\psi \in \mathrm{Hom}_{\mathcal{C}_2}(P, R)$, for any generalized quaternion group R such that $|R| \geq |Q|$. Now Corollary 6.7 and Theorem 5.12 of [8] show that $H_Q(P)$ has an $\mathbb{F}_2$-basis consisting of the images $\overline{V}$ of the rational irreducible representations V of P, whose type is generalized quaternion of order at least equal to $|Q|$, up to isomorphism. Moreover, if V is such a representation, and (T, S) is a genetic section of P for V, then $\gamma(V, u) = \gamma(\Phi_{T/S}, \mathrm{Defres}^P_{T/S}u)$, by Lemma 4.2 of [8]. It follows that $\gamma(V, u) = 0$ for any $V \in \mathrm{Irr}_{\mathbb{Q}}(P)$ of quaternion type R with $|R| \geq |Q|$. Hence $u = 0$, as was to be shown for Lemma 10.4. □

10.7. End of proof of Theorem 10.2 : since $H_Q(P) \subseteq \mathrm{Im}\ \pi_P$ by 10.3, the map π_P is surjective. Now $F(P) \cap D^\Omega(P) \subseteq \mathrm{Ker}\ \pi_P$: indeed if $\varphi \in \mathrm{Hom}_{\mathcal{C}_2}(Q, P)$ is such that $u = \varphi(\eta_Q) \in D^\Omega(P)$, then $\psi(\varphi(\eta_Q)) \in D^\Omega(R)$, for any generalized quaternion group R and any $\psi \in \mathrm{Hom}_{\mathcal{C}_p}(P, R)$. The same argument as above, using Lemma 10.5, shows that $\varphi(\overline{\Phi_Q}) = 0$, i.e. $\pi_P(u) = 0$.

Now $F(P)/(F(P) \cap D^\Omega(P)) \cong (F(P) + D^\Omega(P))/D^\Omega(P)$ is isomorphic to a subgroup of $D(P)/D^\Omega(P)$, which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{q_P}$, where q_P is equal to the number of isomorphism classes of rational irreducible representations of P whose type is generalized quaternion of order at least equal to $|Q|$. Thus by the above remarks $D(P)/D^\Omega(P) \cong H_Q(P)$, and the surjection

$$F(P)/(F(P) \cap D^\Omega(P)) \to H_Q(P)$$

induced by π_P has to be an isomorphism. Hence $F(P) + D^\Omega(P) = D(P)$, and $F(P) \cap D^\Omega(P) = \mathrm{Ker}\ \pi_P$. This gives a well defined map

$$\sigma_P : D(P) \to H_Q(P)$$

sending the element $u+w$ of $D(P)$, where $u \in F(P)$ and $w \in D^\Omega(P)$, to $\pi_P(u)$.

Suppose that $u = \varphi(\eta_Q)$, for $\varphi \in \mathrm{Hom}_{\mathcal{C}_p}(Q,P)$. Let P' be a 2-group, and $f \in \mathrm{Hom}_{\mathcal{C}_2}(P,P')$. Then $w' = f(\varphi(\eta_Q)) - (f\varphi)(\eta_Q)$ is in $D^\Omega(P')$, so

$$\begin{aligned}\sigma_{P'}D(f)(u+w) &= \sigma_{P'}\Big(f(\varphi(\eta_Q)) + f(w)\Big)\\ &= \sigma_{P'}\Big((f\varphi)(\eta_Q) + w' + f(w)\Big)\\ &= f\varphi(\overline{\Phi_Q}) = H_Q(f)\pi_P(u) = H_Q(f)\sigma_P(u+w) \quad .\end{aligned}$$

This shows that the square on right hand side in the theorem is commutative. This completes the proof of Theorem 10.2, since the square on the left hand side is also commutative. □

References

[1] J. L. Alperin. Invertible modules for groups. *Notices Amer. Math. Soc.*, 24:A–64, 1977.

[2] J. L. Alperin. Construction of endo-permutation modules. *J. Group Theory*, 4:3–10, 2001.

[3] J. L. Alperin. Lifting endo-trivial modules. *J. of Group Theory*, 4:1–2, 2001.

[4] S. Bouc. Biset functors, rational representations, and the Dade group. Darstellungen endlicher Gruppen, 23-03/29-03 2003, M.F.O. Tagungsbericht 14/2003 `http://www.mfo.de/programme/schedule/2003/13/Report14_2003.pdf`.

[5] S. Bouc. Foncteurs d'ensembles munis d'une double action. *J. of Algebra*, 183(0238):664–736, 1996.

[6] S. Bouc. Tensor induction of relative syzygies. *J. reine angew. Math.*, 523:113–171, 2000.

[7] S. Bouc. A remark on a theorem of Ritter and Segal. *J. of Group Theory*, 4:11–18, 2001.

[8] S. Bouc. The functor of rational representations for p-groups. *Advances in Mathematics*, 186:267–306, 2004.

[9] S. Bouc. A remark on the Dade group and the Burnside group. *J. of Algebra*, 279(1):180–190, 2004.

[10] S. Bouc. Biset functors and genetic sections for p-groups. *Journal of Algebra*, 284(1):179–202, 2005.

[11] S. Bouc and N. Mazza. The Dade group of (almost) extraspecial p-groups. *J. Pure and Applied Algebra*, 192:21–51, 2004.

[12] S. Bouc and J. Thévenaz. The group of endo-permutation modules. *Invent. Math.*, 139:275–349, 2000.

[13] J. Carlson and J. Thévenaz. Torsion endo-trivial modules. *Algebras and Representation Theory*, 3:303–335, 2000.

[14] J. Carlson and J. Thévenaz. The classification of torsion endo-trivial modules. *Ann. of Math.*, 162(2), September 2005.

[15] J. F. Carlson. Endo-trivial modules over (p, p)-groups. *Illinois J. of Math.*, 24:287–295, 1980.

[16] J. F. Carlson. A characterization of endo-trivial modules over p-groups. *Manuscripta Math.*, 97:303–307, 1998.

[17] J. F. Carlson and J. Thévenaz. The classification of endo-trivial modules. *Invent. Math.*, 158(2):389–411, 2004.

[18] E. Dade. Une extension de la théorie de Hall et Higman,. *J. Algebra*, 20:570–609, 1972.

[19] E. Dade. Endo-permutation modules over p-groups I. *Ann. of Math.*, 107:459–494, 1978.

[20] E. Dade. Endo-permutation modules over p-groups II. *Ann. of Math.*, 108:317–346, 1978.

[21] W. Gorenstein. *Finite groups.* Chelsea, 1968.

[22] L. Puig. Nilpotent blocks and their source algebras. *Invent. Math.*, 93:77–116, 1988.

[23] L. Puig. Affirmative answer to a question of Feit. *J. of Algebra*, 131:513–526, 1990.

[24] L. Puig. *On the local structure of Morita and Rickard equivalences between Brauer blocks.* Number 178 in Progress in Math. Birkhäuser Verlag, Basel, 1999.

[25] J. Thévenaz. *G-algebras and modular representation theory.* Clarendon Press, Oxford, 1995.