A conjecture on B-groups

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Abstract : In this note, I propose the following conjecture : a finite group G is nilpotent if and only if its largest quotient B-group $\beta(G)$ is nilpotent. I give a proof of this conjecture under the additional assumption that G be solvable. I also show that this conjecture is equivalent to the following : the kernel of restrictions to nilpotent subgroups is a biset-subfunctor of the Burnside functor.

AMS Subject classification : 18B99, 19A22, 20J15.

Keywords : B-group, Burnside ring, biset functor.

1. Introduction

In the study of the lattice of biset-subfunctors of the Burnside functor $\mathbb{K}B$ over a field $\mathbb{K}$ of characteristic 0 (cf. Section 7.2 of [2], or Chapter 5 of [3]), a special class of finite groups, called *B-groups* (see Definition 2.2), plays an important role : indeed, the simple subquotients of the biset functor $\mathbb{K}B$ are exactly the functors $S_{H,\mathbb{K}}$, where H is such a B-group.

It was shown in particular in [2] Proposition 9 (see also [3] Theorem 5.4.11) that any finite group G admits a largest quotient in this class, well-defined up to isomorphism, and denoted by $\beta(G)$. A few properties of B-groups were proved in [2], some of which will be recalled in this paper, but almost no progress was made since, until the following theorem proved recently by Mélanie Baumann ([1]) : when p is a prime number, recall that a finite group G is called *cyclic modulo p* (or *p-hypo-elementary*) if the group $G/O_p(G)$ is cyclic.

1.1. Theorem : [M. Baumann] *Let p be a prime number, and G be a finite group. Then G is cyclic modulo p if and only if $\beta(G)$ is cyclic modulo p.*

In this note I propose the following similar looking conjecture :

Conjecture A : *Let G be a finite group. Then $\beta(G)$ is nilpotent if and only if G is nilpotent.*

1.2. Remark : It was shown in [2] (Proposition 14) that the nilpotent B-groups are the groups of the form $C_n \times C_n$, where C_n is a cyclic group of *square free* order n.

After recalling the basic definitions and properties of B-groups, I will give a proof of Conjecture A under the additional assumption that G be *solvable*.

2. B-groups

Let $\mathbb{K}$ be a field of characteristic 0. Let G be a finite group, let s_G denote the set of subgroups of G, and let $[s_G]$ be a set of representatives of G-conjugacy classes on s_G.

Denote by $\mathbb{K}B(G)$ the Burnside algebra of G over $\mathbb{K}$. It is a split semisimple commutative $\mathbb{K}$-algebra, with two natural $\mathbb{K}$-bases : the first one consists of the isomorphism classes of transitive G-sets, i.e. the set $\{[G/H] | H \in [s_G]\}$. The second one consists of the primitive idempotents of $\mathbb{K}B(G)$, i.e. the set $\{e_H^G | \ H \in [s_G]\}$. The transition matrix from the first basis to the second one has been described explicitly by Gluck ([4]) and Yoshida ([6]), as follows

$$e_H^G = \frac{1}{|N_G(H)|} \sum_{X \leq H} |X| \mu(X, H) \, [G/X] \ \ ,$$

where μ is the Möbius function of the poset of subgroups of G.

The correspondence $G \mapsto \mathbb{K}B(G)$ is a biset functor : when G and H are finite groups, and U is a finite (H, G)-biset, the functor $S \mapsto U \times_G S$ from the category of finite G-sets to the category of finite H-sets induces a map $\mathbb{K}B(U) : \mathbb{K}B(G) \to \mathbb{K}B(H)$, which is well behaved with respect to disjoint union and composition of bisets.

This involves in a single formalism the usual operations of restriction, induction, inflation, and transport by isomorphism between the corresponding Burnside groups. It also involves the less usual operation of *deflation* : when $N \trianglelefteq G$, the deflation homomorphism $\mathrm{Def}_{G/N}^G : B(G) \to B(G/N)$ corresponds to the $(G/N, G)$-biset G/N, and it induced by the functor $S \mapsto N \backslash S$ from G-sets to (G/N)-sets.

These elementary operations can be expressed explicitly in each of the above bases. In particular ([3] Theorem 5.2.4), the effect of these operations on the "top" idempotents e_G^G is given as follows :

2.1. Theorem : *Let G be a finite group.*

1. *If H is a subgroup of G, then* $\mathrm{Ind}_H^G e_H^H = |N_G(H) : H| e_H^G$.
2. *If H is a proper subgroup of G, then* $\mathrm{Res}_H^G e_G^G = 0$.

3. *If $N \trianglelefteq G$, then*

$$\mathrm{Inf}_{G/N}^{G} e_{G/N}^{G/N} = \sum_{\substack{X \in [s_G] \\ XN = G}} e_X^G \ .$$

4. *When $N \trianglelefteq G$, set*

$$m_{G,N} = \frac{1}{|G|} \sum_{\substack{X \leq G \\ XN = G}} |X| \mu(X, G) \ .$$

   Then $\mathrm{Def}_{G/N}^{G} e_G^G = m_{G,N} e_{G/N}^{G/N}$.

5. *If $\varphi : G \to G'$ is a group isomorphism, then* $\mathrm{Iso}(\varphi)(e_G^G) = e_{G'}^{G'}$.

This leads to the notion of B-group : the group G is a B-group if any proper deflation of e_G^G is equal to 0. In other words :

2.2. Definition : *The finite group G is called a B-group if $m_{G,N} = 0$ for any non-trivial normal subgroup N of G.*

2.3. Notation : *When G is a finite group, and $N \trianglelefteq G$ is maximal such that $m_{G,N} \neq 0$, set $\beta(G) = G/N$.*

There may be several normal subgroups N with the required properties, but the group G/N does not depend on the choice of N, up to isomorphism. More precisely ([3] Theorem 5.4.11) :

2.4. Theorem : *Let G be a finite group.*

1. *The group $\beta(G)$ is a B-group.*
2. *If a B-group H is isomorphic to a quotient of G, then H is isomorphic to a quotient of $\beta(G)$.*
3. *Let $M \trianglelefteq G$. The following conditions are equivalent :*

   (a) $m_{G,M} \neq 0$.
   (b) The group $\beta(G)$ is isomorphic to a quotient of G/M.
   (c) $\beta(G) \cong \beta(G/M)$.

2.5. Proposition : *Let G be a finite group.*

1. *The group G is a B-group if and only if $m_{G,N} = 0$ for any minimal (non-trivial) normal subgroup of G.*
2. *Let N be a minimal (non-trivial) normal subgroup of G. If N is abelian, then*
$$m_{G,N} = 1 - \frac{|K_G(N)|}{|N|} \ ,$$
where $K_G(N)$ is the set of complements of N in G.

Proof : Assertion 1 follows from the transitivity of deflations. Assertion 2 is Proposition 5.6.4 of [3]. □

3. Proof of Conjecture A in the solvable case

3.1. Theorem : *Let G be a solvable finite group. Then $\beta(G)$ is nilpotent if and only if G is nilpotent.*

Proof : If G is nilpotent, then $\beta(G)$ is nilpotent, for it is a quotient of G. The converse follows from an induction argument on the order of G : assume that if G' is a finite solvable group of order $|G'| < |G|$, and if $\beta(G')$ is nilpotent, then G' is nilpotent. Assume that $\beta(G)$ is nilpotent, and let N be a non-trivial normal subgroup of G. Since $\beta(G/N)$ is a quotient of $\beta(G)$, it is nilpotent. Hence G/N is nilpotent. In particular, if $Z(G) \neq \mathbf{1}$, then $G/Z(G)$ is nilpotent, hence G is nilpotent. So we can assume that $Z(G) = \mathbf{1}$.

Now suppose that M and N are non trivial normal subgroups of G, such that $M \cap N = \mathbf{1}$. Then G is nilpotent : indeed, the group $G/(M \cap N)$, isomorphic to G, maps injectively into $(G/M) \times (G/N)$, which is nilpotent.

It follows that we can assume that G has a unique (non trivial) minimal normal subgroup N. Since G is solvable, the group N is elementary abelian, isomorphic to $(C_p)^k$, for some prime number p and some integer $k \geq 1$. Let Q be a Sylow p-subgroup of G. Then $Q \geq N$. Since the group G/N is nilpotent, and since Q/N is a Sylow p-subgroup of G/N, it follows that $Q/N \trianglelefteq G/N$, i.e. $Q \trianglelefteq G$.

Now N is a non trivial normal subgroup of Q, thus $N \cap Z(Q) \neq \mathbf{1}$. But $N \cap Z(Q)$ is a normal subgroup of G, and by minimality of N, it follows that $N \leq Z(Q)$.

The group G splits as a semidirect product $G = Q \rtimes H$, where H is a (nilpotent) p'-subgroup of G. The group H acts on the p-group Q, thus $Q = C_Q(H)[H, Q]$ (by [5] Theorem 3.5 Chapter 5).

Since G/N is nilpotent, it follows that $G/N \cong (Q/N) \times H$. It follows that $[H, Q] \leq N$. Thus $Q = C_Q(H)N$. Now $N \cap C_Q(H)$ is centralized by H, and by Q, since $N \leq Z(Q)$. Thus $N \cap C_Q(H) \leq Z(G) = \mathbf{1}$, and it follows that $Q = N \times C_Q(H)$. Then $C_Q(H)$ is normalized by Q, and centralized by H. Thus $C_Q(H) \trianglelefteq G$, and as $N \cap C_Q(H) = \mathbf{1}$, it follows that $C_Q(H) = \mathbf{1}$, thus $N = Q$.

But now $G = N \rtimes H$, where $N \cong (C_p)^k$, and H is a p'-group. Since N is minimal normal in G, it follows that H acts irreducibly on N, and that H is a maximal subgroup of G. Since H is not normal in G (as N is the only minimal normal subgroup of G, and $N \not\leq H$), it follows that $N_G(H) = H$. Finally, since H is a p'-group, all the complements of N in G are conjugate, hence $|K_G(N)| = |G : N_G(H)| = |N|$. Thus $m_{G,N} = 1 - \frac{|K_G(N)|}{|N|} = 0$. Hence G is a B-group, thus $G \cong \beta(G)$ is nilpotent. □

3.2. Remark : Actually, in the situation of the end of the proof, the group G is trivial : indeed, it is a nilpotent B-group, hence isomorphic to $C_n \times C_n$, where n is a square free integer. As G has a unique minimal normal subgroup by assumption, the only possibility is $n = 1$.

4. Comments

The following conjecture doesn't mention B-groups :

Conjecture B : *For any group G, let ν_G denote the restriction map*

$$\nu_G = \prod_{H \in \mathcal{N}(G)} \mathrm{Res}_H^G : B(G) \to \prod_{H \in \mathcal{N}(G)} B(H) \ ,$$

where $\mathcal{N}(G)$ is the set of nilpotent subgroups of G.

Then the correspondence $G \mapsto L(G) = \mathrm{Ker}\,\nu_G$ is a biset subfunctor of B.

Still :

4.1. Theorem : *Conjecture B is equivalent to Conjecture A.*

Proof : Since $B(G)$ is a free $\mathbb{Z}$-module, it maps injectively in $\mathbb{K}B(G)$. Let

$u \in B(G)$. Then u can be written

$$u = \sum_{H \in [s_G]} |u^H| e_H^G$$

in $\mathbb{K}B(G)$. Thus $u \in L(G)$ if and only if $|u^H| = 0$ for any $H \in \mathcal{N}(G)$.

Suppose that Conjecture A holds. Proving Conjecture B amounts to proving that L is invariant under the elementary biset operations of induction, restriction, inflation, deflation, and transport by isomorphism.

The latter case is clear : if $\varphi : G \to G'$ is a group isomorphism, then $\mathrm{Iso}(\varphi)\big(L(G)\big) \leq L(G')$.

Now let G be a group, and let K be a subgroup of G. As nilpotent subgroups of K are nilpotent subgroups of G, the transitivity of restrictions implies that $\mathrm{Res}_K^G L(G) \subseteq L(K)$. Conversely, if $u \in L(K)$ and $H \in \mathcal{N}(G)$, then by the Mackey formula

$$\mathrm{Res}_H^G \mathrm{Ind}_K^G u = \sum_{g \in [H \backslash G / K]} \mathrm{Ind}_{H \cap {}^g K}^H c_g \mathrm{Res}_{H^g \cap K}^K u = 0 \ \ ,$$

(where c_g denote conjugation by G), since $H^g \cap K \in \mathcal{N}(K)$. It follows that $\mathrm{Ind}_K^G L(K) \subseteq L(G)$.

Suppose now that $N \trianglelefteq G$, and that $u \in L(G/N)$. Then for any $H \in \mathcal{N}(G)$

$$\mathrm{Res}_H^G \mathrm{Inf}_{G/N}^G u = \mathrm{Inf}_{H/H\cap N}^H \mathrm{Iso}_{HN/N}^{H/H\cap N} \mathrm{Res}_{HN/N}^{G/N} u = 0 \ \ ,$$

since $HN/N \in \mathcal{N}(G/N)$. Hence $\mathrm{Inf}_{G/N}^G L(G/N) \subseteq L(G)$.

Finally, in the same situation, let $v \in L(G)$, and $K/N \in \mathcal{N}(G/N)$. Then

$$\mathrm{Res}_{K/N}^{G/N} \mathrm{Def}_{G/N}^G v = \mathrm{Def}_{K/N}^K \mathrm{Res}_K^G v \ \ .$$

Moreover

$$\mathrm{Res}_K^G v = \sum_{X \in [s_K]} |v^X| e_X^K \ \ ,$$

and $|v^X| = 0$ for $X \in \mathcal{N}(X)$. Since moreover (Theorem 2.1)

$$e_X^K = \frac{1}{|N_K(X) : X|} \mathrm{Ind}_X^K e_X^X \ \ ,$$

it follows that

$$\begin{aligned}
\mathrm{Res}^{G/N}_{K/N}\mathrm{Def}^{G}_{G/N}v &= \sum_{X\in[s_K]}\frac{|v^X|}{|N_K(X):X|}\mathrm{Def}^{K}_{K/N}\mathrm{Ind}^{K}_{X}e^{X}_{X}\\
&= \sum_{X\in[s_K]}\frac{|v^X|}{|N_K(X):X|}\mathrm{Ind}^{K/N}_{XN/N}\mathrm{Iso}^{XN/N}_{X/X\cap N}\mathrm{Def}^{X}_{X/X\cap N}e^{X}_{X}\\
&= \sum_{X\in[s_K]}\frac{|v^X|}{|N_K(X):X|}m_{X,X\cap N}\mathrm{Ind}^{K/N}_{XN/N}\mathrm{Iso}^{XN/N}_{X/X\cap N}e^{X/X\cap N}_{X/X\cap N}\\
&= \sum_{X\in[s_K]}\frac{|v^X|}{|N_K(X):X|}m_{X,X\cap N}\mathrm{Ind}^{K/N}_{XN/N}e^{XN/N}_{XN/N}\quad .
\end{aligned}$$

If $X\leq K$ is such that $m_{X,X\cap N}\neq 0$, then $\beta(X)\cong\beta(X/X\cap N)\cong\beta(XN/N)$. The group XN/N is a subgroup of K/N, hence it is nilpotent, hence $\beta(X)\cong\beta(XN/N)$ is nilpotent. If Conjecture A is true, then X is nilpotent, hence $|v^X|=0$. It follows that $\mathrm{Res}^{G/N}_{K/N}\mathrm{Def}^{G}_{G/N}v=0$, hence $\mathrm{Def}^{G}_{G/N}L(G)\leq L(G/N)$.

Observe that one can still conclude that $\mathrm{Res}^{G/N}_{K/N}\mathrm{Def}^{G}_{G/N}v=0$ without assuming Conjecture A, in the case where N is solvable : indeed in this case, the group K is solvable (as N is solvable and K/N is nilpotent), so X is solvable, and one can conclude by Theorem 3.1.

Conversely, assume that Conjecture B holds, and let G be a finite group. Then $|G|e^G_G$ is an element of $B(G)$, whose restrictions to all proper subgroups of G are 0. If G is not nilpotent, then $|G|e^G_G\in L(G)$. Hence for any normal subgroup N of G, the element

$$\mathrm{Def}^{G}_{G/N}|G|e^G_G=m_{G,N}|G|e^{G/N}_{G/N}$$

is in $L(G/N)$. If $G/N\cong\beta(G)$, then $m_{G,N}\neq 0$, hence $|G|e^{G/N}_{G/N}\in L(G/N)$. Since it is a non-zero element, it follows that $G/N\notin\mathcal{N}(G/N)$, i.e. that $G/N\cong\beta(G)$ is not nilpotent. Hence Conjecture A holds. □

4.2. Remark : The above proof shows that the correspondence $G\mapsto L(G)$ is a biset functor on the full subcategory of the biset category consisting of *solvable* groups. Actually, it proves a little more : the correspondence $G\mapsto L(G)$ is a biset functor on the category of *all* finite groups, if we only allow as morphisms those bisets for which *left stabilizers are solvable* (or equivalently, if we only allow deflations by solvable normal subgroups).

4.3. Remark : Let G be a minimal counterexample to Conjecture A. Then G is non solvable, and as above G has a unique minimal normal subgroup

N, non-central in G. Thus $N \cong S^k$, where S is a non-abelian simple group, and $C_G(N) = \mathbf{1}$. The group $H = G/N$ is nilpotent, and it is the largest solvable quotient of G. In particular $\beta(G)$ is a quotient of H, hence of $\beta(H)$. Since $\beta(G)$ cannot be a p-group (for otherwise G would be cyclic modulo p, by Theorem 1.1, hence solvable, hence nilpotent by Theorem 3.1), it follows that H is not p-elementary for any prime p (that is, there are at least two different primes p such that the Sylow p-subgroups of H are non cyclic) : indeed, if P is a p-group, then $\beta(P) = \mathbf{1}$ if P is cyclic, and $\beta(P) = C_p \times C_p$ otherwise (cf. Remark 1.2).

Since the Frattini subgroup $\Phi(G)$ is nilpotent, it follows that $\Phi(G) = \mathbf{1}$. Let X be a minimal subgroup of G such that $XN = G$. Then $X \cap N \leq \Phi(X)$, thus $m_{X,X\cap N} = 1$. Hence $\beta(X) \cong \beta(X/X\cap N) \cong \beta(G)$ is nilpotent. Moreover $X < G$ (as $N \not\leq \Phi(G) = \mathbf{1}$), so X is nilpotent.

4.4. Remark : In view of Conjecture A and Theorem 3.1, Jacques Thévenaz has proposed the following :

Conjecture : *Let G be a finite group. Then $\beta(G)$ is solvable if and only if G is solvable.*

This conjecture implies Conjecture A, by Theorem 3.1. A straightforward modification of the proof of Theorem 4.1 shows that this conjecture is equivalent to saying that the correspondence sending a finite group G to the kernel of the restriction map

$$\rho_G = \prod_{H\in\mathcal{R}(G)} \mathrm{Res}_H^G : B(G) \to \prod_{H\in\mathcal{R}(G)} B(H) \ ,$$

where $\mathcal{R}(G)$ is the set of solvable subgroups of G, is a biset functor.

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