

Complexity and cohomology of cohomological Mackey functors

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Abstract : Let k be a field of characteristic $p > 0$. Call a finite group G a *poco group over k* if any finitely generated cohomological Mackey functor for G over k has polynomial growth. The main result of this paper is that G is a poco group over k if and only if the Sylow p -subgroups of G are cyclic, when $p > 2$, or have sectional rank at most 2, when $p = 2$.

A major step in the proof is the case where G is an elementary abelian p -group. In particular, when $p = 2$, all the extension groups between simple functors can be determined completely, using a presentation of the graded algebra of self extensions of the simple functor S_1^G , by explicit generators and relations.

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1. Introduction

This paper addresses the question of the cohomology and rate of growth of cohomological Mackey functors for a finite group G over a field k . The three main results are the following :

1.1. Theorem : *Let G be a finite group and k be a field of positive characteristic p . Then the following conditions are equivalent :*

- 1. Every finitely generated cohomological Mackey functor for G over k has a projective resolution with polynomial growth.*
- 2. Let S be a Sylow p -subgroup of G . Then S is cyclic if $p > 2$, or S has sectional 2-rank at most 2, if $p = 2$.*

The key argument in the proof of this theorem is a reduction to the case where G is an elementary abelian p -group. The case of elementary abelian 2-groups can be described quite completely :

1.2. Theorem : *Let $G = (C_2)^m$ be an elementary abelian 2-group of rank m , and k be a field of characteristic 2. Let $M_k^c(G)$ denote the category of cohomological Mackey functors for G over k . Let S_1^G denote the simple*

cohomological Mackey functor defined by

$$\forall Q \leq G, \quad S_1^G(Q) = \begin{cases} k & \text{if } Q = \mathbf{1} \\ \{0\} & \text{otherwise} \end{cases} .$$

Then the algebra $\text{Ext}_{M_k^c(G)}^*(S_1^G, S_1^G)$ is finitely generated by elements of degree 2, and its Poincaré series

$$P(t) = \sum_{j \in \mathbb{N}} \dim_k \text{Ext}_{M_k^c(G)}^j(S_1^G, S_1^G) t^j$$

is equal to

$$P(t) = \frac{1}{(1-t^2)(1-3t^2)(1-7t^2)\dots(1-(2^{m-1}-1)t^2)} .$$

1.3. Theorem : Let $G = (C_2)^m$ be an elementary abelian 2-group of rank m , and k be a field of characteristic 2. Then the algebra $\mathcal{E} = \text{Ext}_{M_k^c(G)}^*(S_1^G, S_1^G)$ admits the following presentation :

- The generators γ_x , of degree 2, are indexed by the elements x of $G - \{0\}$.
- The relations are the following :

1. Whenever H is a subgroup of index 2 of G ,

$$\sum_{x \notin H} \gamma_x = 0 .$$

2. For any distinct elements x and y of $G - \{0\}$,

$$[\gamma_x + \gamma_y, \gamma_{x+y}] = 0 ,$$

where $[a, b] = ab + ba$ denotes the commutator of two elements a and b in \mathcal{E} .

1.4. This paper is divided in two parts : the first one focuses on complexity, and the second one on cohomology. Section 2 of Part I quickly recalls the definitions and basic results on the rate of growth of a module for a finite dimensional algebra over a field. In Section 3, the categories of cohomological Mackey functors are introduced, from different points of view, which are equivalent thanks to Yoshida's theorem. Next some functors associated to

bisets between categories of cohomological Mackey functors are defined, with nice adjunction properties. In Section 4, it is shown how to reduce the question of complexity of cohomological Mackey functors for a finite group G over a field of characteristic p to the same question for a Sylow p -subgroup of G . Section 5 exposes a sketch of the proof of Theorem 1.1. In Sections 6, 7, and 8, some simple cohomological functors and extensions between them are discussed. In Section 9, the case of cyclic p -groups is recalled from Samy Modeliar's thesis ([7]). In Section 10, the case of elementary abelian p -groups is settled, and Section 11 handles the case of 2-groups with sectional 2 rank (at most) equal to 2.

The first section of Part II states further results on extensions of simple cohomological functors for elementary abelian p -groups. This leads to the proof of Theorem 1.3, in Section 13. In Section 14, a similar partial result is stated, for $p = 3$, which is conjectured for any odd prime p . Finally, Section 15 exposes some results on extensions of simple functors for an arbitrary finite p -group G , which show in particular how to reduce the computation of these extensions to the computation of self extensions for simple functors indexed by the trivial subgroup of some subquotients of G .

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I - Complexity

2. Polynomial growth

2.1. Definition : *Let A be a finite dimensional (unital) algebra over a field k . A finitely generated A -module M is said to have polynomial growth if there exists a resolution*

$$P_* : \cdots \longrightarrow P_n \longrightarrow P_{n-1} \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of M by projective A -modules, and constants c , d and e such that

$$\forall n \in \mathbb{N}, \dim_k P_n \leq c n^d + e .$$

The module M is said to have exponential growth if for any projective reso-

lution P_* of M , there are constants c, d , and e , with $c > 0$ and $d > 1$, such that

$$\forall n \in \mathbb{N}, \dim_k P_n \geq c d^n + e ,$$

The module M is said to have intermediate growth if M has neither polynomial nor exponential growth.

2.2. Remark : If S is a generating set of M as an A -module, then there is a surjective map of A -modules $A^{|S|} \rightarrow M$. In particular, the projective cover of M has dimension at most $d \dim_k M$, where $d = \dim_k A$. By induction, this shows that there exists a projective resolution as above such that

$$\dim_k P_n \leq (d - 1)^{n-1} d \dim_k M .$$

In particular, this dimension is always bounded by some exponential function of n .

2.3. Lemma : Let A be a finite dimensional algebra over a field k , and M be a finitely generated A -module.

1. If

$$\cdots \rightarrow P_n \rightarrow P_{n-1} \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

is a minimal projective resolution of M , then

$$P_n \cong \bigoplus_{S \in \text{Irr}(A)} P_S^{\dim_k \text{Ext}_A^n(M, S) / \dim_k \text{End}_A(S)} ,$$

where $\text{Irr}(A)$ is a set of representatives of isomorphism classes of simple A -modules, and P_S denotes a projective cover of S .

2. In particular M has polynomial growth if and only if for any simple A -module S , there exists constants c, d and e such that

$$\forall n \in \mathbb{N}, \dim_k \text{Ext}_A^n(M, S) \leq c n^d + e .$$

Proof : Assertion 1 follows by *décalage* from the case $n = 0$, and from the fact that the largest semisimple quotient of M is isomorphic to

$$\bigoplus_{S \in \text{Irr}(A)} S^{\dim_k \text{Ext}_A^0(M, S) / \dim_k \text{End}_A(S)} .$$

Assertion 2 is a straightforward consequence of Assertion 1, since there are finitely many simple A -modules, up to isomorphism. \square

2.4. Remark : In particular, if M has polynomial growth, then any direct summand of M has polynomial growth.

Conversely, the class of modules with polynomial growth is closed under extensions. More precisely :

2.5. Lemma : *Let A be a finite dimensional algebra over a field k .*

1. *If $L \xrightarrow{f} M \xrightarrow{g} N$ is an exact sequence of finite dimensional k -vector spaces, then*

$$\dim_k M \leq \dim_k L + \dim_k N ,$$

with equality if and only if f is injective and g is surjective.

2. *Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence of finitely generated A -modules. If two of the modules L , M , and N have polynomial growth, so does the third.

Proof : For Assertion 1, there is a short exact sequence

$$0 \longrightarrow \text{Im } f \longrightarrow M \longrightarrow \text{Im } g \longrightarrow 0 ,$$

hence $\dim_k M = \dim_k \text{Im } f + \dim_k \text{Im } g \leq \dim_k L + \dim_k N$. Equality holds if and only if $\dim_k \text{Im } f = \dim_k L$ and $\dim_k \text{Im } g = \dim_k N$, i.e. if f is injective and g is surjective.

For Assertion 2, let S be a finitely generated A -module. Consider the long exact sequence of Ext-groups

$$\cdots \rightarrow \text{Ext}_A^n(N, S) \rightarrow \text{Ext}_A^n(M, S) \rightarrow \text{Ext}_A^n(L, S) \rightarrow \text{Ext}_A^{n+1}(N, S) \rightarrow \cdots$$

It follows from Assertion 1 that

$$\dim_k \text{Ext}_A^n(M, S) \leq \dim_k \text{Ext}_A^n(N, S) + \dim_k \text{Ext}_A^n(L, S) ,$$

hence if L and N have polynomial growth, so does M . Similarly,

$$\dim_k \text{Ext}_A^n(N, S) \leq \dim_k \text{Ext}_A^n(M, S) + \dim_k \text{Ext}_A^{n-1}(L, S) ,$$

It follows that N has polynomial growth if L and M have. Finally,

$$\dim_k \text{Ext}_A^n(L, S) \leq \dim_k \text{Ext}_A^n(M, S) + \dim_k \text{Ext}_A^{n+1}(N, S) ,$$

hence L has polynomial growth if M and N do. □

2.6. Corollary : *Let A be a finite dimensional algebra over a field k . The following conditions are equivalent :*

1. *Every finitely generated A -module has polynomial growth.*
2. *Every simple A -module has polynomial growth.*

Proof : Obviously Condition 1 implies Condition 2. The converse follows by induction on the length of a finitely generated A -module. \square

3. Cohomological Mackey functors

3.1. Definition. Let R denote an arbitrary commutative unital ring. Recall that a *Mackey functor M for G over R* consists of the assignment $H \mapsto M(H)$ of an R -module to each subgroup H of G , together with maps of R -modules

$$t_H^K : M(H) \longrightarrow M(K) \ ,$$

called *transfer maps*, and

$$r_H^K : M(K) \longrightarrow M(H) \ ,$$

called *restriction maps*, whenever $H \leq K \leq G$, and maps of R modules

$$c_{x,H} : M(H) \longrightarrow M({}^xH) \ ,$$

for each $x \in G$, subject to a list of compatibility conditions, in particular the Mackey formula (cf. [9] for details). A Mackey functor M is called *cohomological* if the additional conditions

$$t_H^K r_H^K = |K : H| \text{Id}_{M(H)}$$

are fulfilled, for any $H \leq K \leq G$.

There is an obvious notion of morphism of (cohomological) Mackey functors, and this yields the category $\mathbf{M}_R^c(G)$ of cohomological Mackey functors for G over R .

3.2. Example (fixed points functors) : In particular, when V is an RG -module, then the fixed point functor FP_V is the Mackey functor defined by $FP_V(H) = V^H$, i.e. the set of elements of V which are invariant by H . When $H \leq K \leq G$, the transfer map $V^H \longrightarrow V^K$ is *the relative trace map* Tr_H^K , defined by

$$\text{Tr}_H^K(v) = \sum_{x \in [K/H]} yv \ ,$$

for any $v \in V^H$, where $[K/H]$ is a transversal of H in K . The restriction map $V^K \longrightarrow V^H$ is the inclusion map, and for $x \in G$, the conjugation map $V^H \longrightarrow V^{xH}$ is the map $v \mapsto xv$. The functor FP_V is obviously cohomological.

The correspondence $V \mapsto FP_V$ is a functor from $RG\text{-Mod}$ to $\mathbf{M}_R^c(G)$. This functor is fully faithful : if V and W are RG -modules, then any morphism $\varphi : FP_V \rightarrow FP_W$ in $\mathbf{M}_R^c(G)$ is equal to FP_f , where f is the morphism of RG -modules from $V = FP_V(\mathbf{1})$ to $W = FP_W(\mathbf{1})$ obtained by evaluating φ at the trivial subgroup of G .

3.3. The cohomological Mackey algebra. It was shown by Thévenaz and Webb (cf. [9]) that $\mathbf{M}_R^c(G)$ is equivalent to the category of modules over the *cohomological Mackey algebra* $co\mu_R(G)$ for G over R . More precisely, if M is a cohomological Mackey functor for G over R , then the corresponding $co\mu_R(G)$ -module is equal to $\bigoplus_{H \leq G} M(H)$.

The algebra $co\mu_R(G)$ is a finitely generated free R -module, so in particular when R is a field, it is a finite dimensional R -algebra. In this case, there is a natural notion of projective cover of a finitely generated $co\mu_R(G)$ -module, hence a natural notion of *minimal projective resolution* of a cohomological Mackey functor.

3.4. Yoshida's theorem. Let $\text{perm}_R(G)$ denote the full subcategory of the category $RG\text{-mod}$ of finitely generated RG -modules, consisting of *permutation* RG -modules, i.e. modules admitting a (globally) G -invariant R -basis. It is an R -linear category, which is naturally equivalent to the opposite category : indeed, the dual $V^* = \text{Hom}_R(V, R)$ of a finitely generated permutation RG -module is again a finitely generated permutation RG -module, and the correspondence $\delta : V \mapsto V^*$ is an equivalence of categories from $\text{perm}_R(G)$ to $\text{perm}_R(G)^{op}$, which is its own inverse (up to a slight abuse of notation).

When M is a cohomological Mackey functor for G over R , then the correspondence

$$V \mapsto \text{Hom}_{\mathbf{M}_R^c(G)}(FP_V, M)$$

is an R -linear contravariant functor \tilde{M} from $\text{perm}_R(G)$ to $R\text{-mod}$. This yields in turn an R -linear functor $M \mapsto \tilde{M}$ from $\mathbf{M}_R^c(G)$ to $\text{Fun}_R(G)$, where $\text{Fun}_R(G)$ denotes the category of R -linear contravariant functors from $\text{perm}_R(G)$ to $R\text{-mod}$.

Conversely, if F is such a functor, and H is a subgroup of G , one can define $\hat{F}(H) = F(R(G/H))$, where $R(G/H)$ is the free R -module with basis the G -set G/H of H -cosets in G . If $H \leq K \leq G$, then the projection map $p_H^K : G/K \longrightarrow G/H$ gives a map of RG -modules $Rp_H^K : R(G/K) \longrightarrow R(G/H)$, and taking image by F gives a transfer map $t_H^K = F(p_H^K) : \hat{F}(H) \longrightarrow \hat{F}(K)$.

Similarly, applying first the equivalence δ gives a map $r_H^K = F(\delta(Rp_H^K)) : \hat{F}(K) \longrightarrow \hat{F}(H)$. Finally, if $x \in G$, then the map $g^x H \mapsto gxH$ induce a map $R(G/xH) \longrightarrow R(G/H)$, whose image by F yields a conjugation map $c_{x,H} : \hat{F}(H) \longrightarrow \hat{F}(^xH)$. With these definitions \hat{F} becomes a cohomological Mackey functor for G over R , and the correspondence $F \mapsto \hat{F}$ is a functor from $\text{Fun}_R(G)$ to $\mathbf{M}_R^c(G)$. The following theorem is essential :

3.5. Theorem : [Yoshida [10]] *The functors $M \mapsto \tilde{M}$ and $F \mapsto \hat{F}$ are mutual inverse equivalences of categories between $\mathbf{M}_R^c(G)$ and $\text{Fun}_R(G)$.*

3.6. Remark : One checks easily that if V is an RG -module, then the functor FP_V is mapped to the functor $\text{Hom}_{RG}(-, V)$ by this equivalence. For this reason, this functor will also be denoted by FP_V . More generally, Yoshida's equivalence allows for an identification of $\mathbf{M}_R^c(G)$ with $\text{Fun}_R(G)$, that will be used freely throughout the rest of this paper.

3.7. Remark : In particular, if V is a permutation RG -module, then the Yoneda functor $FP_V = \text{Hom}_{\text{perm}_R(G)}(-, V)$ is a projective object in $\text{Fun}_R(G)$. More precisely, if M is a cohomological Mackey functor for G over R , and H is a subgroup of G , there is an isomorphism of R -modules

$$(3.8) \quad \text{Hom}_{\mathbf{M}_R^c(G)}(FP_{R(G/H)}, M) \cong M(H) .$$

It follows more generally that if V is a direct summand of a permutation RG -module, then the functor FP_V is a projective object in $\mathbf{M}_R^c(G)$. Thévenaz and Webb (cf. [9]) have shown conversely that any projective object in $\mathbf{M}_R^c(G)$ is isomorphic to FP_V , where V is a direct summand of a permutation RG -module.

3.9. Remark : It also follows that the category $\text{Fun}_R(G)$ is equivalent to the category of modules over the Hecke algebra

$$Y_R(G) = \text{End}_{RG} \left(\bigoplus_{H \leq G} R(G/H) \right) .$$

One can show easily that this algebra is actually isomorphic to the cohomological Mackey algebra $co\mu_R(G)$ (cf. [4]). The $Y_R(G)$ -module corresponding to the object F of $\text{Fun}_R(G)$ (resp. to the cohomological Mackey functor M for G over R) under these equivalences, is equal to $\bigoplus_{H \leq G} F(R(G/H))$ (resp. to $\bigoplus_{H \leq G} M(H)$). In particular F is finitely generated (resp. M is finitely generated) if and only if $F(W)$ is a finitely generated R -module, for any finitely generated permutation RG -module W (resp. $M(H)$ is a finitely generated R -module, for any $H \leq G$).

3.10. The dual of a Mackey functor. If M is a Mackey functor for G over R , then the *dual* Mackey functor M^* is defined by

$$\forall H \leq G, \quad M^*(G) = \text{Hom}_R(M(G), R) .$$

The transfer, restriction, and conjugation maps for M^* are defined by

$$r_H^K = \tau(t_H^K) , \quad t_H^K = \tau(r_H^K) , \quad c_{x,H} = \tau(c_{x^{-1},xH}) ,$$

for any $H \leq K \leq G$ and any $x \in G$, where the exponent τ denotes transposed maps.

If M is cohomological, then M^* is also cohomological. Through the equivalence given by Yoshida's Theorem 3.5, this duality maps the functor F of $\text{Fun}_R(G)$ to the functor F^* defined as the composition

$$\text{perm}_R(G) \xrightarrow{*} \text{perm}_R(G)^{op} \xrightarrow{F} R\text{-Mod} \xrightarrow{*} R\text{-Mod}^{op} .$$

3.11. Remark : The correspondence $M \mapsto M^*$ is a functor from $\mathbf{M}_R^c(G)$ to the opposite category. The canonical morphism from M to its bidual $(M^*)^*$ is functorial in M , and it is an isomorphism when R is a field k and M is finitely generated. In other words, the correspondence $M \mapsto M^*$ induces an equivalence from the category of finitely generated cohomological Mackey functors for G over k to the opposite category. Thus, for any finitely generated cohomological Mackey functors M and N for G over k , there is a natural isomorphism

$$\text{Hom}_{\mathbf{M}_k^c(G)}(M, N) \cong \text{Hom}_{\mathbf{M}_k^c(G)}(N^*, M^*) .$$

The functor M is a finitely generated projective object in $\mathbf{M}_k^c(G)$ if and only if M^* is a finitely generated injective object in $\mathbf{M}_k^c(G)$, and the previous isomorphism extends to natural isomorphisms

$$\text{Ext}_{\mathbf{M}_k^c(G)}^n(M, N) \cong \text{Ext}_{\mathbf{M}_k^c(G)}^n(N^*, M^*) ,$$

for any $n \in \mathbb{N}$.

3.12. Construction of functors. Let G and H be finite groups. If U is a finite (H, G) -biset consider the R -linear functor

$$\mathbf{t}_U : V \mapsto RU \otimes_{RG} V$$

from $RG\text{-mod}$ to $RH\text{-mod}$ maps permutation RG -modules to permutation RH -modules. By composition, this induces a functor

$$\mathbf{L}_U : F \mapsto F \circ \mathbf{t}_U$$

from $\text{Fun}_R(H)$ to $\text{Fun}_R(G)$. Since the right adjoint to the functor \mathbf{t}_U is the functor

$$\mathbf{h}_U : W \mapsto \text{Hom}_{RH}(RU, W) \ ,$$

it follows from standard results of category theory that the functor

$$\mathbf{R}_U : F \mapsto F \circ \mathbf{h}_U$$

from $\text{Fun}_R(G)$ to $\text{Fun}_R(H)$, is right adjoint to \mathbf{L}_U .

3.13. Remark : The functors \mathbf{L}_U and \mathbf{R}_U are a generalization of functors considered by Tambara (see [8] Section 4).

3.14. Proposition :

1. Let G and H be finite groups, and U be a finite (H, G) -biset. Then the functors \mathbf{R}_U and \mathbf{L}_U are exact.
2. Let G and H be finite groups. If U and U' are isomorphic finite (H, G) -bisets, then there are isomorphisms of functors $\mathbf{L}_U \cong \mathbf{L}_{U'}$ and $\mathbf{R}_U \cong \mathbf{R}_{U'}$.
3. Let G , H , and K be finite groups. Let U be a finite (H, G) -biset, and V be a finite (K, H) -biset. Then there are isomorphisms of functors

$$\mathbf{R}_V \circ \mathbf{R}_U \cong \mathbf{R}_{V \times_H U} \ , \ \mathbf{L}_U \circ \mathbf{L}_V \cong \mathbf{L}_{V \times_H U} \ .$$

4. Let G be a finite group, and let Id_G denote the identity biset for G , i.e. the set G for its (G, G) -biset structure given by multiplication. Then the functor \mathbf{L}_{Id_G} and \mathbf{R}_{Id_G} are isomorphic to the identity functor.
5. Let G and H be finite groups. If U and U' are finite (H, G) -bisets, there are isomorphisms of functors

$$\mathbf{L}_{U \sqcup U'} \cong \mathbf{L}_U \oplus \mathbf{L}_{U'} \ , \ \mathbf{R}_{U \sqcup U'} \cong \mathbf{R}_U \oplus \mathbf{R}_{U'} \ .$$

6. Let G and H be finite groups, and U be a finite (H, G) -biset. Then for any object F of $\text{Fun}_R(G)$, there is an isomorphism

$$\mathbf{R}_{U \circ p}(F)^* \cong \mathbf{L}_U(F^*)$$

in $\text{Fun}_R(H)$, which is functorial in F .

Proof : Assertion 1 is obvious, since the functors \mathbf{R}_U and \mathbf{L}_U are obtained by pre-composition with some functor (in other words, they are both *restriction functors* along a suitable functor).

Assertion 2 is a consequence of the isomorphisms of functors $\mathbf{t}_U \cong \mathbf{t}_{U'}$, and $\mathbf{h}_U \cong \mathbf{h}_{U'}$, which both follow from the isomorphism $RU \cong RU'$ of (RH, RG) -bimodules.

For Assertion 3, the associativity of tensor product gives an isomorphism of functor

$$\mathbf{t}_V \circ \mathbf{t}_U \cong \mathbf{t}_{V \times_H U} \ ,$$

which by adjunction, gives the isomorphism of functors

$$\mathbf{h}_U \circ \mathbf{h}_V \cong \mathbf{h}_{V \times_H U} \ .$$

The isomorphisms of Assertion 3 follow by composition.

Assertion 4 follows from the fact that the functors \mathbf{t}_{Id_G} and \mathbf{t}_{Id_G} are both isomorphic to the identity functor. Similarly, Assertion 5 follows from similar additivity properties of the functors t_U and h_U with respect to U .

For Assertion 6, let W be an object of $\text{perm}_R(G)$. Then

$$\begin{aligned} \mathbf{R}_{U^{op}}(F)^*(W) &= \text{Hom}_R(\mathbf{R}_{U^{op}}(F)(W^*), R) \\ &= \text{Hom}_R\left(F\left(\text{Hom}_{RG}(RU^{op}, \text{Hom}_R(W, R))\right), R\right) \\ &\cong \text{Hom}_R\left(F\left(\text{Hom}_R(RU \otimes_{RG} W, R)\right), R\right) \\ &= \mathbf{L}_U(F^*)(W) \ , \end{aligned}$$

and these isomorphisms are functorial with respect to W . □

3.15. Proposition : *Let G and H be finite groups, and U be a finite (H, G) -biset.*

1. *If V is an RH -module, then*

$$\mathbf{L}_U(FP_V) = FP_{\text{Hom}_{RH}(RU, V)} \ .$$

In particular, the functor \mathbf{L}_U maps projective objects to projective objects.

2. *If G acts freely on U , then $\mathbf{L}_U \cong \mathbf{R}_{U^{op}}$, where U^{op} denotes the opposite biset of U .*

Proof : For Assertion 1, there is an isomorphism of functors

$$\text{Hom}_{RG}(-, \text{Hom}_{RH}(RU, V)) \cong \text{Hom}_{RH}(RU \otimes_{RG} -, V) \ .$$

In other words $FP_V \circ \mathbf{t}_U \cong FP_{\text{Hom}_{RH}(RU, V)}$.

The last part of Assertion 1 follows from the facts that if V is a permutation RH -module, then $\text{Hom}_{RH}(RU, V) = \mathfrak{h}_U(V)$ is a permutation RG -module. An alternative proof consists in observing that since \mathbf{L}_U is left adjoint to an exact functor, it maps projective objects to projective objects.

For Assertion 2, for any RG -module W , there is an isomorphism of RH -modules

$$RU \otimes_{RG} W \longrightarrow \text{Hom}_{RG}(RU^{op}, W) \ ,$$

defined by sending $u \otimes w$, for $u \in U$ and $w \in W$, to the map sending $v \in U^{op}$ (recall that $U = U^{op}$ as a set) to gw , if there exists an element $g \in G$ such that $v = ug^{-1}$ (and in this case there is a unique such g , since G acts freely on U), and to 0 otherwise. This isomorphism is obviously functorial in W , and this completes the proof. \square

3.16. Proposition : *Let G and H be finite groups, and U be a finite (H, G) -biset. Then for any $n \in \mathbb{N}$, the adjunction of the pair $(\mathbf{L}_U, \mathbf{R}_U)$ induces an isomorphism of bifunctors*

$$\text{Ext}_{\text{Fun}_R(G)}^n(\mathbf{L}_U(-), -) \cong \text{Ext}_{\text{Fun}_R(H)}^n(-, \mathbf{R}_U(-)) \ .$$

Proof : This follows from the fact that \mathbf{L}_U and \mathbf{R}_U are both exact functors. \square

3.17. Remark : In terms of Yoneda extensions, this isomorphism can be viewed as follows : the adjunction of the pair $(\mathbf{L}_U, \mathbf{R}_U)$ is equivalent to the existence of natural transformation of functors

$$\eta : \text{Id} \longrightarrow \mathbf{R}_U \circ \mathbf{L}_U \quad \text{and} \quad \varepsilon : \mathbf{L}_U \circ \mathbf{R}_U \longrightarrow \text{Id} \ ,$$

called respectively the *unit* and *counit* of the adjunction, with the property that for each object M in $\text{Fun}_R(G)$ and each object N in $\text{Fun}_R(H)$,

$$\varepsilon_{\mathbf{L}_U(N)} \circ \mathbf{L}_U(\eta_N) = \text{Id}_{\mathbf{L}_U(N)} \quad \text{and} \quad \mathbf{R}_U(\varepsilon_M) \circ \eta_{\mathbf{R}_U(M)} = \text{Id}_{\mathbf{R}_U(M)} \ .$$

The bijection $\text{Hom}_{\text{Fun}_R(G)}(\mathbf{L}_U(N), M) \longrightarrow \text{Hom}_{\text{Fun}_R(H)}(N, \mathbf{R}_U(M))$ is given by $\alpha \mapsto \mathbf{R}_U(\alpha) \circ \eta_N$. The inverse bijection is given by $\beta \mapsto \varepsilon_M \circ \mathbf{L}_U(\beta)$.

In other words, these bijections consist in taking images by one of the functors, and then compose on the suitable side with unit or counit.

Now interpreting the extension group $\text{Ext}_{\text{Fun}_R(G)}^n(\mathbf{L}_U(N), M)$ as the set of equivalence classes of exact sequences in $\text{Fun}_R(G)$ of the form

$$0 \longrightarrow M \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow \mathbf{L}_U(N) \longrightarrow 0 \ ,$$

the procedure is the same : first apply the functor $\mathbf{R}_U(-)$, to get an exact sequence

$$0 \rightarrow \mathbf{R}_U(M) \rightarrow \mathbf{R}_U(X_{n-1}) \rightarrow \cdots \rightarrow \mathbf{R}_U(X_1) \rightarrow \mathbf{R}_U(X_0) \rightarrow \mathbf{R}_U \circ \mathbf{L}_U(N) \rightarrow 0 ,$$

and then compose with the map η_N , i.e. complete the cartesian square at the right of the following diagram

$$\begin{array}{ccccccccccc} 0 & \rightarrow & \mathbf{R}_U(M) & \rightarrow & \mathbf{R}_U(X_{n-1}) & \rightarrow & \cdots & \rightarrow & \mathbf{R}_U(X_1) & \rightarrow & \mathbf{R}_U(X_0) & \rightarrow & \mathbf{R}_U \circ \mathbf{L}_U(N) & \rightarrow & 0 \\ & & \parallel & & \parallel & & & & \parallel & & \uparrow & & \uparrow & \eta_N & \\ 0 & \rightarrow & \mathbf{R}_U(M) & \rightarrow & \mathbf{R}_U(X_{n-1}) & \rightarrow & \cdots & \rightarrow & \mathbf{R}_U(X_1) & \rightarrow & Y_0 & \rightarrow & N & \rightarrow & 0 . \end{array}$$

The inverse bijection is obtained similarly by first applying the functor \mathbf{L}_U , and composing with the map ε_M , i.e. completing a cocartesian square at the left of the resulting diagram.

3.18. Remark : It follows easily that the isomorphisms of functors

$$\alpha_n : \text{Ext}_{\mathbf{Fun}_R(G)}^n(\mathbf{L}_U(-), -) \xrightarrow{\cong} \text{Ext}_{\mathbf{Fun}_R(H)}^n(-, \mathbf{R}_U(-)) .$$

of Proposition 3.16, are compatible with the Yoneda product, in the following sense : if P is an object of $\mathbf{Fun}_R(H)$, and if M and N are objects of $\mathbf{Fun}_R(G)$, if $e \in \text{Ext}_{\mathbf{Fun}_R(G)}^n(\mathbf{L}_U(P), N)$ and $f \in \text{Ext}_{\mathbf{Fun}_R(G)}^m(N, M)$, then

$$\alpha_{m+n}(f \circ e) = \mathbf{R}_U(f) \circ \alpha_n(e) \in \text{Ext}_{\mathbf{Fun}_R(H)}^{m+n}(P, \mathbf{R}_U(M)) .$$

3.19. Example (induction and restriction) : Let G be a subgroup of H . Set $U = H$, viewed as an (H, G) -biset by left and right multiplication. Then the functor $V \mapsto RU \otimes_{RG} V$ from $RG\text{-mod}$ to $RH\text{-mod}$ is isomorphic to the induction functor $V \mapsto \text{Ind}_G^H V$. It follows that the functor \mathbf{L}_U is isomorphic to the *restriction* functor $\text{Res}_G^H : \mathbf{M}_R^c(H) \rightarrow \mathbf{M}_R^c(G)$ in this case.

In the same situation, the functor $W \mapsto \text{Hom}_{RH}(RU, W)$ from $RH\text{-mod}$ to $RG\text{-mod}$ is isomorphic to the restriction functor Res_G^H . It follows that the functor \mathbf{R}_U is isomorphic to the *induction* functor $\text{Ind}_G^H : \mathbf{M}_R^c(G) \rightarrow \mathbf{M}_R^c(H)$.

3.20. Example (restriction and induction) : Suppose now that H is a subgroup of G , and consider $U = G$ as an (H, G) -biset by left and right multiplication. Then the functor $V \mapsto RU \otimes_{RG} V$ from $RG\text{-mod}$ to $RH\text{-mod}$ is isomorphic to the restriction functor $V \mapsto \text{Res}_H^G V$. It follows that the

functor \mathbf{L}_U is isomorphic to the *induction* functor $\text{Ind}_H^G : \mathbf{M}_R^c(H) \longrightarrow \mathbf{M}_R^c(G)$ in this case.

In the same situation, the functor $W \mapsto \text{Hom}_{RH}(RU, W)$ from $RH\text{-mod}$ to $RG\text{-mod}$ is isomorphic to the induction functor Res_G^H . It follows that the functor \mathbf{R}_U is isomorphic to the *induction* functor $\text{Ind}_G^H : \mathbf{M}_R^c(G) \longrightarrow \mathbf{M}_R^c(H)$.

This yields another proof of the well known fact that the induction and restriction functors between categories of cohomological Mackey functors are left and right adjoint to each other (cf. [9] for details).

3.21. Example (the functor $\rho_{G/N}^G$) : Let H be a finite group, and N be a normal subgroup of H , and set $G = H/N$. Also set $U = G$, viewed as an (H, G) -biset in the obvious way. In this case, the functor \mathbf{t}_U is the inflation functor from $RG\text{-mod}$ to $RH\text{-mod}$. Using the equivalences of categories of Theorem 3.5, the functor \mathbf{L}_U gives a functor denoted by $\rho_{H/N}^H$, from $\mathbf{M}_R^c(H)$ to $\mathbf{M}_R^c(H/N)$. One checks easily that if M is a cohomological Mackey functor for H over R , then, denoting by $x \mapsto \bar{x}$ the projection map $H \longrightarrow H/N$

$$(\rho_{H/N}^H(M))(\bar{K}) = M(K) \ .$$

Similarly, the transfer, restriction, and conjugation maps for the functor $\rho_{H/N}^H(M)$ are obtained by just “removing the bars”, i.e.

$$t_{\bar{K}}^{\bar{L}} = t_K^L \ , r_{\bar{K}}^{\bar{L}} = r_K^L \ , c_{\bar{x}, \bar{K}} = c_{x, K} \ .$$

The right adjoint to $\rho_{H/N}^H$ is the functor \mathbf{R}_U , hereafter denoted by $\mathcal{J}_{G/N}^G$.

Moreover Assertion 3 of Proposition 3.15 shows that the functor $\rho_{G/N}^G$ is also equal to the functor $\mathbf{R}_{U^{op}}$. In particular, the left adjoint to $\rho_{G/N}^G$ is the functor $\mathbf{L}_{U^{op}}$, hereafter denoted by $\mathcal{I}_{G/N}^G$.

3.22. Example (the functor $\mathcal{I}_{G/N}^G$) : The left adjoint to the functor $\rho_{G/N}^G$ will be denoted by $\mathcal{I}_{G/N}^G$. It is obtained as follows : if M is a cohomological Mackey functor for G/N , and K is a subgroup of G , then

$$(\mathcal{I}_{G/N}^G(M))(K) = M(KN/N) \ .$$

If $K \leq L \leq G$, then the transfer, restriction, and conjugation maps for the functor $\mathcal{I}_{G/N}^G(M)$ are given by

$$t_K^L = |L \cap N : K \cap N| t_{KN/N}^{LN/N} \ , r_K^L = r_{KN/N}^{LN/N} \ , c_{x, K} = c_{xN, KN/N} \ .$$

3.23. Example (the functor $j_{G/N}^G$) : The right adjoint to the functor $\rho_{G/N}^G$ will be denoted by $j_{G/N}^G$. It is obtained as follows : if M is a cohomological Mackey functor for G/N , and K is a subgroup of G , then

$$(j_{G/N}^G(M))(K) = M(KN/N) .$$

If $K \leq L \leq G$, then the transfer, restriction, and conjugation maps for the functor $j_{G/N}^G(M)$ are given by

$$t_K^L = t_{KN/N}^{LN/N} , \quad r_K^L = |L \cap N : K \cap N| r_{KN/N}^{LN/N} , \quad c_{x,K} = c_{xN,KN/N} .$$

3.24. Remark : The functors $i_{G/N}^G$ and $j_{G/N}^G$ should not be confused with the inflation functor $\text{Inf}_{G/N}^G$: recall (cf. [9]) that if N is a normal subgroup of a finite group G , and M is a Mackey functor for G/N over R , then the functor $\text{Inf}_{G/N}^G M$ is the Mackey functor for G over R defined by

$$(\text{Inf}_{G/N}^G M)(H) = \begin{cases} M(H/N) & \text{if } H \geq N \\ \{0\} & \text{otherwise} \end{cases} .$$

The transfer, restriction, and conjugation maps are the obvious ones. In general, this inflation procedure does not preserve cohomological Mackey functors : indeed, if M is cohomological, and if $K < H$ are subgroups of G such that $H \geq N$ but $K \not\geq N$, then the composition

$$t_K^H r_K^H : (\text{Inf}_{G/N}^G M)(H) \longrightarrow (\text{Inf}_{G/N}^G M)(H)$$

is equal to 0, so it is not equal in general to the multiplication by $|H : K|$.

However, if N is a p -group, for some prime number p , then p divides $|KN : K| = |N : K \cap N|$, hence p divides $|H : K|$. If moreover R has characteristic p , then $|H : K| = 0$ in R in this situation, and one checks then that $\text{Inf}_{G/N}^G M$ is cohomological. In this case, the inflation functor $\mathbf{M}_R^c(G/N) \longrightarrow \mathbf{M}_R^c(G)$ corresponds via Yoshida's equivalence of categories to the functor $\text{Fun}_R(G/N) \longrightarrow \text{Fun}_R(G)$ obtained by composition with the Brauer quotient functor $V \mapsto V[N]$ from $\text{perm}_R(G)$ to $\text{perm}_R(G/N)$.

3.25. Example : group isomorphism : Let $f : G \rightarrow H$ be a group isomorphism. Then the set $U = H$ has a natural structure of (H, G) -biset, for which $h \in H$ acts by left multiplication by h , and $g \in G$ acts by right multiplication by $f(g)$. It is clear in this case that the functor \mathbf{R}_U is the transport by isomorphism via f . This functor will be denoted by $\text{Iso}(f)$. The functor \mathbf{L}_U is isomorphic to $\text{Iso}(f^{-1})$, by Assertion 3 of Proposition 3.15.

Recall that a *section* (B, A) of G is a pair of subgroups of G such that $A \trianglelefteq B$.

3.26. Lemma : *Let G be a finite group, and (B, A) be a section of G . Then there is an isomorphism*

$$\text{Ind}_B^G \iota_{B/A}^B \cong \mathbb{L}_{A \setminus G}$$

of functors from $\mathbf{M}_R^c(B/A)$ to $\mathbf{M}_R^c(G)$, where $A \setminus G$ is endowed with its natural $(B/A, G)$ -biset structure. Similarly, there is an isomorphism

$$\rho_{B/A}^B \text{Res}_B^G \cong \mathbb{L}_{G/A}$$

of functors from $\mathbf{M}_R^c(G)$ to $\mathbf{M}_R^c(B/A)$, where G/A is endowed with its natural $(G, B/A)$ -biset structure.

Proof : The functor $\text{Ind}_B^G \iota_{B/A}^B$ is equal to the composition of $\text{Ind}_B^G = \mathbb{L}_V$, where V is the set G for its natural (B, G) -biset structure, and $\iota_{B/A}^B = \mathbb{L}_U$, where U is the set B/A , for its natural $(B/A, B)$ -biset structure. It follows from Proposition 3.14 that $\text{Ind}_B^G \iota_{B/A}^B \cong \mathbb{L}_W$, where $W = (B/A) \times_B G$ is clearly isomorphic to the $(B/A, G)$ -biset $A \setminus G$.

By adjunction, it follows that $\rho_{B/A}^B \text{Res}_B^G \cong \mathbb{R}_{A \setminus G}$. But $\mathbb{R}_{A \setminus G} \cong \mathbb{L}_{G/A}$ by Assertion 3 of Proposition 3.15, since B/A acts freely on G/A . \square

3.27. Proposition : *Let G be a finite group, and $(B, A), (D, C)$ be two sections of G . Then there is an isomorphism*

$$\rho_{D/C}^D \text{Res}_D^G \text{Ind}_B^G \iota_{B/A}^B \cong \bigoplus_{g \in [D \setminus G/B]} \text{Ind}_{\bar{D}_g}^{\bar{D}} \iota_{\bar{D}_g/\bar{C}_g}^{\bar{D}_g} \text{Iso}(f_g) \rho_{\bar{B}_g/\bar{A}_g}^{\bar{B}_g} \text{Res}_{\bar{B}_g}^{\bar{B}}$$

of functors from $\mathbf{M}_k^c(B/A)$ to $\mathbf{M}_k^c(D/C)$, where

$$\bar{D} = D/C, \quad \bar{D}_g = (D \cap {}^g B)C/C, \quad \bar{C}_g = (D \cap {}^g A)C/C, \quad ,$$

$$\bar{B} = B/A, \quad \bar{B}_g = (D^g \cap B)A/A, \quad \bar{A}_g = (C^g \cap B)A/A, \quad ,$$

and where $f_g : \bar{B}_g/\bar{A}_g \rightarrow \bar{D}_g/\bar{C}_g$ is the group isomorphism sending $x\bar{A}_g$ to ${}^g x\bar{C}_g$, for $x \in D^g \cap B$.

Proof : (see Proposition 7.1 of [5] for details) By Lemma 3.26 and Proposi-

tion 3.14, there are isomorphisms of functors

$$\begin{aligned} \rho_{D/C}^D \text{Res}_D^G \text{Ind}_B^G \iota_{B/A}^B &\cong \mathbf{L}_{G/C} \circ \mathbf{L}_{A \setminus G} \\ &\cong \mathbf{L}_{A \setminus G \times_G G/C} \cong \mathbf{L}_{A \setminus G/C} . \end{aligned}$$

Now the $(B/A, D/C)$ -biset $A \setminus G/C$ splits as a disjoint union of transitive ones

$$A \setminus G/C \cong \bigsqcup_{g \in [D \setminus G/B]} A \setminus Bg^{-1}D/C .$$

Moreover for each $g \in [D \setminus G/B]$, with the above notation, there is an isomorphism of (\bar{B}, \bar{D}) -bisets

$$A \setminus Bg^{-1}D/C \cong (\bar{B}/\bar{A}_g) \times_{\bar{B}_g/\bar{A}_g} \text{Iso}(f_g) \times_{\bar{D}_g/\bar{C}_g} (\bar{C}_g \setminus \bar{D}) .$$

By Lemma 3.26 and Proposition 3.14 again, there is an isomorphism of functors

$$\mathbf{L}_{A \setminus Bg^{-1}D/C} \cong \text{Ind}_{\bar{D}_g}^{\bar{D}} \circ \iota_{\bar{D}_g/\bar{C}_g}^{\bar{D}_g} \circ \text{Iso}(f_g) \circ \rho_{\bar{B}_g/\bar{A}_g}^{\bar{B}_g} \circ \text{Res}_{\bar{B}_g}^{\bar{B}} ,$$

as was to be shown. \square

4. Reduction to p -groups

4.1. Lemma : *Let k be a field, let G and H be finite groups, and U be a finite (H, G) -biset.*

1. *If F is an object of $\text{Fun}_k(H)$, then*

$$\dim_k \mathbf{L}_U(F) \leq C_U \dim_k F ,$$

where $C_U = \sum_{K \leq G} |H \setminus U/K|$.

2. *If F is an object of $\text{Fun}_k(H)$, and if F has polynomial growth, then so does $\mathbf{L}_U(F)$.*

Proof : Let F be an object of $\text{Fun}_k(H)$. The $\text{co}\mu_k(G)$ -module corresponding

to $\mathbb{L}_U(F)$ is equal to $\bigoplus_{K \leq G} \mathbb{L}_U(F)(k(G/K))$, by Remark 3.9. Thus

$$\begin{aligned} \dim_k \mathbb{L}_U(F) &= \sum_{K \leq G} \dim_k F(kU \otimes_{kG} k(G/K)) \\ &= \sum_{K \leq G} \dim_k F(k(U/K)) \\ &= \sum_{K \leq G} \sum_{u \in [H \setminus U/K]} \dim_k F(k(H/H_{uK})) , \end{aligned}$$

where $H_{uK} = \{h \in H \mid \exists g \in K, hu = ug\}$. Now for each $K \leq G$ and each $u \in [H \setminus U/K]$

$$\dim_k F(k(H/H_{uK})) \leq \dim_k F = \sum_{L \leq H} \dim_k F(k(H/L)) ,$$

thus

$$\dim_k \mathbb{L}_U(F) \leq \left(\sum_{K \leq G} \sum_{u \in [H \setminus U/K]} 1 \right) \dim_k F ,$$

showing Assertion 1.

Now if

$$\cdots P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow F \longrightarrow 0$$

is a projective resolution with polynomial growth, there are constants c, d , and e , such that $\dim_k P_n \leq cn^d + e$, for all $n \in \mathbb{N}$. Now the complex

$$\cdots \mathbb{L}_U(P_n) \longrightarrow \mathbb{L}_U(P_{n-1}) \longrightarrow \cdots \longrightarrow \mathbb{L}_U(P_0) \longrightarrow \mathbb{L}_U(F) \longrightarrow 0$$

is a projective resolution $\mathbb{L}_U(F)$, since \mathbb{L}_U is exact and maps projective objects to projective objects. Moreover

$$\dim_k \mathbb{L}_U(P_n) \leq C_U \dim_k P_n \leq C_U cn^d + C_U e ,$$

so $\mathbb{L}_U(F)$ has polynomial growth. □

4.2. Proposition : *Let G be a finite group, and (B, A) be a section of G . Let k be a field, and N be a cohomological Mackey functor for B/A over k . Then the following conditions are equivalent :*

1. *The functor N has polynomial growth.*
2. *The functor $\text{Ind}_B^G \iota_{B/A}^B(N)$ has polynomial growth.*

Proof : The functor $\text{Ind}_B^G \iota_{B/A}^B$ is isomorphic to L_W , where W is the $(B/A, G)$ biset $A \setminus G$, by Lemma 3.26, so Condition 1 implies Condition 2, by Lemma 4.1.

Conversely, if $L_W(N)$ has polynomial growth, then so does $L_{W^{op}} \circ L_W(N) \cong L_{W \times_G W^{op}}(N)$, by Lemma 4.1. But $W \times_G W^{op} \cong A \setminus G / A$, as a $(B/A, B/A)$ -biset. In particular, it is the disjoint union of the identity biset $\text{Id}_{B/A} = B/A$ and some other $(B/A, B/A)$ -biset. By Assertions 4 and 5 of Proposition 3.14, it follows that the identity functor is a direct summand of $L_{W \times_G W^{op}}$. In particular N is isomorphic to a direct summand of $L_{W \times_G W^{op}}(N)$, and N has polynomial growth, by Remark 2.4. Thus Condition 2 implies Condition 1. \square

4.3. Lemma : *Let G be a finite group and M be a cohomological Mackey functor for G over R . Then for any subgroup H of G the composition*

$$M \longrightarrow \text{Ind}_H^G \text{Res}_H^G M \longrightarrow M$$

of the unit and counit morphisms of the adjoint pairs of functors $(\text{Res}_H^G, \text{Ind}_H^G)$ and $(\text{Ind}_H^G, \text{Res}_H^G)$ is equal to the multiplication by $|G : H|$.

Proof : This is because the same is true for the categories of RG -modules and RH -modules, and the induction and restriction functors between them. \square

4.4. Proposition : *Let k be a field of positive characteristic p . Let G be a finite group, and S be a subgroup of G , containing a Sylow p -subgroup of G . Then for any cohomological Mackey functor M for G over k , the following conditions are equivalent :*

1. *The functor M has polynomial growth.*
2. *The functor $\text{Res}_S^G M$ has polynomial growth.*

Proof : Condition 1 implies Condition 2, by Lemma 4.1. Conversely, if $\text{Res}_S^G M$ has polynomial growth, so does $\text{Ind}_S^G \text{Res}_S^G M$, by Lemma 4.1 again. Now Lemma 4.3 shows that M is a direct summand of $\text{Ind}_S^G \text{Res}_S^G M$, since $|G : S|$ is non zero in k . Hence M has polynomial growth, by Remark 2.4. \square

4.5. Definition : *Let k be a field. A finite group G is called a poco group over k if every finitely generated cohomological Mackey functor for G over k has polynomial growth.*

4.6. Corollary : *Let k be a field of characteristic p . Let G be a finite group,*

and S be a Sylow p -subgroup of G . Then G is a poco group over k if and only if S is a poco group over k .

Proof : Indeed, if G is a poco group, and N is a cohomological Mackey functor for S over k , then $\text{Ind}_S^G N$ has polynomial growth, and N has polynomial growth, by Proposition 4.2, applied to the section $(S, \mathbf{1})$ of G . So S is a poco group.

Conversely, if S is a poco group, and M is a cohomological Mackey functor for G over k , the $\text{Res}_S^G M$ has polynomial growth, so M has polynomial growth, by Proposition 4.4. \square

4.7. Proposition : Let k be a field, and G be a finite group. The following conditions are equivalent :

1. The group G is a poco group over k .
2. For any finitely generated kG -module V , the functor FP_V has a polynomial growth.

Proof : Obviously Condition 1 implies Condition 2. Conversely, observe that since any morphism from FP_L to FP_N , where L and N are kG -modules, is determined by a morphism of kG -modules from L to N , for any cohomological Mackey functor M for G over k , there is a short exact sequence

$$0 \longrightarrow FP_V \longrightarrow FP_L \longrightarrow FP_N \longrightarrow M \longrightarrow 0 ,$$

where L and N are permutation kG -modules, and V is the kernel of a morphism of kG -modules from L to N . Now if FP_V has polynomial growth, so does M . \square

5. Proof of Theorem 1.1

Corollary 4.6 shows that G is a poco group if and only if S is a poco group, where S is a Sylow p -subgroup of G . Now by Proposition 4.2, if (B, A) is a section of S , the factor group B/A is also a poco group.

It will be shown in Sections 9 and 10 that an elementary abelian p -group is a poco group if and only if it has rank at most 1, when $p > 2$, or at most 2, when $p = 2$. It will follow that if G is a poco group, then S has sectional rank at most 1 if $p > 2$, which implies that S is cyclic, or at most 2 if $p = 2$. In other words, Condition 1 of Theorem 1.1 implies Condition 2.

Conversely, it will be shown in Section 9 that cyclic p -groups are poco groups. So Condition 2 of Theorem 1.1 implies Condition 1, when $p > 2$. The

corresponding assertion for $p = 2$ will be stated in Section 11, completing the proof of Theorem 1.1.

6. Simple cohomological functors for p -groups

Recall (see [9] for details) that if G is a p -group, the simple cohomological Mackey functors for G over k , up to isomorphism, are in one to one correspondence with the subgroups of G , up to conjugation. The simple functor S_Q^G (also denoted by S_Q if G is clear from context) corresponding to the subgroup Q is defined by

$$\forall T \leq G, \quad S_Q^G(T) = \begin{cases} k & \text{if } T =_G Q \\ \{0\} & \text{otherwise.} \end{cases}$$

The projective cover of the functor S_Q^G is the fixed point functor $FP_{kG/Q}$. Moreover $\text{End}_{\mathbf{M}_k^c(G)}(S_Q^G) \cong k$, and S_Q^G is self dual (i.e. $(S_Q^G)^* \cong S_Q^G$).

6.1. Lemma : *Let k be a field of characteristic p , and G be a p -group. There is a short exact sequence in $\mathbf{M}_k^c(G)$*

$$0 \longrightarrow FP_{\Omega_G} \longrightarrow FP_{kG} \longrightarrow S_{\mathbf{1}}^G \longrightarrow 0 \quad ,$$

where Ω_G is the kernel of the augmentation map $\varepsilon : kG \rightarrow k$.

Proof : Consider the exact sequence of kG -modules

$$0 \longrightarrow \Omega_G \longrightarrow kG \xrightarrow{\varepsilon} k \longrightarrow 0 \quad .$$

Since fixed point functors are left exact, the inclusion $\Omega_G \subseteq kG$ yields an inclusion $FP_{\Omega_G} \subseteq FP_{kG}$. Let S denote the quotient functor. In particular $S(\mathbf{1}) \cong k$. And if Q is a non-trivial subgroup of G , then $(kG)^Q = \text{Tr}_{\mathbf{1}}^Q(kG) \subseteq \Omega_G$. Thus $(\Omega_G)^Q = (kG)^Q$, and $S(Q) = \{0\}$. Hence $S \cong S_{\mathbf{1}}^G$. \square

6.2. Theorem : [Samy Modeliar [7]] *Let k be a field of characteristic p , and G be a finite p -group. Then*

$$\text{Ext}_{\mathbf{M}_k^c(G)}^1(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G) \cong \text{Hom}_{\mathbb{Z}}\left(G/(\Phi(G)I(G)), k^+\right) \quad ,$$

where k^+ is the additive group of k , where $\Phi(G)$ is the Frattini subgroup of G , and $I(G)$ is the subgroup of G generated by elements of order 2 (so $I(G) = \mathbf{1}$ if $p > 2$).

Proof : By Lemma 6.1, there is an exact sequence

$$0 \longrightarrow k \longrightarrow \text{Hom}_{M_k^c(G)}(FP_{\Omega_G}, S_1^G) \longrightarrow \text{Ext}_{M_k^c(G)}^1(S_1^G, S_1^G) \longrightarrow 0 ,$$

hence $\text{Ext}_{M_k^c(G)}^1(S_1^G, S_1^G) \cong \text{Hom}_{M_k^c(G)}(FP_{\Omega_G}, S_1^G)$.

Now a morphism $\varphi : FP_{\Omega_G} \rightarrow S_1^G$ is entirely determined by its evaluation at the trivial group, which is a morphism of kG -modules from Ω_G to k . Conversely, a morphism of kG -modules $f : \Omega_G \rightarrow k$ is the evaluation at $\mathbf{1}$ of a morphism of Mackey functors from FP_{Ω_G} to S_1^G if and only if it maps $(\Omega_G)^Q = r_1^Q FP_{\Omega_G}(Q)$ to $r_1^Q S_1^G(Q) = \{0\}$, for any non-trivial subgroup Q of G , or equivalently, for any subgroup Q of order p of G .

In other words f is a G -invariant linear form on the space

$$\Omega_G / \sum_{\substack{Q \leq G \\ |Q|=p}} \text{Tr}_1^Q kG ,$$

i.e. a linear form on Ω_G such that

$$(6.3) \quad f(h(g-1) - (g-1)) = 0, \quad \forall g, h \in G ,$$

and

$$(6.4) \quad f(1+x+\cdots+x^{p-1}) = 0, \quad \forall x \in G, |x| = p .$$

Now Ω_G is generated as a k -vector space by the elements $d_g = g-1$, for $g \in G$, and the only relations between these generators are $d_1 = 0$. Hence a linear form f on Ω_G is determined by the values $u(g) = f(g-1)$, subject to $u(1) = 0$. Since $d_{hg} = (hd_g - d_g) + d_h + d_g$, for any $g, h \in G$, Equation 6.3 is equivalent to

$$u(gh) = u(g) + u(h), \quad \forall g, h \in G .$$

In other words u is a group homomorphism from G to k^+ (note that this implies $u(1) = 0$, as required). Equivalently u factors through a group homomorphism $G/\Phi(G) \rightarrow k^+$.

Now if x is an element of order p of G

$$1+x+\cdots+x^{p-1} = d_1 + d_x + d_{x^2} + \cdots + d_{x^{p-1}} ,$$

so Equation 2 is equivalent to

$$u(1) + u(x) + u(x^2) + \cdots + u(x^{p-1}) = 0, \quad \forall x \in G, |x| = p .$$

If u is a group homomorphism from G to k^+ , this is equivalent to

$$(0 + 1 + 2 + \cdots + (p-1))u(x) = \binom{p}{2}u(x) = 0, \quad \forall x \in G, |x| = p .$$

Now if p is odd, the integer $\binom{p}{2}$ is a multiple of p , so this condition is satisfied whenever u is a group homomorphism to k^+ . And if $p = 2$, this condition is equivalent to

$$u(x) = 0, \quad \forall x \in G, |x| = 2 .$$

This shows finally that $\text{Hom}_{\mathcal{M}_k^c(G)}(FP_{\Omega_G}, S_1^G)$ is isomorphic to the group of homomorphisms from $G/(\Phi(G)I(G))$ to k^+ , and this completes the proof. \square

6.5. Lemma : *Let G be a finite p -group, and Q be a subgroup of G . Set $B = N_G(Q)$, and denote by $A = \Phi(Q)$ the Frattini subgroup of Q . Then*

$$S_Q^G \cong \text{Ind}_B^G i_{B/A}^B(S_{Q/A}^{B/A}) \cong L_{A \setminus G}(S_{Q/A}^{B/A}) ,$$

where $A \setminus G$ is viewed as a $(B/A, G)$ -biset.

Proof : The second isomorphism follows from Lemma 3.26. As for the first one, recall (see [9]) that $S_Q^G = \text{Ind}_B^G S_Q^B$, so it suffices to show that $S_Q^G = i_{G/A}^G(S_{Q/A}^{G/A})$, when $Q \trianglelefteq G$. In this case, if $H \leq G$

$$i_{G/A}^G(S_{Q/A}^{G/A})(H) = S_{Q/A}^{G/A}(HA/A) .$$

This is equal to k if $HA = Q$, and to zero otherwise. But $HA = Q$ if and only if $H = Q$, and this completes the proof. \square

6.6. Corollary : *With the same notation,*

$$S_Q^G \cong \text{Ind}_B^G j_{B/A}^B(S_{Q/A}^{B/A}) \cong R_{G/A}(S_{Q/A}^{B/A}) .$$

Proof : This follows from Lemma 6.5, using Proposition 3.14, and the fact that S_Q^G is self dual. \square

6.7. Lemma : *Let G be a p -group, let Q be a subgroup of G , and (B, A) be a section of G . Then*

$$\rho_{B/A}^B \text{Res}_B^G(S_Q^G) \cong \bigoplus_{Q'} S_{Q'/A}^{B/A} ,$$

where Q' runs through the set of G -conjugates of Q which contain A and are contained in B , modulo B -conjugation.

Proof : Let H/A be a subgroup of B/A . Then

$$(\rho_{B/A}^B \text{Res}_B^G(S_Q^G))(H/A) = (\text{Res}_B^G(S_Q^G))(H) = S_Q^G(H) ,$$

and this is equal to k if H is conjugate to Q in G , and to $\{0\}$ otherwise. \square

7. Some cohomological functors for p -groups

7.1. Notation : Let G be a finite group. A subset \mathcal{S} of the set of subgroups of G will be called *convex* if

$$\forall H \leq K \leq L \leq G, H, L \in \mathcal{S} \Rightarrow K \in \mathcal{S} .$$

The set \mathcal{S} will be called *G -stable* if it is invariant by G -conjugation.

The following is an extension of Assertion (ii) of Corollary 15.3 of [9] :

7.2. Proposition : Let k be a field of characteristic p , and G be a finite group. The correspondence

$$M \mapsto \text{Supp}_p(M) = \{H \leq G \mid H \text{ is a } p\text{-group and } M(H) \neq \{0\}\}$$

induces a one to one correspondence between the set of isomorphism classes of subquotients of the functor FP_k and the set of G -stable convex subsets of the set of p -subgroups of G . The inverse bijection maps the G -stable convex subset \mathcal{S} of p -subgroups of G to the class of the functor $k_{\mathcal{S}}$ defined by

$$\forall H \leq G, k_{\mathcal{S}}(H) = \begin{cases} k & \text{if } H_p \in \mathcal{S} \\ \{0\} & \text{otherwise} , \end{cases}$$

where H_p denotes a Sylow p -subgroup of H .

When $H \leq K \leq G$, the restriction map r_H^K is equal to 0, except if $H_p \in \mathcal{S}$ and $K_p \in \mathcal{S}$, in which case $r_H^K = 1$. The transfer map t_H^K is multiplication by $|K : H|$ for all $H \leq K \leq G$. The conjugation map $c_{x,H}$ is always the identity map $k_{\mathcal{S}}(H) \rightarrow k_{\mathcal{S}}(xH)$.

Proof : By Corollary 15.3 of [9], the correspondences

$$F \mapsto \text{Supp}_p(F) = \{Q \mid Q \text{ is a } p\text{-subgroup of } G \text{ and } F(Q) \neq \{0\}\}$$

and

$$\mathcal{T} \mapsto k_{\mathcal{T}} = \langle FP_k(Q) \rangle_{Q \in \mathcal{T}}$$

are mutual inverse bijections between the set of subfunctors of FP_k and the set of G -stable subsets of the set $s_p(G)$ of p -subgroups of G , which are closed under taking subgroups.

By Proposition 2.4 of [9], the value of $k_{\mathcal{T}}$ at some subgroup H of G is equal to

$$k_{\mathcal{T}} = \sum_{\substack{Q \in \mathcal{T} \\ Q \leq H}} |H : Q|k .$$

This is equal to k if $H_p \in \mathcal{T}$, and to zero otherwise. The restriction maps between non zero values of $k_{\mathcal{T}}$ are equal to the identity map of k . The transfer map t_H^K is multiplication by $|K : H|$, and the conjugation maps are identity maps (possibly zero).

It follows that any subquotient M of FP_k is equal to $k_{\mathcal{T}}/k_{\mathcal{T}'}$, where \mathcal{T} and \mathcal{T}' are G -stable subsets of $s_p(G)$, which are closed under taking subgroups, and such that $\mathcal{T}' \subseteq \mathcal{T}$. With the notation of Proposition 7.2, this means that $M \cong k_{\mathcal{S}}$, where $\mathcal{S} = \mathcal{T} - \mathcal{T}'$ is a G -stable convex subset of $s_p(G)$, equal to $\text{Supp}_p(M)$.

Conversely, let \mathcal{S} be a G -stable convex subset of $s_p(G)$. Set

$$\mathcal{T} = \{Q \in s_p(G) \mid \exists S \in \mathcal{S}, Q \leq S\}$$

$$\mathcal{T}' = \mathcal{T} - \{Q \in s_p(G) \mid \exists S \in \mathcal{S}, S \leq Q\} .$$

Then \mathcal{T} and \mathcal{T}' are G -stable, and closed under taking subgroups. Moreover

$$\mathcal{T} - \mathcal{T}' = \{Q \in s_p(G) \mid \exists S, S' \in \mathcal{S}, S' \leq Q \leq S\} .$$

Thus $\mathcal{T} - \mathcal{T}' = \mathcal{S}$, and $k_{\mathcal{S}} = k_{\mathcal{T}}/k_{\mathcal{T}'}$ is a subquotient of FP_k .

These correspondences $M \mapsto \text{Supp}_p(M)$ and $\mathcal{S} \mapsto k_{\mathcal{S}}$ are clearly mutual inverse bijections between the set of isomorphism classes of subquotients of FP_k and the set of G -stable convex subsets of $s_p(G)$. \square

7.3. Corollary : *Let G be a finite p -group, and $Q \leq R$ be subgroups of G . Then there exists a unique object $\Sigma_{Q,R}^G$ of $\mathbf{M}_k^c(G)$ such that*

$$\forall H \leq G, \Sigma_{Q,R}^G(H) = \begin{cases} k & \text{if } Q \leq_G H \leq_G R \\ \{0\} & \text{otherwise} . \end{cases}$$

and such that

$$\forall H < K \leq G, \quad t_H^K = 0, \quad r_H^K = \begin{cases} 1 & \text{if } Q \leq_G H \text{ and } K \leq_G R \\ 0 & \text{otherwise} \end{cases} .$$

Moreover the socle of $\Sigma_{Q,R}^G$ is isomorphic to S_Q^G , and the head of $\Sigma_{Q,R}^G$ is isomorphic to S_R^G .

Proof : Let $\mathcal{S} = \{S \leq G \mid Q \leq_G S \leq_G R\}$. Then \mathcal{S} is a G -stable convex set of subgroups of G , and the corresponding functor $k_{\mathcal{S}}$ fulfills the required conditions, so the functor $\Sigma_{Q,R}^G$ exists. The uniqueness follows from the fact that the values of $\Sigma_{Q,R}^G$ are given, as well as the transfer and restriction maps. The non zero conjugation maps $c_{x,Q}$ are determined by elements of $H^1(G, \text{Ind}_Q^G k^\times) \cong \text{Hom}_{\mathbb{Z}}(Q, k^\times)$, which is equal to zero since G is a p -group. So the conjugation maps are all identity maps (possibly zero).

Now a morphism φ from $\Sigma_{Q,R}^G$ to some functor N is entirely determined by its evaluation $\varphi_R : \Sigma_{Q,R}^G(R) = k \rightarrow N(R)$: indeed, if $S \leq G$ and $\Sigma_{Q,R}^G(S) \neq \{0\}$, then $S^g \leq R$ for some $g \in G$, and then the map $\psi = c_{g,S^g} r_{S^g}^Q$ is an isomorphism from $\Sigma_{Q,R}^G(R)$ to $\Sigma_{Q,R}^G(S)$, such that $\varphi_Q = c_{g,S^g} r_{S^g}^Q \varphi_R \psi^{-1}$. Hence the only simple quotient of $\Sigma_{Q,R}^G$ is S_R^G , with multiplicity one.

By a similar argument, a morphism from N to $\Sigma_{Q,R}^G$ is determined by its value at Q , so the only simple subfunctor of $\Sigma_{Q,R}^G$ is S_Q^G , with multiplicity one. \square

7.4. Notation : Let G be a finite p -group. If $Q < R$ are subgroups of G such that $|R : Q| = p$, the functor $\Sigma_{Q,R}^G$ will be denoted by $\begin{pmatrix} S_R^G \\ S_Q^G \end{pmatrix}$. The dual functor $(\Sigma_{Q,R}^G)^*$ will be denoted by $\begin{pmatrix} S_Q^G \\ S_R^G \end{pmatrix}$. With this notation, there are non split exact sequences in $\mathbf{M}_k^c(G)$

$$(7.5) \quad D_{Q,R} : 0 \longrightarrow S_Q^G \longrightarrow \begin{pmatrix} S_R^G \\ S_Q^G \end{pmatrix} \longrightarrow S_R^G \longrightarrow 0 .$$

$$(7.6) \quad D_{Q,R}^* : 0 \longrightarrow S_R^G \longrightarrow \begin{pmatrix} S_Q^G \\ S_R^G \end{pmatrix} \longrightarrow S_Q^G \longrightarrow 0 .$$

In particular, if X is a subgroup of order p of G , set $D_X = D_{\mathbf{1},X}$ and $D_X^* = D_{\mathbf{1},X}^*$, and denote by γ_X^G (or γ_X if G is clear from the context) the

element of $\text{Ext}_{\mathbb{M}_k^c(G)}^2(S_1^G, S_1^G)$ represented by the exact sequence

$$\Gamma_X : 0 \longrightarrow S_1^G \longrightarrow \begin{pmatrix} S_X^G \\ S_1^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_1^G \\ S_X^G \end{pmatrix} \longrightarrow S_1^G \longrightarrow 0 ,$$

obtained by splicing the sequences D_X and D_X^* .

8. Extensions of simple functors for p -groups

8.1. Lemma : *Let G be a p -group, and Z be a central subgroup of order p of G . Then there are isomorphisms*

$$i_{G/Z}^G(S_1^{G/Z}) \cong \begin{pmatrix} S_Z^G \\ S_1^G \end{pmatrix} , \quad j_{G/Z}^G(S_1^{G/Z}) \cong \begin{pmatrix} S_1^G \\ S_Z^G \end{pmatrix}$$

in $\mathbb{M}_k^c(G)$. In particular, there are non-split exact sequences in $\mathbb{M}_k^c(G)$

$$D_Z : 0 \longrightarrow S_1^G \longrightarrow i_{G/Z}^G(S_1^{G/Z}) \longrightarrow S_Z^G \longrightarrow 0 .$$

$$D_Z^* : 0 \longrightarrow S_Z^G \longrightarrow j_{G/Z}^G(S_1^{G/Z}) \longrightarrow S_1^G \longrightarrow 0 .$$

Proof : Let H be a subgroup of G . Then $i_{G/Z}^G(S_1^{G/Z})(H) = S_1^{G/Z}(HZ/Z)$ is equal to k if $HZ = Z$, i.e. if $H \leq Z$, and to zero otherwise. The conjugation maps are all identity maps (possibly zero), the restriction map r_1^Z is the identity map of k , and the transfer map t_1^Z is zero. Thus $i_{G/Z}^G(S_1^{G/Z}) \cong \begin{pmatrix} S_Z^G \\ S_1^G \end{pmatrix}$. The other isomorphism follows by duality, and the two exact sequences are special cases of Sequences 7.5 and 7.6. \square

8.2. Corollary : *For any $n \in \mathbb{N}$, composition by D_Z induces a group isomorphism*

$$\text{Ext}_{\mathbb{M}_k^c(G)}^{n-1}(S_1^G, S_1^G) \cong \text{Ext}_{\mathbb{M}_k^c(G)}^n(S_Z^G, S_1^G)$$

Proof : Indeed, for $n \in \mathbb{N}$,

$$\text{Ext}_{\mathbb{M}_k^c(G)}^n\left(\begin{pmatrix} S_Z^G \\ S_1^G \end{pmatrix}, S_1^G\right) \cong \text{Ext}_{\mathbb{M}_k^c(G/Z)}^n(S_1^{G/Z}, \rho_{G/Z}^G(S_1^G)) = \{0\}$$

by Proposition 3.16 and Lemma 6.7. \square

8.3. Corollary : *Let k be a field of characteristic p , and G be a p -group. If X is a subgroup of order p of G , then*

$$\mathrm{Ext}_{\mathbf{M}_k^c(G)}^1(S_X^G, S_{\mathbf{1}}^G) \cong k \cong \mathrm{Ext}_{\mathbf{M}_k^c(G)}^1(S_{\mathbf{1}}^G, S_X^G) .$$

Proof : Indeed $N_G(X)$ is equal to the centralizer C of X , since $|X| = p$, and $S_X^G = \mathrm{Ind}_C^G S_X^C$. Thus for any $n \in \mathbb{N}$

$$\begin{aligned} \mathrm{Ext}_{\mathbf{M}_k^c(G)}^n(S_X^G, S_{\mathbf{1}}^G) &\cong \mathrm{Ext}_{\mathbf{M}_k^c(C)}^n(S_X^C, \mathrm{Res}_C^G S_{\mathbf{1}}^G) \\ &\cong \mathrm{Ext}_{\mathbf{M}_k^c(C)}^n(S_X^C, S_{\mathbf{1}}^C) \\ &\cong \mathrm{Ext}_{\mathbf{M}_k^c(C)}^{n-1}(S_{\mathbf{1}}^C, S_X^C) \end{aligned}$$

by Corollary 8.2, since X is a central subgroup of order p of C . Corollary 8.3 follows, taking $n = 1$.

The isomorphism $k \cong \mathrm{Ext}_{\mathbf{M}_k^c(G)}^1(S_{\mathbf{1}}^G, S_X^G)$ follows from Remark 3.11, since the simple functors are self dual. \square

8.4. Notation : *If N is a normal subgroup of a group G , contained in the subgroup H of G , denote by $K_H(N)$ the set of complements of N in H , i.e. the set of subgroups X of G such that $NX = H$ and $N \cap X = \mathbf{1}$.*

Denote by $[N_G(H) \backslash K_H(N)]$ a set of representatives of $N_G(H)$ -conjugacy classes of subgroups in $K_H(N)$.

The following proposition is a generalization of Lemma 8.1 :

8.5. Proposition : *Let G be a p -group, let Q be a subgroup of G , and Z be a central subgroup of order p of G contained in Q . Then there are non-split exact sequences in $\mathbf{M}_k^c(G)$*

$$\begin{aligned} 0 \longrightarrow \bigoplus_{X \in [N_G(Q) \backslash K_Q(Z)]} S_X^G \longrightarrow i_{G/Z}^G(S_{Q/Z}^{G/Z}) \longrightarrow S_Q^G \longrightarrow 0 , \\ 0 \longrightarrow S_Q^G \longrightarrow j_{G/Z}^G(S_{Q/Z}^{G/Z}) \longrightarrow \bigoplus_{X \in [N_G(Q) \backslash K_Q(Z)]} S_X^G \longrightarrow 0 . \end{aligned}$$

Proof : Since the simple functors are self dual, the second sequence is obtained from the first by applying duality, by definition of the functor $j_{G/Z}^G$

and Assertion 6 of Proposition 3.14. So it suffices to prove the existence of the first one.

Denote by I the functor $\iota_{G/Z}^G(S_{Q/Z}^{G/Z})$, and by R its radical. If H is a subgroup of G , then $I(H) = S_{Q/Z}^{G/Z}(HZ/Z)$ is equal to k if $HZ =_G Q$, and to zero otherwise. Moreover $HZ = Q$ if and only if $H = Q$, or $H \in K_Q(Z)$, since Z has order p . Equation 3.8 shows that the composition factors of I are the functors S_Q^G and the functors S_X^G , for $X \in [N_G(Q) \setminus K_Q(Z)]$, each with multiplicity 1.

Moreover, if Y is any subgroup of G , consider

$$\mathcal{H}_Y = \text{Hom}_{\mathbb{M}_k^c(G)}(I, S_Y^G) \cong \text{Hom}_{\mathbb{M}_k^c(G/Z)}(S_{Q/Z}^{G/Z}, \rho_{G/Z}^G(S_Y^G)) .$$

Now $\rho_{G/Z}^G(S_Y^G) = \{0\}$ if $Z \not\leq Y$, and $\rho_{G/Z}^G(S_Y^G) = S_{Y/Z}^{G/Z}$, by Lemma 6.7. Thus $\mathcal{H}_Y = \{0\}$ unless $Y =_G Q$, and $\mathcal{H}_Y \cong k$ in this case.

This means that I/R is simple, isomorphic to S_Q^G . Moreover, if H is a subgroup of G , then $R(H) = \{0\}$ except if $HZ =_G Q$ and $H \cap Z = \mathbf{1}$, i.e. if H is conjugate to some element of $K_Q(Z)$ in G , and $R(H) \cong k$ in this case. Hence R is isomorphic to the direct sum of the simple functors S_H^G , where H runs in a set of representatives of G -conjugacy classes of such subgroups, i.e. equivalently $H \in [N_G(Q) \setminus K_Q(Z)]$. \square

8.6. Theorem : *Let k be a field of characteristic $p > 0$, let G be a finite p -group. Let Q and R be subgroups of G , and let Z be a subgroup of order p of $Q \cap R \cap Z(G)$. Set $\bar{G} = G/Z$, $\bar{Q} = Q/Z$, $\bar{R} = R/Z$, and $\mathcal{K} = [N_G(Q) \setminus K_Q(Z)]$.*

Then, for any $n \in \mathbb{N}$, there are short exact sequences of vector spaces

$$0 \rightarrow \bigoplus_{X \in \mathcal{K}} \text{Ext}_{\mathbb{M}_k^c(G)}^{n-1}(S_X^G, S_R^G) \rightarrow \text{Ext}_{\mathbb{M}_k^c(G)}^n(S_Q^G, S_R^G) \xrightarrow{\pi_n} \text{Ext}_{\mathbb{M}_k^c(\bar{G})}^n(S_{\bar{Q}}^{\bar{G}}, S_{\bar{R}}^{\bar{G}}) \rightarrow 0 ,$$

where the map π_n is induced by $\rho_{G/Z}^G$.

Proof : Applying $\text{Hom}_{\mathbb{M}_k^c(G)}(-, S_R^G)$ to the first exact sequence of Proposition 8.5 gives a long exact sequence

$$\begin{array}{c} \cdots \longrightarrow \bigoplus_{X \in \mathcal{K}} \text{Ext}_{\mathbb{M}_k^c(G)}^{n-1}(S_X^G, S_R^G) \longrightarrow \text{Ext}_{\mathbb{M}_k^c(G)}^n(S_Q^G, S_R^G) \xrightarrow{\pi_n} \text{Ext}_{\mathbb{M}_k^c(\bar{G})}^n(S_{\bar{Q}}^{\bar{G}}, S_{\bar{R}}^{\bar{G}}) \longrightarrow \cdots \\ \searrow \hspace{10em} \swarrow \\ \bigoplus_{X \in \mathcal{K}} \text{Ext}_{\mathbb{M}_k^c(G)}^n(S_X^G, S_R^G) \longrightarrow \text{Ext}_{\mathbb{M}_k^c(G)}^{n+1}(S_Q^G, S_R^G) \xrightarrow{\pi_{n+1}} \cdots \end{array}$$

where the image of an extension $u \in \text{Ext}_{\mathbb{M}_k^c(G)}^n(S_Q^G, S_R^G)$ under the map π_n is

obtained by first taking the image under the map

$$\varphi : i_{G/Z}^G(S_{Q/Z}^{G/Z}) \rightarrow S_Q^G \in \text{Ext}_{\mathcal{M}_k^c(G)}^0(i_{G/Z}^G(S_{Q/Z}^{G/Z}), S_Q^G) ,$$

and then using the adjunction isomorphisms

$$\text{Ext}_{\mathcal{M}_k^c(G)}^n(i_{G/Z}^G(S_{Q/Z}^{G/Z}), S_R^G) \cong \text{Ext}_{\mathcal{M}_k^c(\bar{G})}^n(S_{\bar{Q}}^{\bar{G}}, \rho_{G/Z}^G(S_R^G)) \cong \text{Ext}_{\mathcal{M}_k^c(\bar{G})}^n(S_{\bar{Q}}^{\bar{G}}, S_{\bar{R}}^{\bar{G}}) .$$

In other words, with the notation of Remark 3.18

$$\pi_n(u) = \alpha_n(u \circ \varphi) = \rho_{G/Z}^G(u) \circ \alpha_0(\varphi) = \rho_{G/Z}^G(u) ,$$

since the map $\alpha_0(\varphi) : S_{Q/Z}^{G/Z} \rightarrow \rho_{G/Z}^G(S_Q^G) \cong S_{Q/Z}^{G/Z}$ obtained from φ by adjunction is the identity map. This shows that π_n is induced by $\rho_{G/Z}^G$.

Now the inflation functor $\text{Inf}_{G/Z}^G : \text{Fun}_k(G/Z) \rightarrow \text{Fun}_k(G)$ is an exact functor such that $\rho_{G/Z}^G \circ \text{Inf}_{G/Z}^G$ is isomorphic to the identity functor. Since $S_Q^Z \cong \text{Inf}_{G/Z}^G(S_{Q/Z}^{G/Z})$ and $S_R^Z \cong \text{Inf}_{G/Z}^G(S_{R/Z}^{G/Z})$, this inflation functor induces a map

$$\sigma_n : \text{Ext}_{\mathcal{M}_k^c(\bar{G})}^n(S_{\bar{Q}}^{\bar{G}}, S_{\bar{R}}^{\bar{G}}) \rightarrow \text{Ext}_{\mathcal{M}_k^c(G)}^n(S_Q^G, S_R^G)$$

such that $\pi_n \circ \sigma_n = \text{Id}$. In particular π_n is surjective, so the long exact sequence above splits as a series of short exact sequences. \square

8.7. Proposition : *Let k be a field of characteristic $p > 0$, let G be a finite p -group, and H be a subgroup of index p in G . Set $I = \text{Ind}_H^G S_1^H$, and let R and S denote respectively the radical and the socle of I as an object of $\mathcal{M}_k^c(G)$. Then $I \supset R \supseteq S \supset \{0\}$. Moreover $I/R \cong S \cong S_1^G$, and*

$$R/S \cong L \oplus \bigoplus_{X \in [G \setminus K_G(H)]} S_X^G ,$$

where L is a functor all of whose composition factors are isomorphic to S_1^G , with multiplicity $p - 2$.

Proof : Let Q be a subgroup of G . Then by Proposition 3.16

$$\text{Hom}_{\mathcal{M}_k^c(G)}(I, S_Q^G) \cong \text{Hom}_{\mathcal{M}_k^c(H)}(S_1^H, \text{Res}_H^G S_Q^G) .$$

Moreover, by Lemma 6.7, the functor $\text{Res}_H^G S_Q^G$ is isomorphic to the direct sum of functors $S_{Q'}^H$, where Q' runs through a set of H -conjugacy classes of G -conjugates of Q which are contained in H .

Thus $\text{Hom}_{\mathcal{M}_k^c(G)}(I, S_Q^G) = \{0\}$ if $Q \neq 1$, and $\text{Hom}_{\mathcal{M}_k^c(G)}(I, S_1^G) \cong k$. This shows that $I/R \cong S_1^G$.

A similar argument shows that $\text{Hom}_{\mathbb{M}_k^e(G)}(S_Q^G, I)$ is equal to $\{0\}$ if $Q \neq \mathbf{1}$, and isomorphic to k if $Q = \mathbf{1}$. Thus $S \cong S_{\mathbf{1}}^G$.

Let X be a subgroup of G . Then

$$I(X) \cong \bigoplus_{x \in G/HX} S_{\mathbf{1}}^H(H \cap {}^x X) .$$

This is isomorphic to k^p if $X = \mathbf{1}$, to k if X is non trivial and $H \cap X = \mathbf{1}$ (which implies that $HX = G$), and to $\{0\}$ otherwise. In particular $I \neq S$ (since $I(\mathbf{1}) = k^p \neq k = S(\mathbf{1})$), so $S \subseteq R$.

Moreover, the short exact sequence

$$0 \longrightarrow R \longrightarrow I \longrightarrow S_{\mathbf{1}}^G \longrightarrow 0$$

shows that $\text{Ext}_{\mathbb{M}_k^e(G)}^n(S_X^G, R) \cong \text{Ext}_{\mathbb{M}_k^e(G)}^{n-1}(S_X^G, S_{\mathbf{1}}^G)$ for any $n \in \mathbb{N}$, since $\text{Ext}_{\mathbb{M}_k^e(G)}^n(S_X^G, I) \cong \text{Ext}_{\mathbb{M}_k^e(H)}^n(\text{Res}_H^G S_X^G, S_{\mathbf{1}}^H)$, and since $\text{Res}_H^G S_X^G = \{0\}$. In particular $\text{Ext}_{\mathbb{M}_k^e(G)}^1(S_X^G, R) \cong \text{Hom}_{\mathbb{M}_k^e(G)}(S_X^G, S_{\mathbf{1}}^G) = \{0\}$.

Now the short exact sequence

$$0 \longrightarrow S_{\mathbf{1}}^G \longrightarrow R \longrightarrow R/S \longrightarrow 0$$

yields a long exact sequence of $\text{Ext}_{\mathbb{M}_k^e(G)}(S_X^G, -)$ groups starting with

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \text{Hom}(S_X, R/S) \longrightarrow \text{Ext}^1(S_X, S_{\mathbf{1}}) \longrightarrow \text{Ext}^1(S_X, R) = \{0\} .$$

It follows that $\text{Hom}(S_X, R/S) \cong k$, by Corollary 8.3. Moreover I is self dual, since $S_{\mathbf{1}}^H$ is self dual and Ind_H^G commutes with duality, by Proposition 3.14. Thus

$$\text{Hom}(R/S, S_X) \cong \text{Hom}(S_X, R/S) \cong k ,$$

by Remark 3.11, since S_X is also self dual.

In particular, there exists an injection $i : S_X \longrightarrow R/S$ and a surjection $s : R/S \longrightarrow S_X$. If $s \circ i = 0$, then $(R/S)(X)$ has dimension at least equal to 2. But $(R/S)(X) \cong I(X) \cong k$, so $s \circ i \neq 0$. Hence $s \circ i$ is invertible, and S_X is a direct summand of R/S .

It follows that $\bigoplus_{X \in [G \setminus K_G(H)]} S_X^G$ is also a direct summand of R/S , for it is a direct sum of non isomorphic simple summands of R/S . Hence there exists a subfunctor L of R/S such that

$$R/S \cong L \oplus \bigoplus_{X \in [G \setminus K_G(H)]} S_X^G .$$

Now if Q is a subgroup of G , then $(R/S)(Q)$ is equal to $\{0\}$, unless $Q = \mathbf{1}$ or Q is a complement of H in G , in which case $(R/S)(Q) \cong k$. It follows that $L(Q)$ is equal to $\{0\}$, except if $Q = \mathbf{1}$ (and $L(\mathbf{1})$ has dimension $p - 2$ over k). This completes the proof. \square

9. The case of cyclic p -groups

9.1. It has been shown by M. Samy Modeliar ([7]) that cyclic p -groups are poco groups over a field k of characteristic p . This result relies on the construction of periodic projective resolutions for the fixed points functors, which can be seen as follows : if G is cyclic of order p^m , then the group algebra kG is isomorphic to a truncated polynomial algebra $A = k[X]/(X^{p^m})$, via the map sending $X \in A$ to $g - 1 \in kG$, where g is some chosen generator of G .

The indecomposable A -modules are the modules $E_d = k[X]/(X^d)$, for $1 \leq d \leq p^m$, so E_d , where the chosen generator g of G acts by multiplication by $1 + X$, is the unique indecomposable kG -module of dimension d , up to isomorphism. The indecomposable permutation kG -modules have dimension equal to a power of p , so they are the modules E_{p^j} , for $0 \leq j \leq m$.

This means that the functors $FP_{E_{p^j}}$ are the projective indecomposable objects, in $M_k^c(G)$, and they are their own projective resolution.

Now if $1 \leq d \leq p^m$ and d is not a power of p , there exists a unique integer h in $\{0, \dots, m-1\}$ such that $p^h < d < p^{h+1}$. Let

$$e_d : E_{p^h} \oplus E_{p^{h+1}} \longrightarrow E_d$$

denote the morphism of A -modules induced by the map

$$(Q, R) \in k[X] \times k[X] \mapsto X^{d-p^h}Q + R \in E_d .$$

This map is well defined, and surjective, and it is a morphism of kG -modules.

The unique subgroup G_l of G of index p^l , where $0 \leq l \leq m$, is the subgroup generated by g^{p^l} . If W is a kG -module, then the subspace W^{G_l} of W on which G_l acts trivially is equal to

$$W^{G_l} = \{w \in W \mid g^{p^l}w = w\} = \{w \in W \mid (g-1)^{p^l}w = 0\} .$$

So viewing W as an A -module, the space W^{G_l} is equal to the kernel of X^{p^l} on W . In particular, for any $j \in \{1, \dots, m\}$, the space $(E_j)^{G_l}$ is equal to $k[X]X^r/(X^j)$, where $r = \max(j - p^l, 0)$. This shows that

$$e_d((E_{p^h} \oplus E_{p^{h+1}})^{G_l}) = (k[X]X^{d-p^h+\max(0, p^h-p^l)} + k[X]X^{\max(0, p^{h+1}-p^l)})/(X^d) .$$

If $l \leq h$, this is equal to $k[X]X^{d-p^l}/(X^d)$, and if $l > h$, this is equal to $k[X]/(X^d)$. In other words

$$e_d((E_{p^h} \oplus E_{p^{h+1}})^{G_l}) = k[X]X^{\max(0, d-p^l)}/(X^d) = (E_d)^{G_l} .$$

It follows that the restriction of the map e_d to the spaces of fixed points by any subgroup of G is surjective : in the terminology of Samy Modeliar ([7]), the map e_d is *supersurjective*. Equivalently, it induces a surjection

$$FP_{e_d} : FP_{E_{p^h}} \oplus FP_{E_{p^{h+1}}} \longrightarrow FP_{E_d}$$

in $\mathbf{M}_k^c(G)$.

Now the kernel of e_d consists of the images in $E_{p^h} \oplus E_{p^{h+1}}$ of the pairs $(Q, R) \in k[X] \times k[X]$ such that $X^{d-p^h}Q + R$ is a multiple of X^d , i.e. of the pairs $(-T, X^{d-p^h}T)$, for $T \in k[X]$. Hence the map

$$T \in E_{p^h+p^{h+1}-d} \mapsto (-T, X^{d-p^h}T) \in \text{Ker } e_d$$

is well defined (note that $p^h < p^h + p^{h+1} - d < p^{h+1}$), and it is an isomorphism of A -modules. This yields a short exact sequence

$$0 \longrightarrow E_{p^h+p^{h+1}-d} \longrightarrow E_{p^h} \oplus E_{p^{h+1}} \xrightarrow{e_d} E_d \longrightarrow 0$$

of kG -modules, leading to a short exact sequence

$$0 \longrightarrow FP_{E_{p^h+p^{h+1}-d}} \longrightarrow FP_{E_{p^h}} \oplus FP_{E_{p^{h+1}}} \xrightarrow{FP_{e_d}} FP_{E_d} \longrightarrow 0$$

in $\mathbf{M}_k^c(G)$. This shows in particular that FP_{E_d} has a projective resolution in $\mathbf{M}_k^c(G)$, which is periodic of period 2, of the form

$$\cdots \xrightarrow{FP_{\alpha_d}} FP_{E_{p^h} \oplus E_{p^{h+1}}} \xrightarrow{FP_{\beta_d}} FP_{E_{p^h} \oplus E_{p^{h+1}}} \xrightarrow{FP_{\alpha_d}} FP_{E_{p^h} \oplus E_{p^{h+1}}} \xrightarrow{FP_{e_d}} FP_{E_d} \longrightarrow 0$$

where α_d and β_d are the endomorphisms of $E_{p^h} \oplus E_{p^{h+1}}$ given by

$$\begin{aligned} \alpha_d(Q, R) &= (-X^{p^{h+1}-d}Q - R, X^{p^{h+1}-p^h}Q + X^{d-p^h}R) \\ \beta_d(Q, R) &= (-X^{d-p^h}Q - R, X^{p^{h+1}-p^h}Q + X^{p^{h+1}-d}R) . \end{aligned}$$

Hence all the functors FP_V , where V is an indecomposable kG -module, have a periodic resolution. In particular G is a poco group, by Proposition 4.7.

9.2. If $p^m \geq 3$, this applies to the case $d = p^m - 1$ (which is not a power of p), and $h = m - 1$, consequently. In this case $E_d \cong \Omega_G$, and $E_{p^{h+1}} \cong kG$. Splicing the above resolution with the short exact sequence

$$0 \longrightarrow FP_{E_{p^{m-1}}} \longrightarrow FP_{E_{p^m}} \longrightarrow S_1^G \longrightarrow 0$$

of Lemma 6.1 gives a projective resolution of S_1^G of the form

$$\cdots \xrightarrow{FP_{\beta}} FP_{E_{p^{m-1}} \oplus E_{p^m}} \xrightarrow{FP_{\alpha}} FP_{E_{p^{m-1}} \oplus E_{p^m}} \xrightarrow{FP_{\gamma}} FP_{E_{p^m}} \longrightarrow S_1^G \longrightarrow 0 ,$$

where α and β are the above maps α_d and β_d , for $d = p^m - 1$ (and $h = m - 1$), and $\gamma : E_{p^{m-1}} \oplus E_{p^m} \rightarrow E_{p^m}$ is the map defined by

$$\gamma(Q, R) = X^{p^m - p^{m-1}}Q + XR .$$

Now $FP_{E_{p^{m-1}} \oplus E_{p^m}} \cong FP_{E_{p^{m-1}}} \oplus FP_{E_{p^m}}$. Moreover, by Equation 3.8,

$$\mathrm{Hom}_{\mathbf{M}_k^c(G)}(FP_{E_{p^{m-1}}}, S_1^G) \cong S_1^G(G_{p^{m-1}}) = \{0\} ,$$

and

$$\mathrm{Hom}_{\mathbf{M}_k^c(G)}(FP_{E_{p^m}}, S_1^G) \cong S_1^G(G_{p^m}) = S_1^G(\mathbf{1}) = k .$$

More precisely, a morphism $FP_{E_{p^m}} \rightarrow S_1^G$ is determined by its evaluation at the trivial subgroup, which is a scalar multiple of the augmentation morphism ε .

It follows easily that the groups $\mathrm{Ext}_{\mathbf{M}_k^c(G)}^n(S_1^G, S_1^G)$ are the cohomology groups of the complex whose terms are all isomorphic to k , with zero differentials. Thus $\mathrm{Ext}_{\mathbf{M}_k^c(G)}^n(S_1^G, S_1^G) \cong k$, for $n \in \mathbb{N}$.

9.3. Proposition : *Let k be a field of characteristic p , and G be a cyclic p -group. Then :*

$$\forall n \in \mathbb{N} - \{0\}, \quad \mathrm{Ext}_{\mathbf{M}_k^c(G)}^n(S_1^G, S_1^G) \cong \begin{cases} k & \text{if } |G| \geq 3 \\ \{0\} & \text{if } |G| \leq 2 \end{cases} .$$

Proof : The case $|G| \geq 3$ follows from the above discussion. The case $|G| \leq 2$ follows essentially from the fact that in this case, any kG -module is a permutation module : if G is trivial, there is nothing to prove. And if G has order 2, then S_1^G has a projective resolution

$$0 \longrightarrow FP_k \longrightarrow FP_{kG} \longrightarrow S_1^G \longrightarrow 0 ,$$

and moreover $\mathrm{Hom}_{\mathbf{M}_k^c(G)}(FP_k, S_1^G) = \{0\}$, by Equation 3.8. \square

10. The case of elementary abelian p -groups

10.1. Proposition : *Let k be a field of characteristic p , and G be an elementary abelian p -group. Let H be a subgroup of index p in G , and $T \in K_G(H)$. Then the functor $I = \mathrm{Ind}_H^G S_1^H$ of $\mathbf{M}_k^c(G)$ has a subfunctor J*

where $E_G(n) = \text{Ext}_{\mathbf{M}_k^c(G)}^n(S_1^G, S_1^G)$, $E_H(n) = \text{Ext}_{\mathbf{M}_k^c(H)}^n(S_1^H, S_1^H)$, $L(n) = \text{Ext}_{\mathbf{M}_k^c(G)}^n(L, S_1^G)$, and $\mathcal{X} = K_G(H) - \{T\}$.

Proof : There is a short exact sequence

$$0 \longrightarrow J \longrightarrow I \longrightarrow I/J \longrightarrow 0$$

in $\mathbf{M}_k^c(G)$. Since $\text{Ext}_{\mathbf{M}_k^c(G)}^n(J, S_1^G) \cong \text{Ext}_{\mathbf{M}_k^c(G/T)}^n(S_1^{G/T}, \rho_{G/T}^G(S_1^G)) = \{0\}$ for any $n \in \mathbb{N}$, by Proposition 3.16 and Lemma 6.7, and since $\text{Res}_H^G S_1^G = S_1^H$, it follows that

$$\text{Ext}_{\mathbf{M}_k^c(G)}^n(I/J, S_1^G) \cong \text{Ext}_{\mathbf{M}_k^c(G)}^n(I, S_1^G) \cong \text{Ext}_{\mathbf{M}_k^c(H)}^n(S_1^H, S_1^H) = E_H(n) .$$

Now by Proposition 10.1, there is a short exact sequence

$$0 \longrightarrow R/J \longrightarrow I/J \longrightarrow S_1^G \longrightarrow 0 ,$$

and $R/J \cong L \oplus \bigoplus_{X \in K_G(H) - \{T\}} S_X^G$. Applying the functor $\text{Hom}_{\mathbf{M}_k^c(G)}(-, S_1^G)$ gives the long exact sequence of extension groups in $\mathbf{M}_k^c(G)$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(S_1^G, S_1^G) & \longrightarrow & \text{Hom}(I/J, S_1^G) & \longrightarrow & \text{Hom}(R/J, S_1^G) \\ & & \longleftarrow & & \longleftarrow & & \longleftarrow \\ & & \text{Ext}^1(S_1^G, S_1^G) & \longrightarrow & \text{Ext}^1(I/J, S_1^G) & \longrightarrow & \text{Ext}^1(R/J, S_1^G) \\ & & \longleftarrow & & \longleftarrow & & \longleftarrow \\ & & \dots & & \dots & & \dots \\ & & \longleftarrow & & \longleftarrow & & \longleftarrow \\ & & \text{Ext}^n(S_1^G, S_1^G) & \longrightarrow & \text{Ext}^n(I/J, S_1^G) & \longrightarrow & \text{Ext}^n(R/J, S_1^G) \\ & & \longleftarrow & & \longleftarrow & & \longleftarrow \\ & & \text{Ext}^{n+1}(S_1^G, S_1^G) & \longrightarrow & \text{Ext}^{n+1}(I/J, S_1^G) & \longrightarrow & \text{Ext}^{n+1}(R/J, S_1^G) \dots \end{array}$$

Now

$$\begin{aligned} \text{Ext}^n(R/J, S_1^G) &\cong \text{Ext}^n(L, S_1^G) \oplus \bigoplus_{X \in K_G(H) - \{T\}} \text{Ext}^n(S_X^G, S_1^G) \\ &\cong \text{Ext}^n(L, S_1^G) \oplus \bigoplus_{X \in K_G(H) - \{T\}} \text{Ext}^{n-1}(S_1^G, S_1^G) \\ &= L(n) \oplus \bigoplus_{X \in \mathcal{X}} E_G(n-1) , \end{aligned}$$

by Corollary 8.2. This completes the proof. \square

10.4. The case $p > 2$.

10.5. Theorem : *Let k be a field of odd characteristic p . Let G be an elementary abelian p -group of rank 2. Then the simple functor S_1^G of $M_k^c(G)$ has exponential growth.*

Proof : Let H be a subgroup of index p in G , and consider the long exact sequence of Corollary 10.3. The set $K_G(H)$ consists of p subgroups of order p in H , so \mathcal{X} has cardinality $p - 1$. Moreover

$$E_H(n) = \text{Ext}_{M_k^c(H)}^n(S_1^H, S_1^H) \cong k \ ,$$

by Proposition 9.3, since H is cyclic of order $p \geq 3$.

This yields a long exact sequence of the form

$$\cdots \rightarrow k \rightarrow L(n) \oplus E_G(n-1)^{p-1} \rightarrow E_G(n+1) \rightarrow k \rightarrow L(n+1) \oplus E_G(n)^{p-1} \rightarrow \cdots \ .$$

In particular, by Lemma 2.5, this shows that

$$\dim_k (L(n) \oplus E_G(n-1)^{p-1}) \leq 1 + \dim_k E_G(n+1) \ ,$$

thus $\dim_k E_G(n+1) \geq (p-1) \dim_k E_G(n-1) - 1$, for any $n \geq 1$. Hence, by induction on n ,

$$\dim_k E_G(2n) - \frac{1}{p-2} \geq (p-1)^n \left(\dim_k E_G(0) - \frac{1}{p-2} \right) = \frac{(p-1)^n (p-3)}{p-2} \ ,$$

$$\dim_k E_G(2n+1) - \frac{1}{p-2} \geq (p-1)^n \left(\dim_k E_G(1) - \frac{1}{p-2} \right) = \frac{(p-1)^n (2p-5)}{p-2} \ ,$$

since $\dim_k E_G(1) = 2$ by Lemma 6.2. Thus S_1^G has exponential growth if $p > 3$.

Now if $p = 3$, the functor L is simple, and isomorphic to S_1^G , by Proposition 10.1. It follows that in this case

$$\dim_k E_G(n) + (p-1) \dim_k E_G(n-1) - 1 \leq \dim_k E_G(n+1) \ .$$

In particular $\dim_k E_G(2) \geq 2+2-1 = 3$, since $\dim_k E_G(1) = 2$ by Lemma 6.2. Since $\dim_k E_G(n+1) \geq 2 \dim_k E_G(n-1) - 1$, it follows that

$$\dim_k E_G(2n) - 1 \geq 2^{n-1} (\dim_k E_G(2) - 1) \geq 2^n \ ,$$

$$\dim_k E_G(2n+1) - 1 \geq 2^n (\dim_k E_G(1) - 1) \geq 2^n \ ,$$

hence S_1^G has exponential growth in this case also. \square

10.6. The case $p = 2$.

10.7. Theorem : *Let k be a field of characteristic 2, and G be an elementary abelian 2-group. Then*

$$\forall j \in \mathbb{N}, \quad \text{Ext}_{M_k^e(G)}^{2j+1}(S_1^G, S_1^G) = \{0\} .$$

Proof : By induction on the rank m of G , starting with $m = 0$, i.e. $G = \mathbf{1}$, where the result is trivial. Suppose $m > 0$, and that the result holds for all elementary abelian 2-groups of rank smaller than m . Choose a subgroup H of index p in G , and a complement $T \in K_G(H)$, and consider the long exact sequence of Corollary 10.3.

In this case by Proposition 10.1, since $p - 2 = 0$, the functor L is equal to zero. Thus $L(n) = \{0\}$ for any $n \in \mathbb{N}$. Moreover $\text{Ext}_{M_k^e(G)}^1(S_1^G, S_1^G) = \{0\}$, by Theorem 6.2, since G is generated by involutions. This starts an induction argument on j : suppose that $\text{Ext}_{M_k^e(G)}^{2r+1}(S_1^G, S_1^G) = \{0\}$, for $0 \leq r < j$. The exact sequence of Corollary 10.3 becomes

$$(10.8) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & \bigoplus_{X \in \mathcal{X}} E_G(n-2) & \longrightarrow & E_G(n) & \longrightarrow & E_H(n) \\ & & & & & & \uparrow \\ & & & & & & \text{Ext}_{M_k^e(G)}^{2j+1}(S_1^G, S_1^G) \\ & & & & & & \downarrow \\ & & & & & & \text{Ext}_{M_k^e(H)}^{2j+1}(S_1^H, S_1^H) \\ & & & & & & \uparrow \\ & & & & & & E_H(n+1) \\ & & & & & & \uparrow \\ & & & & & & \text{Ext}_{M_k^e(H)}^1(S_1^H, S_1^H) \\ & & & & & & \uparrow \\ & & & & & & E_H(n+1) \cdots \end{array} ,$$

where $E_G(n) = \text{Ext}_{M_k^e(G)}^n(S_1^G, S_1^G)$ and $E_H(n) = \text{Ext}_{M_k^e(H)}^n(S_1^H, S_1^H)$, and $\mathcal{X} = K_G(H) - \{T\}$.

Set $n = 2j + 1$ in this sequence. By induction hypothesis on m

$$E_H(n) = E_H(2j + 1) = \text{Ext}_{M_k^e(H)}^{2j+1}(S_1^H, S_1^H) \cong \{0\} ,$$

since H is elementary abelian of rank $m - 1$. Also

$$E_G(n-2) = E_G(2j-1) = \text{Ext}_{M_k^e(G)}^{2j-1}(S_1^G, S_1^G) = \{0\} ,$$

by induction hypothesis on j .

It follows that $E_G(n) = \text{Ext}_{M_k^e(G)}^{2j+1}(S_1^G, S_1^G) = \{0\}$, and this completes the inductive step on j , hence the inductive step on m . \square

10.9. Theorem : *Let k be a field of characteristic 2. Let G be an elementary abelian 2-group, and H be a subgroup of index 2 of G . Choose*

$T \in K_G(H)$, and set $\mathcal{S} = K_G(H) - \{T\}$.

Then for any $j \in \mathbb{N}$, there is a short exact sequence of extension groups

$$0 \rightarrow \bigoplus_{X \in \mathcal{S}} \text{Ext}^{2j-2}(S_1^G, S_1^G) \xrightarrow{\gamma} \text{Ext}^{2j}(S_1^G, S_1^G) \xrightarrow{r_H^G} \text{Ext}^{2j}(S_1^H, S_1^H) \longrightarrow 0$$

where r_H^G is induced by restriction to H , and γ is the direct sum of the maps induced by the Yoneda product with the element γ_X of $\text{Ext}^2(S_1^G, S_1^G)$, for $X \in \mathcal{S}$.

Proof : This follows from the exact sequence 10.8, in which all the terms $E_G(2j - 1)$, $E_G(2j + 1)$, and $E_H(2j + 1)$ are equal to zero. The exact sequence 10.8 splits as a series of short exact sequences

$$0 \rightarrow \bigoplus_{X \in \mathcal{S}} \text{Ext}^{2j-2}(S_1^G, S_1^G) \xrightarrow{\gamma} \text{Ext}^{2j}(S_1^G, S_1^G) \xrightarrow{r} \text{Ext}^{2j}(S_1^H, S_1^H) \longrightarrow 0 ,$$

for each $j \in \mathbb{N}$. It remains to show that the morphisms γ and r are as stated in Theorem 10.9.

First, with the notation of Proposition 10.1, the morphism

$$t : \text{Ext}^{2j-1}(R/J, S_1^G) \longrightarrow \text{Ext}^{2j}(S_1^G, S_1^G)$$

is the transition morphism associated to the short exact sequence

$$T : 0 \longrightarrow R/J \longrightarrow I/J \longrightarrow S_1^G \longrightarrow 0 ,$$

so it is given by Yoneda product with T .

Via the isomorphism $R/J \cong \bigoplus_{X \in \mathcal{S}} S_X^G$, the morphism t can be viewed as the direct sum of morphisms $t_X : \text{Ext}^{2j-1}(S_X^G, S_1^G) \rightarrow \text{Ext}^{2j}(S_1^G, S_1^G)$, where t_X is given by composition with the sequence T_X obtained from T by the projection π_X to the summand S_X^G , i.e. by completing the diagram

$$\begin{array}{ccccccc} T & : & 0 & \longrightarrow & \bigoplus_{X \in \mathcal{S}} S_X^G & \longrightarrow & I/J \longrightarrow S_1^G \longrightarrow 0 \\ & & & & \downarrow \pi_X & & \downarrow f & & \parallel \\ T_X & : & 0 & \longrightarrow & S_X^G & \longrightarrow & Y & \longrightarrow & S_1^G \longrightarrow 0 . \end{array}$$

Now f is surjective, since π_X and $\text{Id}_{S_1^G}$ are. Since S_X^G is not a quotient of I/J , it follows that Y is a non-split extension of S_1^G by S_X^G . Thus $Y \cong j_{G/X}^G(S_1^{G/X})$, by Lemma 8.3. More precisely T_X is isomorphic to the sequence D_X^* of Lemma 8.1.

Moreover, by Corollary 8.2, the isomorphism

$$\mathrm{Ext}^{2j-2}(S_1^G, S_1^G) \cong \mathrm{Ext}^{2j-1}(S_X^G, S_1^G)$$

is given by Yoneda composition with D_X . Hence the component index by $X \in \mathcal{S}$ of the morphism

$$\gamma : \bigoplus_{X \in \mathcal{S}} \mathrm{Ext}^{2j-2}(S_1^G, S_1^G) \longrightarrow \mathrm{Ext}^{2j}(S_1^G, S_1^G)$$

is given by composition with $\Gamma_X = D_X \circ D_X^*$, i.e. by Yoneda product with γ_X , as claimed.

Now the morphism $r_H^G : \mathrm{Ext}_{\mathcal{M}_k^c(G)}^{2j}(S_1^G, S_1^G) \rightarrow \mathrm{Ext}_{\mathcal{M}_k^c(H)}^{2j}(S_1^H, S_1^H)$ is composed of two steps : first, taking the image by the projection map $q : I \rightarrow S_1^G$, i.e. taking the Yoneda product with $q \in \mathrm{Ext}_{\mathcal{M}^c(G)}^0(\mathrm{Ind}_H^G S_1^H, S_1^G)$, and then using the adjunction $(\mathrm{Ind}_H^G, \mathrm{Res}_H^G)$, which gives a map

$$\mathrm{Ext}_{\mathcal{M}_k^c(G)}^{2j}(I, S_1^G) \cong \mathrm{Ext}_{\mathcal{M}_k^c(H)}^{2j}(S_1^H, \mathrm{Res}_H^G S_1^G) \cong \mathrm{Ext}_{\mathcal{M}_k^c(H)}^{2j}(S_1^H, S_1^H) .$$

In other words, with the notation of Remark 3.18, for $u \in \mathrm{Ext}_{\mathcal{M}_k^c(G)}^{2j}(S_1^G, S_1^G)$

$$r_H^G(u) = \alpha_{2j}(u \circ q) = \mathrm{Res}_H^G(u) \circ \alpha_0(q) = \mathrm{Res}_H^G(u) ,$$

since the map $\alpha_0(q) : S_1^H \rightarrow \mathrm{Res}_H^G S_1^G \cong S_1^H$ obtained by adjunction is the identity map. Hence r_H^G is induced by Res_H^G , and this completes the proof. \square

The following corollary is Theorem 1.2 of Section 1 :

10.10. Corollary : *Let G be an elementary abelian 2-group of rank m . Then the algebra $\mathrm{Ext}_{\mathcal{M}_k^c(G)}^*(S_1^G, S_1^G)$ is finitely generated by the elements γ_X , where X is a subgroup of order 2 of G . Its Poincaré series*

$$P(t) = \sum_{j \in \mathbb{N}} \dim_k \mathrm{Ext}_{\mathcal{M}_k^c(G)}^j(S_1^G, S_1^G) t^j$$

is equal to

$$P(t) = \frac{1}{(1-t^2)(1-3t^2)(1-7t^2)\dots(1-(2^{m-1}-1)t^2)} .$$

Proof : By induction on m , the case $m = 0$ being trivial. In Theorem 10.9, one can assume that the algebra $\mathrm{Ext}^*(S_1^H, S_1^H)$ is generated by the elements γ_X^H , where X is a subgroup of order 2 of H . This means that for

$n \in \mathbb{N}$, any element in $\text{Ext}^{2n}(S_1^H, S_1^H)$ is a k -linear combination of products of the form

$$\gamma_{X_1}^H \gamma_{X_2}^H \cdots \gamma_{X_n}^H ,$$

where X_1, \dots, X_n are subgroups of order 2 of H .

Denote by Γ the k -linear subspace of $\text{Ext}^{2n}(S_1^G, S_1^G)$ generated by the similar products $\gamma_{X_1}^G \gamma_{X_2}^G \cdots \gamma_{X_n}^G$.

Now for $X \leq H$, by Proposition 3.14, there are isomorphisms of functors

$$\text{Res}_H^G \circ \iota_{G/X}^G \cong \mathbf{L}_G \circ \mathbf{L}_{G/X} \cong \mathbf{L}_{G \times_G (G/X)} ,$$

where G is viewed as an (H, G) -biset and G/X as a $(G, G/X)$ -biset. Now $G \times_G (G/X) \cong (H/X) \times_{H/X} (G/X)$ as $(H, G/X)$ -bisets, thus

$$\text{Res}_H^G \circ \iota_{G/X}^G \cong \iota_{H/X}^H \circ \text{Res}_{H/X}^{G/X} .$$

It follows that $\text{Res}_H^G \begin{pmatrix} S_X^G \\ S_1^G \end{pmatrix} \cong \begin{pmatrix} S_X^H \\ S_1^H \end{pmatrix}$. Similarly $\text{Res}_H^G \begin{pmatrix} S_1^G \\ S_X^G \end{pmatrix} \cong \begin{pmatrix} S_1^H \\ S_X^H \end{pmatrix}$. Thus $r_H^G \gamma_X^G = \gamma_X^H$, because Res_H^G respects the Yoneda composition.

It follows that

$$\gamma_{X_1}^H \gamma_{X_2}^H \cdots \gamma_{X_n}^H = r_H^G (\gamma_{X_1}^G \gamma_{X_2}^G \cdots \gamma_{X_n}^G) ,$$

thus $\text{Ext}^{2n}(S_1^G, S_1^G) = \Gamma + \text{Ker } r_H^G$. But

$$\text{Ker } r_H^G = \gamma(\text{Ext}^{2n-2}(S_1^G, S_1^G)) = \sum_{X \in \mathcal{S}} \text{Ext}^{2n-2}(S_1^G, S_1^G) \circ \gamma_X^G ,$$

and by induction on n , this shows that $\Gamma = \text{Ext}^{2n}(S_1^G, S_1^G)$.

Denote by $P_m(t)$ the Poincaré series of the algebra $\text{Ext}_{M_k^c(G)}^*(S_1^G, S_1^G)$, where G is an elementary abelian 2-group of rank m . The short exact sequence of Theorem 10.9 gives the relation

$$P_m(t) = |\mathcal{S}| t^2 P_m(t) + P_{m-1}(t) ,$$

and moreover $|\mathcal{S}| = 2^{m-1} - 1$. Thus

$$P_m(t) = P_{m-1}(t) \cdot \frac{1}{1 - (2^{m-1} - 1)t^2} ,$$

and the claimed formula follows by induction on m . □

10.11. Theorem : *Let k be a field of characteristic 2, and G be an elementary abelian 2-group of rank m .*

1. [Samy Modeliar [7]] *If $m \leq 2$, the group G is a poco group over k .*
2. *If $m \geq 3$, the group G is not a poco group over k . More precisely, the simple functor S_1^G of $\mathbf{M}_k^c(G)$ has exponential growth.*

For Assertion 2, observe that

$$\frac{1}{(1-t^2)(1-3t^2)\dots(1-(2^{m-1}-1)t^2)} = \prod_{j=1}^{m-1} \left(\sum_{n_j=0}^{\infty} (2^j-1)^{n_j} t^{2n_j} \right) ,$$

Thus

$$\begin{aligned} \dim_k \text{Ext}^{2n}(S_1^G, S_1^G) &= \sum_{n_1+\dots+n_{m-1}=n} \prod_{j=1}^{m-1} (2^j-1)^{n_j} \\ &\geq (2^{m-1}-1)^n . \end{aligned}$$

By Corollary 8.2, since moreover $\text{Ext}^{2n+1}(S_Z^G, S_1^G) \cong \text{Ext}^{2n+1}(S_1^G, S_Z^G)$ as simple functors are self dual, it follows that

$$\dim_k \text{Ext}^{2n+1}(S_1^G, S_Z^G) \geq (2^{m-1}-1)^n ,$$

thus S_1^G has exponential growth, by Lemma 2.3, if $2^{m-1}-1 > 1$, i.e. if $m \geq 3$. \square

Assertion 1 is trivial if $m = 0$. It is straightforward if $m = 1$: since every kG -module for a group G of order 2 is a permutation module, it follows that all the fixed points functors are projective in this case, so $\mathbf{M}_k^c(G)$ has finite projective dimension in this case.

For $m = 2$, M. Samy Modeliar has described explicit eventually periodic resolutions of the fixed points functors FP_V , for all indecomposable kG -modules V . An alternative proof can be sketched as follows : the Poincaré series of $\text{Ext}^*(S_1^G, S_1^G)$ is equal to $\frac{1}{1-t^2}$ in this case. It follows that $\text{Ext}^n(S_1^G, S_1^G)$ is equal to $\{0\}$ if n is odd, and one dimensional if n is even. If Q is a non trivial subgroup of G , then there is an exact sequence

$$(10.12) \quad 0 \longrightarrow \bigoplus_{X \in K_Q(Z)} S_X^G \longrightarrow i_{G/Z}^G(S_{Q/Z}^{G/Z}) \longrightarrow S_Q^G \longrightarrow 0 ,$$

where Z is a subgroup of order 2 of Q . Moreover $i_{G/Z}^G(S_{Q/Z}^{G/Z})$ has finite projective dimension, since G/Z has order 2, and since $i_{G/Z}^G$ is exact and

preserves projectives. It follows easily by induction on $|Q|$ that there exists a constant c_Q such that $\dim_k \text{Ext}^j(S_Q^G, S_1^G) \leq c_Q$ for any $j \in \mathbb{N}$.

Thus $\dim_k \text{Ext}^j(S_1^G, S_R^G) \leq c_R$ for any $j \in \mathbb{N}$, and any $R \leq G$. Using again the exact sequence 10.12, one can show by induction on $|Q|$ that for any $Q, R \leq G$, there is a constant $c_{Q,R}$ such that $\dim_k \text{Ext}^j(S_Q^G, S_R^G) \leq c_{Q,R}$ for any $j \in \mathbb{N}$. By Lemma 2.3 and Corollary 2.6, the group G is a poco group. \square

11. The case of 2-groups of sectional rank at most 2

11.1. Proposition : *Let k be a field of characteristic 2, let G be a 2-group, and let H be a subgroup of index 2 of G . If the functor S_1^H (over k) has polynomial growth, and if the functor $S_X^{C_G(X)}$ has polynomial growth, for any $X \in K_G(H)$, then the functor S_1^G has polynomial growth.*

Proof : Consider the functor $I = \text{Ind}_H^G S_1^H$. By Proposition 8.7, there is a filtration

$$I \supset R \supseteq S \supset \{0\} ,$$

where R is the radical and S is the socle of I , such that $I/R \cong S \cong S_1^G$, and $R/S \cong \bigoplus_{X \in [G \setminus K_G(H)]} S_X^G$. This gives two short exact sequences in $\mathbf{M}_k^c(G)$

$$(11.2) \quad 0 \longrightarrow S_1^G \longrightarrow R \longrightarrow \bigoplus_{X \in [G \setminus K_G(H)]} S_X^G \longrightarrow 0$$

$$(11.3) \quad 0 \longrightarrow R \longrightarrow I \longrightarrow S_1^G \longrightarrow 0 .$$

Let M be a finitely generated cohomological Mackey functor for G over k . Applying the functor $\text{Hom}_{\mathbf{M}_k^c(G)}(-, M)$ to the sequence (11.2) gives the following long exact sequence of Ext groups in $\mathbf{M}_k^c(G)$

$$\cdots \longrightarrow \bigoplus_{X \in [G \setminus K_G(H)]} \text{Ext}^n(S_X^G, M) \longrightarrow \text{Ext}^n(R, M) \longrightarrow \text{Ext}^n(S_1^G, M) \longrightarrow \cdots$$

By Lemma 2.5, this gives

$$r_n \leq e_n + \sum_{X \in [G \setminus K_G(H)]} e_{X,n} ,$$

where $r_n = \dim_k \text{Ext}^n(R, M)$, $e_{X,n} = \dim_k \text{Ext}^n(S_X^G, M)$, for $X \in K_G(H)$, and $e_n = \dim_k \text{Ext}^n(S_1^G, M)$.

Now applying the functor $\text{Hom}_{\mathbf{M}_k^c(G)}(-, M)$ to the sequence (11.3) gives the following long exact sequence of Ext groups in $\mathbf{M}_k^c(G)$

$$\cdots \longrightarrow \text{Ext}^n(R, M) \longrightarrow \text{Ext}^{n+1}(S_1^G, M) \longrightarrow \text{Ext}^{n+1}(I, M) \longrightarrow \cdots$$

By Lemma 2.5, this gives

$$e_{n+1} \leq r_n + i_{n+1} \ ,$$

where $i_n = \dim_k \text{Ext}^n(I, M)$. Thus

$$(11.4) \quad e_{n+1} \leq e_n + \left(\sum_{X \in [G \backslash K_G(H)]} e_{X,n} \right) + i_{n+1} \ .$$

Now $S_X^G = \text{Ind}_{C_G(X)}^G S_X^{C_G(X)}$, for any $X \in K_G(H)$, thus

$$e_{X,n} = \dim_k \text{Ext}_{\mathbf{M}_k^c(C_G(X))}^n(S_X^{C_G(X)}, \text{Res}_{C_G(X)}^G M) \ .$$

Hence if $S_X^{C_G(X)}$ has polynomial growth, there are constants c_X , d_X , and e_X , such that

$$\forall n \in \mathbb{N}, \quad e_{X,n} \leq c_X n^{d_X} + e_X \ .$$

Thus

$$\forall n \in \mathbb{N}, \quad \sum_{X \in [G \backslash K_G(H)]} e_{X,n} \leq C n^D + E \ ,$$

where $C = \sum_{X \in [G \backslash K_G(H)]} c_X$, $D = \max_{X \in [G \backslash K_G(H)]} d_X$, and $E = \sum_{X \in [G \backslash K_G(H)]} e_X$.

Similarly,

$$i_n = \dim_k \text{Ext}_{\mathbf{M}_k^c(H)}^n(S_1^H, \text{Res}_H^G M) \ ,$$

so if S_1^H has polynomial growth, there are constants c , d , and e such that

$$\forall n \in \mathbb{N}, \quad i_n \leq c n^d + e \ .$$

Thus

$$\forall n \in \mathbb{N}, \quad i_{n+1} \leq c|(n+1)^d - 1| + c + e \leq c(2n)^d + c + e = c2^d n^d + c + e \ .$$

Inequality 11.4 now gives

$$\forall n \in \mathbb{N}, \quad e_{n+1} \leq e_n + \gamma n^\delta + \epsilon$$

where $\gamma = c2^d + C$, $\delta = \max(D, d)$, and $\epsilon = E = c + e$. By induction, it follows that

$$\forall n \in \mathbb{N}, \quad e_n \leq \gamma \sum_{j=1}^n j^\delta + n\epsilon + e_0 \leq \gamma n^{\delta+1} + n\epsilon + e_0 \leq (\gamma + \epsilon)n^{\delta+1} + e_0 \ ,$$

so S_1^G has polynomial growth. \square

11.5. Theorem : *Let k be a field of characteristic 2, and G be a 2-group of sectional 2-rank at most 2. Then G is a poco group over k .*

Proof : By induction on the order of G , one may assume that for any section $(B, A) \neq (G, \mathbf{1})$, the group B/A is a poco group. By Corollary 2.6, proving that G is a poco group is equivalent to proving that for any subgroup Q of G , the simple functor S_Q^G has polynomial growth. First consider the case where $Q = \mathbf{1}$.

Let H be a subgroup of index 2 of G . Then H is a poco group by induction hypothesis. Similarly, if $X \in K_G(H)$ and $C_G(X) \neq G$, then $C_G(X)$ is a poco group. Thus, if H has no central complement in G , by Proposition 11.1, the functor S_1^G has polynomial growth. This holds in particular if H has no complement at all in G , i.e. if H contains all involutions of G . Thus, if G is not generated by involutions, then S_1^G has polynomial growth. So one can assume that G is generated by involutions.

Now if H has some central complement X in G , then $G \cong H \times X$. Since G has sectional 2-rank at most 2, the group H has sectional 2-rank at most 1, hence it is cyclic. Thus G is abelian, and generated by involutions, hence elementary abelian, and of sectional 2-rank at most 2. So the group G is elementary abelian of rank at most 2, and the results of Section 10 show that S_1^G has polynomial growth.

In any case S_1^G has polynomial growth. By induction on the order of Q , it follows that S_Q^G has polynomial growth, for any subgroup Q of G : indeed, setting $B = N_G(Q)$ and $A = \Phi(Q)$, Lemma 6.5 shows that

$$S_Q^G \cong \mathbf{L}_U(S_{Q/A}^{B/A}) ,$$

where U is the set $A \backslash G$, for its natural structure of $(B/A, G)$ -biset structure. If $(B, A) \neq (G, \mathbf{1})$, then $S_{Q/A}^{B/A}$ has polynomial growth, hence S_Q^G has polynomial growth, by Lemma 4.1.

So it remains to consider the case where $(B, A) = (Q, \mathbf{1})$, i.e. the case where Q is a non trivial normal elementary abelian subgroup of G . Let Z be a subgroup of order 2 of $Q \cap Z(G)$.

By Proposition 8.5, there is a short exact sequence

$$0 \longrightarrow \bigoplus_{X \in [G \backslash K_Q(Z)]} S_X^G \longrightarrow i_{G/Z}^G(S_{Q/Z}^{G/Z}) \longrightarrow S_Q^G \longrightarrow 0 .$$

By the induction hypothesis, all the functors S_X^G , for $X \in K_Q(Z)$, have polynomial growth. Moreover, the functor $i_{G/Z}^G(S_{Q/Z}^{G/Z})$ has polynomial growth,

by Lemma 4.1. Hence S_Q^G has polynomial growth, by Lemma 2.5, and this completes the proof. \square

11.6. Remark : The 2-groups of sectional rank at most 1 are the cyclic 2-groups. In a recent preprint, Andersen, Oliver and Ventura have shown that the 2-groups of sectional rank 2 are the metacyclic 2-groups (see Lemma 10.2 of [1]). For 2-groups of order at least 32, this also follows from a theorem of Blackburn (see [2], or Satz 11.13 in [6]).

II - Cohomology

12. Extension of simple functors for elementary abelian p -groups

In this section k is a field of characteristic p , the group $G \cong (C_p)^m$ is an elementary abelian p -group of rank m , in additive notation, and all the cohomological Mackey functors have values in k -vector spaces.

Recall from Notation 7.4 that $\gamma_X \in \text{Ext}_{M_k^c(G)}^2(S_1^G, S_1^G)$ is the element represented by the 2-fold extension

$$0 \longrightarrow S_1^G \longrightarrow \begin{pmatrix} S_X^G \\ S_1^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_1^G \\ S_X^G \end{pmatrix} \longrightarrow S_1^G \longrightarrow 0 .$$

12.1. Notation : If $G = (C_p)^m$ and $x \in G - \{0\}$, set $\gamma_x = \gamma_{\langle x \rangle}^G$. Thus $\gamma_x \in \text{Ext}^2(S_1^G, S_1^G)$.

12.2. Lemma : Let A be a ring. If $L \supseteq M \supseteq N \supseteq \{0\}$ is a filtration of an A -module L , then the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow L/N \longrightarrow L/M \longrightarrow 0$$

obtained by splicing the short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & L/N \longrightarrow L/M \longrightarrow 0 \\ & & & & \searrow & & \nearrow \\ & & & & & M/N & \\ & & & & \nearrow & & \searrow \\ & & 0 & & & & 0 \end{array}$$

represents the zero class of $\text{Ext}_A^2(L/M, N)$.

Proof : There is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & L/N & \xrightarrow{p} & L/M & \longrightarrow & 0 \\
& & \parallel & & \downarrow i & & \downarrow (\text{Id}, p) & & \parallel & & \\
0 & \longrightarrow & N & \longrightarrow & L & \xrightarrow{(q, 0)} & L/N \oplus L/M & \xrightarrow{\binom{0}{\text{Id}}} & L/M & \longrightarrow & 0
\end{array}$$

where i is the inclusion map, and p, q are projection maps. Moreover, the bottom sequence represents zero in $\text{Ext}_A^2(L/M, N)$, since the map $\binom{0}{\text{Id}}$ is split surjective. \square

12.3. Lemma : *Let G be an elementary abelian p -group. If X and Y are distinct subgroups of order p of G , set $Q = XY$. Then the sequences*

$$0 \longrightarrow S_X^G \longrightarrow \begin{pmatrix} S_{\mathbf{1}}^G \\ S_X^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_Y^G \\ S_{\mathbf{1}}^G \end{pmatrix} \longrightarrow S_Y^G \longrightarrow 0$$

and

$$0 \longrightarrow S_X^G \longrightarrow \begin{pmatrix} S_Q^G \\ S_X^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_Y^G \\ S_Q^G \end{pmatrix} \longrightarrow S_Y^G \longrightarrow 0$$

represent opposite elements of $\text{Ext}_{M_k^c(G)}^2(S_Y^G, S_X^G)$.

Proof : Indeed, the sum of the corresponding elements in $\text{Ext}_{M_k^c(G)}^2(S_Y^G, S_X^G)$ is represented by the sequence

$$(12.4) \quad 0 \longrightarrow S_X^G \longrightarrow \begin{pmatrix} S_{\mathbf{1}}^G & S_Q^G \\ & S_X^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_Y^G \\ S_{\mathbf{1}}^G & S_Q^G \end{pmatrix} \longrightarrow S_Y^G \longrightarrow 0 ,$$

where the functor $L_X = \begin{pmatrix} S_{\mathbf{1}}^G & S_Q^G \\ & S_X^G \end{pmatrix}$ is the Mackey functor for G over k whose values at $\mathbf{1}, X$ and Q are equal to k , and other values are zero. The conjugation maps for this functor are all identity maps (possibly zero), and the possibly non zero transfer and restriction maps are given in the following diagram

$$\begin{array}{c}
L_X(Q) = k \\
r_X^Q = 1 \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) t_X^Q = 0 \\
L_X(X) = k \\
r_{\mathbf{1}}^X = 0 \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) t_{\mathbf{1}}^X = 1 \\
L_X(\mathbf{1}) = k
\end{array}$$

It follows in particular that S_X^G is isomorphic to a subfunctor of L_X , and this yields an exact sequence

$$(12.5) \quad 0 \longrightarrow S_X^G \longrightarrow \begin{pmatrix} S_1^G & S_Q^G \\ S_X^G & \end{pmatrix} \longrightarrow S_1^G \oplus S_Q^G \longrightarrow 0$$

in $M_k^c(G)$.

The functor $\begin{pmatrix} S_Y^G \\ S_1^G & S_Q^G \end{pmatrix}$ is isomorphic to the dual L_Y^* of $L_Y = \begin{pmatrix} S_1^G & S_Q^G \\ S_Y^G & \end{pmatrix}$: its non zero values are at $\mathbf{1}$, Y and Q , and they are equal to k . The conjugation maps for L_Y^* are all identity maps (possibly zero), and the possibly non zero transfer and restriction maps are given in the following diagram

$$\begin{array}{c} L_Y^*(Q) = k \\ \left. \begin{array}{c} \uparrow \\ r_Y^Q=0 \\ \downarrow \end{array} \right\} t_Y^Q=1 \\ L_Y^*(Y) = k \\ \left. \begin{array}{c} \uparrow \\ r_1^Y=1 \\ \downarrow \end{array} \right\} t_1^Y=0 \\ L_Y^*(\mathbf{1}) = k \end{array}$$

There is an exact sequence

$$(12.6) \quad 0 \longrightarrow S_1^G \oplus S_Q^G \longrightarrow \begin{pmatrix} S_Y^G \\ S_1^G & S_Q^G \end{pmatrix} \longrightarrow S_Y^G \longrightarrow 0$$

in $M_k^c(G)$, and the exact sequence 12.4 is obtained by splicing the exact sequences 12.5 and 12.6.

Let $M_{X,Y}$ denote the functor defined for $H \leq G$ by $M_{X,Y}(H) = k$ if $H \in \{\mathbf{1}, X, Y, Q\}$, and by $M_{X,Y}(H) = \{0\}$ otherwise. The conjugation maps for $M_{X,Y}$ are all identity maps (possibly zero), and the possibly non zero transfer and restriction maps for $M_{X,Y}$ are given in the following diagram

$$(12.7) \quad \begin{array}{ccccc} & & M_{X,Y}(Q) = k & & \\ & \swarrow r_X^Q=1 & & \nwarrow t_Y^Q=1 & \\ M_{X,Y}(X) = k & & & & M(Y) = k \\ & \searrow t_X^Q=0 & & \swarrow r_Y^Q=0 & \\ & & & & \\ & \swarrow t_1^X=1 & & \nwarrow r_1^Y=1 & \\ & & M_{X,Y}(\mathbf{1}) = k & & \\ & \searrow r_1^X=0 & & \swarrow t_1^Y=0 & \end{array}$$

One checks easily that $M_{X,Y}$ is a cohomological Mackey functor for G over k , that L_X is a subfunctor of $M_{X,Y}$ (represented by the left half of diagram 12.7),

and that L_Y^* is a quotient functor of $M_{X,Y}$ (represented by the right half of 12.7). Hence there is a filtration

$$M_{X,Y} \supset L_X \supset S_X^G \supset \{0\} ,$$

such that $M_{X,Y}/L_X \cong S_Y^G$ and $M_{X,Y}/S_X^G \cong L_Y^*$. Now Lemma 12.2 shows that the exact sequence 12.4 represents zero in $\text{Ext}_{M_k^c(G)}^2(S_Y^G, S_X^G)$, and this completes the proof. \square

12.8. Lemma : *Let G be an elementary abelian p -group, and let $Q < R$ be subgroups of G such that $|R:Q| = p^2$. Let M denote the functor $\Sigma_{Q,R}^G$ of Corollary 7.3. Then there is a filtration $M \supset J \supset S \supset \{0\}$, where J is the radical of M and S is its socle. Moreover*

$$M/J \cong S_R^G , \quad S \cong S_Q^G , \quad J/M \cong \bigoplus_{Q < X < R} S_X^G .$$

Proof : Recall that $\Sigma_{Q,R}^G$ is the subquotient of FP_k whose value at $H \leq G$ is equal to k if $Q \leq H \leq R$, and to zero otherwise. By Corollary 7.3, the head of $\Sigma_{Q,R}^G$ is simple, isomorphic to S_R^G , and its socle is simple, isomorphic to S_Q^G . In particular $S \leq J$. Moreover, if $H \leq G$, then $(J/S)(H)$ is equal to zero, except if $Q < H < R$, and in this case $(J/S)(H) = k$. It follows that

$$J/M \cong \bigoplus_{Q < X < R} S_X^G ,$$

as was to be shown. \square

12.9. Proposition : *Let G be an elementary abelian p -group, and Q be a subgroup of order p^2 of G . Then for any subgroup Y of order p of Q*

$$[\sum_{1 < X < Q} \gamma_X, \gamma_Y] = 0 ,$$

in $\text{Ext}_{M_k^c(G)}^4(S_1^G, S_1^G)$, where $[a, b] = ab - ba$, for $a, b \in \text{Ext}_{M_k^c(G)}^2(S_1^G, S_1^G)$.

Proof : Recall that if X is a subgroup of order p of G , the element γ_X of $\text{Ext}_{M_k^c(G)}^2(S_1^G, S_1^G)$ is represented by the sequence

$$0 \longrightarrow S_1^G \longrightarrow \begin{pmatrix} S_X^G \\ S_1^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_1^G \\ S_X^G \end{pmatrix} \longrightarrow S_1^G \longrightarrow 0 .$$

It follows that if Y is another subgroup of order p of G , the Yoneda product $\gamma_X \gamma_Y$ is represented by the sequence

$$0 \longrightarrow S_1^G \longrightarrow \begin{pmatrix} S_X^G \\ S_1^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_1^G \\ S_X^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_Y^G \\ S_1^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_1^G \\ S_Y^G \end{pmatrix} \longrightarrow S_1^G \longrightarrow 0 .$$

This sequence is obtained by splicing the following three exact sequences :

$$(12.10) \quad \begin{aligned} & 0 \longrightarrow S_1^G \longrightarrow \begin{pmatrix} S_X^G \\ S_1^G \end{pmatrix} \longrightarrow S_X^G \longrightarrow 0 \\ & 0 \longrightarrow S_X^G \longrightarrow \begin{pmatrix} S_1^G \\ S_X^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_Y^G \\ S_1^G \end{pmatrix} \longrightarrow S_Y^G \longrightarrow 0 \\ & 0 \longrightarrow S_Y^G \longrightarrow \begin{pmatrix} S_Y^G \\ S_1^G \end{pmatrix} \longrightarrow S_1^G \longrightarrow 0 . \end{aligned}$$

By Lemma 12.3, if $XY = Q$, i.e. if X and Y are distinct subgroups of order p of Q , the sequence 12.10 and the sequence

$$(12.11) \quad 0 \longrightarrow S_X^G \longrightarrow \begin{pmatrix} S_Q^G \\ S_X^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_Y^G \\ S_Q^G \end{pmatrix} \longrightarrow S_Y^G \longrightarrow 0$$

represent opposite elements of $\text{Ext}_{M_k(G)}^2(S_Y^G, S_X^G)$. It follows that for a given subgroup Y of order p of Q

$$\left(\sum_{\substack{1 < X < Q \\ X \neq Y}} \gamma_X \right) \gamma_Y = - \left(\sum_{\substack{1 < X < Q \\ X \neq Y}} f_{X,Y}^Q \right) \gamma_Y ,$$

where $f_{X,Y}^Q$ is the element represented by the sequence

$$0 \longrightarrow S_1^G \longrightarrow \begin{pmatrix} S_X^G \\ S_1^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_Q^G \\ S_X^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_Y^G \\ S_Q^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_1^G \\ S_Y^G \end{pmatrix} \longrightarrow S_1^G \longrightarrow 0 .$$

This in turn is the splice of the following two exact sequences

$$\begin{aligned} U_X^Q & : 0 \longrightarrow S_1^G \longrightarrow \begin{pmatrix} S_X^G \\ S_1^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_Q^G \\ S_X^G \end{pmatrix} \longrightarrow S_Q^G \longrightarrow 0 \\ (U_Y^Q)^* & : 0 \longrightarrow S_Q^G \longrightarrow \begin{pmatrix} S_Y^G \\ S_Q^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_1^G \\ S_Y^G \end{pmatrix} \longrightarrow S_1^G \longrightarrow 0 . \end{aligned}$$

The sum of the elements u_X^Q of $\text{Ext}_{M_k^c(G)}^2(S_Q^G, S_1^G)$ represented by the sequences U_X^Q , for $1 < X < Q$, is represented by the sequence

$$0 \longrightarrow S_1^G \longrightarrow \begin{pmatrix} \Sigma_Q^G \\ S_1^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_1^G \\ \Sigma_Q^G \end{pmatrix} \longrightarrow S_Q^G \longrightarrow 0 ,$$

where $\begin{pmatrix} \Sigma_Q^G \\ S_1^G \end{pmatrix}$ has simple socle isomorphic to S_1^G , and head isomorphic to $\Sigma_Q^G = \bigoplus_{1 < X < Q} S_X^G$, and $\begin{pmatrix} S_1^G \\ \Sigma_Q^G \end{pmatrix}$ is isomorphic to the dual of $\begin{pmatrix} \Sigma_Q^G \\ S_1^G \end{pmatrix}$. By Lemma 12.2 and Lemma 12.8, it follows that

$$\sum_{1 < X < Q} u_X^Q = 0 .$$

Hence $\sum_{\substack{1 < X < Q \\ X \neq Y}} u_X^Q = -u_Y^Q$, so

$$\left(\sum_{\substack{1 < X < Q \\ X \neq Y}} \gamma_X \right) \gamma_Y = u_Y^Q \gamma_Y .$$

This is represented by the sequence

$$0 \longrightarrow S_1^G \longrightarrow \begin{pmatrix} S_Y^G \\ S_1^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_Q^G \\ S_Y^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_Y^G \\ S_Q^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_1^G \\ S_Y^G \end{pmatrix} \longrightarrow S_1^G \longrightarrow 0 ,$$

which is obviously self dual. It follows that

$$\left(\sum_{\substack{1 < X < Q \\ X \neq Y}} \gamma_X \right) \gamma_Y = \left(\left(\sum_{\substack{1 < X < Q \\ X \neq Y}} \gamma_X \right) \gamma_Y \right)^* = \gamma_Y \left(\sum_{\substack{1 < X < Q \\ X \neq Y}} \gamma_X \right) .$$

Adding γ_Y^2 to both side, this shows that γ_Y and $\sum_{1 < X < Q} \gamma_X$ commute, which completes the proof. \square

13. The algebra $\text{Ext}_{M_k^c(G)}^*(S_1^G, S_1^G)$ for $G \cong (C_2)^m$

In this section k is a field of characteristic 2, the group $G \cong (C_2)^m$ is an elementary abelian 2-group of rank m , in additive notation, and all the cohomological Mackey functors have values in k -vector spaces.

By Corollary 10.10 and Notation 12.1, the algebra $\text{Ext}_{M_k^c(G)}^*(S_1^G, S_1^G)$ is generated by the elements γ_x of $\text{Ext}^2(S_1^G, S_1^G)$, for $x \in G - \{0\}$.

13.1. Proposition : *Let H be a subgroup of index 2 of G . Then*

$$\sum_{x \notin H} \gamma_x = 0 \text{ in } \text{Ext}_{M_k^c(G)}^2(S_1^G, S_1^G) .$$

Proof : By Proposition 10.1, there is a filtration $I \supset R \supset S \supset \{0\}$, where $I = \text{Ind}_H^G S_1^H$, such that $S \cong I/R \cong S_1^G$, and i

$$R/S \cong \bigoplus_{X \in \mathcal{K}} S_X^G ,$$

where $\mathcal{K} = K_G(H)$. By Lemma 12.2, the sequence

$$0 \longrightarrow S_1^G \longrightarrow R \longrightarrow I/S \longrightarrow S_1^G \longrightarrow 0$$

represents 0 in $\text{Ext}^2(S_1^G, S_1^G)$.

For each $X \in \mathcal{K}$, the functors $\rho_{G/X}^G(I)$, $\rho_{G/X}^G(R)$, and $\rho_{G/X}^G(I/S)$ are all isomorphic to $S_1^{G/X}$, since their non zero evaluation is at the trivial subgroup, where it is isomorphic to k . By adjunction, this gives morphisms

$$p_X : \begin{pmatrix} S_X^G \\ S_1^G \end{pmatrix} \rightarrow R \text{ and } q_X : I/S \rightarrow \begin{pmatrix} S_1^G \\ S_X^G \end{pmatrix} .$$

This gives a commutative diagram with exact lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{X \in \mathcal{K}} S_1^G & \longrightarrow & \bigoplus_{X \in \mathcal{K}} \begin{pmatrix} S_X^G \\ S_1^G \end{pmatrix} & \longrightarrow & \bigoplus_{X \in \mathcal{K}} \begin{pmatrix} S_1^G \\ S_X^G \end{pmatrix} & \longrightarrow & \bigoplus_{X \in \mathcal{K}} S_1^G & \longrightarrow & 0 \\ & & \downarrow \Sigma & & \downarrow p & & \parallel & & \parallel & & \\ 0 & \longrightarrow & S_1^G & \longrightarrow & R & \longrightarrow & \bigoplus_{X \in \mathcal{K}} \begin{pmatrix} S_1^G \\ S_X^G \end{pmatrix} & \longrightarrow & \bigoplus_{X \in \mathcal{K}} S_1^G & \longrightarrow & 0 \\ & & \parallel & & \parallel & & \uparrow q & & \uparrow \Delta & & \\ 0 & \longrightarrow & S_1^G & \longrightarrow & R & \longrightarrow & I/S & \longrightarrow & S_1^G & \longrightarrow & 0 \end{array} ,$$

where p is the sum of the maps p_X and q is the sum of the maps q_X , for $X \in \mathcal{K}$, where Σ is the summation map, and Δ the diagonal inclusion. The top line of this diagram is the direct sum of the sequences Γ_X , for $X \in \mathcal{K}$ (see Notation 7.4).

One checks easily that the top left square in this diagram is cocartesian, and that the bottom right square is cartesian. It follows that the bottom line represents the sum in $\text{Ext}^2(S_1^G, S_1^G)$ of the 2-fold extensions in the top line, i.e. the sum $\sum_{X \in \mathcal{K}} \gamma_X$.

Hence this sum is equal to 0, i.e. equivalently $\sum_{x \notin H} \gamma_x = 0$. \square

13.2. Proof of Theorem 1.3. Let $G \cong (C_2)^m$ be an elementary abelian 2-group of rank m , in additive notation, and k be a field of characteristic 2. Denote by \mathcal{E} the graded algebra $\text{Ext}_{M_k^*(G)}^*(S_1^G, S_1^G)$.

• By Corollary 10.10 and Notation 12.1, the algebra \mathcal{E} is generated by the elements γ_x , for $x \in G - \{0\}$, where γ_x has degree 2.

Let x and y be distinct elements in $G - \{0\}$. Then x and y generate a subgroup Q of order 4 of G , and the non zero elements of Q are x , y , and $x + y$. By Proposition 12.9, the commutator

$$[\gamma_x + \gamma_y + \gamma_{x+y}, \gamma_{x+y}]$$

in \mathcal{E} is equal to 0. Equivalently

$$(13.3) \quad [\gamma_x + \gamma_y, \gamma_{x+y}] = 0 \ .$$

Now if H is a subgroup of index 2 of G , Proposition 13.1 shows that

$$(13.4) \quad \sum_{x \notin H} \gamma_x = 0$$

in \mathcal{E} .

• Conversely, let $\tilde{\mathcal{E}}$ denote the graded associative k -algebra with generators $\tilde{\gamma}_x$ in degree 2, for $x \in G - \{0\}$, subject to the relations

$$\begin{cases} \forall H \leq G, |G : H| = 2, \sum_{x \notin H} \tilde{\gamma}_x = 0 \ , \\ \forall x, y \in G - \{0\}, x \neq y, [\tilde{\gamma}_x + \tilde{\gamma}_y, \tilde{\gamma}_{x+y}] = 0 \ . \end{cases}$$

Then there is a unique surjective homomorphism $s : \tilde{\mathcal{E}} \rightarrow \mathcal{E}$ of graded k -algebras such that $s(\tilde{\gamma}_x) = \gamma_x$, for all $x \in G - \{0\}$. Thus showing that s is an isomorphism is equivalent to showing that for any integer $n \in \mathbb{N}$, the restriction s_n of s to the subspace $\tilde{\mathcal{E}}_n$ of elements of degree n in $\tilde{\mathcal{E}}$ is an isomorphism onto the corresponding subspace \mathcal{E}_n of \mathcal{E} . Since s_n is surjective, this amounts to showing that $\dim_k \tilde{\mathcal{E}}_n \leq \dim_k \mathcal{E}_n$.

13.5. Lemma : *Let B be an \mathbb{F}_2 -basis of G . Then the set*

$$\{\tilde{\gamma}_x \mid x \in G - (B \sqcup \{0\})\}$$

is a k -basis of $\tilde{\mathcal{E}}_2$, and s_2 is an isomorphism.

Proof : By Corollary 10.10

$$\dim_k \mathcal{E}_2 = (2^1 - 1) + (2^2 - 1) + \cdots + (2^{m-1} - 1) = 2^m - m - 1 .$$

Thus $\dim_k \tilde{\mathcal{E}}_2 \geq 2^m - m - 1$, since s_2 is surjective. As $2^m - m - 1$ is precisely equal to the cardinality of $G - (B \sqcup \{0\})$, it is enough to show that the subspace $\tilde{\mathcal{E}}'_2$ of $\tilde{\mathcal{E}}_2$ generated by the elements $\tilde{\gamma}_x$, for $x \in G - (B \sqcup \{0\})$, is equal to $\tilde{\mathcal{E}}_2$.

Let $b \in B$, and denote by H the subgroup of G generated by $B - \{b\}$. Then $|G : H| = 2$, so

$$\sum_{x \notin H} \tilde{\gamma}_x = 0 .$$

Since $G - H = H + b$, the only element of $G - H$ which is also in B is b itself. This gives

$$\tilde{\gamma}_b = \sum_{y \in H - \{0\}} \tilde{\gamma}_{b+y} ,$$

so $\tilde{\gamma}_b \in \tilde{\mathcal{E}}'_2$. Since this holds for any $b \in B$, and since $\tilde{\mathcal{E}}_2$ is the set of k -linear combinations of elements $\tilde{\gamma}_x$, for $x \in G - \{0\}$, it follows that $\tilde{\mathcal{E}}'_2 = \tilde{\mathcal{E}}_2$, as was to be shown. \square

13.6. Notation : *Fix a linear ordering $b_1 < b_2 < \cdots < b_m$ on B . If $x \in G - \{0\}$, let $d_B(x)$ denote the least integer $i \in \{1, \dots, m\}$ such that $x \in \langle b_1, \dots, b_i \rangle$.*

Then $d_B(x) = i$ if and only if $x = y + b_i$, for some $y \in \langle b_1, \dots, b_{i-1} \rangle$.

13.7. Definition : *Let $n \in \mathbb{N}$. If $M = \tilde{\gamma}_{x_1} \tilde{\gamma}_{x_2} \cdots \tilde{\gamma}_{x_n} \in \tilde{\mathcal{E}}_{2n}$, where $x_i \in G - \{0\}$ for $1 \leq i \leq n$, set $w(M) = \sum_{i=1}^n d_B(x_i)$. The monomial M is called special if $x_i \in G - (B \sqcup \{0\})$, for $1 \leq i \leq n$, and ordered if $d_B(x_i) \leq d_B(x_{i+1})$, for $1 \leq i < n$.*

13.8. Lemma : *Let x_1, \dots, x_n be elements of $G - (B \sqcup \{0\})$. Then for any*

$i \in \{1, \dots, n-1\}$ such that $d_B(x_i) \neq d_B(x_{i+1})$, the sum

$$\tilde{\gamma}_{x_1} \cdots \tilde{\gamma}_{x_{i-1}} \tilde{\gamma}_{x_i} \tilde{\gamma}_{x_{i+1}} \tilde{\gamma}_{x_{i+2}} \cdots \tilde{\gamma}_{x_n} + \tilde{\gamma}_{x_1} \cdots \tilde{\gamma}_{x_{i-1}} \tilde{\gamma}_{x_{i+1}} \tilde{\gamma}_{x_i} \tilde{\gamma}_{x_{i+2}} \cdots \tilde{\gamma}_{x_n}$$

is a linear combination of special monomials M' with $w(M') > w(M)$.

Proof : Since the assertion is symmetric in x_i and x_{i+1} , one can assume that $r = d_B(x_i) > d_B(x_{i+1}) = s$. In this case $x_i = u + b_r$, where $u \in \langle b_1, \dots, b_{r-1} \rangle - \{0\}$, and $x_{i+1} = v + b_s$, where $v \in \langle b_1, \dots, b_{s-1} \rangle - \{0\}$. Let $t = x_i + x_{i+1} = (v + b_s + u) + b_r$, so $d_B(t) = r$. Now the relation $[\tilde{\gamma}_{x_{i+1}} + \tilde{\gamma}_t, \tilde{\gamma}_{x_i}] = 0$ gives

$$\tilde{\gamma}_{x_i} \tilde{\gamma}_{x_{i+1}} + \tilde{\gamma}_{x_{i+1}} \tilde{\gamma}_{x_i} = \tilde{\gamma}_t \tilde{\gamma}_{x_i} + \tilde{\gamma}_{x_i} \tilde{\gamma}_t ,$$

so the sum in the lemma is equal to

$$(13.9) \quad S = \tilde{\gamma}_{x_1} \cdots \tilde{\gamma}_{x_{i-1}} \tilde{\gamma}_t \tilde{\gamma}_{x_i} \tilde{\gamma}_{x_{i+2}} \cdots \tilde{\gamma}_{x_n} + \tilde{\gamma}_{x_1} \cdots \tilde{\gamma}_{x_{i-1}} \tilde{\gamma}_{x_i} \tilde{\gamma}_t \tilde{\gamma}_{x_{i+2}} \cdots \tilde{\gamma}_{x_n} .$$

If $t \notin B$, this is a sum of two special monomials M' with $w(M') > w(M)$, for $d_B(t) = d_B(x_i) > d_B(x_{i+1})$. And if $t \in B$, then $t = b_r$, so

$$(13.10) \quad \tilde{\gamma}_t = \sum_{u \in H - \{0\}} \tilde{\gamma}_{b_r + u} ,$$

where H is the subgroup of index 2 of G generated by $B - \{b_r\}$. But $d_B(b_r + u) \geq r$, for any $u \in H$. Replacing $\tilde{\gamma}_t$ in 13.9 by the right hand side of 13.10 gives an expression of S as a sum of special monomials M' with $w(M') > w(M)$. \square

13.11. Proposition : *The set of special ordered monomials is a k -basis of $\tilde{\mathcal{E}}$, and the map s is an isomorphism.*

Proof : This is equivalent to saying that for any $n \in \mathbb{N}$, the set of special ordered monomials of degree $2n$ is a k -basis of $\tilde{\mathcal{E}}_{2n}$, and that the map s_{2n} is an isomorphism.

The first step consists in showing that the special ordered monomials of degree $2n$ generate $\tilde{\mathcal{E}}_{2n}$. In other words, using (13.10), any special monomial

$$M = \tilde{\gamma}_{x_1} \tilde{\gamma}_{x_2} \cdots \tilde{\gamma}_{x_n}$$

should be equal to a linear combination of special ordered monomials of degree $2n$.

For such an arbitrary monomial $w(M) \leq nm$, and this allows for a proof by induction on $j = nm - w(M)$: if $j = 0$, then $d_B(x_i) = m$, for $1 \leq i \leq n$,

and the monomial M is a special ordered monomial, so there is nothing to prove.

Otherwise, if M is not ordered, there is a least integer $i \in \{1, \dots, n-1\}$ such that $d_B(x_i) > d_B(x_{i+1})$. By Lemma 13.8, the monomial M is equal to the monomial obtained by exchanging $\tilde{\gamma}_{x_i}$ and $\tilde{\gamma}_{x_{i+1}}$, up to a linear combination of monomials M' with $w(M') > w(M)$, which are equal to a linear combination of special ordered monomials, by induction hypothesis.

By repeated application of this procedure, the term $\tilde{\gamma}_{x_{i+1}}$ can be moved to the left, until it sits between $\tilde{\gamma}_{x_j}$ and $\tilde{\gamma}_{x_{j+1}}$ such that

$$d_B(x_j) \leq d_B(x_{i+1}) < d_B(x_{j+1}) \quad ,$$

(possibly $j = 0$, in which case $\tilde{\gamma}_{x_{i+1}}$ is moved to the first place on the left), and the monomial M is equal to the monomial

$$\tilde{\gamma}_{x_1} \tilde{\gamma}_{x_2} \cdots \tilde{\gamma}_{x_j} \tilde{\gamma}_{x_{i+1}} \tilde{\gamma}_{x_{j+1}} \cdots \tilde{\gamma}_{x_{i-1}} \tilde{\gamma}_{x_i} \tilde{\gamma}_{x_{i+2}} \cdots \tilde{\gamma}_{x_n}$$

up to a linear combination of special ordered monomials. In this monomial, the $i+1$ first values

$$d_B(x_1), d_B(x_2), \dots, d_B(x_j), d_B(x_{i+1}), d_B(x_{j+1}) \dots, d_B(x_{i-1}), d_B(x_i)$$

are linearly ordered. By induction on $n-i$, this monomial is equal to a linear combination of special ordered monomials. This shows that the special ordered monomials of degree $2n$ generate $\tilde{\mathcal{E}}_{2n}$.

The second step consists in counting the special ordered monomials : such a monomial is a product

$$\underbrace{\tilde{\gamma}_{x_1} \cdots \tilde{\gamma}_{x_{j_1}}}_{d_B(x_i)=1} \underbrace{\tilde{\gamma}_{x_{j_1+1}} \cdots \tilde{\gamma}_{x_{j_1+j_2}}}_{d_B(x_i)=2} \cdots \underbrace{\tilde{\gamma}_{x_{j_1+\dots+j_{m-1}+1}} \cdots \tilde{\gamma}_{x_{j_1+\dots+j_m}}}_{d_B(x_i)=m}$$

of j_1 elements γ_x with $x \in G - (B \sqcup \{0\})$ and $d_B(x) = 1$, followed by j_2 elements γ_x with $x \in G - (B \sqcup \{0\})$ and $d_B(x) = 2$, and so on, up to a product of j_m elements γ_x with $x \in G - (B \sqcup \{0\})$ and $d_B(x) = m$, where $j_1 + j_2 + \cdots + j_m = n$. An element x of $G - (B \sqcup \{0\})$ such that $d_B(x) = i$ is an element of the form $y + b_i$, where y is a non zero element of $\langle b_1, \dots, b_{i-1} \rangle - \{0\}$. It follows that the number of special ordered monomials of degree $2n$ is equal to

$$\sum_{j_1+j_2+\dots+j_m=n} \prod_{i=1}^m (2^{i-1} - 1)^{j_i} \quad .$$

The only element x in $G - \{0\}$ with $d_B(x) = 1$ is $b_1 \in B$, and this forces $j_1 = 0$. So the number of ordered monomials of degree $2n$ is equal to

$$\sum_{l_1+l_2+\dots+l_{m-1}=n} \prod_{i=1}^{m-1} (2^i - 1)^{l_i} ,$$

where $l_i = j_{i+1}$, for $1 \leq i \leq m-1$. But this is precisely equal to the coefficient of the term of degree $2n$ in the Poincaré series for \mathcal{E} , by Corollary 10.10. It follows that $\dim_k \tilde{\mathcal{E}}_{2n} \leq \dim_k \mathcal{E}_{2n}$, hence $\dim_k \tilde{\mathcal{E}}_{2n} = \dim_k \mathcal{E}_{2n}$, and the map s_{2n} is an isomorphism. This completes the proof of Corollary 13.11, and also the proof of Theorem 1.3. \square

14. Partial results for $G \cong (C_p)^m$, $p > 2$

Let $G \cong (C_p)^m$ be an odd order elementary abelian p -group. The main difference with the case $p = 2$ is that Theorem 10.7 no longer holds, as can be seen from Theorem 6.2 : if $\varphi : G \rightarrow k^+$ is a group homomorphism, let E_φ^G denote the kG -module $k \oplus k$, where the G -action is defined by

$$\forall g \in G, \quad \forall (x, y) \in k^2, \quad g(x, y) = (x + y\varphi(g), y) ,$$

and let T_φ^G denote the unique Mackey functor for G over k such that $T_\varphi^G(H)$ is equal to zero if $\mathbf{1} \neq H \leq G$, and such that $T_\varphi^G(\mathbf{1}) \cong E_\varphi$. Then T_φ^G is a cohomological Mackey functor, and there is a non split exact sequence

$$0 \longrightarrow S_{\mathbf{1}}^G \longrightarrow T_\varphi^G \longrightarrow S_{\mathbf{1}}^G \longrightarrow 0 ,$$

whose class is an element $\tau_\varphi^G \in \text{Ext}_{\mathcal{M}_k^e(G)}^1(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G)$.

If X is a subgroup of order p of G , recall that there is a 2-fold extension $\gamma_X^G \in \text{Ext}_{\mathcal{M}_k^e(G)}^2(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G)$ represented by the short exact sequence

$$0 \longrightarrow S_{\mathbf{1}}^G \longrightarrow \begin{pmatrix} S_X^G \\ S_{\mathbf{1}}^G \end{pmatrix} \longrightarrow \begin{pmatrix} S_{\mathbf{1}}^G \\ S_X^G \end{pmatrix} \longrightarrow S_{\mathbf{1}}^G \longrightarrow 0 .$$

In this case, I propose the following conjecture :

14.1. Conjecture : *Let k be a field of odd characteristic p , and $G \cong (C_p)^m$. Then :*

1. *The algebra $\mathcal{E} = \text{Ext}_{\mathcal{M}_k^e(G)}^*(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G)$ is generated by the elements τ_φ^G in degree 1, for $\varphi \in \text{Hom}_{\mathbb{Z}}(G, k^+)$, and by the elements γ_X^G in degree 2,*

compatible with the Yoneda product. By induction hypothesis, any element in $E_H(n)$ is a linear combination of Yoneda products of elements τ_ψ^H and elements γ_Y^H , where $\psi \in \text{Hom}_{\mathbb{Z}}(H, k^+)$ and $Y \leq H$ with $|Y| = p$.

To show that r_n is surjective, it suffices to show that for any such ψ , there exists $\varphi \in \text{Hom}_{\mathbb{Z}}(G, k^+)$ such that $\tau_\psi^H = r_1(\tau_\varphi^G)$, and that for any such Y , there exists $X \leq G$ with $|X| = p$, such that $\gamma_Y^H = r_2(\gamma_X^G)$. For the latter, the argument of the proof of Corollary 10.10 applies, and one can take $X = Y$. Now if $\psi \in \text{Hom}_{\mathbb{Z}}(H, k^+)$, there exists $\varphi \in \text{Hom}_{\mathbb{Z}}(G, k^+)$ whose restriction to H is equal to ψ , and it is straightforward to check that the restriction to H of the extension T_φ^G defining τ_φ^G is isomorphic to T_ψ^H . It follows that $r_1(\tau_\varphi^G) = \tau_\psi^H$, so r_n is surjective, for any $n \in \mathbb{N}$. Actually this proves more : the submodule $E'_G(n)$ of $E_G(n)$ generated by products of elements τ_φ^G and γ_X^G maps surjectively by r_n on $E_H(n)$. Thus $E'_G(n) + \text{Ker } r_n = E_G(n)$.

Finally, in the case $p = 3$, the long exact sequence 14.3 splits as a series of short ones

$$(14.5) \quad 0 \longrightarrow E_G(n-1) \oplus \bigoplus_{X \in \mathcal{X}} E_G(n-2) \longrightarrow E_G(n) \xrightarrow{r_n} E_H(n) \longrightarrow 0 .$$

The image of each component $E_G(n-2)$ in $E_G(n)$ is obtained by taking Yoneda product with some γ_X^G , and the image of $E_G(n-1)$ is obtained by taking the Yoneda product with the sequence 14.4, which for $p = 3$, is of the form

$$0 \longrightarrow S_1^G \longrightarrow M \longrightarrow S_1^G \longrightarrow 0 .$$

It follows easily that it is isomorphic to a sequence T_φ , where $\varphi \in \text{Hom}_{\mathbb{Z}}(G, k^+)$ has kernel H . So the image of $E_G(n-1)$ in $E_G(n)$ is obtained by taking Yoneda product with τ_φ^G .

An easy induction argument now shows that the kernel of r_n is contained in $E'_G(n)$, so $E'_G(n) = E_G(n)$, completing the inductive step for Assertion 1 of Conjecture 14.1.

Assertion 2 now follows from Sequence 14.5 : this sequence shows that if $P_m(t)$ denotes the Poincaré series for the algebra $\text{Ext}_{\mathbb{M}_k^*(G)}^*(S_1^G, S_1^G)$, where $G \cong (C_p)^m$, then

$$P_m(t) = tP_m(t) + |\mathcal{X}|t^2P_m(t) + P_{m-1}(t) .$$

Thus

$$P_m(t) = \frac{P_{m-1}(t)}{1 - t - (p^{m-1} - 1)t^2} ,$$

and Assertion 2 follows easily by induction. \square

14.6. Remark : The main reason for proposing Conjecture 14.1, apart from Theorem 14.2, is a computer calculation for $p = 5$ or $p = 7$, using GAP

software (<http://www.gap-system.org>), showing that the terms of lower degree of the Poincaré series for the algebra $\text{Ext}_{M_k^c(G)}^*(S_1^G, S_1^G)$ (up to degree 5 for $p = 5$ and $m = 2$) are as predicted by Assertion 2 of Conjecture 14.1.

15. More on extension of simple functors

15.1. Proposition : *Let k be a field of characteristic p , and G be a finite p -group. Let Q and R be normal subgroups of G , and set $N = [Q, R]$. Then for each $n \in \mathbb{N}$, the functor $\rho_{G/N}^G$ induces an isomorphism*

$$\text{Ext}_{M_k^c(G)}^n(S_Q^G, S_R^G) \xrightarrow{\cong} \text{Ext}_{M_k^c(G/N)}^n(S_{Q/N}^{G/N}, S_{R/N}^{G/N}) .$$

Proof : By induction on the order of N : if N is trivial, there is nothing to show. Otherwise, the subgroup N is a non-trivial normal subgroup of G , contained in $Q \cap R$, so N contains a central subgroup Z of order p of G . If X is a complement of Z in Q , then for any $n \in \mathbb{N}$

$$\begin{aligned} \text{Ext}_{M_k^c(G)}^n(S_X^G, S_R^G) &\cong \text{Ext}_{M_k^c(G)}^n(\text{Ind}_{N_G(X)}^G S_X^{N_G(X)}, S_R^G) \\ &\cong \text{Ext}_{M_k^c(N_G(X))}^n(S_X^{N_G(X)}, \text{Res}_{N_G(X)}^G S_R^G) , \end{aligned}$$

and $\text{Res}_{N_G(X)}^G S_R^G = \{0\}$ if $R \not\leq N_G(X)$, by Lemma 6.7. But if $R \leq N_G(X)$, then $[R, X] \leq X$, hence $[R, Q] \leq X$, for $Q = X \cdot Z$ and $Z \leq Z(G)$. It follows that $Z \leq X$, and this contradiction shows that $\text{Ext}_{M_k^c(G)}^n(S_X^G, S_R^G) = \{0\}$ for any $X \in K_Q(Z)$. Then by Theorem 8.6, for any $n \in \mathbb{N}$, the map

$$\pi_n : \text{Ext}_{M_k^c(G)}^n(S_Q^G, S_R^G) \longrightarrow \text{Ext}_{M_k^c(\bar{G})}^n(S_{\bar{Q}}^{\bar{G}}, S_{\bar{R}}^{\bar{G}})$$

induced by $\rho_{G/Z}^G$ is an isomorphism. Now \bar{Q} and \bar{R} are normal subgroups of \bar{G} , and $[\bar{Q}, \bar{R}] = [Q, R]/Z$ has order smaller than $|N|$. By induction, the functor $\rho_{G/N}^{G/Z}$ induces an isomorphism

$$\text{Ext}_{M_k^c(\bar{G})}^n(S_{\bar{Q}}^{\bar{G}}, S_{\bar{R}}^{\bar{G}}) \longrightarrow \text{Ext}_{M_k^c(G/N)}^n(S_{Q/N}^{G/N}, S_{R/N}^{G/N}) ,$$

and the result follows by composition with π_n , since $\rho_{G/N}^{G/Z} \circ \rho_{G/Z}^G \cong \rho_{G/N}^G$. \square

The following result shows that for a p -group G and a field k of characteristic p , the computation of extension groups $\text{Ext}_{M_k^c(G)}^j(S_Q^G, S_R^G)$ comes down

to the case where Q and R are elementary abelian normal subgroups of G , which centralize each other. First a notation :

15.2. Notation : Let G be a finite p -group. If Q and R are subgroups of G , set

$$\mathcal{S}_{R,Q}^G = \{g \in G \mid \Phi(R) \cdot {}^g\Phi(Q) \cdot [R, {}^gQ] \leq R \cap {}^gQ\} .$$

Denote by $\bar{\mathcal{S}}_{R,Q}^G$ the set of $(N_G(R), N_G(Q))$ -double cosets in $\mathcal{S}_{R,Q}^G$, and by $[\bar{\mathcal{S}}_{R,Q}^G]$ a set of representatives of $\bar{\mathcal{S}}_{R,Q}^G$.

Let $\bar{\mathbb{N}} = \mathbb{N} \sqcup \{\infty\}$ be the linearly ordered set obtained by adding to \mathbb{N} a largest element ∞ . Denote by $\nu_G(R, Q) \in \bar{\mathbb{N}}$ the element defined by

$$p^{\nu_G(R,Q)} = \min_{g \in \mathcal{S}_{R,Q}^G} |R \cdot {}^gQ : R \cap {}^gQ| = \min_{g \in [\bar{\mathcal{S}}_{R,Q}^G]} |R \cdot {}^gQ : R \cap {}^gQ| ,$$

if $\mathcal{S}_{R,Q}^G \neq \emptyset$, and by $\nu_G(R, Q) = \infty$ otherwise.

15.3. Theorem : Let k be a field of characteristic p , let G be a finite p -group, and let Q and R be subgroups of G . Then for $g \in \mathcal{S}_{R,Q}^G$:

1. the groups R and gQ normalize each other.
2. the group $\Phi(R) \cdot {}^g\Phi(Q) \cdot [R, {}^gQ]$ is equal to $\Phi(R \cdot {}^gQ)$.
3. the groups $\widehat{R} = R/\Phi(R \cdot {}^gQ)$ and $\widehat{{}^gQ} = {}^gQ/\Phi(R \cdot {}^gQ)$ are elementary abelian normal subgroups of $\widehat{N}_G(R, {}^gQ) = N_G(R, {}^gQ)/\Phi(R \cdot {}^gQ)$, which centralize each other.

Moreover for any $j \in \mathbb{N}$,

$$\text{Ext}_{M_k^c(G)}^j(S_Q, S_R) \cong \bigoplus_{g \in [\bar{\mathcal{S}}_{R,Q}^G]} \text{Ext}_{M_k^c(\widehat{N}_G(R, {}^gQ))}^j \left(S_{\widehat{{}^gQ}}^{\widehat{N}_G(R, {}^gQ)}, S_{\widehat{R}}^{\widehat{N}_G(R, {}^gQ)} \right) ,$$

Proof : If $g \in \mathcal{S}_{R,Q}^G$, then $[R, {}^gQ] \leq R \cap {}^gQ$, so R and gQ normalize each other. Let $T = \Phi(R) \cdot {}^g\Phi(Q) \cdot [R, {}^gQ]$. Then the groups R/T and ${}^gQ/T$ are elementary abelian subgroups of $N_G(R, {}^gQ)/T$, which centralize each other. Thus $(R \cdot {}^gQ)/T$ is elementary abelian, and $T \geq \Phi(R \cdot {}^gQ)$.

Conversely $\Phi(R) \leq \Phi(R \cdot {}^gQ)$, since $R \cdot {}^gQ$ is a p -group, and similarly $\Phi({}^gQ) = {}^g\Phi(Q) \leq \Phi(R \cdot {}^gQ)$. Moreover $[R, {}^gQ] \leq [R \cdot {}^gQ, R \cdot {}^gQ] \leq \Phi(R \cdot {}^gQ)$, so finally $T \leq \Phi(R \cdot {}^gQ)$, hence $T = \Phi(R \cdot {}^gQ)$. Now $\widehat{R} = R/T$ and $\widehat{{}^gQ} = {}^gQ/T$ are elementary abelian normal subgroups of $\widehat{N}_G(R, {}^gQ)$, which centralize each

other.

Set $A = \Phi(Q)$ and $B = N_G(Q)$. Set similarly $C = \Phi(R)$ and $D = N_G(R)$. Then $S_Q^G \cong \text{Ind}_B^G \iota_{B/A}^B(S_{Q/A}^{B/A})$ and $S_R^G = \text{Ind}_D^G j_{D/C}^D(S_{R/C}^{D/C})$, by Lemma 6.5 and Corollary 6.6. Thus, by Proposition 3.16, for any $j \in \mathbb{N}$,

$$\text{Ext}_{M_k^c(G)}^j(S_Q, S_R) \cong \text{Ext}_{M_k^c(D/C)}^j(\rho_{D/C}^D \text{Res}_D^G \text{Ind}_B^G \iota_{B/A}^B(S_{Q/A}^{B/A}), S_{R/C}^{D/C}) .$$

Now by Proposition 3.27,

$$\rho_{D/C}^D \text{Res}_D^G \text{Ind}_B^G \iota_{B/A}^B(S_{Q/A}^{B/A}) \cong \bigoplus_{g \in [D \setminus G/B]} \text{Ind}_{\bar{D}_g}^{\bar{D}} \iota_{\bar{D}_g/\bar{C}_g}^{\bar{D}_g} \text{Iso}(f_g) \rho_{\bar{B}_g/\bar{A}_g}^{\bar{B}_g} \text{Res}_{\bar{B}_g}^{\bar{B}}(S_{Q/A}^{B/A}) ,$$

where

$$\bar{D} = D/C, \quad \bar{D}_g = (D \cap {}^g B)C/C, \quad \bar{C}_g = (D \cap {}^g A)C/C ,$$

$$\bar{B} = B/A, \quad \bar{B}_g = (D^g \cap B)A/A, \quad \bar{A}_g = (C^g \cap B)A/A ,$$

and where $f_g : \bar{B}_g/\bar{A}_g \rightarrow \bar{D}_g/\bar{C}_g$ is the group isomorphism sending $x\bar{A}_g$ to ${}^g x\bar{C}_g$, for $x \in D^g \cap B$.

By Lemma 6.7, the functor $\rho_{\bar{B}_g/\bar{A}_g}^{\bar{B}_g} \text{Res}_{\bar{B}_g}^{\bar{B}}(S_{Q/A}^{B/A})$ is isomorphic to the direct sum of the simple functors $S_{X/\bar{A}_g}^{\bar{B}_g/\bar{A}_g}$ corresponding to subgroup X such that X/A is conjugate to $\bar{Q} = Q/A$ in B/A , up to conjugation by \bar{B}_g/\bar{A}_g . Since $Q \trianglelefteq B$, the only possible such subgroup is Q itself, if $\bar{A}_g \leq \bar{Q} \leq \bar{B}_g$. This gives

$$\text{Ext}_{M_k^c(G)}^j(S_Q, S_R) \cong \bigoplus_{\substack{g \in [D \setminus G/B] \\ \bar{A}_g \leq \bar{Q} \leq \bar{B}_g}} \text{Ext}_{M_k^c(\bar{D})}^j(\text{Ind}_{\bar{D}_g}^{\bar{D}} \iota_{\bar{D}_g/\bar{C}_g}^{\bar{D}_g} \text{Iso}(f_g)(S_{\bar{Q}/\bar{A}_g}^{\bar{B}_g/\bar{A}_g}), S_{R/C}^{D/C}) .$$

By adjunction, this gives

$$\text{Ext}_{M_k^c(G)}^j(S_Q, S_R) \cong \bigoplus_{\substack{g \in [D \setminus G/B] \\ \bar{A}_g \leq \bar{Q} \leq \bar{B}_g}} \text{Ext}_{M_k^c(\bar{D})}^j(\text{Iso}(f_g)(S_{\bar{Q}/\bar{A}_g}^{\bar{B}_g/\bar{A}_g}), \rho_{\bar{D}_g/\bar{C}_g}^{\bar{D}_g} \text{Res}_{\bar{D}_g}^{\bar{D}}(S_{\bar{R}}^{\bar{D}})) ,$$

where $\bar{R} = R/C$. By the same argument, the functor $\rho_{\bar{D}_g/\bar{C}_g}^{\bar{D}_g} \text{Res}_{\bar{D}_g}^{\bar{D}}(S_{\bar{R}}^{\bar{D}})$ is equal to zero, unless $\bar{C}_g \leq \bar{R} \leq \bar{D}_g$, in which case it is isomorphic to $S_{\bar{R}/\bar{C}_g}^{\bar{D}_g/\bar{C}_g}$. Thus

$$\text{Ext}_{M_k^c(G)}^j(S_Q, S_R) \cong \bigoplus_{\substack{g \in [D \setminus G/B] \\ \bar{A}_g \leq \bar{Q} \leq \bar{B}_g \\ \bar{C}_g \leq \bar{R} \leq \bar{D}_g}} \text{Ext}_{M_k^c(\bar{D}_g/\bar{C}_g)}^j(\text{Iso}(f_g)(S_{\bar{Q}/\bar{A}_g}^{\bar{B}_g/\bar{A}_g}), S_{\bar{R}/\bar{C}_g}^{\bar{D}_g/\bar{C}_g}) .$$

Now the condition $\bar{Q} \leq \bar{B}_g$ is equivalent to $Q \leq (D^g \cap B)A$, i.e. to $Q = A(D^g \cap Q)$, i.e. to $D^g \cap Q = Q$, since $A = \Phi(Q)$. In other words $Q \leq D^g$. Similarly, the condition $\bar{R} \leq \bar{D}_g$ is equivalent to $R \leq (D \cap {}^g B)C$, i.e. to $R = (R \cap {}^g B)C$, i.e. to $R = R \cap {}^g B$, since $C = \Phi(R)$. In other words $R \leq {}^g B$.

Then the condition $\bar{A}_g \leq \bar{Q}$ is equivalent to $C^g \cap B \leq Q$. Since moreover $C^g \leq R^g \leq B$, it follows that $C^g \cap B = C^g \leq Q$.

Similarly, the condition $\bar{C}_g \leq \bar{R}$ is equivalent to $D \cap {}^g A \leq R$. But ${}^g A \leq {}^g Q \leq D$, thus $D \cap {}^g A = {}^g A \leq R$.

In this situation

$$\bar{D}_g = (D \cap {}^g B)/C, \quad \bar{C}_g = {}^g A \cdot C/C, \quad \bar{B}_g = (D^g \cap B)/A, \quad \bar{A}_g = C^g \cdot A/A, \quad ,$$

and the isomorphism

$$f_g : \bar{B}_g/\bar{A}_g = (D^g \cap B)/(C^g \cdot A) \longrightarrow (D \cap {}^g B)/(C \cdot {}^g A) = \bar{D}_g/\bar{C}_g$$

is induced by conjugation by g . In particular $f_g(\bar{Q}/\bar{A}_g) = {}^g Q/(C \cdot {}^g A)$, thus finally

$$\text{Ext}_{\mathbb{M}_k^c(G)}^j(S_Q, S_R) \cong \bigoplus_{g \in \mathcal{S}} \text{Ext}_{\mathbb{M}_k^c((D \cap {}^g B)/(C \cdot {}^g A))}^j \left(S_{{}^g Q/(C \cdot {}^g A)}^{(D \cap {}^g B)/(C \cdot {}^g A)}, S_{R/(C \cdot {}^g A)}^{(D \cap {}^g B)/(C \cdot {}^g A)} \right),$$

where

$$\begin{aligned} \mathcal{S} &= \{g \in [D \backslash G/B] \mid C \cdot {}^g A \leq R \cap {}^g Q \leq R \cdot {}^g Q \leq D \cap {}^g B\} \\ &= \{g \in [D \backslash G/B] \mid C \cdot {}^g A \cdot [R, {}^g Q] \leq R \cap {}^g Q\} = [\bar{\mathcal{S}}_{R,Q}^G]. \end{aligned}$$

Moreover, by Proposition 15.1, for $g \in \mathcal{S}$, the group

$$\text{Ext}_{\mathbb{M}_k^c((D \cap {}^g B)/(C \cdot {}^g A))}^j \left(S_{{}^g Q/(C \cdot {}^g A)}^{(D \cap {}^g B)/(C \cdot {}^g A)}, S_{R/(C \cdot {}^g A)}^{(D \cap {}^g B)/(C \cdot {}^g A)} \right)$$

is isomorphic to

$$\text{Ext}_{\mathbb{M}_k^c(\hat{N}_G(R, {}^g Q))}^j \left(S_{{}^g \hat{Q}}^{\hat{N}_G(R, {}^g Q)}, S_{\hat{R}}^{\hat{N}_G(R, {}^g Q)} \right),$$

and this completes the proof. \square

15.4. Proposition : *Let k be a field of characteristic p , let G be a finite p -group, and let Q and R be subgroups of G . Then*

$$n < \nu_G(R, Q) \Rightarrow \text{Ext}_{\mathbb{M}_k^c(G)}^n(S_Q^G, S_R^G) = \{0\} .$$

Proof : If $\nu_G(R, Q) = \infty$, i.e. if $\mathcal{S}_{R,Q}^G = \emptyset$, then $\text{Ext}_{\mathcal{M}_k^c(G)}^n(S_Q^G, S_R^G) = \{0\}$ for any $n \in \mathbb{N}$, by Theorem 15.3. So one can assume $\nu_G(R, Q) \in \mathbb{N}$.

The result is trivial if $G = \mathbf{1}$, so by induction on $|G|$, one can assume that it holds for any group of order less than $|G|$. The group G being given with this property, one can proceed by induction on n , to show that $\text{Ext}_{\mathcal{M}_k^c(G)}^n(S_Q^G, S_R^G) = \{0\}$ if Q and R are subgroups of G with $\nu_G(R, Q) > n$. This starts with the case $n = -1$, where there is nothing to prove, since $\nu_G(R, Q) \geq 0$ and $\text{Ext}_{\mathcal{M}_k^c(G)}^n(S_Q^G, S_R^G) = \{0\}$ for $n < 0$.

For the inductive step, let $n \geq 0$, let Q and R be subgroups of G such that $\nu_G(R, Q) > n$, and let $g \in \mathcal{S}_{R,Q}^G$. Then the index $|R \cdot {}^g Q : R \cap {}^g Q|$ is equal to the index $|R' \cdot Q' : R' \cap Q'|$, where $R' = R / (\Phi(R) \cdot {}^g \Phi(Q))$ and $Q' = {}^g Q / (\Phi(R) \cdot {}^g \Phi(Q))$. By Theorem 15.3, and by induction hypothesis on G , it suffices to consider the case where Q and R are elementary abelian normal subgroups of G , which centralize each other.

In this case, set $j = \nu_G(R, Q)$, so $p^j = |R \cdot Q : R \cap Q|$. If $Q \cap R \neq \mathbf{1}$, let Z be a subgroup of order p of $Q \cap R \cap Z(G)$. By Theorem 8.6, for any integer n , there is a short exact sequence

$$0 \rightarrow \bigoplus_{X \in \mathcal{K}} \text{Ext}_{\mathcal{M}_k^c(G)}^{n-1}(S_X^G, S_R^G) \rightarrow \text{Ext}_{\mathcal{M}_k^c(G)}^n(S_Q^G, S_R^G) \rightarrow \text{Ext}_{\mathcal{M}_k^c(\bar{G})}^n(S_{\bar{Q}}^G, S_{\bar{R}}^G) \rightarrow 0 \quad ,$$

where \mathcal{K} is a set of representatives of G -conjugacy classes of complements of Z in Q , and \bar{Q} and \bar{R} denote Q/Z and R/Z , respectively.

If $X \in \mathcal{K}$, since $R \trianglelefteq G$, the set $N_G(X) \setminus G/N_G(R)$ has cardinality 1. Moreover $X \cap R \neq Q \cap R$, since $Z \leq Q \cap R$ and $Z \not\leq X$, so $|Q \cap R : X \cap R| = p$, and

$$|X \cdot R : X \cap R| = \frac{|Q||R|}{p|X \cap R|^2} = p^{j+1} \quad .$$

In other words $\nu_G(R, X) = j + 1 > n - 1$, so $\text{Ext}_{\mathcal{M}_k^c(G)}^{n-1}(S_X^G, S_R^G) = \{0\}$ by induction hypothesis on n .

On the other hand $\text{Ext}_{\mathcal{M}_k^c(\bar{G})}^n(S_{\bar{Q}}^G, S_{\bar{R}}^G) = 0$ since $n < j$, by induction hypothesis on $|G|$. It follows that $\text{Ext}_{\mathcal{M}_k^c(G)}^n(S_Q^G, S_R^G) = \{0\}$.

It remains to consider the case where $Q \cap R = \mathbf{1}$. Since $\nu_G(R, Q) > n \geq 0$, it follows that at least one of the groups Q or R is non trivial. By symmetry on Q and R , one can assume that $Q \neq \mathbf{1}$, and choose a subgroup Z of order p in $Q \cap Z(G)$.

Now for any $n \in \mathbb{N}$

$$\text{Ext}_{\mathcal{M}_k^c(G)}^n(\iota_{G/Z}^G(S_{Q/Z}^G), S_R^Q) \cong \text{Ext}_{\mathcal{M}_k^c(G/Z)}^n(S_{Q/Z}^{G/Z}, \rho_{G/Z}^G(S_R^G)) = \{0\} \quad ,$$

by Proposition 6.7, since $Z \not\subseteq R$. Applying $\text{Hom}_{M_k^c(G)}(-, S_R^G)$ to the first exact sequence of Proposition 8.5 gives the isomorphisms

$$(15.5) \quad \text{Ext}_{M_k^c(G)}^n(S_Q^G, S_R^G) \cong \bigoplus_{X \in \mathcal{K}} \text{Ext}_{M_k^c(G)}^{n-1}(S_X^G, S_R^G) ,$$

for any $n \in \mathbb{N}$, where \mathcal{K} is a set of representatives of G -conjugacy classes of complements of Z in Q .

Let $X \in \mathcal{K}$, and set $l = \nu_G(R, X)$. Then

$$p^l = |X \cdot R : X \cap R| = |X \cdot R| = |X||R| ,$$

so $l = j - 1$.

If $n < j$, then $n - 1 < l$, hence $\text{Ext}_{M_k^c(G)}^{n-1}(S_X^G, S_R^G) = \{0\}$ by induction hypothesis on n . It follows that $\text{Ext}_{M_k^c(G)}^n(S_Q^G, S_R^G) = \{0\}$, as was to be shown. \square

15.6. Notation : *Let Q be an elementary abelian p -group. Denote by $\text{St}(Q)$ the only non-zero reduced integral homology group of the poset $]\mathbf{1}, Q[$ of proper non trivial subgroups of Q , ordered by inclusion.*

It is well known (see e.g. [3]) that, if Q has rank r , then $\text{St}(Q) = \tilde{H}_{r-2}(]\mathbf{1}, Q[, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{\binom{r}{2}}$. The automorphism group $A \cong \text{GL}(r, \mathbb{F}_p)$ of Q acts on $\text{St}(Q)$, and the module $\text{St}(Q)$ is isomorphic to the Steinberg module of A . The restriction of $\text{St}(Q)$ to a Sylow p -subgroup P of A is a free $\mathbb{Z}P$ -module of rank 1.

When G is an elementary abelian p -group, the following theorem was proved by Tambara ([8] Theorem 4.1) :

15.7. Theorem : *Let k be a field of characteristic p , and G be a finite p -group. Let Q be an elementary abelian normal subgroup of rank q of G , and H be a subgroup of rank h of Q . Then $\nu_G(H, Q) = q - h$ and :*

1. *There is a group isomorphism*

$$\text{Ext}_{M_k^c(G)}^{q-h}(S_Q^G, S_H^G) \cong \text{Ext}_{M_k^c(N_G(H)/H)}^{q-h}(S_{Q/H}^{N_G(H)/H}, S_{H/H}^{N_G(H)/H}) .$$

2. *These groups are isomorphic to the space of coinvariants $k\text{St}(Q/H)_{N_G(H)}$ of $N_G(H)$ on $k \otimes_{\mathbb{Z}} \text{St}(Q/H)$.*

3. *Let $H = H_0 < H_1 < \dots < H_{q-h} = Q$ be a maximal $N_G(H)$ -invariant flag. Then $\text{Ext}_{M_k^c(G)}^{q-h}(S_Q^G, S_H^G)$ has a k -basis indexed by the $N_G(H)$ -conjugacy classes of flags $X_1 > X_2 > \dots > X_{q-h-1}$ of Q such that*

$X_i + H_i = Q$ and $X_i \cap H_i = H$, for $1 \leq i \leq q - h - 1$. In particular

$$\dim_k \text{Ext}_{\mathbf{M}_k^c(G)}^{q-h}(S_Q^G, S_H^G) = \frac{p^{\binom{q-h}{2}}}{|N_G(H) : Z_G(H, Q)|} ,$$

where $Z_G(H, Q) = \{g \in G \mid [g, Q] \leq H\}$ (in other words $Z_G(H, Q)$ is the preimage in $N_G(H)$ of the centralizer of Q/H in $N_G(H)/H$).

Proof : Since $H \leq Q$ and $\Phi(Q) = \mathbf{1}$, it follows that $\Phi(H)\Phi(Q)[H, Q] = \mathbf{1} \leq H \cap Q$, so $\nu_G(H, Q) < \infty$. Moreover $p^{\nu_G(H, Q)} = |Q \cdot H : Q \cap H| = |Q : H|$, since $Q \trianglelefteq G$, so $\nu_G(H, Q) = q - h$.

For Assertion 1, observe first that

$$\text{Ext}_{\mathbf{M}_k^c(G)}^n(S_Q^G, S_H^G) \cong \text{Ext}_{\mathbf{M}_k^c(N_G(H))}^n(S_Q^{N_G(H)}, S_H^{N_G(H)}) ,$$

for any $n \in \mathbb{N}$, and $\nu_{N_G(H)}(H, Q) = q - h$ since $Q \trianglelefteq N_G(H)$. By induction on $|G|$, if $N_G(H) < G$, there is an isomorphism

$$\text{Ext}_{\mathbf{M}_k^c(N_G(H))}^n(S_Q^{N_G(H)}, S_H^{N_G(H)}) \cong \text{Ext}_{\mathbf{M}_k^c(N_G(H)/H)}^n(S_{Q/H}^{N_G(H)/H}, S_{H/H}^{N_G(H)/H}) ,$$

so one can suppose $H \trianglelefteq G$.

If $H = \mathbf{1}$, there is nothing to prove. Otherwise, let Z be a subgroup of order p of $H \cap Z(G)$. By Theorem 8.6, there is an exact sequence

$$0 \rightarrow \bigoplus_{X \in \mathcal{K}} \text{Ext}_{\mathbf{M}_k^c(G)}^{q-h-1}(S_X^G, S_H^G) \rightarrow \text{Ext}_{\mathbf{M}_k^c(G)}^{q-h}(S_Q^G, S_H^G) \xrightarrow{\pi_{q-h}} \text{Ext}_{\mathbf{M}_k^c(\bar{G})}^{q-h}(S_{\bar{Q}}^{\bar{G}}, S_{\bar{H}}^{\bar{G}}) \rightarrow 0 ,$$

in $\mathbf{M}_k^c(G)$, where \mathcal{K} is a set of representatives of G -conjugacy classes of complements of Z in Q , and overlines denote quotients by Z . If $X \in \mathcal{K}$, then

$$p^{\nu_G(H, X)} = |X \cdot H : X \cap H| = \frac{|X||H|}{|X \cap H|^2} = \frac{|Q||H|}{p|X \cap H|^2} = p^{q-h-1}|H : X \cap H|^2 .$$

Since $Z \leq H$ and $Z \cap X = \mathbf{1}$, it follows that $H \not\leq X$, thus $|H : X \cap H| = p$, and $\nu_G(H, X) = q - h + 1$. By Proposition 15.4, the group $\text{Ext}_{\mathbf{M}_k^c(G)}^{q-h-1}(S_X^G, S_H^G)$ is equal to zero. Hence the map π_{q-h} is an isomorphism. By induction on the the order of G , there is an isomorphism

$$\text{Ext}_{\mathbf{M}_k^c(\bar{G})}^{q-h}(S_{\bar{Q}}^{\bar{G}}, S_{\bar{H}}^{\bar{G}}) \cong \text{Ext}_{\mathbf{M}_k^c(G/H)}^{q-h}(S_{Q/H}^{G/H}, S_{H/H}^{G/H}) ,$$

which completes the proof of Assertion 1.

It follows that for Assertions 2 and 3, it suffices to consider the case $H = \mathbf{1}$, i.e. $h = 0$. In this case $H_i \trianglelefteq G$, for $0 \leq i \leq q$. If $q = 0$, i.e. if $Q = \mathbf{1}$,

there is nothing to prove, since $\text{Ext}_{\mathbb{M}_k^c(G)}^0(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G) \cong k$. This allows to start a proof by induction on q .

If $q > 0$, by Proposition 8.5, there is an exact sequence

$$0 \longrightarrow \bigoplus_{X_1 \in \mathcal{K}} S_{X_1}^G \longrightarrow \iota_{G/H_1}^G(S_{Q/H_1}^{G/H_1}) \longrightarrow S_Q^G \longrightarrow 0 ,$$

where \mathcal{K} is a set of representatives of G -conjugacy classes of complements of H_1 in Q . Since $\rho_{G/H_1}^G(S_{\mathbf{1}}^G) = \{0\}$, this gives an isomorphism

$$\text{Ext}_{\mathbb{M}_k^c(G)}^q(S_Q^G, S_{\mathbf{1}}^G) \cong \bigoplus_{X_1 \in \mathcal{K}} \text{Ext}_{\mathbb{M}_k^c(G)}^{q-1}(S_{X_1}^G, S_{\mathbf{1}}^G) .$$

Moreover, for $X_1 \in \mathcal{K}$,

$$\text{Ext}_{\mathbb{M}_k^c(G)}^{q-1}(S_{X_1}^G, S_{\mathbf{1}}^G) \cong \text{Ext}_{\mathbb{M}_k^c(N_G(X_1))}^{q-1}(S_{X_1}^{N_G(X_1)}, S_{\mathbf{1}}^{N_G(X_1)}) .$$

The flag $\mathbf{1} < H_2 \cap X_1 < H_3 \cap X_1 < \dots < H_{q-1} \cap X_1 < X_1$ is a maximal $N_G(X_1)$ -invariant flag in X_1 . By induction hypothesis, the space

$$\text{Ext}_{\mathbb{M}_k^c(N_G(X_1))}^{q-1}(S_{X_1}^{N_G(X_1)}, S_{\mathbf{1}}^{N_G(X_1)})$$

has a k -basis indexed by $N_G(X_1)$ -conjugacy classes of flags

$$X_2 > X_3 > \dots > X_{q-1}$$

such that $X_i \oplus (H_i \cap X_1) = X_1$, for $2 \leq i \leq q-1$. Equivalently $X_2 < X_1$, and $X_i \oplus H_i = Q$, for $2 \leq i \leq q-1$.

Now X_1 runs through a set of representatives of G -conjugacy classes of complements of H_1 in Q , and $X_2 > \dots > X_{q-1}$ runs through a set of representatives of $N_G(X_1)$ -conjugacy classes of flags such that $X_i \oplus H_i = Q$, for $2 \leq i \leq q-1$. Equivalently $X_1 > X_2 > \dots > X_{q-1}$ runs through a set of conjugacy classes of flags of Q such that $X_i \oplus H_i = Q$, for $1 \leq i \leq q-1$. This completes the inductive step, and the first part of Assertion 3 follows.

Now the stabilizer C in G of a flag $X_1 > X_2 > \dots > X_{q-1}$ such that $X_i \oplus H_i = Q$, for $1 \leq i \leq q-1$ stabilizes two opposite maximal flags of Q . Since G is a p -group, it follows that the image of C in the automorphism group of Q is trivial (since it consists of matrices which are both upper and lower triangular with 1's on the diagonal). In other words $C = C_G(Q)$.

There are $p^{\binom{q}{2}}$ maximal flags opposite to a given maximal flag in Q , and the cardinality of each G -conjugacy class of such flags is $|G : C_G(Q)|$. This completes the proof of Assertion 3.

Assertion 2 follows from the fact that $k\text{St}(Q)$ has a k -basis indexed by maximal flags opposite to a given G -invariant maximal flag of Q , and this

basis is permuted by G (see [3] for details). In other words $k\text{St}(Q)$ is a free $k(G/C_G(Q))$ -module. \square

Together with Theorem 15.3, the following theorem allows to compute any extension group between simple functors for a p -group G , from the self extension groups for the simple functor S_1^C , where C runs through some subquotients of G :

15.8. Theorem : *Let k be a field of characteristic p , let G be a finite p -group, and let Q and R be elementary abelian normal subgroups of G which centralize each other. Let q denote the rank of Q and r the rank of R .*

1. *For any $n \in \mathbb{N}$, the group $\text{Ext}_{\mathbb{M}_k^c(G)}^n(S_Q^G, S_R^G)$ is isomorphic to*

$$\left(\bigoplus_{\substack{h \in \mathbb{N} \\ H \leq Q \cap R \\ \text{rank}(H)=h}} k\text{St}(R/H) \otimes_k \text{Ext}_{\mathbb{M}_k^c(Z_H)}^{n+2h-q-r}(S_1^{Z_H}, S_1^{Z_H}) \otimes_k k\text{St}(Q/H) \right)_G,$$

where $Z_H = Z_G(H, Q \cdot R)/H = \{g \in G \mid [g, Q \cdot R] \leq H\}/H$, and the G -subscript denotes the space of coinvariants.

2. *In particular, the dimension of the space $\text{Ext}_{\mathbb{M}_k^c(G)}^n(S_Q^G, S_R^G)$ is equal to*

$$\sum_{\substack{h \in \mathbb{N} \\ H \leq Q \cap R, \text{ mod. } G \\ \text{rank}(H)=h}} \frac{p^{\binom{q-h}{2} + \binom{r-h}{2}}}{|N_G(H) : Z_G(H, Q \cdot R)|} \dim_k \text{Ext}_{\mathbb{M}_k^c(Z_H)}^{n+2h-q-r}(S_1^{Z_H}, S_1^{Z_H}).$$

Proof : For Assertion 1, when Q and R are both trivial, there is nothing to prove. This allows for a proof by induction on $|Q||R|$. By symmetry on Q and R , one can assume that $Q \neq \mathbf{1}$, and choose a subgroup Z of order p of $Q \cap Z(G)$. There are two cases :

• if $Z \not\leq R$, then $\rho_{G/Z}^G(S_R^G) = \{0\}$ by Lemma 6.7, and the exact sequence of Proposition 8.5 gives isomorphisms

$$\text{Ext}_{\mathbb{M}_k^c(G)}^n(S_Q^G, S_R^G) \cong \bigoplus_{X \in \mathcal{K}} \text{Ext}_{\mathbb{M}_k^c(G)}^{n-1}(S_X^G, S_R^G),$$

for any $n \in \mathbb{N}$, where $\mathcal{K} = [G \setminus K_Q(Z)]$ is a set of representatives of G conjugacy classes of complements of Z in Q . Now for each $X \in \mathcal{K}$

$$\text{Ext}_{\mathbb{M}_k^c(G)}^{n-1}(S_X^G, S_R^G) \cong \text{Ext}_{\mathbb{M}_k^c(N_G(X))}^{n-1}(S_X^{N_G(X)}, S_R^{N_G(X)}).$$

Since $|X||R| = |Q||R|/p$, by induction hypothesis, this group is isomorphic to

$$\left(\bigoplus_{\substack{h \in \mathbb{N} \\ H \leq X \cap R \\ \text{rank}(H)=h}} k\text{St}(R/H) \otimes_k \text{Ext}_{\mathcal{M}_k^c(Z_H^X)}^{n-1+2h-x-r}(S_1^{Z_H^X}, S_1^{Z_H^X}) \otimes_k k\text{St}(X/H) \right)_{N_G(X)},$$

where $x = q - 1$ is the rank of X , and

$$Z_H^X = \{g \in N_G(X) \mid [g, X \cdot R] \leq H\} / H .$$

But if $g \in G$, then $[g, X \cdot R] \leq H$ if and only if $[g, Q \cdot R] \leq H$, since $Q = X \cdot Z$ and $Z \leq Z(G)$. And if $[g, Q \cdot R] \leq H$, then

$$[g, X] \leq [g, Q] \leq [g, Q \cdot R] \leq H \leq X ,$$

so $g \in N_G(X)$. It follows that

$$Z_H^X = \{g \in G \mid [g, Q \cdot R] \leq H\} / H = Z_H .$$

Finally, the group $\text{Ext}_{\mathcal{M}_k^c(G)}^n(S_Q^G, S_R^G)$ is isomorphic to

$$\left(\bigoplus_{\substack{X \in K_Q(Z) \\ h \in \mathbb{N} \\ H \leq X \cap R \\ \text{rank}(H)=h}} k\text{St}(R/H) \otimes_k \text{Ext}_{\mathcal{M}_k^c(Z_H)}^{n+2h-q-r}(S_1^{Z_H}, S_1^{Z_H}) \otimes_k k\text{St}(X/H) \right)_G .$$

Exchanging the order of summation, this gives

$$\left(\bigoplus_{\substack{h \in \mathbb{N} \\ H \leq Q \cap R \\ \text{rank}(H)=h}} k\text{St}(R/H) \otimes_k \text{Ext}_{\mathcal{M}_k^c(Z_H)}^{n+2h-q-r}(S_1^{Z_H}, S_1^{Z_H}) \otimes_k \left(\bigoplus_{\substack{X \in K_Q(Z) \\ X \geq H}} k\text{St}(X/H) \right) \right)_G .$$

Now $Z \cap H = \mathbf{1}$ since $Z \cap R = \mathbf{1}$, hence the conditions $X \in K_Q(Z)$ and $X \geq H$ are equivalent to $X \cdot (ZH) = Q$ and $X \cap (ZH) = H$, i.e. to $H \leq X \leq Q$ and $X/H \in K_{Q/H}(ZH/H)$. By classical combinatorial results,

$$\bigoplus_{X/H \in K_{Q/H}(ZH/H)} k\text{St}(X/H) \cong \text{St}(Q/H) ,$$

thus $\text{Ext}_{\mathbb{M}_k^c(G)}^n(S_Q^G, S_R^G)$ is isomorphic to

$$\left(\bigoplus_{\substack{h \in \mathbb{N} \\ H \leq Q \cap R \\ \text{rank}(H)=h}} k\text{St}(R/H) \otimes_k \text{Ext}_{\mathbb{M}_k^c(Z_H)}^{n+2h-q-r}(S_1^{Z_H}, S_1^{Z_H}) \otimes_k k\text{St}(Q/H) \right)_G,$$

as was to be shown.

- If $Z \leq R$, then by Theorem 8.6, for any $n \in \mathbb{N}$, there is an exact sequence

$$0 \rightarrow \underbrace{\bigoplus_{X \in \mathcal{K}} \text{Ext}_{\mathbb{M}_k^c(G)}^{n-1}(S_X^G, S_R^G)}_{\Sigma} \rightarrow \text{Ext}_{\mathbb{M}_k^c(G)}^n(S_Q^G, S_R^G) \rightarrow \text{Ext}_{\mathbb{M}_k^c(\bar{G})}^n(S_{\bar{Q}}^{\bar{G}}, S_{\bar{R}}^{\bar{G}}) \rightarrow 0,$$

in $\mathbb{M}_k^c(G)$, where $\mathcal{K} = [G \setminus K_Q(Z)]$ as above, and overlines denote quotients by Z . It follows that $\text{Ext}_{\mathbb{M}_k^c(G)}^n(S_Q^G, S_R^G) \cong \Sigma \oplus \text{Ext}_{\mathbb{M}_k^c(\bar{G})}^n(S_{\bar{Q}}^{\bar{G}}, S_{\bar{R}}^{\bar{G}})$.

The same argument as above shows that the space Σ is isomorphic to **(15.9)**

$$\left(\bigoplus_{\substack{h \in \mathbb{N} \\ H \leq Q \cap R, H \cap Z = 1 \\ \text{rank}(H)=h}} k\text{St}(R/H) \otimes_k \text{Ext}_{\mathbb{M}_k^c(Z_H)}^{n+2h-q-r}(S_1^{Z_H}, S_1^{Z_H}) \otimes_k k\text{St}(Q/H) \right)_G.$$

On the other hand, by induction hypothesis, the space $\text{Ext}_{\mathbb{M}_k^c(\bar{G})}^n(S_{\bar{Q}}^{\bar{G}}, S_{\bar{R}}^{\bar{G}})$ is isomorphic to

$$\left(\bigoplus_{\substack{\bar{h} \in \mathbb{N} \\ \bar{H} \leq \bar{Q} \cap \bar{R} \\ \text{rank}(\bar{H})=\bar{h}}} k\text{St}(\bar{R}/\bar{H}) \otimes_k \text{Ext}_{\mathbb{M}_k^c(Z_{\bar{H}})}^{n+2\bar{h}-\bar{q}-\bar{r}}(S_1^{Z_{\bar{H}}}, S_1^{Z_{\bar{H}}}) \otimes_k k\text{St}(\bar{Q}/\bar{H}) \right)_{\bar{G}},$$

where $\bar{q} = q - 1$ and $\bar{r} = r - 1$. Summing over \bar{H} is equivalent to summing over $H \leq Q \cap R$ with $H \geq Z$. If H has rank h , then $\bar{H} = H/Z$ has rank $\bar{h} = h - 1$, and $n + 2\bar{h} - \bar{q} - \bar{r} = n + 2h - q - r$. Moreover $\bar{R}/\bar{H} \cong R/H$, and $\bar{Q}/\bar{H} \cong Q/H$. Also

$$Z_{\bar{H}} = \{\bar{g} \in \bar{G} \mid [\bar{g}, \bar{Q} \cdot \bar{R}] \leq \bar{H}\} / \bar{H} \cong \{g \in G \mid [g, Q \cdot R] \leq H\} / H = Z_H.$$

Thus $\text{Ext}_{M_k^c(\bar{G})}^n(S_{\bar{Q}}^G, S_{\bar{R}}^G)$ is isomorphic to

$$(15.10) \quad \left(\bigoplus_{\substack{h \in \mathbb{N} \\ Z \leq H \leq Q \cap R \\ \text{rank}(H)=h}} k\text{St}(R/H) \otimes_k \text{Ext}_{M_k^c(Z_H)}^{n+2h-q-r}(S_{\mathbf{1}}^{Z_H}, S_{\mathbf{1}}^{Z_H}) \otimes_k k\text{St}(Q/H) \right)_G .$$

Now the direct sum of 15.9 and 15.10 is isomorphic to the expression in Theorem 15.8, as was to be shown for Assertion 1.

It follows that the dimension of the space $\text{Ext}_{M_k^c(G)}^n(S_Q^G, S_R^G)$ is equal to

$$\sum_{\substack{h \in \mathbb{N} \\ H \leq Q \cap R, \text{ mod } G \\ \text{rank}(H)=h}} \dim_k \left(k\text{St}(R/H) \otimes_k \text{Ext}_{M_k^c(Z_H)}^{n+2h-q-r}(S_{\mathbf{1}}^{Z_H}, S_{\mathbf{1}}^{Z_H}) \otimes_k k\text{St}(G/H) \right)_{N_G(H)} .$$

The image of the group $N_G(H)$ in the automorphism group of R/H is isomorphic to $N_G(H)/Z_G(H, R)$, and $k\text{St}(R/H)$ is a free $kN_G(H)/Z_G(H, R)$ -module. Similarly the module $k\text{St}(Q/H)$ is a free $kN_G(H)/Z_G(H, Q)$ -module. It follows that the module $V = k\text{St}(R/H) \otimes_k k\text{St}(Q/H)$ is a free $kN_G(H)/Z$ -module, where $Z = Z_G(H, R) \cap Z_G(H, Q) = Z_G(H, Q \cdot R)$.

Moreover since $Z_H = Z/H$, the group Z acts trivially on

$$W = \text{Ext}_{M_k^c(Z_H)}^{n+2h-q-r}(S_{\mathbf{1}}^{Z_H}, S_{\mathbf{1}}^{Z_H}) ,$$

so W is a $kN_G(H)/Z$ -module. Hence $V \otimes W$ is a free $kN_G(H)/Z$ -module, and it follows that $\dim_k(V \otimes W)_{N_G(H)} = \frac{\dim_k(V \otimes W)}{|N_G(H)/Z|}$, which completes the proof of Assertion 2. \square

In the case where G is elementary abelian, the following has been proved by Tambara ([8] Theorem B) :

15.11. Corollary : *Let k be a field of characteristic p , let G be a finite p -group, and let Q and R be subgroups of G , such that $\nu_G(R, Q) < \infty$. Then $\text{Ext}_{M_k^c(G)}^{\nu_G(R, Q)}(S_Q^G, S_R^G)$ is isomorphic to*

$$\bigoplus_{g \in \mathcal{M}_{Q,R}^G} \left(k\text{St}(R/(R \cap {}^g Q)) \otimes_k k\text{St}({}^g Q/(R \cap {}^g Q)) \right)_{N_G(R, {}^g Q)} ,$$

where $\mathcal{M}_{Q,R}^G = \{g \in [\bar{\mathcal{S}}_{R,Q}^G] \mid |R \cdot {}^g Q : R \cap {}^g Q| = p^{\nu_G(R, Q)}\}$.

In particular $\nu_G(R, Q) = \min\{n \in \mathbb{N} \mid \text{Ext}_{M_k^c(G)}^n(S_Q^G, S_R^G) \neq \{0\}\}$.

Proof : If $n < \nu_G(R, Q)$, then $\text{Ext}_{\mathbf{M}_k^c(G)}^n(S_Q^G, S_R^G) = \{0\}$, by Proposition 12.4. Suppose that $n = \nu_G(R, Q)$, and let $g \in [\bar{S}_{R,Q}^G]$ such that, with the notation of Theorem 15.3,

$$\text{Ext}_{\mathbf{M}_k^c(\widehat{N}_G(R, {}^gQ))}^n(S_{\widehat{gQ}}^{\widehat{N}_G(R, {}^gQ)}, S_{\widehat{R}}^{\widehat{N}_G(R, {}^gQ)}) \neq 0 .$$

It follows that $p^n \geq p^{\nu_{\widehat{N}_G(R, {}^gQ)}(\widehat{R}, \widehat{gQ})} = |R \cdot {}^gQ : R \cap {}^gQ| \geq p^{\nu_G(R, Q)}$. Hence $n = \nu_G(R, Q) = |R \cdot {}^gQ : R \cap {}^gQ|$.

Moreover, in the expression of $\text{Ext}_{\mathbf{M}_k^c(\widehat{N}_G(R, {}^gQ))}^n(S_{\widehat{gQ}}^{\widehat{N}_G(R, {}^gQ)}, S_{\widehat{R}}^{\widehat{N}_G(R, {}^gQ)})$ given by Theorem 15.8, the non zero terms correspond to subgroups H of $\widehat{R} \cap \widehat{gQ}$ such that $n + 2h - q - r \geq 0$, where h , q , and r are the ranks of H , \widehat{gQ} and \widehat{R} respectively. Thus

$$|H|^2 \geq |{}^g\widehat{Q}||\widehat{R}|/p^n = |{}^g\widehat{Q}||\widehat{R}||/\widehat{R} \cdot {}^g\widehat{Q}| = |\widehat{R} \cap {}^g\widehat{Q}|^2 .$$

It follows that $H = \widehat{R} \cap {}^g\widehat{Q}$, that $\widehat{R}/H \cong R/R \cap {}^gQ$, that ${}^g\widehat{Q}/H \cong {}^gQ/R \cap {}^gQ$, and that $n + 2h - q - r = 0$, so $\text{Ext}_{\mathbf{M}_k^c(Z_H)}^0(S_1^{Z_H}, S_1^{Z_H}) \cong k$. This completes the proof. \square

15.12. Remark : Let G be an abelian p -group, and Q, R be elementary abelian subgroups of G , of rank q and r , respectively. In this case, by Assertion 1 of Theorem 15.3, the group $\text{Ext}_{\mathbf{M}_k^c(G)}^n(S_Q^G, S_R^G)$ is isomorphic to

$$\bigoplus_{\substack{h \in \mathbb{N} \\ H \leq Q \cap R \\ \text{rank}(H)=h}} k\text{St}(R/H) \otimes_k \text{Ext}_{\mathbf{M}_k^c(G/H)}^{n+2h-q-r}(S_1^{G/H}, S_1^{G/H}) \otimes_k k\text{St}(Q/H) .$$

Fix $H \leq Q \cap R$, of rank h , and denote by i the inclusion map

$$k\text{St}(R/H) \otimes_k \text{Ext}_{\mathbf{M}_k^c(G/H)}^{n+2h-q-r}(S_1^{G/H}, S_1^{G/H}) \otimes_k k\text{St}(Q/H) \hookrightarrow \text{Ext}_{\mathbf{M}_k^c(G)}^n(S_Q^G, S_R^G).$$

It is easy to check that this map can be explicitly described as follows : if $y \in k\text{St}(R/H) \cong \text{Ext}_{\mathbf{M}_k^c(G)}^{r-h}(S_H^G, S_R^G)$, if $e \in \text{Ext}_{\mathbf{M}_k^c(G/H)}^{n+2h-q-r}(S_1^{G/H}, S_1^{G/H})$, and if $x \in k\text{St}(Q/H) \cong \text{Ext}_{\mathbf{M}_k^c(G)}^{q-h}(S_Q^G, S_H^G)$, then $i(y \otimes e \otimes x)$ is equal to the Yoneda composition $y \circ \text{Inf}_{G/H}^G e \circ x$.

In terms of morphisms in the derived category of $\mathbf{M}_k^c(G)$, it means that any morphism from S_Q^G to some translate of S_R^G is a linear combination of morphisms which factor through a morphism of the form $\text{Inf}_{G/H}^G e$, where $H \leq Q \cap R$, and e is a morphism from $S_1^{G/H}$ to some of its translate.

In the case $p = 2$ and G is elementary abelian, this leads to an explicit description of all the Yoneda products of extensions of simple cohomological Mackey functors.

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