Classification of functors between categories of G-sets

S. Bouc

Abstract. Let G and H be finite groups. I give a one to one correspondence between the isomorphism classes of functors from G-**sets** to H-**sets** which preserve disjoint unions ans cartesian squares, and isomorphism classes of H-sets-G. The composition of functors leads to a new product on sets with a double action, which is a subset of the usual one. The associated double Burnside rings and categories are better behaved than the usual ones with respect to semi-simplicity. I give some applications to Hochschild (co)-homology and to Thévenaz-Webb's version of Alperin's conjecture.

1. Introduction

Let G be a finite group, and R be a commutative ring. There are several equivalent definitions of Mackey functors for G with values in the category R-**mod** of left R-modules (cf. [**5**]), and one of them is in terms of the category G-**set** of finite sets with a left G-action: a Mackey functor M for G with values in R-**mod** is a bifunctor from G-**set** to R-**mod**, i.e. a couple $M = (M_*, M^*)$ of functors with M_* covariant and M^* contravariant, which coïncide on objets (i.e. $M_*(X) = M^*(X)$ for any G-set X). This bifunctor is supposed to have the following two properties:

1. It transforms disjoint unions into direct sums: if $X \coprod Y$ is the disjoint union of X and Y, and i_X and i_Y are the canonical injections from X and Y into $X \coprod Y$, then the morphisms $M_*(i_X) \oplus M_*(i_Y)$ and $M^*(i_X) \oplus M^*(i_Y)$ are mutual inverse isomorphisms from $M(X) \oplus M(Y)$ onto $M(X \coprod Y)$.
2. It commutes on cartesian squares: if

$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ Z & \xrightarrow[d]{} & T \end{array}$$

is a cartesian (or pullback) square, then

$$M_*(b)M^*(a) = M^*(d)M_*(c)$$

A morphism of Mackey functors is a natural transformation of bifunctors. The Mackey functors for G in R-**mod** form an abelian category $Mack_R(G)$.

This definition suggests the following question: if G and H are two groups, are there functors F from G-**set** to H-**set** such that if M is a Mackey functor for H,

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then the composite functor $M \circ F$ is a Mackey functor for G? There are well known examples of that situation: for instance, if H is a subgroup of G, and M a Mackey functor for H, then the induced functor $\mathrm{Ind}_H^G M$ is defined over the G-set X as

$$(\mathrm{Ind}_H^G M)(X) = M(\mathrm{Res}_H^G X)$$

and so it is composed of M and the restriction functor from G-**set** to H-**set**.

In order to generalize that situation, it seems natural to look for functors F from G-**set** to H-**set** with the following two properties:

1. The functor F transforms disjoint unions into disjoint unions: with the same notation as above, the map $F(i_X)\coprod F(i_Y)$ is a bijection of $F(X)\coprod F(Y)$ into $F(X\coprod Y)$.
2. The functor F transforms a cartesian square into a cartesian square.

The object of the present work is to give a classification of such functors F.

2. Sets with a double action

2.1. Definitions. *Let G and H be two groups. A set with a double action, or G-set-H, is a set U with a left G-action and a right H-action which commute, i.e. are such that*

$$(g.u).h = g.(u.h) \qquad \forall (g,u,h) \in G \times U \times H$$

It turns out that the (isomorphism classes of) functors F from G-**set** to H-**set** which preserve disjoint unions and pullback squares are in one to one correspondence whith the (isomorphism classes of) finite H-sets-G. In order to give a precise statement, I need to define the following product on sets with a double action: if G, H and K are three groups, if U is a G-set-H, and V a H-set-K, then let

$$U \circ V = \{(u,v) \in U \times V \mid \forall h \in H,\ u.h = u \Rightarrow \exists k \in K,\ h.v = v.k\}$$

It is the set of couples (u,v) such that the left orbit of v under the right stabilizer of u is contained in the right orbit of v.

The group H acts on the right on $U \circ V$ by

$$(u,v).h = (u.h, h^{-1}.v)$$

and I define $U \circ_H V$ as the quotient set

$$U \circ_H V = (U \circ V)/H$$

The set $U \circ_H V$ is a G-set-K with the following double action

$$g.(u,v).k = (g.u, v.k)$$

It is a subset of the usual reduced product $U \times_H V$, but it is a strict subset in general.

2.2. Classification of functors. Let G and H be two groups, and U be a finite H-set-G. If X is a G-set, or a G-set-$\{1\}$, then $U \circ_G X$ is a H-set-$\{1\}$, i.e. an H-set. This provides a functor $U \circ_H -$ from G-**set** to H-**set**. The precise statement mentioned above is the following:

THEOREM 2.1. *Let G and H be two groups. If F is a functor from G-**set** to H-**set** which preserves disjoint unions and cartesian squares, then there exists an H-set-G U, unique up to isomorphism of H-sets-G, such that F is isomorphic to the functor $U \circ_G -$. Conversally, if U is a H-set-G, then the functor $U \circ_G -$ preserves disjoint unions and cartesian squares.*

The following corollary is probably well-known:

Corollary 2.2. *Let G and H be finite groups. Then the categories G-**set** and H-**set** are equivalent if and only if G and H are isomorphic.*

2.3. Mackey formula.

Notation 2.3. If G and H are groups, and L is a subgroup of the product $G\times H$, then let $p_1(L)$ (resp. $p_2(L)$) be the projection of L in G (resp. in H), and let $k_1(L)$ (resp. $k_2(L)$) be the (projection of the) intersection of L with $G\times\{1\}$ (resp. with $\{1\}\times H$), i.e.

$$k_1(L)=\{g\in G\mid (g,1)\in L\}$$

Let $(G\times H)/L$ be the set of left classes of L in $G\times H$, viewed as a G-set-H with the following double action

$$g.(x,y)L.h=(gx,h^{-1}y)L$$

If K is another group, and M is a subgroup of $H\times K$, let $L*M$ the subgroup of $G\times K$ defined by

$$L*M=\{(g,k)\in G\times K\mid \exists h\in H,\ (g,h)\in L,\ (h,k)\in M\}$$

With those notations:

Proposition 2.4. *Let G, H and K be finite groups. If L is a subgroup of $G\times H$ and M is a subgroup of $H\times K$, then*

$$(G\times H)/L\circ_H (H\times K)/M=\sum_{\substack{x\in p_2(L)\backslash H/p_1(M)\\ k_2(L)^x\subseteq p_1(M)}}(G\times K)/(L*{}^{(x,1)}M)$$

Remark 2.5. There is an analogue formula for the usual product $\times_H$, obtained by forgetting the second summation condition $k_2(L)^x\subseteq p_1(M)$.

Proposition 2.6. *Let G and H be finite groups, and L be a subgroup of $H\times G$. If X is a G-set, then $X^{k_2(L)}$ has a natural structure of $p_1(L)$-set, and*

$$\Big((H\times G)/L\Big)\circ_G X\simeq \mathrm{Ind}_{p_1(L)}^H X^{k_2(L)}$$

Example 2.7. 1) Let H be a subgroup of G. If U is the group G, viewed as an H-set-G, then the functor $U\circ_G -$ is isomorphic to the restriction functor from G-**set** to H-**set**. It leads to the induction functor between the corresponding categories of Mackey functors. If V is the group G, viewed as a G-set-H, then the functor $V\circ_H -$ is the induction functor from G-**set** to H-**set**, which gives the restriction functor from $Mack_R(G)$ to $Mack_R(H)$.
2) Let N be a normal subgroup of G, and $H=G/N$. If U is the group H, viewed as an H-set-G, then the functor $U\circ_G -$ is isomorphic to the fixed points functors $X\mapsto X^N$ from G-**set** to H-**set**. The corresponding functor between categories of Mackey functors is the inflation functor from $Mack_R(H)$ to $Mack_R(G)$. If V is the group H, viewed as an G-set-H, then the functor $V\circ_H -$ is isomorphic to the inflation functor from H-**set** to G-**set**, and is associated to the co-inflation functor from $Mack_R(G)$ to $Mack_R(H)$.
3) If θ is an isomorphism of G on H, then H is an H-set-G with the right action of G given by θ. It corresponds of course to the transport by θ of G-**set** to H-**set**, and of $Mack_R(H)$ to $Mack_R(G)$.

4) It's easy to see that any functor from G-**set** to H-**set** preserving disjoint unions and cartesian squares is a disjoint union of functors which are composed functors of the previous five types.

3. The associated double Burnside ring

The product on sets with a double action defined in the previous section is associative and left and right distributive with respect to disjoint unions. The products $\circ_G$ gives a ring structure on the Grothendieck group $\Gamma(G)$ of G-sets-G, which is a kind of double Burnside ring. The unit element of that ring is the group G, viewed as a G-set-G by left and right multiplication.

3.1. A subring of $\Gamma(G)$. Let X and M be subgroups of G such that $X \subseteq N_G(M)$. Let

$$\Delta_{X,M} = \{(a,b) \in G \times G \mid a \in X,\ ab^{-1} \in M\}$$

It is a subgroup of $G \times G$ if $X \subseteq N_G(M)$. Let

$$t_{X,M} = (G \times G)/\Delta_{X,M}$$

LEMMA 3.1. *Let (X,M) and (Y,N) be couples of subgroups of G such that $X \subseteq N_G(M)$ and $Y \subseteq N_G(N)$. Then for any $x \in G$, the group $X \cap {}^xY$ normalizes $M.{}^xN$, which is a subgroup of G if $M^x \subseteq Y$. Moreover*

$$t_{X,M} \circ_G t_{Y,N} = \sum_{\substack{x \in X\backslash G/Y \\ M^x \subseteq Y}} t_{X \cap {}^xY, M.{}^xN}$$

This lemma shows that the elements $t_{X,N}$ generate a subring of $\Gamma(G)$.

3.2. Projectors in the Burnside ring. The usual Burnside ring $b(G)$, which is the Grothendieck group of G-**set**, has a natural structure of $\Gamma(G)$-module, obtained by extending the map $X \mapsto U \circ_G X$ by linearity. This gives a natural morphism Φ of rings from $\Gamma(G)$ into $\mathrm{End}_Z\Big(b(G)\Big)$.

In [**1**], I introduced orthogonal projectors of the Burnside ring, associated to some families $\underline{F}$ of subgroups of G having the following properties:

1. The family $\underline{F}$ is stable under G-conjugacy.
2. The family $\underline{F}$ is stable under product: if H and K are in $\underline{F}$, and if $H \subseteq N_G(K)$, then $H.K \in \underline{F}$.
3. The trivial subgroup is in $\underline{F}$.

The projector associated to the element P of $\underline{F}$ could be defined by

$$\pi_P^G(X) = -\sum_{\substack{\inf\ s = P \\ s \in Sd(\underline{F})/N_G(P)}} (-1)^{|s|} \mathrm{Ind}_{N_G(s)}^G X^{\sup\ s}$$

where $Sd(\underline{F})$ is the set of strictly increasing sequences of elements of $\underline{F}$.

A natural question is then to ask if those endomorphisms of $b(G)$ can be lifted through Φ to an orthogonal family of idempotents of $\Gamma(G)$. Of course, the answer is yes:

PROPOSITION 3.2. *Let P an element of $\underline{F}$. Define $E_P^G \in \Gamma(G)$ by*

$$E_P^G = -\sum_{\substack{\inf\ s = P \\ s \in Sd(\underline{F})/N_G(P)}} (-1)^{|s|} t_{N_G(s), \sup s}$$

Then $\Phi(E_P^G) = \pi_P^G$*, and the elements* E_P^G*, for* $P \in \underline{F}/G$*, form an orthogonal family of idempotents of* $\Gamma(G)$*, and their sum is the identity.*

3.3. A morphism from $b(G)$ to $\Gamma(G)$. The natural morphism from $b(G)$ to $\mathrm{End}_Z\Big(b(G)\Big)$ induced by multiplication can also be lifted to a ring morphism from $b(G)$ to $\Gamma(G)$:

LEMMA 3.3. *Let* X *be a* G*-set, and* $\widetilde{X}$ *be the set* $G \times X$*, viewed as a* G*-set-*G *by*

$$g.(a,x).g' = (gag', g.x)$$

Then the map $X \mapsto \widetilde{X}$ *extends to a ring homomorphism from* $b(G)$ *to* $\Gamma(G)$*, and for any* G*-set* Y

$$\widetilde{X} \circ_G Y = \widetilde{X} \times_G Y \simeq X \times Y$$

The main consequence of this lemma is that any decomposition of unity into orthogonal idempotents in $b(G)$ can be transported to $\Gamma(G)$. In particular, if $\mathcal{K}$ is a field of characteristic 0 or coprime to the order of G, then the Burnside ring $b_{\mathcal{K}}(G) = \mathcal{K} \otimes_Z b(G)$ is semi-simple, and there are explicit formulae, due to Gluck ([**3**]), giving its primitive idempotents: they are indexed by the conjugacy classes of subgroups of G, and the idempotent e_H^G associated to the class of the subgroup H is given by

$$e_H^G = \frac{1}{|N_G(H)|}\sum_{K \subseteq H} |K|\tilde{\chi}]K,H[.(G/K)$$

where $\tilde{\chi}]K,H[$ is the reduced Euler-Poincaré characteristic of the "open interval" $]K,H[$ in the poset of subgroups of G.

The family $\widetilde{e_H^G}$ gives then another decomposition of unity in orthogonal idempotents in the ring $\Gamma_{\mathcal{K}}(G) = \mathcal{K} \otimes \Gamma(G)$.

3.4. The idempotents $F_{K,H}^G$. Taking for $\underline{F}$ the family of all subgroups of G, I have now two different decomposition of unity in orthogonal idempotents in $\Gamma_{\mathcal{K}}(G)$ (namely the E_H^G and the $\widetilde{e_H^G}$). It can be shown that the products $\widetilde{e_G^G} \circ_G E_H^G$ are zero unless H is a normal subgroup of G, and a few transformations lead to the following proposition:

PROPOSITION 3.4. *Let* $\mathcal{K}$ *be a field of characteristic 0 or coprime to* $|G|$*. Let* $H \trianglelefteq K$ *be subgroups of* G*. Define*

$$F_{K,H}^G = \frac{1}{|N_G(K,H)|}\sum_{\substack{X \subseteq K \\ H \subseteq N \trianglelefteq K}} |X|\tilde{\chi}]X,K[\tilde{\chi}]H,N[^K.t_{X,N}$$

where $\tilde{\chi}]H,N[^K$ *is the reduced Euler-Poincaré characteristic of the "open interval"* $]H,N[^K$ *in the poset of normal subgroups of* K*.*

Then the $F_{K,H}^G$*, as* (K,H) *runs through a set of representatives of conjugacy classes of couples of subgroups of* G *such that* $H \trianglelefteq K$*, are a set of mutually orthogonal idempotents of* $\Gamma_{\mathcal{K}}(G)$*, and their sum is the identity.*

3.5. Central idempotents of $\Gamma_{\mathcal{K}}(G)$. The sum of idempotents $F^G_{K,H}$ associated to couples (K,H) for which K/H is isomorphic to a given group is a central idempotent of $\Gamma_{\mathcal{K}}(G)$:

PROPOSITION 3.5. *Let $H \trianglelefteq K$ and $H' \trianglelefteq K'$ be subgroups of G. If*

$$F^G_{K',H'} \circ_G \Gamma_{\mathcal{K}}(G) \circ_G F^G_{K,H} \neq 0$$

then the quotients K/H and K'/H' are isomorphic.

COROLLARY 3.6. *Let S be a finite group. Define*

$$B_S = \sum_{\substack{(K,H)\, mod.\, G \\ H \trianglelefteq K \\ K/H \simeq S}} F^G_{K,H}$$

Then the B_S, as S runs through a set of representatives of isomorphism classes of sections of G, are mutually orthogonal idempotent of the center of $\Gamma_{\mathcal{K}}(G)$, and their sum is the identity.

3.6. Identification of $\Gamma_K(G)$. The previous proposition provides a decomposition of $\Gamma_{\mathcal{K}}(G)$ into a direct product of algebras $B_S \circ_G \Gamma_{\mathcal{K}}(G)$. Each of those pieces can be completely decribed up to isomorphism. Let me first fix some notations:

NOTATION 3.7. Let S be a finite group, and R_S be a system of representatives of conjugacy classes of couples (K,H) of subgroups of G such that $H \trianglelefteq K$ and $K/H \simeq S$. For $(K,H) \in R_S$, choose an isomorphism $\iota_{K,H}$ from K/H to S. It induces a morphism $a_{K,H}$ of the group $N_G(K,H)/K$ to the group $Out(S)$ of outer automorphisms of S.

With those notations:

THEOREM 3.8. *Let $\mathcal{K}$ be a field of characteristic 0 or coprime to $|G|$. Then*

1. *The algebra $\Gamma_{\mathcal{K}}(G)$ is isomorphic to the direct product of the algebras $B_S \circ_G \Gamma_{\mathcal{K}}(G)$, as S runs through a set of representatives of isomorphism classes of sections of G.*
2. *For any section S of G, the algebra $B_S \circ_G \Gamma_{\mathcal{K}}(G)$ is isomorphic to the Hecke algebra*

$$\mathrm{End}_{Out(S)}\Big(\oplus_{(K,H)\in R_S} \mathrm{Ind}^{Out(S)}_{a_{K,H}(N_G(K,H)/K)} \mathcal{K} \Big)$$

COROLLARY 3.9. *If the characteristic of $\mathcal{K}$ is coprime to $|G|$ and $|Out(S)|$ for all sections S of G, then the algebra $\Gamma_{\mathcal{K}}(G)$ is semi-simple.*

4. Associated categories

The product $\circ_G$ is associative and posesses a unit. It is natural to try to associate to it a category which objects are the finite groups, the morphisms the sets with a double action, the composition of morphisms being given by the product $\circ_G$. Its actually possible to refine this definition.

4.1. $\mathcal{P}$-free-$\mathcal{Q}$ sets. Let $\mathcal{P}$ be a non-empty family of finite groups which is stable under taking subgroups, taking quotients, and taking extensions. This is equivalent to say that $\mathcal{P}$ is the family of finite groups with composition factors in a given family of simple groups.

If $\mathcal{P}$ and $\mathcal{Q}$ are two such families, if G and H are finite groups, and U is a G-set-H, I will say that U is $\mathcal{P}$-free-$\mathcal{Q}$ if for any $u \in U$, the left stabilizer of u in G is in $\mathcal{P}$ and its right stabilizer is in $\mathcal{Q}$. In the case where $\mathcal{P}$ and $\mathcal{Q}$ are reduced to the trivial group, a $\mathcal{P}$-free-$\mathcal{Q}$ set is just a set which is free on the left and on the right.

LEMMA 4.1. *Let G, H and K be groups. Let A be a G-set-H and B be a H-set-K. If A and B are $\mathcal{P}$-free-$\mathcal{Q}$, then so is $A \circ_H B$.*

4.2. The categories $\mathcal{C}(\mathcal{P},\mathcal{Q})$ and $\mathcal{F}_R(\mathcal{P},\mathcal{Q})$. Let $\mathcal{C}(\mathcal{P},\mathcal{Q})$ be the following category: the objects are the finite groups. A morphism in $\mathcal{C}(\mathcal{P},\mathcal{Q})$ from G to H is an element of the Grothendieck group of the category of H-sets-G which are $\mathcal{P}$-free-$\mathcal{Q}$: such elements are difference of two H-sets-G which are $\mathcal{P}$-free-$\mathcal{Q}$. The product of the morphism $A-B$ from G to H and the morphism $C-D$ from H to K is defined as

$$(C-D)\circ(A-B) = \Big((C\circ_H A)\coprod(D\circ_H B)\Big) - \Big((C\circ_H B)\coprod(D\circ_H A)\Big)$$

More generally, if R is a commutative ring, let $\mathcal{C}_R(\mathcal{P},\mathcal{Q})$ be the category $R\otimes\mathcal{C}(\mathcal{P},\mathcal{Q})$, defined the same way with

$$\mathrm{Hom}_{\mathcal{C}_R(\mathcal{P},\mathcal{Q})}(G,H) = R\otimes \mathrm{Hom}_{\mathcal{C}(\mathcal{P},\mathcal{Q})}(G,H)$$

Finally, let $\mathcal{F}_R(\mathcal{P},\mathcal{Q})$ be the category of R-linear functors from $\mathcal{C}_R(\mathcal{P},\mathcal{Q})$ to R-**mod**.

The category $\mathcal{F}_R(\mathcal{P},\mathcal{Q})$ is an abelian category, and I can talk about simple, projective, injective objects.

PROPOSITION 4.2. *The simple objects of $\mathcal{F}_R(\mathcal{P},\mathcal{Q})$ are in one to one correspondence with the isomorphism classes of couples (H,V), where H is a finite group, and V an $ROut(H)$-simple-module. If $\mathcal{P}\subseteq\mathcal{Q}$, then for any group G the value of the simple functor $S_{H,V}$ on G is given by*

$$S_{H,V}(G)\simeq \oplus_{(G_1,N_1)} Tr_1^{N_G(G_1,N_1)/G_1}(V)$$

where (G_1,N_1) runs through a set of representatives of conjugacy classes of couples of subgroups of G such that $N_1 \trianglelefteq G_1$, $N_1\in\mathcal{P}$, and $G_1/N_1\simeq H$.

4.3. Semi-simplicity. If G is a finite group, there is a largest normal subgroup of G which is in $\mathcal{P}$, which I shall denote by $O_{\mathcal{P}}(G)$.

If K is a subgroup of G, and P a normal subgroup of K which is in $\mathcal{P}$, let

$$\Phi_{K,P}^G = \sum_{\substack{H\trianglelefteq K\\ O_{\mathcal{P}}(H)=P\\ H\ mod.\ N_G(K,P)}} F_{K,H}^G$$

Then it can be shown that the elements $\Phi_{K,P}^G$ are in fact in $\mathrm{End}_{\mathcal{C}_{\mathcal{K}}(\mathcal{P},\mathcal{P})}(G)$, which I shall denote by $\mathrm{End}_{\mathcal{K},\mathcal{P}}(G)$. They give a decomposition of unity in mutually orthogonal idempotents of that algebra, and lead to a complete description of $\mathrm{End}_{\mathcal{K},\mathcal{P}}(G)$, similar to theorem 3.8, proving that those algebras are also semi-simple.

They also provide a good notion of residue for an object M of $\mathcal{F}_{\mathcal{K}}(\mathcal{P},\mathcal{P})$: the residue $\overline{M}(G)$ of M at G is defined by

$$\overline{M}(G) = \mathrm{Im}\, M(\Phi_1^G)$$

It can also be identified with the quotient of $M(G)$ by the submodule $\sum M(f)\Big(M(H)\Big)$, for $|H| < |G|$ and $f \in \mathrm{Hom}_{\mathcal{C}_{\mathcal{K}}(\mathcal{P},\mathcal{P})}(H,G)$.

Those residues provide the following decomposition:

Proposition 4.3. *Let M be an object of $\mathcal{F}_{\mathcal{K}}(\mathcal{P},\mathcal{P})$, and G a finite group. Then*

$$M(G) \simeq \oplus_{(K,P)}\overline{M}(K/P)^{N_G(K,P)/K}$$

where (K,P) runs through a set of representatives of conjugacy classes of couples of subgroups of G such that $P \trianglelefteq K$ and $P \in \mathcal{P}$.

A little more work gives then the following:

Theorem 4.4. *If $\mathcal{K}$ is a field of characteristic 0, then the category $\mathcal{F}_{\mathcal{K}}(\mathcal{P},\mathcal{P})$ is semi-simple: any object M splits as*

$$M \simeq \oplus_{\Gamma} S_{\Gamma,\overline{M}(\Gamma)}$$

where Γ runs in a set of representatives of isomorphism classes of finite groups. A further decomposition of $\overline{M}(\Gamma)$ into a sum of $\mathcal{K}Out(\Gamma)$-simple modules gives a decomposition of M into a sum of simple functors.

5. Applications

5.1. Mackey functors. The main motivation of that work was the construction of functors between categories of Mackey functors, and the sets with a double action provide indeed such constructions: let G and H be finite groups, and U be a H-set-G. If M is a Mackey functor for the group H, and X is a G-set, define

$$(M \circ U)(X) = M(U \circ_G X)$$

For a map of G-sets $f : X \to Y$ define

$$(M \circ U)_*(f) = M_*(U \circ_G f) \qquad (M \circ U)^*(f) = M^*(U \circ_G f)$$

Proposition 5.1. *If M is a Mackey functor for H, and U is a H-set-G, then $M \circ U$ is a Mackey functor for G. The correspondence $M \mapsto M \circ U$ is a functor from $Mack_R(H)$ to $Mack_R(G)$.*

5.2. Hochschild (co)homology. Let G be a finite group, and $[G]$ be a set of representatives of the conjugacy classes of G. Define

$$c_G = \coprod_{g\in[G]} t_{C_G(g),<g>}$$

It is a G-set-G, and can be identified with its image in $\Gamma(G)$. If M is a Mackey functor for G, let $HM = M \circ c_G$ be the Mackey functor for G obtained by composition with c_G. Then:

Proposition 5.2. *Let M is a Mackey functor for G over the ring R, and $HM = M \circ c_G$. Then for any subgroup K of G*

$$HM(K) = \oplus_{k\in[K]} M\Big(C_K(k)\Big)$$

In particular, if M is the i-th functor of cohomology $H^i(-,R)$ (resp. the i-th functor of homology $H_i(-,R)$), then $HM(K)$ is isomorphic to the i-th group of Hochschild cohomology $HH^i(K,R)$ (resp. the i-th group of Hochschild homology $HH_i(K,R)$).

5.3. Mackey functors and Alperin's conjecture. J.Thévenaz and P.Webb have proposed the following equivalent version of Alperin's conjecture (cf. [**4**])

CONJECTURE 5.3. *For any finite group G, and any prime number p, there exists Mackey functors M_1 and M_2, over a field R of characteristic 0 or coprime to $|G|$, such that*

1. *For any subgroup H of G, the restrictions $\mathrm{Res}_H^G M_1$ and $\mathrm{Res}_H^G M_2$ are projective with respect to p-local subgroups of H.*
2. *For any subgroup H of G*

$$\dim_R M_1(H) - \dim_R M_2(G) = np(H)$$

where $np(H)$ is the number of simple non-projective kH-modules over an algebraically closed field k of characteristic p.

The formalism of sets with a double action provides an explicit form of that conjecture: let s_G be the idempotent E_1^G, when the family $\underline{F} = \underline{s}_p(G)$ is the family of p-subgroups of G, i.e.

$$s_G = -\sum_{\substack{\inf\ s=\{1\} \\ s\in Sd(\underline{s}_p(G))/G}} (-1)^{|s|} t_{N_G(s),\sup\ s}$$

Let

$$c'_G = \coprod_{g\in[G_{p'}]} t_{C_G(g),<g>}$$

where the sum runs over a set or representatives of the conjugacy classes of p-regular elements of G.

Let $Alp_G = c_G \circ_G s_G$. Its possible to show that Alp_G is also equal to $c'_G \circ_G s_G$. Let R be any commutatice ring, and FP_R be the fixed points Mackey functor for the trivial RG-module R. Then:

PROPOSITION 5.4. *The virtual Mackey functor $M = FP_R \circ (c'_G - Alp_G)$ is the difference of Mackey functors M_1 and M_2 such that*

1. *For any subgroup H of G, the functors $\mathrm{Res}_H^G M_1$ and $\mathrm{Res}_H^G M_2$ are projective with respect to p-local subgroups of H.*
2. *For any subgroup H of G, the modules $M_1(H)$ and $M_2(H)$ are free over R and*

$$rank_R M_1(H) - rank_R M_2(H) = l(H) - |H\backslash Alp_H/H| = -\sum_{s\in Sd(s_p(H))/H} (-1)^{|s|} l\Big(N_H(s)\Big)$$

where $s_p(H)$ is the poset of non-trivial p-subgroups of H.

In particular, Alperin's conjecture is equivalent to

$$|G\backslash Alp_G/G| = f_0(G)$$

for any finite group G.

5.4. Steinberg residues. Let M be an object of $\mathcal{F}_R(\mathcal{P},\mathcal{P})$, where $\mathcal{P}$ contains finite p-groups. Define the Steinberg residue of M at the group G by

$$SM(G) = \mathrm{Im}\, M(s_G)$$

Steinberg residues provide decomposition of objects of $\mathcal{F}_R(\mathcal{P},\mathcal{P})$, generalizing those obtained in [**1**]:

PROPOSITION 5.5. *If $\mathcal{P}$ contains p-groups, and M is an object of $\mathcal{F}_R(\mathcal{P},\mathcal{P})$, then for any group G*

$$M(G) \simeq \oplus_{P\in \underline{s}_p(G)/G} SM\Big(N_G(P)/P\Big)$$

5.5. Adjunction and generalized Steinberg modules. Let G and H be finite groups, and U be a H-set-G. The functor $M \mapsto M\circ U$ from $Mack_R(H)$ to $Mack_R(G)$ admits a left adjoint $N \mapsto \mathcal{L}_U(N)$ and a right adjoint $N \mapsto \mathcal{R}_U(N)$, constructed in [**2**]. The functor $\mathcal{L}_U$ is left adjoint to an exact functor, and so it maps projectives to projectives. It is also easy to see that if $\mathcal{P} = \underline{s}_p(G)$, and if U is $\mathcal{P}$-free-$\mathcal{P}$, then $\mathcal{L}_U(N)$ is projective with respect to p-subgroups if N is.

Now if R is a field of characteristic p, then evaluation at the trivial subgroup gives a one to one correspondence between the (isomorphism classes of) projective Mackey functors which are projective with respect to p-subgroups and the (isomorphism classes of) trivial source modules. This allows a translation of the functor $\mathcal{L}_U$ in the category of trivial source modules: if L is such a module, this construction gives a module $U\circ L$, which is a sum of trivial source modules.

It is easy to see that the virtual module $s_G \circ L$ obtained that way is what I called the generalized Steinberg module $St(G,L)$ in [**1**].

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S. BOUC, EQUIPE DES GROUPES FINIS, UFR DE MATHÉMATIQUES, UNIVERSITÉ PARIS 7-DENIS DIDEROT, 2 PLACE JUSSIEU, 75231. PARIS CEDEX 05, FRANCE

E-mail address: bouc@mathp7.jussieu.fr