CHAPTER 1

Burnside rings

Serge Bouc
Université Paris 7-Denis Diderot, 2 place Jussieu, 75251 Paris Cedex 05. France

1. Introduction

[1] Let G be a finite group. The Burnside ring $B(G)$ of the group G is one of the fundamental representation rings of G, namely the ring of *permutation* representations.

It is in many ways the universal object to consider when looking at the category of G-sets. It can be viewed as an analogue of the ring $\mathbb{Z}$ of integers for this category.

It can be studied from different points of view. First $B(G)$ is a *commutative ring*, and one can look at is prime spectrum and primitive idempotents. This leads to various induction theorems (Artin, Conlon, Dress): the typical statement here is that any (virtual) RG-module is a linear combination with suitable coefficients of modules induced from certain subgroups of G (cyclic, hypoelementary, or Dress subgroups).

The Burnside ring is the natural framework to study the *invariants* attached to structured G-sets (such as G-posets, or more generally simplicial G-sets). Those invariants are generalizations for the category of G-sets of classical notions, such as the Möbius function of a poset, or the Steinberg module of a Chevalley group. They have properties of projectivity, which lead to congruences on the values of Euler-Poincaré characteristic of some sets of subgroups of G.

The ring $B(G)$ is also functorial with respect to G and subgroups of G, and this leads to the *Mackey functor* or *Green functor* point of view. There are close connections between the Burnside ring and the Mackey algebra. The Burnside Mackey functor is a typical example of projective Mackey functor. It is also a universal object in the category of Green functors. This leads to decomposition of the category of Mackey functors for G as a directs sum of smaller abelian categories.

Finally $B(G)$ is also functorial with respect to bisets, and this is leads to the definition of *double Burnside rings*. Those rings are connected to stable homotopy

[1]This is a slightly corrected version [date: 16/06/2016] of the original text published in the "Handbook of Algebra", vol 2, pp 739-804 (2000) Elsevier

theory via the Segal conjecture, and they provide tools to study the stable splittings of the classifying spaces of finite groups.

2. Basic properties of G-sets

2.1. Notation. Let G be a finite group. The category of finite G-sets will be denoted by G-**set**. The objects of G-**set** are the finite sets with a left G-action, morphisms are G-equivariant maps, and composition of morphisms is the composition of maps.

If H is a subgroup of G, and x is an element of G, the notation ${}^xH$ stands for $x.H.x^{-1}$, and the notation H^x for $x^{-1}.H.x$. The normalizer of H in G is denoted by $N_G(H)$. If X is a G-set, the stabilizer in G of an element x of X is denoted by G_x.

The set of conjugacy classes of subgroups of G is denoted by s_G, and a set of representatives of s_G is denoted by $[s_G]$. If H and K are subgroups of G, the notation $H =_G K$ (resp. $H \subseteq_G K$) means that there is an element $x \in G$ such that $H^x = K$ (resp. $H^x \subseteq K$).

The cardinality of a set S is denoted by $|S|$.

If p is a prime number, the smallest normal subgroup N of G such that G/N is a p-group is denoted by $O^p(G)$. It is the subgroup of G generated by the p'-elements, i.e. the elements of order coprime to p.

More generally, if π is a set of primes, the notation $O^\pi(G)$ stands for the smallest normal subgroup N of G such that G/N is a *solvable* π-group. The group G is called π-perfect if $O^\pi(G) = G$. If G itself is a π-group, then the group $O^\pi(G)$ is the limit of the derived series of G. In particular, if π is the set of all primes, then a group is π-perfect if and only if it is *perfect*, i.e. equal to its derived subgroup.

The trivial group will be denoted by $\mathbb{1}$.

2.2. Operations on G-sets. When X is a G set, and H is a subgroup of G, one can view X as an H-set by restriction of the action. This H-set is denoted by $\mathrm{Res}_H^G X$. If $f : X \to Y$ is a morphism of G-sets, let $\mathrm{Res}_H^G f$ denote the map f viewed as a morphism of H-sets. This defines a *restriction functor* $\mathrm{Res}_H^G : G\text{-}\mathbf{set} \to H\text{-}\mathbf{set}$.

Now if Z is an H-set, the induced G-set $\mathrm{Ind}_H^G Z$ is defined as $G \times_H X$, i.e. the quotient of the cartesian product $G \times X$ by the right action of H given by $(g, x).h = (gh, h^{-1}x)$ for $g \in G$, $h \in H$, $x \in X$. The left action of G on $G \times_H X$ is induced by its left action on $G \times X$ given by $g'.(g, x) = (g'g, x)$, for $g', g \in G$ and $x \in X$. If $f : Z \to T$ is a morphism of H-sets, then $\mathrm{Ind}_H^G f$ is the morphism of G-sets from $\mathrm{Ind}_H^G Z$ to $\mathrm{Ind}_H^G T$ defined by $(\mathrm{Ind}_H^G f)\big((g, x)\big) = \big(g, f(x)\big)$. This defines an *induction functor* $\mathrm{Ind}_H^G : H\text{-}\mathbf{set} \to G\text{-}\mathbf{set}$.

Note that if Z is isomorphic to H/K for some subgroup K of H, then $\mathrm{Ind}_H^G Z$ is isomorphic to G/K.

The set of fixed points of H on X is denoted by X^H. It is viewed as a $N_G(H)/H$-set. If $f : X \to Y$ is a morphism of G-sets, then the restriction of f to X^H is denoted by f^H. It is a morphism of $N_G(H)/H$-sets from X^H to Y^H, and this defines a *fixed points functor* from G-**set** to $N_G(H)/H$-**set**.

When H is a normal subgroup of G, any G/H-set can be viewed as a G-set by inflation, and this operation defines an *inflation functor* from G/H-**ens** to G-**ens**.

Finally, let H be a subgroup of G, and x be an element of G. If Z is an H-set, then the group ${}^xH$ acts on Z by $h.z = h^x z$, where $h \in {}^xH$ and $z \in Z$, and $h^x z$ is computed in the H-set Z. This gives an ${}^xH$-set denoted by ${}^xZ$. If $f : Z \to T$ is a morphism of H-sets, then $c_{x,H}(f)$ is the map f, viewed as a morphism of ${}^xH$-sets from ${}^xZ$ to ${}^xT$. This defines a *conjugation* functor $c_{x,H} : H\text{-}\mathbf{set} \to {}^xH\text{-}\mathbf{set}$.

Those constructions are connected by various identities. Among them:

PROPOSITION 2.2.1. *Let G be a finite group, and H and K be subgroups of G.*

(1) [Mackey formula] *If Z is an H-set, then there is an isomorphism of K-sets*
$$\mathrm{Res}^G_K\mathrm{Ind}^G_H Z \simeq \bigsqcup_{x\in K\backslash G/H} \mathrm{Ind}^K_{K\cap {}^xH}\ {}^x\mathrm{Res}^H_{K^x\cap H} Z$$

(2) [Frobenius identity] *If X is a G-set and Z is an H-set, then there is an isomorphism of G-sets*
$$X\times \mathrm{Ind}^G_H Z \simeq \mathrm{Ind}^G_H\big((\mathrm{Res}^G_H X).Z\big)$$
and in particular for any G-set X, there is an isomorphism of G-sets
$$X\times (G/H)\simeq \mathrm{Ind}^G_H\mathrm{Res}^G_H X$$

(3) *If Z is an H-set, then there is an isomorphism of $N_G(H)/H$-sets*
$$(\mathrm{Ind}^G_H Z)^K \simeq \bigsqcup_{\substack{x\in N_G(K)\backslash G/H\\ K^x\subseteq H}} \mathrm{Ind}^{N_G(K)/K}_{N_{{}^xH}(K)/K}({}^xZ)^K$$

PROOF. (sketch) (1) Let S be a set of representatives in G of double cosets $K\backslash G/H$, and consider the map
$$\mathrm{Res}^G_K\mathrm{Ind}^G_H Z \to \bigsqcup_{x\in S} \mathrm{Ind}^K_{K\cap {}^xH}\ {}^x\mathrm{Res}^H_{K^x\cap H} Z$$
sending (g,z), with $g\in G$ and $z\in Z$ to the element (k,hz) of the component $x\in S$, if g can be written $g=kxh$, for some $k\in K$ and $h\in H$. This is the required isomorphism of K-sets.

(2) Consider the map
$$X\times \mathrm{Ind}^G_H Z \to \mathrm{Ind}^G_H\big((\mathrm{Res}^G_H X)\times Z\big)$$
sending the element $\big(x,(g,z)\big)$ of the left hand side, with $x\in X$, $g\in G$, and $z\in Z$, to the element $\big(g,(g^{-1}x,z)\big)$ of the right hand side. This is the required isomorphism of G-sets. The other isomorphism in assertion (2) is the special case $Z=H/H$.

(3) Note that $(\mathrm{Ind}^G_H Z)^K = (\mathrm{Res}^G_{N_G(K)}\mathrm{Ind}^G_H Z)^K$, and use the Mackey formula. □

2.3. Characterization of G-sets.

LEMMA 2.3.1. *Let G be a finite group.*

(1) *Any G-set is a disjoint union of transitive ones. If X is a transitive G-set, and if $x\in X$, then the map*
$$gG_x\in G/G_x\mapsto g.x\in X$$
is an isomorphism of G-sets.

(2) *If H and K are subgroups of G, then the map $f\mapsto f(H)$ is a one to one correspondence between the set of G-set homomorphisms from G/H to G/K and the set of cosets $xK\in G/K$ such that $H\subseteq {}^xK$. In particular, the G-sets G/H and G/K are isomorphic if and only if H and K are conjugate in G.*

PROOF. Both assertions are obvious. □

One can characterize a G-set up to isomorphism using the following fundamental theorem of Burnside ([**15**] Chapter XII Theorem I):

THEOREM 2.3.2. [Burnside] *Let G be a finite group, and X and Y be finite G-sets. Then the following are equivalent:*

(1) *The G-sets X and Y are isomorphic.*
(2) *For any subgroup H of G, the sets X^H and Y^H have the same cardinality.*

Proof. It is clear that (1) implies (2), since any G-set isomorphism $X \to Y$ induces a bijection $X^H \to Y^H$ on the sets of fixed points by any subgroup H of G.

To show the converse, observe that it follows from Lemma 2.3.1 that any finite G-set X can be written up to isomorphism as

$$X = \bigsqcup_{K \in [s_G]} a_K(X)\, G/K$$

for some $a_K(X) \in \mathbb{N}$, where $a_K(X)\ G/K$ denotes the disjoint union of $a_K(X)$ copies of G/K.

Now if (2) holds, for any $H \in [s_G]$, there is an equation

$$\sum_{K \in [s_G]} \big(a_K(X) - a_K(Y)\big)|(G/K)^H| = 0$$

The matrix m of this system of equations is given by

$$m(H,K) = |(G/K)^H| = |\{x \in G/K \mid H^x \subseteq K\}|$$

for $K, H \in [s_G]$. In particular the entry $m(H,K)$ is non-zero if and only if some conjugate of H is contained in K.

If the set $[s_G]$ is given a total ordering $\preceq$ such that $H \preceq K$ implies $|H| \leq |K|$, then the matrix m is upper triangular, with non-zero diagonal coefficient $m(H,H) = |N_G(H) : H|$. In particular m is non-singular, and it follows that $a_K(X) = a_K(Y)$, for any $K \in [s_G]$, and the G-sets X and Y are isomorphic. □

Definition 2.3.3. *The above matrix m (or sometimes its transpose) is called the table of marks of the group G.*

3. The ring structure

3.1. Definition. The following definition of the Burnside ring of the group G appears in an article of Solomon ([**37**]):

Definition 3.1.1. [Solomon] *The Burnside ring $B(G)$ of G is the Grothendieck group of the category G-**set**, for relations given by decomposition in disjoint union of G-sets. The multiplication on $B(G)$ is induced by the direct product of G-sets.*

It means that $B(G)$ is the free $\mathbb{Z}$-module with basis the set of equivalence classes of finite G-sets, quotiented by relations identifying the class of the disjoint union $X \sqcup Y$ of two G-sets X and Y to the sum of the class of X and the class of Y.

The direct product of G-sets is commutative and distributive with respect to disjoint union, up to canonical isomorphisms. Hence it induces by bilinearity a commutative ring structure on $B(G)$. The class of a set $\bullet$ of cardinality 1 is a unit for this ring structure.

Two finite G-sets A and B have the same image in $B(G)$ if and only if there is a sequence of finite G-sets X_i and Y_i, for $1 \leq i \leq n$, and an isomorphism of G-sets

$$A \sqcup \big(\bigsqcup_{i=1}^{n} X_i\big) \sqcup \big(\bigsqcup_{i=1}^{n} Y_i\big) \simeq B \sqcup \Big(\bigsqcup_{i=1}^{n} (X_i \sqcup Y_i)\Big)$$

Taking fixed points of both sides shows that for any subgroup H of G, one has $|A^H| = |B^H|$, and Burnside's Theorem 2.3.2 now implies that A and B are isomorphic as G-sets. In the sequel, the G-set A and its image in $B(G)$ will be identified.

It follows from Burnside's Theorem 2.3.2 that any finite G-set X can be written uniquely up to isomorphism as

$$X \simeq \bigsqcup_{H\in[s_G]} a_H(X)G/H$$

Hence $B(G)$ is a free $\mathbb{Z}$-module, with basis indexed by elements G/H, for $H \in [s_G]$. In this basis, the multiplication law can be recovered by

$$(G/H).(G/K) = \sum_{x\in H\backslash G/K} G/(H \cap {}^xK) \tag{3.1.2}$$

This follows from Proposition 2.2.1, since by Frobenius identities

$$(G/H) \times (G/K) = \mathrm{Ind}_H^G \mathrm{Res}_H^G \mathrm{Ind}_K^G K/K$$

and since by the Mackey formula

$$\mathrm{Res}_H^G \mathrm{Ind}_K^G K/K = \bigsqcup_{x\in H\backslash G/K} H/(H \cap {}^xK)$$

Finally, the operations on G-sets defined in section 2.2 all commute with disjoint unions. Hence they can be extended to the Burnside ring: the elements of $B(G)$ can be viewed as a formal differences $X - Y$ of two finite G-sets. If $F : G\text{-}\mathbf{set} \to H\text{-}\mathbf{set}$ denotes one of the functors of restriction, induction, fixed points, inflation, or conjugation, then F induces a group homomorphism still denoted by F from $B(G)$ to $B(H)$, defined by

$$F(X - Y) = F(X) - F(Y)$$

for any finite G-sets X and Y.

Thus for example, if H is a subgroup of G, there is a restriction homomorphism

$$\mathrm{Res}_H^G : B(G) \to B(H)$$

This homomorphism is actually a morphism of rings (with unit).

In the special case $H = \mathbb{1}$, since $B(H) \simeq \mathbb{Z}$, this gives an extension of the cardinality to a map $X \mapsto |X| = \mathrm{Res}_{\mathbb{1}}^G X$ from $B(G)$ to $\mathbb{Z}$.

Similarly, there is an induction homomorphism

$$\mathrm{Ind}_H^G : B(H) \to B(G)$$

This morphism is not a ring homomorphism in general.

If H is a subgroup of G, there is a fixed points homomorphism $X \mapsto X^H$ from $B(G)$ to $B\big(N_G(H)/H\big)$, which is actually a ring homomorphism. When H is a normal subgroup of G, there is an inflation homomorphism

$$\mathrm{Inf}_{G/H}^G : B(G/H) \to B(G)$$

which is a ring homomorphism.

Finally, if x is an element of G, there is a conjugation homomorphism $Z \mapsto {}^xZ$ from $B(H)$ to $B({}^xH)$, which is a ring isomorphism.

The following is an obvious extension of Proposition 2.2.1:

Proposition 3.1.3. *Let G be a finite group, and H and K be subgroups of G.*

(1) [Mackey formula] *If $Z \in B(H)$, then in $B(K)$*

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G Z = \sum_{x\in K\backslash G/H} \mathrm{Ind}_{K\cap {}^xH}^K \, {}^x\mathrm{Res}_{K^x\cap H}^H Z$$

(2) [Frobenius identity] *If $X \in B(G)$ and $Z \in B(H)$, then in $B(G)$*

$$X.\mathrm{Ind}_H^G Z = \mathrm{Ind}_H^G\big((\mathrm{Res}_H^G X).Z\big)$$

and in particular for any $X \in B(G)$

$$X.(G/H) = \mathrm{Ind}_H^G \mathrm{Res}_H^G X$$

(3) *If $Z \in B(H)$, then in $B\big(N_G(H)/H\big)$*

$$(\mathrm{Ind}_H^G Z)^K = \sum_{\substack{x \in N_G(K)\backslash G/H \\ K^x \subseteq H}} \mathrm{Ind}_{N_{^xH}(K)/K}^{N_G(K)/K} ({}^xZ)^K$$

3.2. Fixed points as ring homomorphisms. The ring $B(G)$ is finitely generated as $\mathbb{Z}$-module, hence it is a noetherian ring. Burnside's Theorem 2.3.2 can be interpreted as follows: each subgroup H of G defines a *ring* homomorphism $\phi_H^G : B(G) \to \mathbb{Z}$ by $\phi_H^G(X) = |X^H|$. The kernel of ϕ_H^G is a prime ideal, since $\mathbb{Z}$ is an integral domain, and the intersection of all those kernels for subgroups H of G is zero. In particular, the ring $B(G)$ is reduced.

Since $\phi_H^G = \phi_K^G$ if H and K are conjugate in G, it follows that the product map

$$\Phi = \prod_{H \in [s_G]} \phi_H^G : B(G) \to \prod_{H \in [s_G]} \mathbb{Z}$$

is injective. Moreover this map Φ is a map between free $\mathbb{Z}$-modules having the same rank. Hence the cokernel of Φ is finite.

The matrix m of Φ with respect to the basis $\{G/H\}_{H \in [s_G]}$ and to the canonical basis $\{u_H\}_{H \in [s_G]}$ of $\prod_{H \in [s_G]} \mathbb{Z}$ is the table of marks of the group G. Recall from Definition 2.3.3 that for $H, K \in [s_G]$

$$m(H,K) = |G/K^H| = |\{x \in G/K \mid H^x \subseteq K\}|$$

The cardinality of the cokernel of Φ is the determinant of m, hence it is equal to

$$|\mathrm{Coker}(\Phi)| = \prod_{H \in [s_G]} |N_G(H) : H|$$

This cokernel has been described by Dress ([**19**]):

Theorem 3.2.1. [Dress] *Let G be a finite group. For H and K in $[s_G]$, set*

$$n(K,H) = |\{x \in N_G(K)/K \mid \ <x,K> =_G H\}|$$

Then the element $y = \sum_{H \in [s_G]} y_H u_H$ of $\prod_{H \in [s_G]} \mathbb{Z}$ is in the image of Φ if and only if for any $K \in [s_G]$

$$\sum_{H \in [s_G]} n(K,H) y_H \equiv 0 \ (|N_G(K)/K|)$$

Proof. First let X be a finite G-set, and let $y = \Phi(X)$. Then with the notation of the theorem, one has $y_H = |X^H|$ for all $H \in [s_G]$, thus for any $K \in [s_G]$

$$\begin{aligned}
\sum_{H \in [s_G]} n(K,H) y_H &= \sum_{H \in [s_G]} |X^H| |\{x \in N_G(K)/K \mid \ <x,K> =_G H\}| \\
&= \sum_{x \in N_G(K)/K} |X^{<x,K>}| \\
&= \sum_{x \in N_G(K)/K} |(X^K)^x|
\end{aligned}$$

Now for any finite group L acting on a finite set Z one has

$$\begin{aligned}|L| \times |L\backslash Z| &= |L|\sum_{z\in Z}\frac{|L_z|}{|L|} = |\{(l,z)\in L\times Z \mid l.z = z\}| \\ &= \sum_{l\in L}|Z^l|\end{aligned}$$

Applying this for $L = N_G(K)/K$ and $Z = X^K$ gives

$$\sum_{H\in[s_G]} n(K,H)y_H = |N_G(K)/K|\,|N_G(K)\backslash X^K| \equiv 0\ (|N_G(K)/K|)$$

By linearity, this proves the "only if" part of the theorem.

Applying this for $X = G/M$, for some $M \in [s_G]$, shows that there is a matrix t indexed by $[s_G]\times[s_G]$, with entries in $\mathbb{Z}$, such that for $K \in [s_G]$

$$\sum_{H\in[s_G]} n(K,H)m(H,M) = |N_G(K)/K|t(K,M)$$

Now the matrix n is upper triangular, and its diagonal coefficients are equal to 1. Thus the matrix t is upper triangular, and

$$t(K,K) = m(K,K)/|N_G(K)/K| = 1$$

In particular t is invertible (over $\mathbb{Z}$).

Now suppose that the element $y = \sum_{H\in[s_G]} y_H u_H \in \prod_{H\in[s_G]}\mathbb{Z}$ satisfies all the congruences of the theorem. Since $\mathrm{Coker}(\Phi)$ is finite, there exist rational numbers r_M, for $M \in [s_G]$, such that

$$y = \sum_{M\in[s_G]} r_M\Phi(G/M)$$

In other words for each $H \in [s_G]$

$$y_H = \sum_{M\in[s_G]} |G/M^H| r_M$$

Thus for each $K \in [s_G]$

$$\begin{aligned}\sum_{H\in[s_G]} n(K,H)y_H &= \sum_{H\in[s_G]}\sum_{M\in[s_G]} n(K,H)m(H,M)r_M \\ &= |N_G(K):K|\sum_{M\in[s_G]} t(K,M)r_M\end{aligned}$$

The left hand side is a multiple of $|N_G(K):K|$ by assumption, hence there exist integers z_K such that for all $K \in [s_G]$

$$\sum_{M\in[s_G]} t(K,M)r_M = z_K$$

Since t is invertible, it follows that $r_M \in \mathbb{Z}$ for all M, and $y \in \mathrm{Im}(\Phi)$. □

3.3. Idempotents. The map Φ of the previous section is an injective map between free $\mathbb{Z}$-modules having the same rank. Hence tensoring with $\mathbb{Q}$ gives a $\mathbb{Q}$-algebra isomorphism

$$\mathbb{Q}\otimes_{\mathbb{Z}}\Phi : \mathbb{Q}\otimes_{\mathbb{Z}} B(G) \overset{\cong}{\to} \prod_{H\in[s_G]}\mathbb{Q}$$

and in particular the algebra $\mathbb{Q}\otimes_{\mathbb{Z}} B(G)$ is semi-simple. It will be denoted by $\mathbb{Q}B(G)$. The component $\mathbb{Q}\phi_H^G$ of $\mathbb{Q}\Phi$ will still be written $X \mapsto |X^H|$, so in general

$|X^H|$ will be a rational number for $X \in \mathbb{Q}B(G)$. More generally, all the notations defined for $B(G)$ will be extended without change to $\mathbb{Q}B(G)$.

The inverse image by $\mathbb{Q}\Phi$ of the canonical $\mathbb{Q}$-basis of $\prod_{H\in[s_G]}\mathbb{Q}$ indexed by H is denoted by e_H^G. If H' is conjugate to H in G, one also sets $e_{H'}^G = e_H^G$. With this notation for any subgroups H and K of G, one has

$$|(e_H^G)^K| = \begin{cases} 1 & \text{if } H =_G K \\ 0 & \text{otherwise} \end{cases}$$

The set of elements e_H^G, for $H \in [s_G]$, is the set of primitive idempotents of $\mathbb{Q}B(G)$. Those idempotents have been computed explicitly by Gluck ([**22**]), and later independently by Yoshida ([**49**]). This can be done using the following lemma:

LEMMA 3.3.1. *Let G be a finite group.*

(1) *Let H be a subgroup of G. Then for any $X \in \mathbb{Q}B(G)$*

$$X.e_H^G = |X^H|e_H^G$$

Conversely, if $Y \in \mathbb{Q}B(G)$ is such that $X.Y = |X^H|Y$ for any $X \in \mathbb{Q}B(G)$, then $Y \in \mathbb{Q}e_H^G$.

(2) *Let H be a proper subgroup of G. Then $\mathrm{Res}_H^G e_G^G = 0$. Conversely, if $Y \in \mathbb{Q}B(G)$ is such that $\mathrm{Res}_H^G Y = 0$ for any proper subgroup H of G, then $Y \in \mathbb{Q}e_G^G$.*

(3) *Let H be a subgroup of G. Then*

$$e_H^G = \frac{1}{|N_G(H) : H|}\mathrm{Ind}_H^G e_H^H$$

PROOF. (1) The set of elements e_H^G, for $H \in [s_G]$, is a $\mathbb{Q}$-basis of $\mathbb{Q}B(G)$, thus for any $X \in \mathbb{Q}B(G)$, there are rational numbers r_H, for $H \in [s_G]$, such that

$$X = \sum_{H\in[s_G]} r_H e_H^G$$

Taking fixed points of both sides by a subgroup $K \in [s_G]$ shows that $r_K = |X^K|$. It follows that for all $K \in [s_G]$

$$X.e_K^G = |X^K|e_K^G$$

Now let Y be an element of $\mathbb{Q}B(G)$ verifying $X.Y = |X^H|Y$ for any $X \in \mathbb{Q}B(G)$. Then in particular $e_K^G.Y = 0$ if $K \neq_G H$, thus $Y = |Y^H|e_H^G$ is a rational multiple of e_H^G.

(2) Let H be a proper subgroup of G. Then for any subgroup of K of H

$$|(\mathrm{Res}_H^G e_G^G)^K| = |(e_G^G)^K| = 0$$

since obviously $|(\mathrm{Res}_H^G X)^K| = |X^K|$ for any G-set X, hence for any X in $\mathbb{Q}B(G)$. It shows that the restriction of e_G^G to any proper subgroup of G is zero.

Conversely, if the restriction of an element Y to any proper subgroup H of G is zero, then in particular $|Y^H| = 0$ for such a subgroup, and $Y = |Y^G|e_G^G$.

(3) Consider next the element $\mathrm{Ind}_H^G e_H^H$. If X is any element of $\mathbb{Q}B(G)$, then by Frobenius identity

$$X.\mathrm{Ind}_H^G e_H^H = \mathrm{Ind}_H^G\big((\mathrm{Res}_H^G X).e_H^H\big) = \mathrm{Ind}_H^G(|X^H|.e_H^H) = |X^H|\mathrm{Ind}_H^G e_H^H$$

It follows that there is a rational number r_H^G such that

$$\mathrm{Ind}_H^G e_H^H = r_H^G e_H^G \tag{3.3.2}$$

By the Mackey formula, the restriction of the left hand side to H is equal to

$$\mathrm{Res}_H^G \mathrm{Ind}_H^G e_H^H = \sum_{x\in H\backslash G/H} \mathrm{Ind}_{H\cap {}^xH}^H {}^x\mathrm{Res}_{H^x\cap H}^H e_H^H = \sum_{x\in N_G(H)/H} {}^x e_H^H$$

since the restriction of e_H^H to $H^x \cap H$ is zero if $H^x \neq H$. Moreover if $y \in G$, then for any subgroup K of ${}^yH$

$$|({}^y e_H^H)^K| = |(e_H^H)^{K^y}| = \begin{cases} 1 & \text{if } K = {}^yH \\ 0 & \text{otherwise} \end{cases}$$

Thus ${}^y e_H^H = e_{{}^yH}^{{}^yH}$, and finally

$$\mathrm{Res}_H^G \mathrm{Ind}_H^G e_H^H = |N_G(H) : H| e_H^H$$

Equation 3.3.2 gives then

$$|N_G(H) : H| e_H^H = r_H^G \mathrm{Res}_H^G e_H^G$$

Taking fixed points of both sides under H gives $r_H^G = |N_G(H) : H|$, and

$$e_H^G = \frac{1}{|N_G(H) : H|} \mathrm{Ind}_H^G e_H^H \tag{3.3.3}$$

as was to be shown. As a consequence

$$\mathrm{Res}_H^G e_H^G = e_H^H \tag{3.3.4}$$

□

THEOREM 3.3.5. [Gluck] *Let G be a finite group. If H is a subgroup of G, then*

$$e_H^G = \frac{1}{|N_G(H)|} \sum_{K\subseteq H} |K|\, \mu(K,H)\; G/K$$

where μ is the Möbius function of the poset of subgroups of G, and G/K denotes $1 \otimes G/K \in \mathbb{Q}B(G)$.

PROOF. One can write

$$e_H^H = \sum_{K\subseteq H} r(K,H)\; H/K$$

where $r(K,H)$ is a rational number. Since ${}^y e_H^H = e_{{}^yH}^{{}^yH}$ for $y \in G$, one can suppose $r({}^yK, {}^yH) = r(K,H)$ for $y \in G$. Taking induction from H to G gives

$$e_H^G = \frac{1}{|N_G(H) : H|} \sum_{K\subseteq H} r(K,H)\; G/K \tag{3.3.6}$$

Now the sum of the elements e_H^G, for $H \in [s_G]$, is equal to $\bullet = G/G$. Summing over all subgroups H of G instead of $H \in [s_G]$ gives

$$G/G = \sum_{H\subseteq G} \frac{|N_G(H)|}{|G|} \frac{1}{|N_G(H) : H|} \sum_{K\subseteq H} r(K,H)\; G/K$$

The coefficient of G/K in the right hand side is equal to the coefficient of G/K', for any conjugate K' of K in G, and it is equal to

$$\sum_{K\subseteq H\subseteq G} \frac{|H|}{|G|} r(K,H)$$

This must be equal to zero if $K \neq G$, and equal to 1 if $K = G$. Setting $r'(K,H) = \frac{|H|}{|K|} r(K,H)$ for $K \subseteq H$, this gives

$$\sum_{K\subseteq H\subseteq G} r'(K,H) = \begin{cases} |G : K| = 1 & \text{if } K = G \\ 0 & \text{otherwise} \end{cases}$$

This shows that $r'(K,H)$ is equal to $\mu(K,H)$, where μ is the Möbius function of the poset of subgroups of G. Thus $m(K,H) = \frac{|K|}{|H|}\mu(K,H)$, and Equation 3.3.6 gives

$$e_H^G = \frac{1}{|N_G(H)|}\sum_{K\subseteq H} |K|\ \mu(K,H)\ G/K$$

as was to be shown. □

The formulae for primitive idempotents in $\mathbb{Q}B(G)$ lead to a natural question: when are these idempotents actually in $B(G)$? More generally, if π is a set of prime numbers, let $\mathbb{Z}_{(\pi)}$ denote the subring of $\mathbb{Q}$ of irreducible fractions with denominator prime to all the elements of π. In other words $\mathbb{Z}_{(\pi)}$ is the localization of $\mathbb{Z}$ with respect to the set $\mathbb{Z} - \cup_{p\in\pi} p\mathbb{Z}$. One may ask when the idempotents of $\mathbb{Q}B(G)$ are actually in $\mathbb{Z}_{(\pi)}B(G)$. The answer is as follows:

THEOREM 3.3.7. [Dress] *Let G be a finite group, and π be a set of primes. Let $\mathcal{F}$ be a family of subgroups of G, closed by conjugation in G. Let $[\mathcal{F}]$ denote the set $\mathcal{F}\cap[s_G]$. Then the following conditions are equivalent:*

1. *The idempotent $\sum_{H\in[\mathcal{F}]} e_H^G$ lies in $\mathbb{Z}_{(\pi)}B(G)$.*
2. *Let H and K be any subgroups of G such that H is a normal subgroup of K and the quotient K/H is cyclic of prime order $p\in\pi$. Then $H\in\mathcal{F}$ if and only if $K\in\mathcal{F}$.*

PROOF. Suppose first that (1) holds, i.e. that the idempotent $e = \sum_{H\in[\mathcal{F}]} e_H^G$ lies in $\mathbb{Z}_{(\pi)}B(G)$. This is equivalent to the existence of an integer m coprime to all the elements of π, such that $me\in B(G)$. Now if H and K are subgroups of G such that $H\trianglelefteq K$ and K/H is cyclic of prime order p, then for any element X of $B(G)$, one has $|X^H|\equiv|X^K|\ (p)$. It follows that

$$m|e^H| \equiv m|e^K|\ (p)$$

But $m|e^H|$ is equal to m if $H\in\mathcal{F}$, and it is equal to zero otherwise. Thus if $p\in\pi$, it follows that assertion (2) holds, because $m\not\equiv 0\ (p)$.

Conversely, suppose that (2) holds. Let m be an integer such that $me\in B(G)$. By Dress's Theorem 3.2.1 this is equivalent to require that for each subgroup K of G

$$\sum_{H\in[s_G]} n(K,H)m|e^H| = m\sum_{H\in[\mathcal{F}]} n(K,H) \equiv 0\ (|N_G(K):K|)$$

This can also be written as

$$m\sum_{H\in[\mathcal{F}]} |\{x\in N_G(K)/K \mid <x,K> =_G H\}| \equiv 0\ (|N_G(K):K|)$$

i.e.

$$m|\{x\in N_G(K)/K \mid <x,K>\in\mathcal{F}\}| \equiv 0\ (|N_G(K):K|) \tag{3.3.8}$$

If (2) holds, then $<x,K>\in\mathcal{F}$ if and only if $<x_{\pi'},K>\in\mathcal{F}$, where $x_{\pi'}$ denotes the π'-part of x. The set Γ of π'-elements y of the group $W = N_G(K)/K$ such that $<y,K>\in\mathcal{F}$ is invariant under conjugation in W. Let $[\Gamma]$ denote a set of representatives of W-conjugacy classes of Γ. Then:

$$\begin{aligned}|\{x\in W \mid x_{\pi'}\in\Gamma\}| &= \sum_{\gamma\in\Gamma}|\{x\in W \mid x_{\pi'}=\gamma\}|\\ &= \sum_{\gamma\in\Gamma}|C_W(\gamma)_\pi|\\ &= \sum_{\gamma\in[\Gamma]}\frac{|W|}{|C_W(\gamma)|}|C_W(\gamma)_\pi|.\end{aligned}$$

Now by a theorem of Frobenius (see Corollaire 1 of Théorème 23 of [**36**]), for each $\gamma\in\Gamma$, there exists an integer l_γ such that $|C_W(\gamma)_\pi| = l_\gamma|C_W(\gamma)|_\pi$. Hence

$$|\{x\in W \mid x_{\pi'}\in\Gamma\}| = \sum_{\gamma\in[\Gamma]}\frac{|W|}{|C_W(\gamma)|_{\pi'}}l_\gamma = |W|_\pi\sum_{\gamma\in[\Gamma]}\frac{|W|_{\pi'}}{|C_W(\gamma)|_{\pi'}}l_\gamma \equiv 0 \text{ (mod. } |W|_\pi).$$

It follows that $|W|_{\pi'}|\{x\in W \mid x_{\pi'}\in\Gamma\}| \equiv 0$ (mod. $|W|$), and in particular Congruence 3.3.8 holds if $m=|G|_{\pi'}$, for any subgroup K of G. Hence $me\in B(G)$, and Assertion (1) holds. □

Corollary 3.3.9. *Let G be a finite group, and π be a set of primes. If J is a π-perfect subgroup of G, set*

$$f_J^G = \sum_{\substack{H\in[s_G]\\ O^\pi(H)=_G J}} e_H^G$$

Then the set of elements f_J^G, for π-perfect elements J of $[s_G]$, is the set of primitive idempotents of $\mathbb{Z}_{(\pi)}B(G)$.

In particular, the set of primitive idempotents of $B(G)$ is in one to one correspondence with the set of conjugacy classes of perfect subgroups of G.

The group G is solvable if and only if G/G is a primitive idempotent of $B(G)$.

Proof. Let J be a π-perfect subgroup of G, and set

$$\mathcal{F} = \{H\subseteq G \mid O^\pi(H) =_G J\}$$

Then clearly the family $\mathcal{F}$ is closed by conjugation in G, and satisfies condition (2) of Theorem 3.3.7. Thus f_J^G lies in $\mathbb{Z}_{(\pi)}B(G)$.

To prove that it is primitive, it suffices to show that the family $\mathcal{F}$ has no proper non-empty subfamily satisfying condition (2) of Theorem 3.3.7. But if $\mathcal{F}'$ is such a non-empty subfamily of $\mathcal{F}$, and if $H\in\mathcal{F}'$, then $O^\pi(H)\in\mathcal{F}'$ since the composition factors of $H/O^\pi(H)$ are cyclic groups of prime order belonging to π. Thus $J\in\mathcal{F}'$ since $\mathcal{F}'$ is closed by conjugation. Now if $H'\in\mathcal{F}$ the group $O^\pi(H')$ is in $\mathcal{F}'$, and $H'\in\mathcal{F}'$ by the same argument. Thus $\mathcal{F}'=\mathcal{F}$, and the idempotent f_J^G is primitive.

The last assertion of the corollary is the case where π is the set of all prime numbers. □

3.4. Prime spectrum. The prime spectrum of $B(G)$ has been determined by Dress ([**20**]):

Theorem 3.4.1. [Dress] *Let G be a finite group, let p denote a prime number or zero, and H be a subgroup of G. Let*

$$I_{H,p}(G) = \{X\in B(G) \mid |X^H|\equiv 0\ (p)\}$$

Then:

1. *The set $I_{H,p}(G)$ is a prime ideal of $B(G)$.*
2. *If I is a prime ideal of $B(G)$, then there is a subgroup H of G and an integer p equal to zero or a prime number, such that $I=I_{H,p}(G)$.*

(3) *If H and K are subgroups of G, and if p and q are prime numbers or zero, then $I_{H,p}(G) \subseteq I_{K,q}(G)$ if and only if one of the following holds:*
   (a) *One has $p = q = 0$, and the subgroups H and K are conjugate in G. In this case moreover $I_{H,p} = I_{K,q}$.*
   (b) *One has $p = 0$ and $q > 0$, and the groups $O^q(H)$ and $O^q(K)$ are conjugate in G. In this case moreover $I_{H,p} \neq I_{K,q}$.*
   (c) *One has $p = q > 0$, and the subgroups $O^p(H)$ and $O^p(K)$ are conjugate in G. In this case moreover $I_{H,p} = I_{K,q}$.*

Proof. (1) Clearly $I_{H,p}$ is the kernel of the ring homomorphism from $B(G)$ to $\mathbb{Z}/p\mathbb{Z}$ mapping X to the class of $|X^H|$. Assertion (1) follows, since $\mathbb{Z}/p\mathbb{Z}$ is an integral domain.

(2) Conversely, if I is a prime ideal of $B(G)$, then the ring $R = B(G)/I$ is an integral domain. Let $\pi : B(G) \to R$ be the canonical projection. Since $R \neq \{0\}$, there is a subgroup H of G minimal subject to the condition $\pi(G/H) \neq 0$. Taking the image of equation 3.1.2 by π and using the minimality of H gives

$$\pi(G/H)\pi(G/K) = \sum_{\substack{x \in G/K \\ H^x \subseteq K}} \pi(G/H)$$

(since moreover $HxK = xK$ if $H^x \subseteq K$). Since R is an integral domain, and since $\pi(G/H) \neq 0$, it follows that

$$\pi(G/K) = |\{x \in G/K \mid H^x \subseteq K\}|1_R = |(G/K)^H|1_R$$

It follows by linearity that $\pi(X) = |X^H|1_R$ for any $X \in B(G)$. Let p denote the characteristic of R. Then p is equal to zero or a prime number. Clearly the kernel of π, equal to I by definition, is also equal to $I_{H,p}(G)$.

(3) Suppose that H and K are conjugate in G. Then for any $X \in B(G)$, one has $|X^H| = |X^K|$, and in particular $I_{H,0}(G) = I_{K,0}(G)$.

Now if p is a prime number, and if X is a finite G-set, since

$$X^H = (X^{O^p(H)})^{H/O^p(H)}$$

and since $H/O^p(H)$ is a p-group, it follows that

$$|X^H| \equiv |X^{O^p(H)}| \ (p)$$

for any $X \in B(G)$. In particular $I_{H,0} \subseteq I_{O^p(H),p}$, and $I_{H,p}(G) = I_{K,p}(G)$ if $O^p(H)$ and $O^p(K)$ are conjugate in G. Thus $I_{H,p}(G) = I_{O^p(H),p}(G)$ for any $H \subseteq G$.

Conversely, suppose that $I_{H,p}(G) \subseteq I_{K,q}(G)$. There is a surjective ring homomorphism from $B(G)/I_{H,p}(G)$ to $B(G)/I_{K,q}(G)$. Since p is the characteristic of the ring $B(G)/I_{H,p}(G)$, there are three possible cases:

(a) Either $p = q = 0$. Then the inclusion $I_{H,0}(G) \subseteq I_{K,0}(G)$ means that if $X \in B(G)$ is such that $|X^H| = 0$, then $|X^K| = 0$. Now $X = |N_G(K)|e_K^G$ is in $B(G)$ by Theorem 3.3.5, and $|X^K| = |N_G(K)| \neq 0$. Thus $|X^H| \neq 0$, hence H is conjugate to K in G. Clearly in this case $I_{H,0}(G) = I_{K,0}(G)$.

(b) The next possible case is $p = 0$ and $q > 0$. In this case if $X \in B(G)$ is such that $|X^H| = 0$, then $|X^K| \equiv 0 \ (q)$. Consider the idempotent

$$f^G_{O^q(K)} = \sum_{\substack{L \in [s_G] \\ O^q(L) =_G O^q(K)}} e_L^G$$

By Corollary 3.3.9, there is an integer m coprime to q such that $X = mf^G_{O^q(K)}$ is in $B(G)$. Now $|X^K| \equiv |X^{O^q(K)}| \equiv m \not\equiv 0 \ (q)$. Thus $|X^H| \neq 0$, and it follows that

$O^q(H) =_G O^q(K)$. The inclusion $I_{H,0}(G) \subseteq I_{K,q}(G)$ is proper since the respective quotient rings have characteristic 0 and q.

(c) The last case is $p = q > 0$. Since $I_{H,0}(G) \subseteq I_{H,p}(G)$, it follows that $I_{H,0}(G) \subseteq I_{K,p}(G)$. Hence $O^p(H) =_G O^p(K)$ by the discussion of the previous case. And in this case $I_{H,p}(G) = I_{K,p}(G)$. □

3.5. Application to induction theorems.

NOTATION 3.5.1. Let G be a finite group. Denote by $\mathcal{C}(G)$ the set of cyclic subgroups of G. Denote by $R_{\mathbb{C}}(G)$ the ring of complex characters of G, and set $\mathbb{Q}R_{\mathbb{C}}(G) = \mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{C}}(G)$.

THEOREM 3.5.2. [Artin] *Let G be a finite group. Then*

$$\mathbb{Q}R_{\mathbb{C}}(G) = \sum_{H \in \mathcal{C}(G)} \mathrm{Ind}_H^G \mathbb{Q}R_{\mathbb{C}}(H)$$

In other words, any complex character of G is a linear combination with rational coefficients of characters induced from cyclic subgroups of G.

PROOF. There is a natural homorphism from the Burnside ring $B(G)$ of G to the ring $R_{\mathbb{C}}(G)$, which maps a G-set X to the associated $\mathbb{C}G$-module $\mathbb{C}X$. This extends to a map of vector spaces $\mathbb{Q}B(G) \mapsto \mathbb{Q}R_{\mathbb{C}}(G)$. Now the value of the character of the permutation module $\mathbb{C}X$ at the element s of G is the trace of s on $\mathbb{C}X$, which is equal to the number of fixed points $|X^s|$ of s on X.

Let H be a subgroup of G. Since $|(e_H^G)^s|$ is equal to 0 if H is not conjugate in G to the subgroup generated by s, it follows that the image of e_H^G in $\mathbb{Q}R_{\mathbb{C}}(G)$ is zero unless H is cyclic. And if H is cyclic, the image of e_H^G is a linear combination with rational coefficients of permutations characters $\mathrm{Ind}_K^G 1$, for subgroups K of G. Taking the image in $\mathbb{Q}R_{\mathbb{C}}(G)$ of the decomposition

$$G/G = \sum_{H \in [s_G]} e_H^G$$

shows that the trivial character is a linear combination with rational coefficients of such characters $\mathrm{Ind}_K^G 1$, which are induced from cyclic subgroups K of G. Then if χ is any character of G

$$\chi \mathrm{Ind}_H^G 1 = \mathrm{Ind}_H^G(\mathrm{Res}_H^G \chi)$$

and the theorem follows. □

NOTATION 3.5.3. Let p be a prime number, and $\mathcal{O}$ be a complete local noetherian commutative ring with maximal ideal $\mathfrak{p}$ and residue field k of characteristic p. If G is a finite group, an *$\mathcal{O}G$-lattice* is a finitely generated $\mathcal{O}$-free $\mathcal{O}G$-module. Also denote by $\mathcal{O}$ the trivial $\mathcal{O}G$-lattice of $\mathcal{O}$-rank 1.

The *Green ring* $A_{\mathcal{O}}(G)$ is the Grothendieck group of the category of $\mathcal{O}G$- lattices, for relations given by direct sums decompositions.

As an additive group, it is the quotient of the free abelian group with basis the set of isomorphism classes of $\mathcal{O}G$-lattices, by the subgroup generated by elements $[M \oplus N] - [M] - [N]$, for $\mathcal{O}G$-lattices M and N, where $[M]$ denotes the class of M. Since Krull-Schmidt theorem holds for $\mathcal{O}G$-lattices, the group $A_{\mathcal{O}}(G)$ is free, with basis the set of indecomposable $\mathcal{O}G$-lattices. The ring structure on $A_{\mathcal{O}}(G)$ is induced by tensor product (over $\mathcal{O}$) of $\mathcal{O}G$-lattices.

There is a natural ring homomorphism $\pi_{\mathcal{O}}$ from $B(G)$ to $A_{\mathcal{O}}(G)$, mapping the (class of the) finite G-set X to the (class of the) permutation lattice $\mathcal{O}X$. It is natural to look at the kernel $I_{\mathcal{O}}(G)$ of this morphism.

Since $A_{\mathcal{O}}(G)$ is torsion-free, the sequence

$$0 \to I_{\mathcal{O}}(G) \to B(G) \overset{\pi_{\mathcal{O}}}{\to} A_{\mathcal{O}}(G)$$

is a split exact sequence of abelian groups.

Hence there is an exact sequence

$$0 \to \mathbb{Q}I_{\mathcal{O}}(G) \to \mathbb{Q}B(G) \overset{\mathbb{Q}\pi_{\mathcal{O}}}{\to} \mathbb{Q}A_{\mathcal{O}}(G)$$

It follows that there exists a set $\mathcal{N}_{\mathcal{O}}(G)$ of subgroups of G, closed by conjugation, such that

$$\mathbb{Q}I_{\mathcal{O}}(G) = \sum_{H \in \mathcal{N}_{\mathcal{O}}(G)/G} \mathbb{Q}e_H^G$$

or equivalently

$$I_{\mathcal{O}}(G) = \{X \in B(G) \mid \forall H \subseteq G,\ H \notin \mathcal{N}_{\mathcal{O}}(G) \Rightarrow |X^H| = 0\}$$

Clearly the morphism $\pi_{\mathcal{O}}$ commutes with induction and restriction. Now formulae 3.3.3 and 3.3.4 show that $e_H^G \in \mathbb{Q}I_{\mathcal{O}}(G)$ if and only if $e_H^H \in \mathbb{Q}I_{\mathcal{O}}(H)$. It follows that there exists a family of finite groups $\mathcal{N}_{\mathcal{O}}$ such that

$$\mathcal{N}_{\mathcal{O}}(G) = \{H \subseteq G \mid H \in \mathcal{N}_{\mathcal{O}}\}$$

The main result of this section is a characterization of the family $\mathcal{N}_{\mathcal{O}}$. First a definition:

Definition 3.5.4. *Let p be a prime number. If G is a finite group, let $O_p(G)$ denote the largest normal p-subgroup of G. The group G is called p-hypoelementary, or cyclic modulo p, if the quotient group $G/O_p(G)$ is cyclic. The family of such finite groups is denoted by $\mathcal{Z}_p$. The set of p-hypoelementary subgroups of a finite group H is denoted by $\mathcal{Z}_p(G)$.*

It turns out that the family $\mathcal{N}_{\mathcal{O}}$ depends only on p, and is equal to the complement of $\mathcal{Z}_p$, by the following theorem (see [**17**]):

Theorem 3.5.5. [Conlon] *Let G be a finite group, and X, Y be finite G-sets. The following conditions are equivalent:*

(1) *The $\mathcal{O}G$-lattices $\mathcal{O}X$ and $\mathcal{O}Y$ are isomorphic.*
(2) *For any p-hypoelementary subgroup H of G, the sets X^H and Y^H have the same cardinality.*

Proof. Recall that if M is an $\mathcal{O}G$-lattice, then a *vertex* of M is a subgroup P of G, minimal such that M is a direct summand of $\mathrm{Ind}_P^G\mathrm{Res}_P^G M$. The deep argument for theorem 3.5.5 relies on the Green correspondence, which, for any p-subgroup P of G, is a bijection between the set of isomorphism classes of indecomposable $\mathcal{O}G$-lattices with vertex P and the set of isomorphism classes of indecomposable $\mathcal{O}N_G(P)$-lattices with vertex P (see [**1**] Chapter 3.12).

In the case of permutation modules, or more generally of their direct summands (or *p-permutation modules*), there is a more elementary approach, due to Broué ([**12**]). It relies on the *Brauer construction*: if M is an $\mathcal{O}G$-module, and P is a subgroup of G, then the Brauer quotient $M[P]$ of M at P is defined by

$$M[P] = M^P / \big(\mathfrak{p}M^P + \sum_{Q \subset P} Tr_Q^P(M^Q)\big)$$

where the sum runs over proper subgroups Q of P, and $Tr_Q^P : M^Q \to M^P$ is the relative trace map or transfer map, defined by

$$\forall m \in M^Q, \quad Tr_Q^P(m) = \sum_{x \in P/Q} xm$$

If S is a Sylow p-subgroup of P, and if $m \in M^P$, then $m = Tr_S^P(\frac{1}{|P:S|}m)$, thus $M[P] = 0$ if P is not a p-subgroup of G. When P is a p-subgroup of G and $M = \mathcal{O}X$ is a permutation lattice associated to a G-set X, the image of the set X^P in $M[P]$ is a k-basis of $M[P]$.

In this case moreover, the value of the Brauer character of M at the element s of G is equal to the number of fixed points of s on X. Hence if $s \in N_G(P)$, the value of the Brauer character of the $kN_G(P)/P$-module $M[P]$ at the element $sP \in N_G(P)/P$ is equal to $|X^{H_{P,s}}|$, where $H_{P,s} = P{<}s{>}$ is the subgroup of G generated by s and P. Note that $H_{P,s}$ is cyclic modulo p, and that conversely, if H is a p-hypoelementary subgroup of G, there is a p-subgroup P of G and an $s \in N_G(P)$ such that $H = H_{P,s}$.

It follows that if $\mathcal{O}X$ and $\mathcal{O}Y$ are isomorphic, then for any p-hypoelementary subgroup H of G, one has $|X^H| = |Y^H|$.

Conversely, if $|X^H| = |Y^H|$ for any p-hypoelementary subgroup H of G, then for any p-subgroup P of G, the modules $kX^P \simeq \mathcal{O}X[P]$ and $kY^P \simeq \mathcal{O}Y[P]$ have the same Brauer character. The next proposition shows that $\mathcal{O}X$ and $\mathcal{O}Y$ are isomorphic, and this completes the proof of Theorem 3.5.5. □

PROPOSITION 3.5.6. *Let G be a finite group, and let M and N be p-permutation $\mathcal{O}G$-lattices. Then the following are equivalent:*

(1) *The $\mathcal{O}G$-lattices M and N are isomorphic.*
(2) *For any p-subgroup P of G, the $kN_G(P)/P$-modules $M[P]$ and $N[P]$ have the same Brauer character.*

PROOF. It is clear that (1) implies (2). The proof of the converse is by induction on the cardinality of the set

$$S(M) = \{P \subseteq G \mid M[P] \neq 0\}$$

Note that if (2) holds, then $S(M) = S(N)$.

If $S(M) = \emptyset$, then $M[1] = M/\mathfrak{p}M = N/\mathfrak{p}N = 0$, hence $M = N = 0$. If $S(M) \neq \emptyset$, choose a maximal element P of $S(M)$. Write $M = M_P \oplus M'$ (resp. $N = N_P \oplus N'$), where all direct summands of M_P (resp. N_P) have non-zero Brauer quotient at P, and where $M'[P] = 0$ (resp. $N'[P] = 0$). Theorem 3.5.7 below shows that $M[P] = M_P[P]$ and $N[P] = N_P[P]$ are projective $kN_G(P)/P$-modules. By assumption, they have the same Brauer character, hence they are isomorphic, and then Theorem 3.5.7 shows that M_P and N_P are isomorphic. Now (2) holds for M' and N', and moreover $|S(M')| < |S(M)|$, since $S(M') \subseteq S(M)$ and $P \in S(M) - S(M')$. Hence $M' \simeq N'$ by induction hypothesis, and $M \simeq N$. □

THEOREM 3.5.7. [Broué] *Let G be a finite group, and P be a p-subgroup of G.*

(1) *Let M be an indecomposable p-permutation $\mathcal{O}G$-module with vertex P. Then for any subgroup Q of G, the module $M[Q]$ is non-zero if and only if Q is conjugate to a subgroup of P.*
(2) *The correspondence $M \mapsto M[P]$ induces a bijection between the set of isomorphism classes of indecomposable p-permutation $\mathcal{O}G$-lattices with vertex P and the set of isomorphism classes of indecomposable projective $kN_G(P)/P$-modules.*

PROOF. (1) Recall (Higman criterion) that if H is a subgroup of G and if M is an $\mathcal{O}G$-module, then M is a direct summand of $\mathrm{Ind}_H^G\mathrm{Res}_H^G M$ if and only if there exists an $\mathcal{O}H$-endomorphism ϕ of M such that $Id_M = Tr_H^G(\phi)$, where $Tr_H^G(\phi) = \sum_{g\in G/H} g\phi g^{-1}$. If S is a Sylow p-subgroup of H, then $|H:S|$ is invertible in $\mathcal{O}$, and $Id_M = Tr_S^G(\phi/|H:S|)$. In particular, any vertex of M is a p-group.

Recall also that if P is a p-group, and Q is a subgroup of P, then the permutation $\mathcal{O}P$-lattice $\mathrm{Ind}_Q^P\mathcal{O}$ is indecomposable, since its reduction modulo $\mathfrak{p}$ is isomorphic to $\mathrm{Ind}_Q^P k$, which has simple socle k. Hence any p-permutation $\mathcal{O}P$-lattice is a permutation $\mathcal{O}P$-lattice.

Let M be an indecomposable p-permutation $\mathcal{O}G$-lattice. Then M is a direct summand of a permutation lattice $\mathcal{O}X$, for some finite G-set X. Let P be a p-subgroup of G such that M is a direct summand of $\mathrm{Ind}_P^G\mathrm{Res}_P^G M$. Then $\mathrm{Res}_P^G M$ is a direct summand of $\mathcal{O}\mathrm{Res}_P^G X$, hence it is a permutation $\mathcal{O}P$-lattice, and there is a set $\mathcal{S}$ of subgroups of P such that

$$\mathrm{Res}_P^G M \simeq \bigoplus_{Q\in\mathcal{S}} \mathrm{Ind}_Q^P\mathcal{O}$$

Thus M is a direct summand of some module $\mathrm{Ind}_Q^G\mathcal{O}$, for $Q\in\mathcal{S}$. Since $\mathrm{Ind}_Q^P\mathcal{O}$ is a direct summand of $\mathrm{Res}_P^G M$, and since $\mathcal{O}$ is a direct summand of $\mathrm{Res}_Q^P\mathrm{Ind}_Q^P\mathcal{O}$, it follows that $\mathcal{O}$ is a direct summand of $\mathrm{Res}_Q^G M$, and that $\mathrm{Ind}_Q^G\mathcal{O}$ is a direct summand of $\mathrm{Ind}_Q^G\mathrm{Res}_Q^G M$.

Now if P is a vertex of M, then $Q=P\in\mathcal{S}$. Hence M is a direct summand of $\mathrm{Ind}_P^G\mathcal{O}$, and $\mathcal{O}$ is a direct summand of $\mathrm{Res}_P^G M$ (in other words the module M has *trivial source*). It follows that $M[Q]\neq 0$ for any subgroup of Q. Conversely, if $M[Q]\neq 0$ for some subgroup Q of G, then $(\mathrm{Ind}_P^G\mathcal{O})[Q]\neq 0$, thus Q has a fixed point on the set G/P, i.e. $Q\subseteq_G P$. Assertion (1) follows. It shows in particular that all the vertices of M are conjugate in G.

(2) Since $\mathrm{Ind}_P^G\mathcal{O}\simeq \mathrm{Ind}_{N_G(P)}^G\mathcal{O}N_G(P)/P$, and since the $\mathcal{O}N_G(P)/P$ is a direct sum of indecomposable projective $\mathcal{O}N_G(P)/P$-lattices, it follows from (1) that if M is an indecomposable p-permutation $\mathcal{O}G$-lattice with vertex P, then there is an indecomposable projective $\mathcal{O}N_G(P)/P$-lattice E such that M is a direct summand of $\mathrm{Ind}_{N_G(P)}^G E$. It is easy to check by Mackey formula that

$$(\mathrm{Ind}_{N_G(P)}^G E)[P]\simeq \sum_{\substack{x\in G/N_G(P)\\ P^x\subseteq N_G(P)}} E[P^xP/P]=E/\mathfrak{p}E$$

since $E[Q/P]=0$ for any non-trivial subgroup Q/P of $N_G(P)/P$.

Now $E/\mathfrak{p}E$ is an indecomposable projective $kN_G(P)/P$-module, having $M[P]$ as a non-zero direct summand. It follows that $M[P]\simeq E/\mathfrak{p}E$ is an indecomposable projective $kN_G(P)/P$-module, and the correspondence of assertion (2) is well defined.

Let E be an indecomposable projective $\mathcal{O}N_G(P)/P$-lattice. Write

$$\mathrm{Ind}_{N_G(P)}^G E\simeq L\oplus L'$$

where all the direct summands of L have vertex conjugate to P in G, and no direct summands of L' have vertex conjugate to P. Since L' is a direct summand of $\mathrm{Ind}_P^G\mathcal{O}$, all the indecomposable direct summands of L' have vertex strictly contained in P (up to G-conjugation). Hence $L'[P]=0$, and $L[P]\simeq E/\mathfrak{p}E$ is indecomposable. It follows that L is indecomposable, and that it is the only indecomposable direct summand of $\mathrm{Ind}_{N_G(P)}^G E$ with vertex P, up to isomorphism.

Thus if M is an indecomposable p-permutation module with vertex P, and if E is the only indecomposable projective $\mathcal{O}N_G(P)/P$-lattice such that $M[P]\simeq E/\mathfrak{p}E$, then M is isomorphic to the only indecomposable direct summand of $\mathrm{Ind}_{N_G(P)}^G E$ with vertex P. This shows that the correspondence $M\mapsto M[P]$ of assertion (2) is injective.

Conversely, if $\overline{E}$ is an indecomposable projective $kN_G(P)/P$-module, then there is a projective $\mathcal{O}N_G(P)/P$-lattice E such that $\overline{E}\simeq E/\mathfrak{p}E$. In particular E is indecomposable. Now $\mathrm{Ind}_{N_G(P)}^G E$ has a unique indecomposable direct summand M with vertex P, and $M[P] \simeq \overline{E}$. This shows that the correspondence $M \mapsto M[P]$ of assertion (2) is surjective, and completes the proof of Theorem 3.5.7. $\square$

Corollary 3.5.8. *Let G be a finite group. Then*

$$\mathbb{Q}A_{\mathcal{O}}(G) = \sum_{H \in \mathcal{Z}_p(G)} \mathrm{Ind}_H^G \mathbb{Q}A_{\mathcal{O}}(H)$$

Two $\mathcal{O}G$-lattices M and N are isomorphic if and only if for any p-hypoelementary subgroup H of G, the restrictions $\mathrm{Res}_H^G M$ and $\mathrm{Res}_H^G N$ are isomorphic.

Proof. The image by $\mathbb{Q}\pi_{\mathcal{O}}(e_H^G)$ of the idempotent e_H^G of $\mathbb{Q}B(G)$ in the ring $\mathbb{Q}A_{\mathcal{O}}(G)$ is zero if H is not p-hypoelementary. Thus

$$\mathcal{O} = \mathbb{Q}\pi_{\mathcal{O}}(G/G) = \sum_{H \in \mathcal{Z}_p(G)} \mathbb{Q}\pi_{\mathcal{O}}(e_H^G)$$

Now e_H^G is a linear combination of elements G/K, for subgroups K of H. Since subgroups of p-hypoelementary groups are p-hypoelementary, it follows that there exists rational numbers r_K such that

$$\mathcal{O} = \sum_{K \in \mathcal{Z}_p(G)} r_K \mathrm{Ind}_K^G \mathcal{O}$$

since moreover $\mathbb{Q}\pi_{\mathcal{O}}(G/K)$ is the (class of) the permutation lattice $\mathrm{Ind}_K^G \mathcal{O}$. Tensoring this identity with M over $\mathcal{O}$, and using Frobenius identity, it follows that

$$M = \sum_{K \in \mathcal{Z}_p(G)} r_K \mathrm{Ind}_K^G \mathrm{Res}_K^G M$$

This proves both assertions of the corollary. $\square$

Remark 3.5.9. A different proof of Theorem 3.5.5 and Corollary 3.5.8 has been given by Dress ([**21**]). It is exposed in Curtis-Reiner ([**18**] Chapter 11.80D).

Definition 3.5.10. *Let p and q be (non-necessarily distinct) prime numbers. A finite group H is called a (p,q)-Dress group if the group $O^q(H)$ is p-hypoelementary. A (p,q)-Dress subgroup H of a finite group G is a subgroup of G which is a (p,q)-Dress group. The set of (p,q)-Dress subgroups of a finite group H is denoted by $\mathcal{D}_{p,q}(G)$.*

Theorem 3.5.11. [Dress] *Let G be a finite group. Then*

$$A_{\mathcal{O}}(G) = \sum_{\substack{H \in \mathcal{D}_{p,q}(G) \\ \text{any } q}} \mathrm{Ind}_H^G A_{\mathcal{O}}(H)$$

Proof. It follows from the expression of the idempotents of the Burnside ring $B(G)$: if q is a prime and J is a q-perfect subgroup of G, then the idempotent f_J^G of Corollary 3.3.9 is mapped to zero by $\mathbb{Z}_{(q)}\pi_{\mathcal{O}}$ if J is not p-hypoelementary.

And if J is p-hypoelementary, the idempotent f_J^G is a linear combination with coefficients in $\mathbb{Z}_{(q)}$ of elements G/K, where K runs through the (p,q)-Dress subgroups of G. This shows that there is an integer m_q coprime to q and integers n_K such that

$$m_q\mathcal{O} = \sum_{K \in \mathcal{D}_{p,q}(G)} n_K \mathrm{Ind}_K^G \mathcal{O} \qquad \text{in} \qquad A_{\mathcal{O}}(G)$$

Setting

$$A'_{\mathcal{O}}(G) = \sum_{\substack{H \in \mathcal{D}_{p,q}(G) \\ \text{any } q}} \mathrm{Ind}_H^G A_{\mathcal{O}}(H)$$

it follows that the quotient $A_{\mathcal{O}}(G)/A'_{\mathcal{O}}(G)$ is a torsion group, with finite exponent coprime to q. Since this holds for any prime q, the groups $A_{\mathcal{O}}(G)$ and $A'_{\mathcal{O}}(G)$ are equal. □

3.6. Further results and references. The group of units of the Burnside ring has been studied by Matsuda ([**29**]), Matsuda-Miyata ([**30**]), and Yoshida ([**50**]).

Examples of non-isomorphic groups having isomorphic Burnside rings have been given by Thévenaz ([**40**]).

The Burnside ring of a compact Lie group has been defined and studied by tom Dieck ([**44**], [**45**], [**46**]) and Schwänzl ([**34**], [**35**]).

General exposition of the properties of Burnside rings can be found in Benson ([**1**] Chapter 5.4), Curtis-Reiner ([**18**] Chapter 11), Karpilovsky ([**24**] Chapter 15), tom Dieck ([**46**]).

4. Invariants

The Burnside ring is an analogue for finite G-sets of the ring $\mathbb{Z}$ for finite sets (and $\mathbb{Z}$ is actually isomorphic to the Burnside ring of the trivial group). One can attach various invariants to structured G-sets, such as G-posets or G-simplicial complexes.

This section is a self-contained algebraic exposition of the properties of those invariants. The original definitions and methods of Quillen ([**32**], [**33**]) are used throughout, avoiding however the topological part of this material. Thus for example no use will be made of the geometric realization of a poset, and the accent will be put on acyclic posets rather than contractible ones.

In other words, in order to define and state properties of the invariants attached to finite G-posets in the Burnside ring, one can forget about the fundamental group of those posets, and consider only homology groups.

4.1. Homology of posets. Let $(X, \leq)$ be a partially ordered set (*poset* for short). As usual, if x, x' are in X, the notation $x < x'$ means $x \leq x'$ and $x \neq x'$. The notation $[x, x']_X$ (resp. $[x, x'[_X$, $]x, x']_X$, $]x, x'[_X$) stands for the set of elements $z \in X$ with $x \leq z \leq x'$ (resp. $x \leq z < x'$, $x < z \leq x'$, $x < z < x'$). The notation $[x, .[_X$ (resp. $]x, .[_X$, $]., x]_X$, $]., x[_X$) stands for the set of elements $z \in X$ with $x \leq z$ (resp. $x < z$, $z \leq x$, $z < x$).

If $n \in \mathbb{N}$, let $Sd_n(X)$ denote the set of chains $x_0 < \ldots < x_n$ of elements of X of cardinality $n+1$. The chain complex $C_*(X, \mathbb{Z})$ is the complex of $\mathbb{Z}$-modules defined as follows: for $n \in \mathbb{N}$, the module $C_n(X, \mathbb{Z})$ is the free $\mathbb{Z}$-module with basis $Sd_n(X)$. The differential $d_n : C_n(X, \mathbb{Z}) \to C_{n-1}(X, \mathbb{Z})$ is given by

$$d_n(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \hat{x_i}, \ldots, x_n)$$

where $(x_0, \ldots, \hat{x_i}, \ldots, x_n)$ denotes the chain $(x_0, \ldots, x_n) - \{x_i\}$.

The *reduced* chain complex $\tilde{C}_*(X, \mathbb{Z})$ is the augmented complex obtained by setting $C_{-1}(X, \mathbb{Z}) = \mathbb{Z}$, the augmentation map $d_0 : C_0(X, \mathbb{Z}) \to C_{-1}(X, \mathbb{Z})$ sending each $x_0 \in X$ to $1 \in \mathbb{Z}$.

The n^{th} homology group of the complex $C_*(X, \mathbb{Z})$ (resp. $\tilde{C}_*(X, \mathbb{Z})$) is denoted by $H_n(X, \mathbb{Z})$, and called the n^{th} *homology group* (resp. *reduced homology group*) of X. A poset X is *acyclic* if all its reduced homology groups are equal to zero.

More generally, if K is a ring, the n^{th} homology group of X *with coefficients in* K is the n^{th} homology group of the complex $K \otimes_{\mathbb{Z}} C_*(X, \mathbb{Z})$. When K is a field, one has $H_n(X, K) = K \otimes H_n(X, \mathbb{Z})$.

The *Euler-Poincaré characteristic* $\chi(X)$ of a finite poset X is defined by

$$\chi(X)=\sum_{n\geq 0}(-1)^n rank_{\mathbb{Z}} C_n(X,\mathbb{Z})=\sum_{n\geq 0}(-1)^n|Sd_n(X)|$$

Similarly, the *reduced* Euler-Poincaré characteristic $\tilde{\chi}(X)$ is defined by

$$\tilde{\chi}(X)=\sum_{n\geq -1}(-1)^n rank_{\mathbb{Z}}\tilde{C}_n(X,\mathbb{Z})=\chi(X)-1$$

If K is a field, then setting $k_n=\dim_K \mathrm{Ker}(K\otimes_{\mathbb{Z}} d_n)$ for any $n\in\mathbb{N}$

$$\begin{aligned}\dim_K H_n(X,K) &= \dim_K K\otimes_{\mathbb{Z}} H_n(X,\mathbb{Z})=k_n-\dim_K \mathrm{Im}(K\otimes d_{n+1})\\ &= k_n+k_{n+1}-\dim_K K\otimes_{\mathbb{Z}} C_{n+1}(X,\mathbb{Z})\end{aligned}$$

It follows that

$$\chi(X)=\sum_{n\geq 0}(-1)^n\dim_K H_n(X,K)\qquad \tilde{\chi}(X)=\sum_{n\geq -1}(-1)^n\dim_K \tilde{H}_n(X,K)$$

In particular, if X is acyclic, then $\tilde{\chi}(X)=0$. Similarly, if X and Y are finite posets, and if there is an homotopy equivalence f_* from the complex $C_*(X,\mathbb{Z})$ to the complex $C_*(Y,\mathbb{Z})$, then $\chi(X)=\chi(Y)$, since f induces a group isomorphism from $H_n(X,\mathbb{Z})$ to $H_n(Y,\mathbb{Z})$, for any $n\in\mathbb{N}$.

If X and Y are posets, a *map of posets* $f:X\to Y$ is a map from X to Y such that $f(x)\leq f(x')$ whenever x and x' are elements of X such that $x\leq x'$. If f is such a map, there is an induced map of chain complexes $C_*(f,\mathbb{Z}):C_*(X,\mathbb{Z})\to C_*(Y,\mathbb{Z})$ defined for $n\in\mathbb{N}$ by

$$C_n(f,\mathbb{Z})(x_0,\ldots,x_n)=\begin{cases}\big(f(x_0),\ldots,f(x_n)\big) & \text{if } f(x_0)<\ldots<f(x_n)\\ 0 & \text{otherwise}\end{cases}$$

One also defines a reduced map $\tilde{C}_*(f,\mathbb{Z}):\tilde{C}_*(X,\mathbb{Z})\to\tilde{C}_*(Y,\mathbb{Z})$ by $\tilde{C}_n(f,\mathbb{Z})=C_n(f,\mathbb{Z})$ if $n\leq 0$, and $\tilde{C}_{-1}=Id_{\mathbb{Z}}$.

If f and g are maps of posets from X to Y, the notation $f\leq g$ means that $f(x)\leq g(x)$ for any $x\in X$. The maps f and g are said to be *comparable* if either $f\leq g$ or $g\leq f$.

LEMMA 4.1.1. *Let f and g be maps of posets from X to Y. If f and g are comparable, then the maps of complexes $C_*(f,\mathbb{Z})$ and $C_*(g,\mathbb{Z})$ are homotopic, as well as the maps $\tilde{C}_*(f,\mathbb{Z})$ and $\tilde{C}_*(g,\mathbb{Z})$.*

PROOF. Suppose for instance that $f\leq g$. Consider the map $h_n:C_n(X,\mathbb{Z})\to C_{n+1}(Y,\mathbb{Z})$ defined for $n\in\mathbb{N}$ by

$$h_n(x_0,\ldots,x_n)=\sum_{i=0}^{n}(-1)^i\big(f(x_0),\ldots,f(x_i),g(x_i),\ldots,g(x_n)\big)$$

where the sequence $\big(f(x_0),\ldots,f(x_i),g(x_i),\ldots,g(x_n)\big)$ is replaced by 0 if it is not strictly increasing. It is easy to check that

$$d_{n+1}\circ h_n+h_{n-1}\circ d_n=C_n(g,\mathbb{Z})-C_n(f,\mathbb{Z})$$

thus the maps $C_n(f,\mathbb{Z})$ and $C_n(g,\mathbb{Z})$ are homotopic. A similar argument can be used for the augmented complexes, the map h_{-1} being the zero map. □

COROLLARY 4.1.2. [Quillen]

(1) *Let X and Y be posets, and let $f:X\to Y$ and $g:Y\to X$ be maps of posets. If $g\circ f$ is comparable to Id_X and if $f\circ g$ is comparable to Id_Y, then the maps of complexes $\tilde{C}_*(f,\mathbb{Z})$ and $\tilde{C}_*(g,\mathbb{Z})$ are mutual inverse homotopy equivalences between $\tilde{C}_*(X,\mathbb{Z})$ and $\tilde{C}_*(Y,\mathbb{Z})$.*

(2) *If the poset X has a biggest element, or a smallest element, the chain complex $\tilde{C}_*(X,\mathbb{Z})$ is contractible.*

Proof. The first assertion is a direct consequence of the previous lemma. For the second one, denote by m the biggest (or the smallest) element of X, and let $\bullet$ denote a poset of cardinality one. Apply assertion (1) to the unique map $f : X \to \bullet$ and to the map $g : \bullet \to X$ sending the unique element of $\bullet$ to m. The result follows, since the complex $\tilde{C}_*(\bullet,\mathbb{Z})$ is clearly contractible. □

4.2. Invariants attached to finite G-posets. The following definition of the Lefschetz invariants is due to Thévenaz ([**39**]):

Definition 4.2.1. *Let G be a finite group. A G-poset X is a G-set equipped with an order relation $\leq$ compatible to the G-action: if $x \leq x'$ are elements of X and if $g \in G$, then $gx \leq gx'$.*

If X and Y are G-posets, a map of G-posets $f : X \to Y$ is a map such that $f(gx) = gf(x)$ if $g \in G$ and $x \in X$, and such that $f(x) \leq f(x')$ in Y, whenever $x \leq x'$ in X. If $y \in Y$, then

$$f^y = \{x \in X \mid f(x) \leq y\} \qquad f_y = \{x \in X \mid f(x) \geq y\}$$

Those sets are sub-G_y-posets of (the restriction of) X to the stabilizer G_y of y in G.

If $x \leq y$ are elements of X, the set $]x,y[_X$ is a $G_{x,y}$-poset, where $G_{x,y}$ is the stabilizer $G_{x,y}$ of the pair $\{x,y\}$. Similarly, the sets $]x,.[_X$ and $].,x[_X$ are G_x-posets.

If X is a G-poset, then for $n \in \mathbb{N}$, the set $Sd_n(X)$ is a G-set. When X is finite, the Lefschetz invariant Λ_X of X is the element of $B(G)$ defined by

$$\Lambda_X = \sum_{n\geq 0} (-1)^n Sd_n(X)$$

The reduced Lefschetz invariant $\tilde{\Lambda}_X$ is the element of $B(G)$ defined by

$$\tilde{\Lambda}_X = \Lambda_X - G/G$$

If $x < y$ are elements of X, the Möbius invariant $\mu_X(x,y)$ is defined as the Lefschetz invariant of the poset $]x,y[_X$. It is an element of the Burnside ring $B(G_{x,y})$. By convention, the Möbius invariant $\mu_X(x,x)$ is equal to G_x/G_x.

It follows from those definitions that $|\Lambda_X|$ is equal to the Euler-Poincaré characteristic of X. One can say more:

Lemma 4.2.2. *Let G be a finite group.*

(1) *If X is a finite G-poset, then for any subgroup H of G*

$$(\Lambda_X)^H = \Lambda_{X^H}$$

in $B\big(N_G(H)/H\big)$. In particular $|(\Lambda_X)^H| = \chi(X^H)$.

(2) *If X and Y are finite G-posets, then $\Lambda_X = \Lambda_Y$ in $B(G)$ if and only if $\chi(X^H) = \chi(Y^H)$ for any subgroup H of G.*

Proof. The first assertion is obvious, since $Sd_n(X)^H = Sd_n(X^H)$ for all $n \in \mathbb{N}$. The second one follows from Burnside's Theorem 2.3.2. □

Definition 4.2.3. *A G-poset X is called G-acyclic if the poset X^H is acyclic for any subgroup H of G.*

The following is a direct consequence of this definition:

Lemma 4.2.4. *Let G be a finite group, and X be a finite G-poset. If X is G-acyclic, then $\tilde{\Lambda}_X = 0$ in $B(G)$.*

Proof. This follows from Burnside's theorem, since $|(\tilde{\Lambda}_X)^H| = \tilde{\chi}(X^H) = 0$ for any subgroup H of G. $\square$

Proposition 4.2.5. *Let G be a finite group.*

(1) *Let X and Y be finite G-posets, and let $f : X \to Y$ and $g : Y \to X$ be maps of G-posets. If $g \circ f$ is comparable to Id_X and if $f \circ g$ is comparable to Id_Y, then $\tilde{\Lambda}_X = \tilde{\Lambda}_Y$ in $B(G)$.*

(2) *If a G-poset X has a biggest element, or a smallest element, then it is G-acyclic.*

Proof. This is a direct consequence of Corollary 4.1.2: for any subgroup H of G, the restrictions of f and g to the posets X^H and Y^H verify the hypotheses of Corollary 4.1.2. Hence $\tilde{\chi}(X^H) = \tilde{\chi}(Y^H)$, and the first assertion follows. For the second, note that the biggest (resp. smallest) element of X is also a biggest (resp. smallest) element of X^H, for any subgroup H of G. $\square$

Example 4.2.6. Let $f : X \to Y$ be a maps of finite G-posets. Denote by $X *_f Y$ the G-poset defined as follows: the underlying G-set is the disjoint union $X \sqcup Y$ of X and Y. The ordering is defined for z and z' in $X *_f Y$ by

$$z \leq z' \Leftrightarrow \begin{cases} z, z' \in X & \text{and } z \leq z' \text{ in } X \\ z, z' \in Y & \text{and } z \leq z' \text{ in } Y \\ z \in X, z' \in Y, & \text{and } f(z) \leq z' \text{ in } Y \end{cases}$$

Let f denote the injection from Y to $X *_f Y$, and g denote the map from $X *_f Y$ to Y defined by

$$g(z) = \begin{cases} f(z) & \text{if } z \in X \\ z & \text{if } z \in Y \end{cases}$$

Then f and g are maps of G-posets, such that $g \circ f = Id_Y$ and $Id_{X *_f Y} \leq f \circ g$. It follows that $\Lambda_{X *_f Y} = \Lambda_Y$.

The consequence is the following relation between $\tilde{\Lambda}_X$ and $\tilde{\Lambda}_Y$:

Proposition 4.2.7. *Let $f : X \to Y$ be a map of finite G-posets. Then in $B(G)$*

$$\tilde{\Lambda}_Y = \tilde{\Lambda}_X + \sum_{y \in G \backslash Y} \mathrm{Ind}_{G_y}^G (\tilde{\Lambda}_{f^y} \tilde{\Lambda}_{]y,\cdot[_Y})$$

$$\tilde{\Lambda}_Y = \tilde{\Lambda}_X + \sum_{y \in G \backslash Y} \mathrm{Ind}_{G_y}^G (\tilde{\Lambda}_{f_y} \tilde{\Lambda}_{]\cdot,y[_Y})$$

In particular, if $\tilde{\Lambda}_{f^y} = 0$ for all $y \in Y$ (for instance if f^y is G_y-acyclic), then $\tilde{\Lambda}_X = \tilde{\Lambda}_Y$ in $B(G)$.

Proof. Let $n \in \mathbb{N}$. The set $Sd_n(X *_f Y)$ is the disjoint union of $Sd_n(X)$ and of the set of sequences $z_0 < \ldots < z_n$ for which $z_n \in Y$. Such a sequence has a smallest element $y = z_i$ in Y, hence it can be written as

$$x_0 < \ldots < x_{i-1} < y < y_0 < \ldots < y_{n-i-1}$$

where $x_0 < \ldots < x_{i-1}$ is in $Sd_{i-1}(f^y)$ (for $i = 0$, the convention is that $Sd_{-1}(f^y)$ is a set of cardinality one), and $y_0 < \ldots < y_{n-i-1}$ is in $Sd_{n-i-1}(]y, .[_Y)$ (for $i = n$, the convention is that $Sd_{-1}(]y, .[_Y)$ has cardinality one).

Keeping track of the action of G, this leads to the following isomorphism of G-sets

$$Sd_n(X *_f Y) = Sd_n(X) \sqcup \bigsqcup_{y \in G \backslash Y} \mathrm{Ind}_{G_y}^G \left(\bigsqcup_{i=0}^{n} \big(Sd_{i-1}(f^y) \times Sd_{n-i-1}(]y, .[_Y)\big) \right)$$

Taking alternating sums gives the first equality of the proposition. The second one follows, by considering the map $f : X^{op} \to Y^{op}$ between the opposite posets of X and Y, since clearly $\tilde{\Lambda}_{X^{op}} = \tilde{\Lambda}_X$ for any finite group G and any finite G-poset X. □

COROLLARY 4.2.8. *Let X be a finite G-poset.*

(1) *The reduced Lefschetz invariant of X is equal to*
$$\tilde{\Lambda}_X = -G/G - \sum_{x \in G\backslash X} \mathrm{Ind}_{G_x}^G \tilde{\Lambda}_{]x,.[_X}$$

(2) *If $x \leq y$ in X, then*
$$\sum_{z \in G_{x,y}\backslash [x,y]} \mathrm{Ind}_{G_{x,y,z}}^{G_{x,y}} \mathrm{Res}_{G_{x,y,z}}^{G_{y,z}} \mu_X(z,y) = \begin{cases} 0 & \text{if } x < y \\ G_x/G_x & \text{if } x = y \end{cases}$$

(3) *If $f : X \to Y$ is a map of finite G-posets, then*
$$\Lambda_X = -\sum_{y \in G\backslash Y} \mathrm{Ind}_{G_y}^G \Lambda_{f^y} \tilde{\Lambda}_{]y,.[_Y} = -\sum_{y \in G\backslash Y} \mathrm{Ind}_{G_y}^G \Lambda_{f_y} \tilde{\Lambda}_{].,y[_Y}$$

Assertion 2) is the reason for the name of the Möbius invariant.

PROOF. Assertion (1) follows from the previous proposition, applied to the inclusion $\emptyset \to X$, since $\tilde{\Lambda}_\emptyset = -G/G$. Assertion (2) follows from assertion (1), applied to the $G_{x,y}$-poset $[x,y[_X$, which has a smallest element x if $x < y$.

Assertion (3) follows from assertion (1) and Proposition 4.2.7:
$$\begin{aligned} \tilde{\Lambda}_Y &= \tilde{\Lambda}_X + \sum_{y \in G\backslash Y} \mathrm{Ind}_{G_y}^G \big((\Lambda_{f^y} - G_y/G_y)\tilde{\Lambda}_{]y,.[_Y}\big) \\ &= \Lambda_X - G/G + \sum_{y \in G\backslash Y} \mathrm{Ind}_{G_y}^G (\Lambda_{f^y}\tilde{\Lambda}_{]y,.[_Y}) - \sum_{y \in G\backslash Y} \mathrm{Ind}_{G_y}^G \tilde{\Lambda}_{]y,.[_Y} \\ &= \Lambda_X + \sum_{y \in G\backslash Y} \mathrm{Ind}_{G_y}^G (\Lambda_{f^y}\tilde{\Lambda}_{]y,.[_Y}) + \tilde{\Lambda}_Y \end{aligned}$$

The second equality in assertion 3) is similar. □

COROLLARY 4.2.9. *Let G be a finite group and X be a finite poset. Denote by $X_\sharp$ (resp. $X^\sharp$) the set of elements $x \in X$ such that $\tilde{\Lambda}_{].,x[_X} \neq 0$ (resp. $\tilde{\Lambda}_{]x,.[_X} \neq 0$) in $B(G_x)$. If Y is a sub-G-poset of X such that $X_\sharp \subseteq Y \subseteq X$ (resp. $X^\sharp \subseteq Y \subseteq X$), then $\tilde{\Lambda}_Y = \tilde{\Lambda}_X$ in $B(G)$.*

PROOF. By induction on the cardinality of X: if $X = \emptyset$, then $X = X_\sharp = X^\sharp$ and there is nothing to prove. For the inductive step, consider the inclusion map $i : X_\sharp \to Y$. It is a map of G-posets. Moreover if $y \in X_\sharp$, then i^y has a biggest element y, hence it is G_y-acyclic, and $\tilde{\Lambda}_{G_y} = 0$ in $B(G_y)$. Now if $y \notin X_\sharp$, then $\tilde{\Lambda}_{].,y[_X} = 0$ in $B(G_y)$ by definition of $X_\sharp$. Moreover in this case
$$\begin{aligned} i^y =].,y[_X \cap X_\sharp &= \{z \in X \mid z < y,\ \tilde{\Lambda}_{].,z[_X} \neq 0 \text{ in } B(G_z)\} \\ &\supseteq \{z \in X \mid z < y,\ \tilde{\Lambda}_{].,z[_X} \neq 0 \text{ in } B(G_z \cap G_y)\} = (].,y[_X)_\sharp \end{aligned}$$

It follows that there are inclusions of G_y-posets
$$(].,y[_X)_\sharp \subseteq i^y \subseteq].,y[_X$$

Moreover $|].,y[_X| < |X|$. By induction hypothesis, it follows that $\tilde{\Lambda}_{i^y} = \tilde{\Lambda}_{].,y[_X} = 0$. Now the corollary follows from Proposition 4.2.7. □

4.3. Steinberg invariants.

DEFINITION 4.3.1. *Let G be a finite group, and p be a prime number. The Steinberg invariant $St_p(G)$ of G at p is the reduced Lefschetz invariant of the poset $s_p(G)$ of non-trivial p-subgroups of G, on which G acts by conjugation.*

The reason for this terminology is that if G is a finite simple Chevalley group in characteristic p, then the (virtual) permutation character associated to $St_p(G)$ is equal up to a sign to the Steinberg character of G.

PROPOSITION 4.3.2. *Let G be a finite group, and p be a prime number. Then $St_p(G) = 0$ in $B(G)$ if and only if G has a non-trivial normal p-subgroup.*

PROOF. If $St_p(G) = 0$, then in particular $\tilde{\chi}\big(s_p(G)^G\big) = 0$. Thus $s_p(G)^G$ is non-empty, and G has a non-trivial normal p-subgroup.

Conversely, suppose that $R \neq \mathbb{1}$ is a non-trivial normal p-subgroup of G. Let f be the map from $s_p(G)$ to $[R, .[_{s_p(G)}$ defined by $f(Q) = Q.R$, and let g denote the inclusion map from $[R, .[_{s_p(G)}$ to $s_p(G)$. Then f and g are maps of G-posets, and moreover $Id \leq g \circ f$ and $f \circ g = Id$. Since $[R, .[_{s_p(G)}$ has a smallest element, it follows from Proposition 4.2.5 that $St_p(G) = 0$ in $B(G)$. □

REMARK 4.3.3. Quillen has conjectured that $s_p(G)$ is contractible if and only if $O_p(G) \neq \mathbb{1}$. The above proof shows actually that $s_p(G)$ is G-contractible if and only if $O_p(G) \neq \mathbb{1}$ (see Thévenaz-Webb [**42**]).

PROPOSITION 4.3.4. *Let G be a finite group, and p be a prime number. Let $a_p(G)$ denote the sub-G-poset of $s_p(G)$ consisting of non-trivial elementary abelian p-subgroups of G, and $b_p(G)$ denote the sub-G-poset of $s_p(G)$ consisting of non-trivial p-subgroups P of G such that $P = O_p\big(N_G(P)\big)$. Then*

$$St_p(G) = \tilde{\Lambda}_{a_p(G)} = \tilde{\Lambda}_{b_p(G)} \ \textit{in } B(G)$$

PROOF. Let P be a non-trivial p-subgroup of G. Suppose P is not elementary abelian, and denote by $\Phi(P)$ the Frattini subgroup of P. Let $f :]., P[_{s_p(G)} \to [\Phi(P), P[_{s_p(G)}$ be the map defined by $f(Q) = Q.\Phi(P)$, and let g be the inclusion map from $[\Phi(P), P[_{s_p(G)}$ to $]., P[_{s_p(G)}$. Then f and g are maps of $N_G(P)$-posets. Moreover $f \circ g = Id$, and $Id \leq g \circ f$. Now $[\Phi(P), P[_{s_p(G)}$ has a smallest element, thus $\tilde{\Lambda}_{]., P[_{s_p(G)}} = 0$ in $B\big(N_G(P)\big)$ by Proposition 4.2.5. In other words

$$\big(s_p(G)\big)_\sharp \subseteq a_p(G) \subseteq s_p(G)$$

and Corollary 4.2.9 shows that $\tilde{\Lambda}_{s_p(G)} = \tilde{\Lambda}_{a_p(G)}$.

The other equality is similar: let P be a non-trivial p-subgroup of G. Set $R = O_p\big(N_G(P)/P\big)$, and suppose $R \neq P$. Let $f :]P, .[_{s_p(G)} \to s_p\big(N_G(P)/P\big)$ defined by $f(Q) = N_Q(P)/P$, and let $g : s_p\big(N_G(P)/P\big) \to]P, .[_{s_p(G)}$ defined by $g(Q/P) = Q$. Then $f \circ g = Id$, and $g \circ f \leq Id$. By Proposition 4.2.5, it follows that

$$\tilde{\Lambda}_{]P, .[_{s_p(G)}} = \mathrm{Inf}_{N_G(P)/P}^{N_G(P)} St_p\big(N_G(P)/P\big)$$

and this is zero since $O_p\big(N_G(P)/P\big) = R/P \neq \mathbb{1}$. Hence $\tilde{\Lambda}_{]P, .[_{s_p(G)}} = 0$ in $B\big(N_G(P)/P\big)$. In other words

$$\big(s_p(G)\big)^\sharp \subseteq b_p(G) \subseteq s_p(G)$$

and the proposition follows from Corollary 4.2.9. □

NOTATION 4.3.5. For the remainder of this section, the group G will be a finite group, and $\underline{\mathcal{F}}$ will denote a family of subgroups of G such that:

(1) $\mathbb{1} \in \underline{\mathcal{F}}$.

(2) $\underline{\mathcal{F}}$ is closed by conjugation.
(3) $\underline{\mathcal{F}}$ is closed by products, i.e. if P and Q are elements of $\underline{\mathcal{F}}$ such that P normalizes Q, then $P.Q \in \underline{\mathcal{F}}$.

If H is a subgroup of G, then $\underline{\mathcal{F}}(H)$ is the set of subgroups of H which are in $\underline{\mathcal{F}}$. It is a family of subgroups of H having the above three properties.

Denote by $\mathcal{F}$ the family $\underline{\mathcal{F}}$ with the trivial group removed. Then $\underline{\mathcal{F}}$ and $\mathcal{F}$ are ordered by inclusion of subgroups, and they are G-posets. When H is a subgroup of G, denote similarly by $\mathcal{F}(H)$ the set of non-trivial subgroups of H which are in $\mathcal{F}$.

Definition 4.3.6. *The Steinberg invariant $St_{\mathcal{F}}(G)$ of G with respect to $\mathcal{F}$ is the reduced Lefschetz invariant of the G-poset $\mathcal{F}$.*

Thus if $\mathcal{F} = s_p(G)$, then $St_{\mathcal{F}}(G) = St_p(G)$.

Lemma 4.3.7. *Let G and $\mathcal{F}$ as in 4.3.5. If $P \in \mathcal{F}$, then $St_{\mathcal{F}}(G)^P = 0$ in $B\big(N_G(P)/P\big)$.*

Proof. Let a denote the inclusion map from $[P,.[_{\mathcal{F}}$ to $\mathcal{F}^P$, and b denote the map from $\mathcal{F}^P$ to $[P,.[_{\mathcal{F}}$ defined by $b(P) = FP$. Then $b \circ a = Id_{[P,.[_{\mathcal{F}}}$, and $Id_{\mathcal{F}^P} \leq a \circ b$. Thus $St_{\mathcal{F}}(G)^P = \tilde{\Lambda}_{\mathcal{F}^P} = \tilde{\Lambda}_{[P,.[_{\mathcal{F}}} = 0$ since $[P,.[_{\mathcal{F}}$ has a smallest element P. □

Theorem 4.3.8. [Bouc] *Let G and $\mathcal{F}$ as in 4.3.5. If $X \in B(G)$, set*

$$St_{\mathcal{F}}(G,X) = \sum_{P \in G\backslash \underline{\mathcal{F}}} \mathrm{Ind}_{N_G(P)}^G\big(\mu_{\underline{\mathcal{F}}}(\mathbb{1},P)\,\mathrm{Inf}_{N_G(P)/P}^{N_G(P)} X^P\big)$$

Then the map $X \mapsto St_{\mathcal{F}}(G,X)$ is an idempotent group endomorphism of $B(G)$, and its image is the set of elements X of $B(G)$ such that $X^P\!=\!0$ in $B\big(N_G(P)/P\big)$ for all $P \in \mathcal{F}$.

Proof. Clearly if $X^P = 0$ for $P \in \mathcal{F}$, then $St_{\mathcal{F}}(G,X) = X$. Hence the only thing to check is that if $P \in \mathcal{F}$, then $St_{\mathcal{F}}(G,X)^P = 0$.

By linearity, one can suppose that X is a G-set, viewed as a G-poset for the discrete ordering. Let Z denote the subposet of $X \times \mathcal{F}$ consisting of pairs (x,P) such that $P \subseteq G_x$. Let $a : Z \to X$ defined by $a\big((x,P)\big) = x$. Then a is a map of G-posets, and a^x is isomorphic to $\mathcal{F}(G_x)$.

Now obviously $\Lambda_X = X$ for any dicrete G-poset X. Moreover all the elements of X are maximal in X, thus $\tilde{\Lambda}_{]x,.[_X} = -G_x/G_x$ for all $x \in X$. It follows from Proposition 4.2.7 that

$$\Lambda_X = X = \Lambda_Z - \sum_{x \in G\backslash X} \mathrm{Ind}_{G_x}^G St_{\mathcal{F}(G_x)}(G_x) \tag{4.3.9}$$

Now let $b : Z \to \mathcal{F}$ be the map defined by $b\big((x,P)\big) = P$. Then b is a map of G-posets, and for $Q \in \mathcal{F}$

$$g_Q = \{(x,P) \in X \times \mathcal{F} \mid Q \subseteq P \subseteq G_x\}$$

The maps $c : q_Q \to X^Q$ and $d : X^Q \to g_Q$ defined by $c\big((x,P)\big) = x$ and $d(x) = (x,Q)$ are maps of $N_G(Q)$-posets, such that $d \circ c \leq Id_{g_Q}$ and $c \circ d = Id_{X^Q}$. Thus by Corollary 4.2.8

$$\Lambda_Z = -\sum_{Q \in G\backslash \mathcal{F}} \mathrm{Ind}_{N_G(Q)}^G\big(\mu_{\underline{\mathcal{F}}}(\mathbb{1},Q)X^Q\big)$$

Hence equation 4.3.9 gives

$$X = -\sum_{Q \in G\backslash \mathcal{F}} \mathrm{Ind}_{N_G(Q)}^G\big(\mu_{\underline{\mathcal{F}}}(\mathbb{1},Q)X^Q\big) - \sum_{x \in G\backslash X} \mathrm{Ind}_{G_x}^G St_{\mathcal{F}(G_x)}(G_x)$$

and finally

$$St_{\mathcal{F}}(G,X) = -\sum_{x\in G\backslash X} \mathrm{Ind}_{G_x}^G St_{\mathcal{F}(G_x)}(G_x)$$

Now for any $x \in X$ and any $P \in \mathcal{F}$, if $P \subseteq G_x$, then $St_{\mathcal{F}(G_x)}(G_x)^P = 0$ by Lemma 4.3.7. Theorem 4.3.8 follows from assertion (3) of Proposition 2.2.1. □

Remark 4.3.10. Another proof of Theorem 4.3.8 can be found in [**5**], where decompositions of $B(G)$ associated to $\mathcal{F}$ are constructed.

Corollary 4.3.11. *Let G and $\underline{\mathcal{F}}$ as in 4.3.5, let p be a prime number, and let $\mathcal{O}$ be a complete local noetherian commutative ring with residue field of characteristic p. Suppose that $\mathcal{F}$ contains the set $s_p(G)$ of non-trivial p-subgroups of G. Then for any finite G-poset X the image of $St_{\mathcal{F}}(G,X)$ in $A_{\mathcal{O}}(G)$ is a linear combination of projective $\mathcal{O}G$-lattices.*

Proof. Indeed, in this case, if Y is an element of $B(G)$ such that $Y^P = 0$ for $P \in \mathcal{F}$, then in particular $Y^P = 0$ for any non-trivial p-subgroup P of G. It follows that $|Y^H| = 0$ whenever H is a subgroup of G with $O_p(H) \neq \mathbb{1}$.

Now $Y = \sum_H |Y^H| e_H^G$ in $\mathbb{Q}B(G)$, and by Theorem 3.5.5, the idempotent e_H^G maps to zero in $\mathbb{Q}A_{\mathcal{O}}(G)$ if H is not p-hypoelementary. If H is hypoelementary and if $O_p(H) = \mathbb{1}$, then H is a cyclic p'-group, and the idempotent e_H^G is a linear combination of elements G/K, for subgroups K of H.

Those remarks show that the image of Y in $A_{\mathcal{O}}(G)$ is a linear combination of permutation lattices $\mathrm{Ind}_K^G\mathcal{O}$, for cyclic p'-subgroups K of G. The corollary follows, since those lattices are $\mathcal{O}G$-projective. □

Remark 4.3.12. If $X = G/G$, then $St_{\mathcal{F}}(G,X) = -St_{\mathcal{F}}(G)$. Hence the image of $St_{\mathcal{F}}(G)$ in $A_{\mathcal{O}}(G)$ is a linear combination of projective $\mathcal{O}G$-lattices. In fact, it has been shown by Webb ([**47**]) that the chain complex $\tilde{C}_*(\mathcal{F},\mathbb{Z}_p)$ is the direct sum of a complex of projective $\mathbb{Z}_pG$-lattices, and of a split acyclic augmented subcomplex.

Corollary 4.3.13. *Let G and $\underline{\mathcal{F}}$ as in 4.3.5, and suppose moreover that $\underline{\mathcal{F}}$ is closed by taking subgroups. Let X be a finite G-poset.*

(1) *If $P \in \underline{\mathcal{F}}$, then there exists an integer m_P such that*

$$\mathrm{Res}_P^G St_{\mathcal{F}}(G,X) = m_P\ P/\mathbb{1}$$

(2) *Denote by $|G|_{\mathcal{F}}$ the l.c.m of the orders of elements of $\mathcal{F}$. Then*

$$\tilde{\chi}(X) + \sum_{P\in\mathcal{F}} \mu_{\underline{\mathcal{F}}}(\mathbb{1},P)\ \tilde{\chi}(X^P) \equiv 0 \quad (|G|_{\mathcal{F}})$$

and in particular

$$\tilde{\chi}(\mathcal{F}) \equiv 0 \quad (|G|_{\mathcal{F}})$$

Proof. Let Y be an element of $B(G)$ such that $Y^P = 0$ for all $P \in \mathcal{F}$. Consider the restriction of Y to an element Q of $\mathcal{F}$. Then

$$\mathrm{Res}_Q^G Y = \sum_R |Y^R| e_R^Q$$

where R runs through a set of representatives of conjugacy classes of subgroups of Q. Since $\underline{\mathcal{F}}$ is closed by taking subgroups, the only non-zero term in this sum is obtained for $R = \mathbb{1}$. Hence

$$\mathrm{Res}_Q^G Y = |Y| e_{\mathbb{1}}^Q = \frac{|Y|}{|Q|}\ Q/\mathbb{1}$$

This shows that $|Y| \equiv 0\ (|Q|)$, and $\mathrm{Res}_Q^G Y$ is an integer multiple of $Q/\mathbb{1}$. The first assertion follows, for $Y = St_{\mathcal{F}}(G,X)$. Taking cardinalities gives the first congruence. The second one follows from Lemma 4.3.7 in the case $X = \mathcal{F}$. □

REMARK 4.3.14. When $\mathcal{F}$ is the poset $s_p(G)$ of non-trivial p-subgroups of G, the second congruence is $\tilde{\chi}\big(s_p(G)\big) \equiv 0 \ (|G|_p)$, and it is due to Brown ([**13**]). Similar congruences have been stated by Thévenaz ([**38**]) and Brown-Thévenaz ([**14**]).

The computation of the Steinberg invariants of the symmetric groups, or more generally of wreath products of a finite group with symmetric groups, can be found in Bouc ([**6**]). It requires the use of a ring of formal power series with coefficients in Burnside rings.

5. The Mackey and Green functor structure

The notion of Mackey functor is a formal generalization of the properties of induction, restriction, and conjugation exposed in section 2.2. The notion of Green functor keeps track moreover of the ring structure. Both have several equivalent definitions, that are quickly recalled hereafter. The Burnside functor is a universal object in this framework also.

5.1. Mackey functors and subgroups. Let G be a finite group, and R be a ring. Let R-**Mod** denote the category of R-modules. The first definition of Mackey functors is due to Green ([**23**]):

DEFINITION 5.1.1. [Green] *A Mackey functor M for the group G over R (or with values in R-**Mod**) consists of the following data:*

- *For each subgroup H of G, an R-module $M(H)$.*
- *Whenever $H \subseteq K$ are subgroups of G with $H \subseteq K$, a map of R-modules $t_H^K : M(H) \to M(K)$, called transfer or induction, and a map of R-modules $r_H^K : M(K) \to M(H)$ called restriction.*
- *For each subgroup H of G and each element $x \in G$, a map of R-modules $c_{x,H} : M(H) \to M({}^xH)$, called conjugation.*

Those maps are subject to four types of conditions:

- *(Triviality conditions) For any subgroup H of G, and any $h \in H$, the maps t_H^H, r_H^H, and $c_{h,H}$ are equal to the identity map of $M(H)$.*
- *(Transitivity conditions) If $H \subseteq K \subseteq L$ are subgroups of G, then $t_K^L \circ t_H^K = t_H^L$ and $r_H^K \circ r_K^L = r_H^L$. If $x, y \in G$, then $c_{y,{}^xH} \circ c_{x,H} = c_{yx,H}$.*
- *(Compatibility conditions) If $H \subseteq K$ are subgroups of G and $x \in G$, then $c_{x,K} \circ t_H^K = t_{{}^xH}^{{}^xK} \circ c_{x,H}$ and $c_{x,H} \circ r_H^K = r_{{}^xH}^{{}^xK} \circ c_{x,K}$.*
- *(Mackey axiom) If $H \subseteq K \supseteq L$ are subgroups of G, then*

$$r_H^K \circ t_L^K = \sum_{x \in H \backslash K / L} t_{H \cap {}^xL}^H \circ c_{x,H^x \cap L} \circ r_{H^x \cap L}^L$$

If M and N are Mackey functors for G over R, then a morphism of Mackey functors $f : M \to N$ is a collection of morphisms of R-modules $f_H : M(H) \to N(H)$, which commute to the maps t_H^K, r_H^K and $c_{x,H}$.

The category of Mackey functors for G over R is denoted by $Mack_R(G)$.

EXAMPLE 5.1.2. The Burnside Mackey functor B is the Mackey functor with values in $\mathbb{Z}$-**Mod** which value at H is the Burnside ring $B(H)$. If $H \subseteq K$ are subgroups of G, then $t_H^K = \mathrm{Ind}_H^K$ and $r_H^K = \mathrm{Res}_H^K$. The conjugation maps $c_{x,H}$ are defined by $c_{x,H}(Z) = {}^xZ$ for $Z \in B(H)$ and $x \in G$.

More generally, if R is a ring, the Burnside functor RB is defined by "tensoring everything with R", i.e. setting $RB(H) = R \otimes_{\mathbb{Z}} B(H)$, and extending the maps t_H^K, r_H^K and $c_{x,H}$ in the obvious way.

Another example of Mackey functor is the Green ring functor $A_{\mathcal{O}}$, which value at H is the Green ring $A_{\mathcal{O}}(H)$ of $\mathcal{O}H$-lattices defined in section 3.5. The transfer is given by induction of lattices, and the operations of restriction and conjugation are the obvious ones. It is clear that the morphism $\pi_{\mathcal{O}}$ actually defines a morphism of Mackey functors from the Burnside functor to the Green ring functor.

5.2. Mackey functors and G-sets. If M is a Mackey functor for G over R, if H and K are subgroups of G, and if $x \in G$ is such that $H^x \subseteq K$, then there are maps of R-modules

$$a_x = t_{H^x}^K \circ c_{x^{-1},H} : M(H) \to M(K) \qquad b_x = c_{x,H^x} \circ r_{H^x}^K : M(K) \to M(H)$$

Moreover, if $k \in K$, then since M is a Mackey functor

$$a_{xk} = t_{H^{xk}}^K \circ c_{(xk)^{-1},H} = c_{k^{-1}x^{-1},K} \circ t_H^{{}^xK} = c_{k^{-1},K} \circ c_{x^{-1},{}^xK} \circ t_H^{{}^xK} = c_{x^{-1},{}^xK} \circ t_H^{{}^xK} = a_x$$

Similarly $b_{xk} = b_x$. Hence a_x and b_x only depend on the class xK. The crucial observation is that the set of classes xK such that $H^x \subseteq K$ is in one to one correspondence with the set of G-sets homomorphisms from G/H to G/K. This leads to the second definition of Mackey functors, due to Dress ([**21**]).

Recall that a *bivariant* functor M from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a pair of functors $M = (M_*, M^*)$, where M_* is a functor from $\mathcal{C}$ to $\mathcal{D}$ and M^* is a functor from $\mathcal{C}$ to $\mathcal{D}^{op}$ (or a cofoncteur from $\mathcal{C}$ to $\mathcal{D}$), which coincide on objects, i.e. $M_*(C) = M^*(C)$ for all objects C of $\mathcal{C}$. This common value is simply denoted by $M(C)$.

Definition 5.2.1. [Dress] *A Mackey functor M for the group G with values in the category R-**Mod** of R-modules is a bivariant functor $M = (M_*, M^*)$ from the category of finite G-sets to R-**Mod**, with the following two properties:*

(1) *Let X and Y be any finite G-sets, and let i_X (resp. i_Y) denote the canonical injection from X (resp. Y) into $X \sqcup Y$. Then the morphisms*

$$\big(M_*(i_X), M_*(i_Y)\big) : M(X) \oplus M(Y) \to M(X \sqcup Y)$$

$$\begin{pmatrix} M^*(i_X) \\ M^*(i_Y) \end{pmatrix} : M(X \sqcup Y) \to M(X) \oplus M(Y)$$

are mutually inverse isomorphisms.

(2) *Let*

$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ Z & \xrightarrow[d]{} & T \end{array}$$

be any cartesian (i.e. pull-back) square of finite G-sets. Then

$$M_*(b) \circ M^*(a) = M^*(d) \circ M_*(c)$$

A morphism of Mackey functors $f : M \to N$ is a natural transformation of bivariant functors, i.e. a collection of morphisms of R-modules $f_X : M(X) \to N(X)$, for finite G-sets X, which commute to the maps $M_*(a)$ and $M^*(a)$ for any morphism of finite G-sets $a : X \to Y$.

If M is a Mackey functor for the first definition, then M yields a Mackey functor $\hat{M}$ for this definition by choosing a set of representatives $[G\backslash X]$, for each finite G-set X, and then setting

$$\hat{M}(X) = \bigoplus_{x \in [G\backslash X]} M(G_x)$$

Let $f : X \to Y$ be a morphism of G-sets. If $x \in [G\backslash X]$ and $y \in [G\backslash Y]$ are such that $f(x) \in Gy$, then there exists $g \in G$ such that $f(x) = gy$. In this case $G_x \subseteq {}^gG_y$, and one can define two maps

$$\alpha_{y,x} = t_{{}^gG_x}^{G_y} \circ c_{g,G_x} : M(G_x) \to M(G_y)$$

$$\beta_{x,y} = c_{g^{-1},{}^gG_x} \circ r^{G_y}_{{}^gG_x} : M(G_y) \to M(G_x)$$

Those maps depend only on x and y, and do not depend on the chosen element g. Define moreover $\alpha_{y,x} = \beta_{x,y} = 0$ for $x \in [G\backslash X]$ and $y \in [G\backslash Y]$ if $f(x) \notin Gy$.

Then the map $\hat{M}_*(f)$ is defined by the block matrix $(\alpha_{y,x})_{y\in[G\backslash Y],x\in[G\backslash X]}$, and the map $\hat{M}^*(f)$ is defined by the block matrix $(\beta_{x,y})_{x\in[G\backslash X],y\in[G\backslash Y]}$. One can check that $\hat{M}$ is a Mackey functor for the second definition.

Conversely, a Mackey functor $\hat{M}$ for the second definition yields a Mackey functor M for the first definition, by setting

$$M(H) = \hat{M}(G/H) \qquad t^K_H = \hat{M}_*(\pi^K_H) \qquad r^K_H = \hat{M}^*(\pi^K_H) \qquad c_{x,H} = \hat{M}_*(\gamma_{x,H})$$

where the morphism π^K_H is the canonical projection $G/H \to G/K$ for $H \subseteq K$, and $\gamma_{x,H}$ is the isomorphism $yH \mapsto yx^{-1x}H$ from G/H to $G/{}^xH$, for $x \in G$.

Example 5.2.2. One can show (see for instance [**9**] Chapter 2.4) that the Mackey functor associated to the Burnside functor B (still denoted by B) can be described as follows: if X is a finite G-set, let G-$\mathbf{set}{\downarrow_X}$ denote the category which objects are the finite G-sets over X, i.e. the pairs (Y, f), where Y is a finite G-set and $f : Y \to X$ is a map of G-sets. A morphism in G-$\mathbf{set}{\downarrow_X}$ from (Y, f) to (Z, g) is a map of G-sets $h : Y \to Z$ such that $g \circ h = f$. The composition of morphisms is the composition of maps.

Then $B(X)$ is the Grothendieck group of the category G-$\mathbf{set}{\downarrow_X}$, for relation given by decomposition in disjoint union. If $\phi : X \to X'$ is a morphism of G-sets, then the map $B_*(\phi)$ sends (Y, f) to $(Y, \phi \circ f)$, and the map $M^*(\phi)$ sends (Z, g) to the G-set over X defined by the pull-back square

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \Big\downarrow & & \Big\downarrow g \\ X & \xrightarrow[\phi]{} & X' \end{array}$$

5.3. Mackey functors as modules. The third definition of Mackey functors is due to Thévenaz and Webb ([**43**]), who defined the following algebra:

Definition 5.3.1. [Thévenaz-Webb] *Let G be a finite group, and R be a commutative ring. The Mackey algebra $\mu_R(G)$ is the R-algebra with generators t^K_H, r^K_H, and $c_{x,H}$, for subgroups $H \subseteq K$ of G, and $x \in G$, subject to the following relations:*

- *If H is a subgroup of G, and if $h \in H$, then $t^H_H = r^H_H = c_{h,H}$. Moreover $\sum_{H\subseteq G} t^H_H = 1_{\mu_R(G)}$, and if $H \neq K$ are subgroups of G, then $t^H_H t^K_K = 0$.*
- *If $H \subseteq K \subseteq L$ are subgroups of G, then $t^L_K \circ t^K_H = t^L_H$ and $r^K_H \circ r^L_K = r^L_H$. If $x, y \in G$, then $c_{y,{}^xH} \circ c_{x,H} = c_{yx,H}$.*
- *If $H \subseteq K$ are subgroups of G and $x \in G$, then $c_{x,K} \circ t^K_H = t^{{}^xK}_{{}^xH} \circ c_{x,H}$ and $c_{x,H} \circ r^K_H = r^{{}^xK}_{{}^xH} \circ c_{x,K}$.*
- *If $H \subseteq K \supseteq L$ are subgroups of G, then*

$$r^K_H \circ t^K_L = \sum_{x\in H\backslash K/L} t^H_{H\cap {}^xL} \circ c_{x,H^x\cap L} \circ r^L_{H^x\cap L}$$

A Mackey functor for G over R is a just a $\mu_R(G)$-module, and a morphism of Mackey functors is a morphism of $\mu_R(G)$-modules.

If M is a Mackey functor over R for the first definition, set

$$\tilde{M} = \mathop{\oplus}_{H\subseteq G} M(H)$$

This is endowed with an obvious structure of $\mu_R(G)$-module: for instance, the action of the generator t_H^K of $\mu_R(G)$ is zero on the component $M(H')$ of $\tilde{M}$, if $H' \neq H$, and it is equal to the map $t_H^K : M(H) \to M(K)$ on the component $M(H)$.

Conversely, being given a $\mu_R(G)$-module $\tilde{M}$, one recovers a Mackey functor for the first definition by setting $M(H) = t_H^H \tilde{M}$. The transfer maps, restriction maps, and conjugation maps for M are obtained by multiplication by the generators of $\mu_R(G)$ with the same name: for instance, the relations of the Mackey algebra show that $t_H^K M(H) \subseteq M(K)$.

Proposition 5.3.2. [Thévenaz-Webb] *Let G be a finite group and R be a commutative ring. Then the algebra $\mu_R(G)$ is a finitely generated free R-module, with basis the set of elements $t_K^H c_{x,K^x} r_{K^x}^L$, for subgroups H and L of G, for $x \in H\backslash G/L$, and a subgroup K of $H \cap {}^xL$ up to $H \cap {}^xL$-conjugacy.*

Proof. See [**43**] Propositions (3.2) and (3.3), or [**9**] Chapter 4.4. □

Remark 5.3.3. It follows in particular that the rank over R of $\mu_R(G)$ does not depend on R. In other words, the algebra $\mu_R(G)$ is isomorphic to $R \otimes_{\mathbb{Z}} \mu_{\mathbb{Z}}(G)$. It is sometimes convenient to define $\mu_R(G) = R \otimes_{\mathbb{Z}} \mu_{\mathbb{Z}}(G)$ for any ring R (not necessarily commutative). The case $R = \mu_S(G)$ for a commutative ring S is of interest (see [**9**] Chapter 1.2). Not that if R is not commutative, then $\mu_R(G)$ is not strictly speaking an R-algebra.

The next corollary states a link between the Burnside ring and the Mackey algebra:

Corollary 5.3.4. *Let G be a finite group, and R be a commutative ring. Define a G-set Ω_G by $\Omega_G = \bigsqcup_{H \subseteq G} G/H$. If H and L are subgroups of G, if $x \in G$ and K is a subgroup of $H \cap {}^xL$, denote by $\pi_{H,x,L}$ the map of G-sets defined by*

$$\pi_{H,x,L} : yK \in G/K \mapsto (yH, y^xL) \in \Omega_G^2$$

Then the R-linear map defined by

$$t_K^H c_{x,K^x} r_{K^x}^L \in \mu_R(G) \mapsto (G/K, \pi_{H,x,L}) \in RB(\Omega_G^2)$$

is an isomorphism of R-modules.

Proof. It is easy to check that the images of the basis elements of $\mu_R(G)$ form an R-basis of $RB(\Omega_G^2)$. □

Remark 5.3.5. Of course, the isomorphism of Corollary 5.3.4 is *not* an algebra isomorphism, since $\mu_R(G)$ is not commutative in general. One can show (see [**9**] Chapter 4.5.1) that the multiplication law it induces on $RB(\Omega_G^2)$ is given by

$$(V, W) \in B(\Omega_G^2) \mapsto V \circ_Y W = B_* \begin{pmatrix} xyz \\ xz \end{pmatrix} B^* \begin{pmatrix} xyz \\ xyyz \end{pmatrix} (V \times W)$$

where $\binom{xyz}{xz}$ is the map $(x, y, z) \in \Omega_G^3 \mapsto (x, z) \in \Omega_G^2$, and $\binom{xyz}{xyyz}$ is the map $(x, y, z) \in \Omega_G^3 \mapsto (x, y, y, z) \in \Omega_G^4$, and $V \times W$ is the product for the Green functor structure, to be defined in the next section, in Example 5.4.4.

5.4. Green functors. Roughly speaking, a *Green functor* for the finite group G over the commutative ring R is a "Mackey functor with a compatible ring structure". More precisely, there are two equivalent definitions of Green functors. The first one is due to Green ([**23**]):

Definition 5.4.1. *A Green functor A for G over R is a Mackey functor for G over R, such that for any subgroup H of G, the R-module $A(H)$ has a structure of R-algebra (associative, with unit), with the following properties:*

(1) *If K is a subgroup of G and $x \in G$, then the map $c_{x,H}$ is a morphism of rings (with unit) from $A(K)$ to $A({}^xK)$.*
(2) *If $H \subseteq K$ are subgroups of G, then the map r_H^K is a morphism of rings (with unit) from $A(K)$ to $A(H)$.*
(3) [Frobenius identities] *In the same conditions, if $a \in A(K)$ and $b \in A(H)$, then*

$$a.t_H^K(b) = t_H^K\big(r_H^K(a).b\big) \qquad t_H^K(b).a = t_H^K\big(b.r_H^K(a)\big)$$

If A and A' are Green functors for G over R, a morphism of Green functors $f : A \to A'$ is a morphism of Mackey functors such that for each subgroup H of G, the map $f_H : A(H) \to A'(H)$ is a ring homomorphism. The morphism is unitary if moreover all the maps f_H are unitary, or equivalently, if f_G is unitary.

Example 5.4.2. The Burnside Mackey functor RB is a Green functor, if the product on $RB(H)$ for $H \subseteq G$ is defined by linearity from the direct product of H-sets. The Frobenius identities follow from Proposition 2.2.1.

The second definition of Green functors is in terms of G-sets, and is detailed in [**9**] Chapter 2.2:

Definition 5.4.3. *A Green functor A for G over R is a Mackey functor for G over R, endowed for any G-sets X and Y with R-bilinear maps $A(X) \times A(Y) \to A(X \times Y)$ with the following properties:*

- *(Bifunctoriality) If $f : X \to X'$ and $g : Y \to Y'$ are morphisms of G-sets, then the squares*

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ {\scriptstyle A_*(f) \times A_*(g)}\downarrow & & \downarrow{\scriptstyle A_*(f \times g)} \\ A(X') \times A(Y') & \xrightarrow[\times]{} & A(X' \times Y') \end{array}$$

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ {\scriptstyle A^*(f) \times A^*(g)}\uparrow & & \uparrow{\scriptstyle A^*(f \times g)} \\ A(X') \times A(Y') & \xrightarrow[\times]{} & A(X' \times Y') \end{array}$$

  are commutative.
- *(Associativity) If X, Y and Z are G-sets, then the square*

$$\begin{array}{ccc} A(X) \times A(Y) \times A(Z) & \xrightarrow{Id_{A(X)} \times (\times)} & A(X) \times A(Y \times Z) \\ {\scriptstyle (\times) \times Id_{A(Z)}}\downarrow & & \downarrow{\scriptstyle \times} \\ A(X \times Y) \times A(Z) & \xrightarrow[\times]{} & A(X \times Y \times Z) \end{array}$$

  is commutative, up to identifications $(X \times Y) \times Z \simeq X \times Y \times Z \simeq X \times (Y \times Z)$.
- *(Unitarity) If $\bullet$ denotes the trivial G-set G/G, there exists an element $\varepsilon_A \in A(\bullet)$, called the unit of A, such that for any G-set X and for any $a \in A(X)$*

$$A_*(p_X)(a \times \varepsilon_A) = a = A_*(q_X)(\varepsilon_A \times a)$$

  denoting by p_X (resp. q_X) the (bijective) projection from $X \times \bullet$ (resp. from $\bullet \times X$) to X.

If A and A' are Green functors for G over R, a morphism of Green functors $f : A \to A'$ is a morphism of Mackey functors such that for any G-sets X and Y, the square

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ {\scriptstyle f_X \times f_Y}\big\downarrow & & \big\downarrow{\scriptstyle f_{X\times Y}} \\ A'(X) \times A'(Y) & \xrightarrow[\times]{} & A'(X \times Y) \end{array}$$

is commutative. The morphism f is unitary if moreover $f_\bullet(\varepsilon_A) = \varepsilon_{A'}$.

Example 5.4.4. If X and Y are finite G-sets, then the product $B(X)\times B(Y) \to B(X \times Y)$ is defined by linearity from the map sending the finite G-set U over X and the finite G-set V over Y to the cartesian product $U \times V$, which is a G-set over $X \times Y$. The element $\varepsilon_B \in B(\bullet)$ is the image of the trivial G-set $\bullet = G/G$.

Remark 5.4.5. There is an obvious notion of module over a Green functor: a (left) module M over the Green functor A is a Mackey functor M endowed for any G-sets X and Y with bilinear maps $A(X) \times M(Y) \to M(X \times Y)$ which are bifunctorial, associative, and unitary. With this definition, a Mackey functor for G over R is just a module over the Green functor RB. This can be viewed as a generalization of the identification of abelian groups with $\mathbb{Z}$-modules.

5.5. Induction, Restriction, Inflation. The second definition of Mackey functors leads to a natural notion of induction, restriction, and inflation for Mackey functors:

Definition 5.5.1. *Let G be a finite group, and R be a commutative ring.*

- *If H is a subgroup of G, and M is a Mackey functor for G over R, then the restriction $\mathrm{Res}_H^G M$ is the Mackey functor for H over R obtained by composition of the functor $M : G$-**set** $\to R$-**Mod** with the induction functor $\mathrm{Ind}_H^G : H$-**set** $\to G$-**set**. If A is a Green functor for G over R, then $\mathrm{Res}_H^G A$ has a natural structure of Green functor for H over R.*
- *If H is a subgroup of G, and N is a Mackey functor for H over R, then the induced Mackey functor $\mathrm{Ind}_H^G M$ is the Mackey functor for G over R obtained by composition of the functor $M : H$-**set** $\to R$-**Mod** with the restriction functor $\mathrm{Res}_H^G : G$-**set** $\to H$-**set**. If A is a Green functor for H over R, then $\mathrm{Ind}_H^G A$ has a natural structure of Green functor for H over R.*
- *If K is a normal subgroup of G, and N is a Mackey functor for G/K, then the inflated Mackey functor $\mathrm{Inf}_{G/K}^G N$ is the Mackey functor for G over R obtained by composition of the functor $M : G/K$-**set** $\to R$-**Mod** with the fixed points functor H-**set** $\to G$-**set**. If A is a Green functor for G/K over R, then $\mathrm{Inf}_H^G A$ has a natural structure of Green functor for G over R.*

Remark 5.5.2. If M is a Mackey functor for G, and if $K \subseteq H$ are subgroups of G, then

$$(\mathrm{Res}_H^G M)(K) = (\mathrm{Res}_H^G M)(H/K) = M(\mathrm{Ind}_H^G H/K) = M(G/K) = M(K)$$

so the above definition of $\mathrm{Res}_H^G M$ coincides with the naive one. In particular, the restriction of the Burnside functor for G to the subgroup H is isomorphic to the Burnside functor for H.

Remark 5.5.3. The constructions of Definition 5.5.1 are examples of functors between categories of Mackey functors, obtained by composition with functors between the corresponding categories of G-sets. A uniform description of this kind of functors is given in [**7**], using the formalism of *bisets*. In [**9**] Chapter 8, it is shown that those constructions also apply to Green functors.

Another fundamental example of this kind of functors is the following:

DEFINITION 5.5.4. [Dress] *If X is a finite G-set, and if M is is a Mackey functor for G over R, then the Mackey functor M_X is the Mackey functor for G over R obtained by composition of the functor $M : G\textbf{-set} \to R\textbf{-Mod}$ with the endofunctor $Y \mapsto Y \times X$ of G-**set**. If A is a Green functor for G over R, then A_X has a natural structure of Green functor for G over R.*

The endofunctor $M \mapsto M_X$ on $Mack_R(G)$ is denoted by $\mathcal{I}_X$.

REMARK 5.5.5. In the case of a transitive G-set $X = G/H$, the isomorphism

$$G/H \times Y \simeq \mathrm{Ind}_H^G \mathrm{Res}_H^G Y$$

of Proposition 2.2.1 leads to an isomorphism of Mackey functors

$$M_{G/H} \simeq \mathrm{Ind}_H^G \mathrm{Res}_H^G M$$

Induction and restriction are mutual left and right adjoints:

PROPOSITION 5.5.6. [Thévenaz-Webb] *Let G be a finite group, let H be a subgroup of G, and R be a commutative ring.*

(1) *The functors*

$$\mathrm{Ind}_H^G : Mack_R(H) \to Mack_R(G) \quad \textit{and} \quad \mathrm{Res}_H^G : Mack_R(G) \to Mack_R(H)$$

are mutual left and right adjoint.

(2) *For any finite G-set X, the endofunctor $\mathcal{I}_X$ is self adjoint.*

PROOF. For assertion (1), see [**41**] Proposition 4.2. Assertion (2) follows trivially. □

5.6. The Burnside functor as projective Mackey functor. The third definition of Mackey functors shows that the category $Mack_R(G)$ is an abelian category, with enough projective objects. A sequence of Mackey functors

$$0 \to L \to M \to N \to 0$$

is exact if and only if for each subgroup H of G, the sequence

$$0 \to L(H) \to M(H) \to N(H) \to 0$$

is an exact sequence of R-modules.

The Burnside functor RB provides the first example of projective Mackey functor:

PROPOSITION 5.6.1. *Let G be a finite group, and R be a commutative ring. Set $\bullet = G/G$.*

(1) *If M is a Mackey functor for G over R, then the map*

$$\Theta_M : f \in \mathrm{Hom}_{Mack_R(G)}(RB, M) \mapsto f_\bullet(\bullet) \simeq M(\bullet) = M(G)$$

is an isomorphism of R-modules.

(2) *The Burnside functor RB is a projective object in $Mack_R(G)$.*

(3) *The map Θ_{RB} is an isomorphism of rings (with unit)*

$$\Theta_{RB} : \mathrm{End}_{Mack_R(G)}(RB) \overset{\simeq}{\to} RB(G)$$

PROOF. Let X be a finite G-set, let (Y,a) be a G-set over X, and denote also by (Y,a) its image in $RB(X)=R\otimes_{\mathbb{Z}}B(X)$. Then clearly $(Y,a)=RB_*(a)(Y,Id_Y)$. Denote by $p_Y: Y\to\bullet$ be the unique morphism of G-sets. Then the square

$$\begin{tikzcd} Y \arrow[r, "p_Y"] \arrow[d, "Id_Y"'] & \bullet \arrow[d, "Id_\bullet"] \\ Y \arrow[r, "p_Y"'] & \bullet \end{tikzcd}$$

is cartesian. Hence denoting by ε_R the element $(\bullet, Id_\bullet)$ of $RB(\bullet)$, it follows that

$$(Y,a)=RB_*(a)RB^*(p_Y)(\varepsilon_R)$$

Now a morphism $f: RB\to M$ is entirely determined by the element $u=f_\bullet(\varepsilon_R)$ of $M(\bullet)=M(G)$, since the map f_X must verify

$$f_X\big((Y,a)\big)=M_*(a)M^*(p_Y)(u)$$

Conversely, if $u\in M(\bullet)$ is given, this equality defines a map of R-modules f_X from $RB(X)$ to $M(X)$. If $g: X\to X'$ is a morphism of G-sets, then obviously

$$\begin{aligned} M_*(g)f_X\big((Y,a)\big) &= M_*(ga)M^*(p_Y)(u)=f_{X'}\big((Y,ga)\big)\\ &= f_{X'}RB_*(g)\big((Y,a)\big) \end{aligned}$$

Similarly, if the square

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "a"'] & Y' \arrow[d, "a'"] \\ X \arrow[r, "g"'] & X' \end{tikzcd}$$

is cartesian, then

$$\begin{aligned} M^*(g)f_{X'}\big((Y',a')\big) &= M^*(g)M_*(a')M^*(p_{Y'})(u)=M_*(a)M^*(h)M^*(p_{Y'})(u)\\ &= M_*(a)M^*(p_Y)(u)=f_X\big((Y,a)\big)\\ &= f_XRB^*\big((Y',a')\big) \end{aligned}$$

This proves that f is a morphism of Mackey functors, and assertion (1) follows. Assertion (2) is now clear, since the evaluation functor $M\mapsto M(G)$ from $Mack_R(G)$ to R-**Mod** is exact.

Now let f and g be two endomorphisms of RB, and suppose that $f_\bullet(\bullet)$ (resp. $g_\bullet(\bullet)$) is equal to (U,p_U) (resp. (V,p_V)) for some G-set U (resp. V). Then

$$\begin{aligned} (g\circ f)_\bullet(\bullet) &= g_\bullet\big((U,p_U)\big)=RB_*(p_U)RB^*(p_U)\big((V,p_V)\big)\\ &= RB_*(p_U)\big((U\times V,p_1)\big)=(U\times V,p_{U\times V}) \end{aligned}$$

where $p_1: U\times V\to U$ is the first projection, since the square

$$\begin{tikzcd} U\times V \arrow[r] \arrow[d, "p_1"'] & V \arrow[d, "p_V"] \\ U \arrow[r, "p_U"'] & \bullet \end{tikzcd}$$

is cartesian.

By linearity, this shows that Θ_{RB} is a ring homomorphism. Clearly Θ_{RB} maps the identity endomorphism to the trivial G-set $\bullet$, which is the unit of $RB(G)$. Assertion (3) follows. □

Colollary 5.3.4 states an isomorphism of R-modules

$$\mu_R(G) \simeq RB(\Omega_G^2)$$

Thus setting $M = RB_{\Omega_G}$, the R-module $\mu_R(G)$ is isomorphic to the module $\tilde{M} = \oplus_{H \subseteq G} M(H)$. The next proposition shows that this is actually an isomorphism of $\mu_R(G)$-modules:

Proposition 5.6.2. [Thévenaz-Webb] *Let G be a finite group, and R be a commutative ring.*

(1) *There is an isomorphism of Mackey functors*

$$\mu_R(G) \simeq RB_{\Omega_G} \simeq \bigoplus_{H \subseteq G} \mathrm{Ind}_H^G \mathrm{Res}_H^G RB$$

(2) *Any Mackey functor is a quotient of a direct sum of induced Burnside Mackey functors.*

Proof. See [**43**] Corollary 8.4. □

5.7. The Burnside functor as universal Green functor. The following proposition shows that the Burnside functor is an initial object in the category of Green functors:

Proposition 5.7.1. *Let G be a finite group, and R be a commutative ring. If A is a Green functor for G over R, then there is a unique unitary morphism of Green functor from RB to A.*

Proof. By Proposition 5.6.1, there is a unique morphism f of Mackey functors from RB to A such that $f_\bullet(\bullet) = \varepsilon_A$. It is easy to check that f is a morphism of Green functors (see [**9**] Proposition 2.4.4 for details). □

Remark 5.7.2. For any Green functor A for G over R, there is a natural algebra structure on $A(\Omega_G^2)$, defined as in Remark 5.3.5 (see [**9**] Chapter 4). Moreover (see [**9**] Chapter 12.2), one can build a Green functor ζ_A for G over R such that $\zeta_A(\bullet) = Z\big(A(\Omega_G^2)\big)$: the evaluation of ζ_A at the finite G-set X is the set of natural transformation from the identity endofunctor $\mathcal{I} = \mathcal{I}_\bullet$ of $Mack_R(G)$ to the endofunctor $\mathcal{I}_X$. The previous proposition shows that for any Green functor A, there is a ring homomorphism from $RB(G)$ to $Z\big(A(\Omega_G^2)\big)$.

Hence there should exist a ring homomorphism from $RB(G)$ to the center of the Mackey algebra. But the center of ring S is isomorphic to the endomorphism algebra of the identity functor on S-**Mod**. Thus, for any Mackey functor M, there should be a morphism from $RB(G)$ to the endomorphism algebra of M, with functorial properties in M.

It can be defined as follows: if X is a finite G-set, then there is a natural transformation $a_X : M \to M_X$ defined for the finite G-set Z by

$$a_{X,Z} = M^* \begin{pmatrix} zx \\ z \end{pmatrix} : M(Z) \to M(Z \times X)$$

where $\binom{zx}{z}$ denotes the projection map $Z \times X \to Z$. Similarly, there is a natural transformation $b_X : M_X \to M$ defined by

$$b_{X,Z} : M_* \begin{pmatrix} zx \\ z \end{pmatrix} : M(Z \times X) \to M(Z)$$

This defines an endomorphism $z(X)_M$ of M by $z(X)_{M,Z} = b_{X,Z} \circ a_{X,Z}$. It is easy to check that for finite G-sets X and Y

$$z(X \sqcup Y)_M = z(X)_M + z(Y)_M \qquad z(X)_M \circ z(Y)_M = z(X \times Y)_M$$

This leads by linearity to an R-algebra homomorphism

$$X \in RB(G) \mapsto z(X)_M \in \mathrm{End}_{Mack_R(G)}(M)$$

for any Mackey functor M for G over R.

Moreover, if $f : M \to N$ is a morphism of Mackey functors, then the square

$$\begin{tikzcd} M \arrow[r, "z(X)_M"] \arrow[d, "f"'] & M \arrow[d, "f"] \\ N \arrow[r, "z(X)_N"'] & N \end{tikzcd}$$

is commutative.

The ring homomorphism from $RB(G)$ to $Z\big(\mu_R(G)\big)$ can be described concretely as follows: if H is a subgroup of G, then the relations of the Mackey algebra show that there is a (non-unitary) ring homomorphism β_H from $RB(H)$ to $\mu_R(G)$ defined by $\beta_H(H/K) = t_K^H r_K^H$. With this notation

PROPOSITION 5.7.3. *Let G be a finite group, and R be a commutative ring. Then the map $z : RB(G) \to \mu_R(G)$ defined by*

$$z(X) = \sum_{H \subseteq G} \beta_H(\mathrm{Res}_H^G X)$$

is an injective unitary ring homomorphism from $RB(G)$ to $Z\big(\mu_R(G)\big)$.

PROOF. By inspection, or see [**43**] Proposition 9.2. $\square$

Taking the image by z of a set of primitive idempotents of $RB(G)$ will give a decomposition of the unit of $\mu_R(G)$ into a sum of orthogonal central idempotents. This in turn will give a decomposition of the category of Mackey functors as a direct sum of abelian categories.

NOTATION 5.7.4. If G is a finite group and R is a commutative ring, denote by $\pi = \pi_R(G)$ be the set of prime factors of $|G|$ which are not invertible in R. If J is a π-perfect subgroup of G, denote by $Mack_R(G, J)$ the subcategory of $Mack_R(G)$ on which the idempotent $z(f_J^G)$ acts trivially. If M is a Mackey functor for G over R, denote by $f_J^G \times M$ the direct summand of M on which $z(f_J^G)$ acts trivially.

THEOREM 5.7.5. [Thévenaz-Webb] *Let G be a finite group, and R be a commutative ring. Set $\pi = \pi_R(G)$.*

(1) *If J is a π-perfect subgroup of G, then the direct summand $f_J^G \times B$ of B has a natural structure of Green functor. Moreover, there are isomorphisms of Green functors*

$$f_J^G \times B \simeq \mathrm{Ind}_{N_G(H)}^G \mathrm{Inf}_{N_G(H)/H}^G (f_{\mathbb{1}}^{N_G(H)/H} \times B)$$

$$RB \simeq \bigoplus_{\substack{J \in [s_G] \\ J = O^\pi(J)}} f_J^G \times RB$$

(2) *The functor*

$$M \mapsto \mathrm{Ind}_{N_G(J)}^G \mathrm{Inf}_{N_G(J)/J}^{N_G(J)} M$$

is an equivalence of categories from $Mack_R\big(N_G(J)/J, \mathbb{1}\big)$ to the category $Mack_R(G, J)$.

PROOF. See [**43**] Section 10 (or [**9**] Proposition 12.1.11 for a generalization to arbitrary Green functors). $\square$

COROLLARY 5.7.6. *Let G be a finite group, and π be a set of prime numbers. If J is a π-perfect subgroup of G, then there is a ring isomorphism*

$$f_J^G B_{(\pi)}(G) \simeq f_{\mathbb{1}}^{N_G(J)/J} B_{(\pi)}\big(N_G(J)/J\big)$$

Proof. By evaluation at the trivial G-set of the first isomorphism of Green functors in Theorem 5.7.5, in the case $R = \mathbb{Z}_{(\pi)}$. Of course, there is also an elementary proof, using the ring homomorphism $X \mapsto X^J$ from $B(G)$ to $B\big(N_G(J)/J\big)$. □

Remark 5.7.7. By definition, the category $Mack_R(G, \mathbb{1})$ is the category of modules over the algebra $\mu_R(G, \mathbb{1}) = z(f_{\mathbb{1}}^G)\mu_R(G)$. One can show that $\mu_R(G, \mathbb{1})$ is equal to the R-submodule of $\mu_R(G)$ generated by the elements $t_P^H c_{x,P^x} r_{P^x}^K$, where H and K are subgroups of G, where $x \in G$, and P is a π-solvable subgroup of $H \cap {}^xK$.

One can also show (see [**10**] Lemme 2.2 for the case $|\pi| = 1$) that this algebra is Morita-equivalent to its subalgebra $\mu_{R,\mathbb{1}}(G)$ generated by the elements $t_P^H c_{x,P^x} r_{P^x}^K$, where $x \in G$ and H and K are π-solvable subgroups of G. The $\mu_{R,\mathbb{1}}(G)$-modules can be viewed as “Mackey functors defined only on π-solvable subgroups”.

Recall (see [**47**] Section 3.2) that a Mackey functor M is said to be *projective relative to a finite G-set X* if the morphism $b_X : M_X \to M$ is split surjective. If $\mathcal{X}$ is a set of subgroups of G, the functor M is said to be projective relative to $\mathcal{X}$ if it is projective relative to the G-set $X = \sqcup_{H\in\mathcal{X}} G/H$. With these definitions, one can show that the Mackey functors in $Mack_R(G, \mathbb{1})$ are exactly the Mackey functors which are projective relative to π-solvable subgroups of G, for $\pi = \pi_R(G)$.

Remark 5.7.8. If R is a field k of characteristic p, there is a nice interpretation of the indecomposable projective objects in $Mack_k(G, \mathbb{1})$, in terms of p-permutation modules:

Theorem 5.7.9. [Thévenaz-Webb] *Let G be a finite group, and k be a field of characteristic $p > 0$. Then evaluation at the trivial subgroup $M \mapsto M(\mathbb{1})$ induces a one to one correspondence between the set of isomorphism classes of indecomposable projective objects in $Mack_k(G, \mathbb{1})$ and the set of isomorphism classes of indecomposable p-permutation kG-modules.*

Proof. See [**43**] Theorem 12.7. □

6. The Burnside ring as biset-functor

6.1. Bisets. In the previous section, the Burnside functor was defined on the subgroups of a given finite group G. The three operations relating the Burnside rings of subgroups of G were induction, restriction, and conjugation.

But there is at least another natural operation on the Burnside ring, which has not yet been considered, namely inflation: if N is a normal subgroup of the group G, then any G/N-set can be viewed as a G-set by inflation. This induces an *inflation functor* $\mathrm{Inf}_{G/N}^G : G/N$-**set** $\to G$-**set** and a ring homomorphism $\mathrm{Inf}_{G/N}^G : B(G/N) \to B(G)$.

There is also a *deflation functor* $\mathrm{Def}_{G/N}^G : G$-**set** $\to G/N$-**set** mapping a G-set X to the set of orbits $N\backslash X$ of N on X, inducing an operation $\mathrm{Def}_{G/N}^G : B(G) \to B(G/N)$.

The common point about all those operations is the following: in all cases, there are two finite groups G and H, and a functor F from G-**set** to H-**set**, preserving disjoint unions, i.e. such that $F(X \sqcup Y) \simeq F(X) \sqcup F(Y)$. I all cases moreover there exists a finite set U with a left H-action and a right G-action, such that

$$F(X) \simeq U \times_G X$$

where $U \times_G X$ is the quotient of $U \times X$ by the right action of G given for $(u, x) \in U \times X$ and $g \in G$ by

$$(u, x)g = (ug, g^{-1}x)$$

The set $U \times X$ is an H-set for the action defined by

$$h(u,x) = (hu,x)$$

for $(u,x) \in U \times X$ and $h \in H$. If the actions of G and H on U commute, i.e. if $(hu)g = h(ug)$ for all $u \in U$, $h \in H$, and $g \in G$, then this action passes down to a well defined left action of H on $U \times_G X$. This leads to the following definition:

DEFINITION 6.1.1. *Let G and H be groups. An H-set-G, or a biset for short, is an $(H \times G^{op})$-set, i.e. a set U with a left H-action and an right G-action which commute.*

If K is another group, if V is a K-set-H, then the product $V \times_H U$ is the quotient of the product $V \times U$ by the right action of H given by $(v,u)h = (vh, h^{-1}u)$ for $v \in V$, $u \in U$, and $h \in H$. The class of (v,u) in $V \times_H U$ is denoted by $(v,_H u)$.

The set $V \times_H U$ is a K-set-G for the action given by

$$k(v,_H u)g = (kv,_H ug)$$

for $k \in K$, $g \in G$, $u \in U$, and $v \in V$.

EXAMPLE 6.1.2. For the induction functor Ind_H^G from a subgroup H of G to G, the set U is the set G, with its natural left G-action and right H-action by multiplication.

For the restriction functor Res_H^G, the set U is the set G, for its left H-action and right G-action.

For the inflation functor $\mathrm{Inf}_{G/N}^G$, the set U is the set G/N, for its left G-action given by projection and multiplication, and its right G/N-action given by multiplication.

For the deflation functor $\mathrm{Def}_{G/N}^G$, the set U is the set G/N, for its left G/N-action and right G-action.

There is still another operation, involving the case of conjugation. It is associated to a group isomorphism $f : G \to H$. Any G-set X can be viewed as an H-set on which $h \in H$ acts as $f^{-1}(h) \in G$. Here the set U is the set G, with right G-action by multiplication, and left H-action of $h \in H$ by left multiplication by $f^{-1}(h)$. This operation is denoted by Iso_G^H (without reference to f, which is generally clear from the context).

NOTATION 6.1.3. Let G and H be finite groups, and L be a subgroup of $H \times G$. Set

$$p_1(L) = \{h \in H \mid \exists g \in G,\ (h,g) \in L\} \qquad p_2(L) = \{g \in G \mid \exists h \in H,\ (h,g) \in L\}$$
$$k_1(L) = \{h \in H \mid (h,1) \in L\} \qquad k_2(L) = \{g \in G \mid (1,g) \in L\}$$

Then $k_i(L)$ is a normal subgroup of $p_i(L)$, for $i = 1,2$. The quotient group is denoted by $q_i(L)$, for $i = 1,2$. There is a canonical group isomorphism $q_1(L) \to q_2(L)$.

The quotient set $(H \times G)/L$ is viewed as an H-set-G by

$$h.(x,y)L.g = (hx, g^{-1}y)L \qquad \forall h,x \in H,\ \forall y,g \in G$$

This biset is transitive, i.e. it is a transitive $(H \times G^{op})$-set. Conversely, any transitive H-set-G is isomorphic to a biset $(H \times G)/L$, for some subgroup L of $H \times G$. Any H-set-G is a disjoint union of transitive bisets.

If G, H, and K are finite groups, if L is a subgroup of $H \times G$ and M is a subgroup of $K \times H$, set

$$M * L = \{(k,g) \in K \times G \mid \exists h \in H,\ (k,h) \in M \text{ and } (h,g) \in L\}$$

It is a subgroup of $K \times G$.

PROPOSITION 6.1.4. (1) *Let G and H be finite groups, and L be a subgroup of $H\times G$. Then there is an isomorphism of bisets*

$$(H\times G)/L \simeq H\times_{p_1(L)} q_1(L)\times_{q_1(L)} q_1(L)\times_{q_2(L)} q_2(L)\times_{p_2(L)} G$$

(2) *Let G, H, and K be finite groups, let L be a subgroup of $H\times G$ and M be a subgroup of $K\times H$. Then there is an isomorphism of K-sets-G*

$$\big((K\times H)/M\big)\times_H \big((H\times G)/L\big) \simeq \bigsqcup_{x\in p_2(M)\backslash H/p_1(L)} (K\times G)/(M*{}^{(x,1)}L)$$

PROOF. See [**8**] Lemme 3 and Proposition 1. □

REMARK 6.1.5. Assertion (1) shows that any transitive biset is a product of bisets associated to a restriction, a deflation, an isomorphism, an inflation, and an induction. Assertion (2) can be viewed as a biset version of Mackey formula (see Proposition 2.2.1).

6.2. Bisets and functors. If G and H are finite groups, then any finite H-set-G U induces a functor $I_U : G$-**set** $\to H$-**set** defined for a finite G-set X by $I_U(X) = U\times_G X$, and for a morphism of finite G-sets $f : X\to Y$ by

$$I_U(f)\big((u,_G x)\big) = \big(u,_G f(x)\big) : I_U(X)\to I_U(Y)$$

If K is another finite group, if V is a K-set-H, and if X is a finite G-set, there is a natural isomorphism of K-sets

$$V\times_H (U\times_G X)\to (V\times_H U)\times_G X$$

mapping $\big(v,_H(u,_G x)\big)$ to $\big((v,_H u),_G x\big)$. This induces an isomorphism of functors $I_V\circ I_U \simeq I_{V\times_H U}$.

The functor I_U preserves disjoint unions, hence it induces a morphism $\mathbb{B}(U)$ from $B(G)$ to $B(H)$ (this morphism should not be confused with the value $B(U)$ of the Burnside functor at the $(H\times G^{op})$-set U!). If U' is an H-set-G, and if U' is isomorphic to U (as biset), then the functors I_U and $I_{U'}$ are clearly isomorphic. Hence $\mathbb{B}(U) = \mathbb{B}(U')$.

With the same notation, one has $\mathbb{B}(V)\circ\mathbb{B}(U) = \mathbb{B}(V\times_H U)$. Moreover, if the biset U is a disjoint union $U_1\sqcup U_2$ of two H-sets-G U_1 and U_2, then the functor I_U is clearly isomorphic to the disjoint union of the functors I_{U_1} and I_{U_2}. It follows that $\mathbb{B}(U) = \mathbb{B}(U_1)+\mathbb{B}(U_2)$.

Those remarks show that there is a well defined morphism from the Burnside group $B(H\times G^{op})$ to the group $\mathrm{Hom}_{\mathbb{Z}}\big(B(G),B(H)\big)$ of group homomorphisms from $B(G)$ to $B(H)$.

If $G = H$, then viewing G as a G-set-G by left and right multiplication, it is clear that the functor I_G is isomorphic to the identity functor. Hence $\mathbb{B}(G)$ is the identity map of $B(G)$. The biset G will be called the *identity biset* for G.

There are many other situations where similar transformations can be associated to bisets: for instance, a finite H-set-G U induces a morphism $R(U) : R_{\mathbb{Q}}(G)\to R_{\mathbb{Q}}(H)$ between the corresponding Grothendieck groups of rational representations, defined for a finite dimensional $\mathbb{Q}G$-module M by

$$R(U)(M) = \mathbb{Q}U\otimes_{\mathbb{Q}G} M$$

where $\mathbb{Q}U$ is the permutation bimodule with basis U.

More generally, if k is a field, one can define a morphism $R(U) : R_k(G)\to R_k(H)$ by $R(U)(M) = kU\otimes_{kG} M$ for a finitely generated kG-module M. However, this map is well defined only if tensoring with kU preserves exact sequences, i.e. if kU is projective as right kG-module. If k has characteristic p, this is equivalent to requiring that for each $u\in U$, the stabilizer in G of u is a p'-group.

This shows that in some natural situations, not all the bisets are allowed, and leads to the following definition:

Definition 6.2.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be classes of finite groups closed under taking sections and extensions. This means that if H is a subgroup or a quotient of an element of $\mathcal{P}$, then H in in $\mathcal{P}$, and conversely that if N is a normal subgroup of H such that N and H/N are in $\mathcal{P}$, then $H \in \mathcal{P}$.*

If G and H are finite groups, then an H-set-G U is said to be $\mathcal{P}$-free-$\mathcal{Q}$ if for any $u \in U$, the stabilizer of u in H is in $\mathcal{P}$, and the stabilizer of u in G is in $\mathcal{Q}$.

One denotes by $B_{\mathcal{P},\mathcal{Q}}(H,G)$ the subgroup of $B(H \times G^{op})$ generated by the H-sets-G which are $\mathcal{P}$-free-$\mathcal{Q}$. If R is a commutative ring, then $RB_{\mathcal{P},\mathcal{Q}}(H,G)$ denotes $R \otimes_R B_{\mathcal{P},\mathcal{Q}}(H,G)$.

Lemma 6.2.2. *Let G, H, and K be finite groups. Let U be an H-set-G, and V be a K-set-H. If U and V are $\mathcal{P}$-free-$\mathcal{Q}$, then so does $V \times_H U$.*

Proof. See [**8**] Lemme 4. □

It follows that if G, H, and K are finite groups, there is a bilinear product

$$\times_H : B_{\mathcal{P},\mathcal{Q}}(K,H) \times B_{\mathcal{P},\mathcal{Q}}(H,G) \to B_{\mathcal{P},\mathcal{Q}}(K,G)$$

extending the product $(X,Y) \mapsto X \times_H Y$. It should be noted moreover that the identity bisets are left and right free, hence they are $\mathcal{P}$-free-$\mathcal{Q}$, for any families $\mathcal{P}$ and $\mathcal{Q}$.

Definition 6.2.3. *Denote by $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$ the category defined as follows:*

- *The objects of $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$ are the finite groups.*
- *If G and H are finite groups, then*
$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{P},\mathcal{Q}}}(G,H) = B_{\mathcal{P},\mathcal{Q}}(H,G)$$
- *The composition of morphisms in $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$ is defined for finite groups G, H, and K by*
$$v \circ u = v \times_H u \qquad \forall u \in B_{\mathcal{P},\mathcal{Q}}(H,G),\ \forall u \in B_{\mathcal{P},\mathcal{Q}}(K,H)$$

*The category $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$ is preadditive (see [**27**] Chapter 1.8), i.e. the sets of morphisms in $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$ are abelian groups, and moreover the composition of morphisms is left and right distributive with respect to addition of morphisms.*

More generally, if R is a commutative ring, then the category $R\mathcal{C}_{\mathcal{P},\mathcal{Q}}$ is the category obtained from $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$ by tensoring with R: the objects are the finite groups, but

$$\mathrm{Hom}_{R\mathcal{C}_{\mathcal{P},\mathcal{Q}}}(G,H) = R \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}_{\mathcal{P},\mathcal{Q}}}(G,H)$$

*Denote by $\mathcal{F}_{\mathcal{P},\mathcal{Q}}$ the category which objects are additive functors from $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$ to $\mathbb{Z}$-**Mod**, morphisms are natural transformations of functors, and composition of morphisms is composition of natural transformations. More generally, denote by $R\mathcal{F}_{\mathcal{P},\mathcal{Q}}$ the category of R-linear functors from $R\mathcal{C}_{\mathcal{P},\mathcal{Q}}$ to R-**Mod**.*

The category $R\mathcal{F}_{\mathcal{P},\mathcal{Q}}$ is abelian. A sequence

$$0 \to L \to M \to N \to 0$$

of objects and morphisms in $R\mathcal{F}_{\mathcal{P},\mathcal{Q}}$ is exact if and only if its evaluation at any finite group is exact.

It is now clear that the correspondence sending a finite group G to its Burnside group $B(G)$, and a finite $\mathcal{P}$-free-$\mathcal{Q}$ H-set-G U, for a finite groups G and H, to the map $\mathbb{B}(U)$, induces an additive functor from $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$ to $\mathbb{Z}$-**Mod**. More generally, the correspondence sending the finite group G to $RB(G)$ and the biset U to the map $R\mathbb{B}(U) : RB(G) \to RB(H)$ induced by $\mathbb{B}(U)$, is an object of the category $R\mathcal{F}_{\mathcal{P},\mathcal{Q}}$. It is called *the Burnside biset-functor* with coefficients in R.

For any finite group G, the set $\mathrm{Hom}_{R\mathcal{C}_{\mathcal{P},\mathcal{Q}}}(\mathbb{1},G)$ identifies with the R-submodule $RB_{\mathcal{P}}(G)$ of $RB(G)$ generated by finite $\mathcal{P}$-free G-sets, or by equivalently by sets G/P, for $P\in\mathcal{P}$. This correspondence $G\mapsto RB_{\mathcal{P}}(G)$ is clearly a subfunctor of the biset functor RB.

The analogue of Proposition 5.6.1 is the following:

PROPOSITION 6.2.4. *Let M be any object of $R\mathcal{F}_{\mathcal{P},\mathcal{Q}}$. Then the map*

$$f\in \mathrm{Hom}_{R\mathcal{F}_{\mathcal{P},\mathcal{Q}}}(RB_{\mathcal{P}},M)\mapsto f_{\mathbb{1}}(\bullet)\in M(\mathbb{1})$$

is an isomorphism of R-modules. In particular, the functor $RB_{\mathcal{P}}$ is a projective object in $R\mathcal{F}_{\mathcal{P},\mathcal{Q}}$, and $\mathrm{End}_{R\mathcal{F}_{\mathcal{P},\mathcal{Q}}}(RB_{\mathcal{P}})\simeq R$.

PROOF. This is essentially Yoneda's Lemma: let G be any finite group, and X be any finite $\mathcal{P}$-free G-set. Then X is also a G-set-$\mathbb{1}$, for the right (trivial!) action of the trivial group, and moreover the G-set X is isomorphic to $X\times_{\mathbb{1}}\bullet$, where $\bullet$ is the trivial set for the trivial group.

Hence if $f:RB_{\mathcal{P}}\to M$ is a natural transformation, the map $f_G:RB_{\mathcal{P}}(G)\to M(G)$ is the map of R-modules sending the G-set X to $M(X)\big(f_{\mathbb{1}}(\bullet)\big)$. Conversely, if $m\in M(\mathbb{1})$ is given, one can define a map f_G from $RB_{\mathcal{P}}(G)$ to $M(G)$ by setting

$$f_G(X)=M(X)(m)$$

for a finite $\mathcal{P}$-free G-set X, and extending f_G by linearity. It is easy to check that this defines a natural transformation from $RB_{\mathcal{P}}$ to M. $\square$

When the ring R is a field, and $\mathcal{P}$ and $\mathcal{Q}$ are both equal to the family $\mathcal{A}ll$ of all finite groups, one can say a little more (see [**8**] Lemme 1 and Proposition 8):

THEOREM 6.2.5. [Bouc] *Let K be a field. Then the functor KB has a unique maximal proper subfunctor J in $K\mathcal{F}_{\mathcal{A}ll,\mathcal{A}ll}$, defined for a finite group G by*

$$J(G)=\{X\in KB(G)\mid \forall Y\in KB(G),\ |G\backslash(Y\times X)|=0_K\}$$

If K has characteristic 0, the quotient functor KB/J is isomorphic to the functor $KR_{\mathbb{Q}}$. It is a simple object in $K\mathcal{F}_{\mathcal{A}ll,\mathcal{A}ll}$.

PROOF. If L is a subfunctor of KB, then $L(\mathbb{1})$ is a K-subspace of $KB(\mathbb{1})=K$. Thus either $L(\mathbb{1})=K$. In this case $\bullet\in L(\mathbb{1})$, thus for any finite group G and any finite G-set X, the G-set $X=X\times_{\mathbb{1}}\bullet=L(X)(\bullet)$ is in $L(G)$. Hence $L=RB$ in this case.

The other case is $L(1)=0$, and then for any finite group G, any $X\in L(G)$, and any morphism Y from G to $\mathbb{1}$ in $K\mathcal{C}_{\mathcal{P},\mathcal{Q}}$, the image $L(Y)(X)$ has to be zero. But clearly $\mathrm{Hom}_{K\mathcal{C}_{\mathcal{P},\mathcal{Q}}}(G,\mathbb{1})\simeq KB(G)$, and with this identification, the element $L(Y)(X)$ of $B(\mathbb{1})$ identifies with $|G\backslash(Y\times X)|$. Thus $L\subseteq J$, and moreover J is clearly a subfunctor of KB.

This shows in particular that the dimension over K of $KB(G)/J(G)$ is equal to the rank of the bilinear form on $KB(G)$ with values in K defined by

$$(X,Y)\in\big(KB(G)\big)^2\mapsto <X,Y>=|G\backslash(Y\times X)|$$

Now if $K\supseteq\mathbb{Q}$, the K-basis $\{e_H^G\}_{H\in[s_G]}$ of $KB(G)$ is orthogonal for this bilinear form, and moreover

$$\begin{aligned}<e_H^G,e_H^G> &= |G\backslash e_H^G|=\frac{1}{|N_G(H)|}\sum_{K\subseteq H}|K|\mu(K,H)\\ &= \frac{1}{|N_G(H)|}\sum_{x\in H}\sum_{<x>\subseteq K\subseteq H}\mu(K,H)=\frac{\phi_1(H)}{|N_G(H)|}\end{aligned}$$

where $\phi_1(H)$ is the number of elements $x \in H$ such that $<x> = H$. This is non-zero if and only if H is cyclic.

Hence the dimension of $KB(G)/J(G)$ is equal to the number $c(G)$ of conjugacy classes of cyclic subgroups of G.

Now the natural morphism $KB(G) \to KR_{\mathbb{Q}}(G)$ mapping a G-set to its permutation module induces clearly a natural transformation of functors

$$\chi : KB \to KR_{\mathbb{Q}} \tag{6.2.6}$$

since for any finite group H and any H-set-G U, there is an isomorphism of $\mathbb{Q}G$-modules

$$\mathbb{Q}(U \times_G X) \simeq \mathbb{Q}U \otimes_{\mathbb{Q}G} \mathbb{Q}X$$

Since χ is non-zero, the kernel of χ is contained in J. Hence the image of $KB(G)$ in $KR_{\mathbb{Q}}(G)$ has dimension at least equal to $c(G)$. But it is well known (see [**36**] Chapitre 13 Théorème 29 Corollaire 1) that the dimension of $KR_{\mathbb{Q}}(G)$ is equal to $c(G)$. The proposition follows. □

Remark 6.2.7. ([**8**] Proposition 2) The set of isomorphism classes of simple objects in $R\mathcal{F}_{\mathcal{P},\mathcal{Q}}$ is in one to one correspondence with the set of isomorphism classes of pairs (H, V), where H is a finite group, and V is a simple $R\mathrm{Out}(H)$-module, where $\mathrm{Out}(H)$ is the group of outer automorphisms of H. The simple functor associated to such a pair (H, V) is denoted by $S_{H,V}$.

For example, when K is a field of characteristic zero, and $\mathcal{P} = \mathcal{Q} = \mathcal{A}ll$, the functor $KR_{\mathbb{Q}}$ of Proposition 6.2.5 is the functor $S_{\mathbb{1},K}$. In this case, the subfunctors of KB are studied in [**8**]. In particular, the composition factors (i.e. the simple sections) of KB are simple functors $S_{H,K}$ associated to the trivial $K\mathrm{Out}(H)$-module, for a class of finite group called *b-groups* ([**8**] Proposition 10).

Remark 6.2.8. In the case where $\mathcal{P}$ and $\mathcal{Q}$ are reduced to the trivial group, the objects of $R\mathcal{F}_{\mathcal{P},\mathcal{Q}}$ are called *globally defined Mackey functors*. They have been studied by Webb ([**48**]). The reason for this denomination is that the operations associated to bisets which are both left and right free only involve induction, restriction and group isomorphisms.

In this paper, Webb also considers the *inflation functors*, which correspond to the case where $\mathcal{Q}$ is reduced to the trivial group, and $\mathcal{P}$ is the class of all finite groups.

Remark 6.2.9. It is also of interest to consider subcategories of the above categories $R\mathcal{C}_{\mathcal{P},\mathcal{Q}}$. For example, let p be a prime number, and $\mathcal{D}_p$ denote the full subcategory of $\mathbb{Q}\mathcal{C}_{\mathcal{A}ll,\mathcal{A}ll}$ consisting of finite p-groups. Denote by $\mathbb{Q}B$ and $\mathbb{Q}R_{\mathbb{Q}}$ the restriction to $\mathcal{D}_p$ of the functors $\mathbb{Q}B$ and $\mathbb{Q}R_{\mathbb{Q}}$ on $\mathbb{Q}\mathcal{C}_{\mathcal{A}ll,\mathcal{A}ll}$.

It has been shown recently (Bouc-Thévenaz [**11**]) that the (restriction to $\mathcal{D}_p$ of the) kernel of the morphism χ defined in 6.2.6 above is isomorphic to the *torsion-free Dade functor* $\mathbb{Q}D$, which value at a p-group is equal to $\mathbb{Q} \otimes_{\mathbb{Z}} D(P)$, where $D(P)$ is the Dade group of endo-permutation kP-modules, for a field k of characteristic p. This functor $\mathbb{Q}D$ is simple, isomorphic to (the restriction to $\mathcal{D}_p$ of) $S_{E_2,K}$, where E_2 is an elementary abelian p-group of order p^2.

6.3. Double Burnside rings. The endomorphism ring of a finite group G in the category $R\mathcal{C}_{\mathcal{P},\mathcal{Q}}$ is called the *double Burnside ring* of G with coefficients in R, for the classes $\mathcal{P}$ and $\mathcal{Q}$.

An essential tool to study those rings is the following lemma:

Lemma 6.3.1. *Let G be a finite group. If X is a finite G-set, denote by $\tilde{X}$ the set $G \times X$, with its G-set-G structure defined by*

$$g(a, x)g' = (gag', gx) \qquad \forall g, a, g' \in G \text{ and } x \in X$$

Then the correspondence $X \mapsto \tilde{X}$ *induces a morphism of rings (with unit) from* $B(G)$ *to* $B_{\mathcal{P},\mathcal{Q}}(G,G)$.

PROOF. See [**8**] Lemme 13. □

In particular, this lemma gives a way to carry the idempotents of the Burnside ring to the double Burnside ring, giving information on the projective and simple modules for those rings. This was used intensively in [**8**].

REMARK 6.3.2. If K is a field of characteristic 0, then the Burnside algebra $KB(G)$ is semi-simple for any finite group G. It is natural to ask if the *double* Burnside algebra $KB_{\mathcal{P},\mathcal{Q}}(G)$ is. The answer is no in general. One can show for instance that if $\mathcal{P} = \mathcal{Q} = \mathcal{A}ll$ is the class of all finite groups, then the algebra $KB_{\mathcal{P},\mathcal{Q}}(G)$ is semi-simple if and only if the group G is *cyclic*.

6.4. Stable maps between classifying spaces. Double Burnside rings have been studied intensively in the case where $\mathcal{P} = \{\mathbb{1}\}$ is reduced to the trivial group, and $\mathcal{Q} = \mathcal{A}ll$ is the class of all finite groups, because they are an essential tool to describe the *stable splitting* of the classifying spaces of finite groups. The origin of this theory is the Segal conjecture, proved by Carlsson ([**16**]), which states an isomorphism between the stable cohomotopy groups of the classifying space BG of a finite group G and the completion of the Burnside ring at the augmentation ideal.

Recall first some notation and definitions (see [**2**] Chapter 2.8):

NOTATION 6.4.1. If X is a pointed CW-complex, denote by SX its (reduced) suspension, and by ΩX its loop space. If X and Y are pointed CW-complexes, denote by $[X;Y]$ the set of homotopy classes of pointed continuous maps from X to Y. If $m \in \mathbb{N}$, there is a map

$$\Omega^m S^m X = [S^m; S^m X] \to [S^{m+1}; S^{m+1} X] = \Omega^{m+1} S^{m+1} X$$

defined by suspension, and one can set

$$\Omega^\infty S^\infty X = \varinjlim_{m\to\infty} \Omega^m S^m X$$

If X and Y are pointed CW-complexes, then the set $\{X;Y\}$ of stable maps from X to Y is defined by

$$\{X;Y\} = [X; \Omega^\infty S^\infty Y]$$

It is an abelian group. If X, Y, and Z are pointed CW-complexes, there is a bilinear map

$$\{Y;Z\} \times \{X;Y\} \to \{X;Z\}$$

The *stable cohomotopy groups* $\pi_s^r(X)$ of a pointed CW-complex X are defined for $r \in \mathbb{Z}$ by

$$\pi_s^r(X) = \{X_+; S^r\} = [X; \varinjlim_{n\to\infty} \Omega^n S^{n+r}]$$

where X_+ is the space X with a disjoint basepoint. Note that this last expression makes sense even for $r < 0$.

The other definition required in the Segal conjecture is the notion of completion. Observe that another consequence of Lemma 6.3.1 is that if G and H are finite groups, then the natural structure of $B_{\mathcal{P},\mathcal{Q}}(H,H)$-module-$B_{\mathcal{P},\mathcal{Q}}(G,G)$ on $B_{\mathcal{P},\mathcal{Q}}(H,G)$ gives by restriction a structure of $B(H)$-module-$B(G)$.

It is easy to check (see [**8**] Lemme 15) that if X is an H-set-G and Y is a G-set, then $X \times_G \tilde{Y}$ identifies with the cartesian product $X \times Y$, for the double action given by

$$h(x,y)g = (hxg, g^{-1}y) \qquad \forall h \in H,\ \forall g \in G,\ \forall x \in X,\ \forall y \in Y$$

NOTATION 6.4.2. Let G be a finite group. Denote by I_G the prime ideal $I_{\mathbb{1},0}(G)$ of $B(G)$ defined in section 3.4. It is called the *augmentation ideal*. The completion of $B(G)$ at the ideal I_G is the inverse limit

$$B(G)\hat{} = \varprojlim_{n\in\mathbb{N}} B(G)/I_G^n$$

More generally, if H and G are finite groups, the *completion* $B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)\hat{}$ of $B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)$ is defined as the inverse limit

$$B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)\hat{} = \varprojlim_{n\in\mathbb{N}} B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)/B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)I_G^n$$

where $B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)I_G^n$ denotes $B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)\times_G \widetilde{I_G^n}$.

If Y is a pointed CW-complex, then the functors $X\mapsto \{X;S^rY\}$ form a *generalized comology theory* (see [**2**] Chapter 2.5). In particular, if H is a subgroup of a finite group G, there is a *transfer map* from $\{BH_+;Y\}$ to $\{BG_+;Y\}$. Taking $Y=BH_+$, the image of the identity of $\{BH_+;BH_+\}$ is an element Tr_H^G of $\{BG_+;BH_+\}$, also called transfer. This element can be composed with the stable map from BH_+ to S^0 obtained by identifying BH to a point. This gives an element τ_H of $\{BG_+,S^0\}=\pi_s^0(BG)$.

The precise statement of Segal's conjecture is now the following:

THEOREM 6.4.3. [Carlsson] *Let G be a finite group. Then the map $H\mapsto\tau_H$ defined above induces a group isomorphism*

$$B(G)\hat{}\simeq \pi_s^0(BG)$$

Furthermore $\pi_s^r(BG)=0$ for $r>0$.

This has been generalized by Lewis, May and McClure ([**25**]): suppose H and G are finite groups. If Q is a subgroup of G and $\phi: Q\to H$ is a group homomorphism, then the stable map $Tr_Q^G\in\{BG_+;BQ_+\}$ can be composed with the element of $\{BQ_+;BH_+\}$ deduced from ϕ, to get an element $\tau_{Q,\phi}\in\{BG_+;BH_+\}$.

This element depends only on the conjugacy class of the subgroup $\Delta_\phi(Q)$ of $H\times G$ consisting of the elements $\big(\phi(l),l\big)$, for $l\in Q$. Moreover, the biset $(H\times G)/\Delta_\phi(Q)$ is transitive, and free on the left. Conversely, any left-free transitive H-set-G is isomorphic to a biset $(H\times G)/\Delta_\phi(Q)$, for some subgroup Q of G and some morphism ϕ from Q to H.

This construction gives all the stable maps from BG_+ to BH_+. More precisely:

THEOREM 6.4.4. [Lewis-May-McClure] *Let G and H be finite groups. The above correspondence sending the left-free transitive H-set-G $(H\times G)/\Delta_\phi(Q)$ to $\tau_{Q,\phi}\in\{BG_+;BH_+\}$ induces an isomorphism*

$$B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)\hat{}\to\{BG_+;BH_+\}$$

This theorem was at the origin of the study of *stable splittings* of the classifying spaces of finite groups. It provides indeed an algebraic translation from the stable endomorphisms of BG_+ for a finite group G, in terms of the completion of the double Burnside ring $B_{\{\mathbb{1}\},\mathcal{A}ll}(G,G)$. For details see the survey paper of Benson ([**3**]), or the articles of Benson and Feschbach ([**4**]), Martino and Priddy ([**28**], Priddy ([**31**]), Webb ([**48**]).

Those two deep theorems on stable homotopy raise some natural questions on the algebraic side. For example, being given three finite groups G, H, and K, the composition of stable maps

$$\{BH_+;BK_+\}\times\{BG_+;BH_+\}\to\{BG_+;BK_+\}$$

shows that there is a map

$$B_{\{\mathbb{1}\},\mathcal{A}ll}(K,H)\times B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)\to B_{\{\mathbb{1}\},\mathcal{A}ll}(K,G)$$

induced by the product $(X,Y)\mapsto X\times_H Y$. Without Theorem 6.4.4, the existence of such a map is not obvious *a priori*, and seems to require the following lemma:

LEMMA 6.4.5. *Let G and H be finite groups. Then for any integer $m\in\mathbb{N}$, there exists an integer $n\in\mathbb{N}$ such that*

$$I_H^n B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)\subseteq B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)I_G^m$$

where $I_H^n B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)$ denotes $\widetilde{I_H^n}\times_H B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)$.

PROOF. Let X be a finite H-set, and Y be an H-set-G. Then it is easy to see that the biset $Z=\tilde{X}\times_H Y$ identifies with the cartesian product $X\times Y$, for the double action given by

$$h(x,y)g=(hx,hyg)\qquad \forall h\in H,\ \forall g\in G,\ \forall x\in X,\ \forall y\in Y$$

In particular for any subgroup L of $H\times G$, the fixed points of L on Z, i.e. the set of $z\in Z$ such that $az=zb$ whenever $(a,b)\in L$, identifies with $X^{p_1(L)}\times Y^L$. It follows that for any $X\in B(H)$ and any $Y\in B(H\times G^{op})$

$$|(\tilde{X}\times_H Y)^L|=|X^{p_1(L)}||Y^L|$$

In particular if $X\in I_G^n$, for an integer $n\geq 1$, then X is a sum of terms $X_1\times\ldots\times X_n$, with $X_1,\ldots,X_n\in I_G$. Hence $|X|=0$, and moreover $|X^P|\equiv 0\ (p^n)$ for any prime p and any p-subgroup P of H, since $|X_i^P|\equiv|X_i|=0\ (p)$, for $i=1,\ldots,n$. Thus

$$I_H^n B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)\subseteq J_n(H,G)$$

where $J_n(H,G)$ is the subset of $B(H,G)=B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)$ defined by

$$J_n(H,G)=\{Z\in B(H,G)\mid |Z|=0,\ |Z^L|\equiv 0\ (p^n)\ \forall L\subseteq H\times G,\ p_1(L)\ p\text{-group}\}$$

For each prime number p, set $m_p=|H|_{p'}|G|_{p'}$. Then for any p-perfect subgroup K of G, the idempotent

$$f_{p,K}^G=\sum_{\substack{M\in[s_G]\\ O^p(M)=_G K}} e_M^G$$

of $\mathbb{Q}B(G)$ lies in $\mathbb{Z}_{(p)}B(G)$, and $m_p f_K^G\in B(G)$. Moreover $|e_M^G|=0$ unless $M=\mathbb{1}$. Thus $m_p f_{p,K}^G\in I_G$ unless $K=\mathbb{1}$.

It follows that for any integer n, the element $m_p^n f_{p,K}^G=(m_p f_{p,K}^G)^n$ is in I_G^n if $K\neq\mathbb{1}$. Now if $Z\in B(H,G)$

$$m_p^n Z=m_p^n Z f_{p,\mathbb{1}}^G+\sum_{\substack{K\in[s_G]\\ O^p(K)=K}} m_p^n Z f_{p,K}^G\equiv m_p^n Z f_{p,\mathbb{1}}^G\quad\big(B(H,G)I_G^n\big)$$

Moreover, the integers m_p, for p dividing $|H||G|$, are relatively prime, and there exist integers $a_{p,n}$, for p prime (equal to 0 for p not dividing $|H||G|$), such that $\sum_p a_{p,n}m_p^n=1$. It follows that

$$Z\equiv\sum_p a_{p,n}Y_p\quad\big(B(H,G)I_G^n\big)$$

where $Y_p=m_p^n Z f_{p,\mathbb{1}}^G$.

Let L be a subgroup of $H\times G$, and M be a subgroup of G. Then in $B(H\times G)$

$$e_L^{H\times G}\times e_M^G=|(e_M^G)^L|e_L^{H\times G}=|(e_M^G)^{p_2(L)}|e_L^{H\times G}=\begin{cases} e_L^{H\times G} & \text{if } p_2(L)=_G M\\ 0 & \text{otherwise}\end{cases}$$

Now for each prime p, the element Y_p is equal to

$$Y_p = Y_p f_{p,\mathbb{1}}^G = \sum_{L\in[s_{H\times G}]} |Y_p^L| e_L^{H\times G}$$

and $e_L^{H\times G} f_{p,\mathbb{1}}^G = 0$ unless $p_2(L)$ is a p-group. Thus

$$Y_p = \sum_{\substack{L\in[s_[H\times G]\\ O^p(p_2(L))=\mathbb{1}}} m_p^n |Z^L| e_L^{H\times G}$$

Moreover, for each subgroup L of $H\times G$, one has $|L| = |k_1(L)||p_2(L)|$, and if $k_1(L)\neq \mathbb{1}$, then $|Z^L| = 0$ since Z is a linear combination of left-free bisets. It follows that if $|Z^L|\neq 0$, and if $p_2(L)$ is a p-group, then then L is a p-group, equal to $\Delta_\phi(Q)$ for some p-subgroup Q of G, and some morphism $\phi : Q\to H$. Hence $p_1(L) = \phi(Q)$ is a p-group. Hence

$$Y_p = \sum_L m_p^n |Z^L| e_L^{H\times G}$$

where L runs through those subgroups $\Delta_\phi(Q)$, up to conjugation by $H\times G$. Finally, for such a subgroup $L = \Delta_\phi(Q)$

$$\begin{aligned} m_p^n |Z^L| e_L^{H\times G} &= |H|_{p'}^n |G|_{p'}^n |Z^L| e_L^{H\times G} e_Q^G \\ &= |H|_{p'}^{n-1} |G|_{p'}^{n-m-1} \frac{|Z^L|}{|H|_p |G|_p^{m+1}} (|H||G| e_L^{H\times G})(|G|^m e_Q^G) \end{aligned}$$

This is zero if $Q = \mathbb{1}$ and $Z\in J_n(H,G)$, because $|Z^L| = |Z| = 0$ in this case. And if $Q\neq \mathbb{1}$, the element $(|H||G|e_L^{H\times G})(|G|^m e_Q^G)$ is in $B(H,G)I_G^m$, and if n is big enough, the quotient $\frac{|Z^L|}{|H|_p|G|_p^{m+1}}$ is an integer for $Z\in J_n(H,G)$, since $p_1(L) = \phi(Q)$ is a p-group. This completes the proof of the lemma. □

Proposition 6.4.6. *Let G, H and K be finite groups. Then the product*

$$(X,Y)\in B_{\{\mathbb{1}\},\mathcal{A}ll}(K,H)\times B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)\mapsto X\times_H Y\in B_{\{\mathbb{1}\},\mathcal{A}ll}(K,G)$$

induces a well defined product

$$B_{\{\mathbb{1}\},\mathcal{A}ll}(K,H)\hat{}\times B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)\hat{}\to B_{\{\mathbb{1}\},\mathcal{A}ll}(K,G)\hat{}$$

Proof. This is clear, since by the previous lemma, the product

$$(X,Y)\in B_{\{\mathbb{1}\},\mathcal{A}ll}(K,H)\times B_{\{\mathbb{1}\},\mathcal{A}ll}(H,G)\mapsto X\times_H Y\in B_{\{\mathbb{1}\},\mathcal{A}ll}(K,G)$$

is continuous for the I_H-adic and I_G-adic topologies. □

Proposition 6.4.7. *Let G be a finite group. Then the correspondence mapping a subgroup H of G to $B(H)\hat{}$ has a natural structure of Green functor.*

Proof. Indeed, for any finite group H, there is an isomorphism

$$B(H)\hat{}\simeq B_{\{\mathbb{1}\},\mathcal{A}ll}(\mathbb{1},H)\hat{}$$

Thus for two finite groups H and K, there are maps

$$B(H)\hat{}\times B_{\{\mathbb{1}\},\mathcal{A}ll}(H,K)\to B(K)\hat{}$$

Now if $H\subseteq K$ are subgroups of G, this gives a transfer map from $B(H)\hat{}$ to $B(K)\hat{}$, using K as a left-free H-set-K, and a restriction map from $B(K)\hat{}$ to $B(H)\hat{}$, using K as a left-free K-set-H. Similarly, for $x\in G$, there is a conjugation map from $B(H)\hat{}$ to $B({}^xH)\hat{}$, induced by the biset H with its obvious structure of H-set-${}^xH$. The axioms of Mackey functors follow from the properties of product of bisets.

In other words, the problem to define a Mackey functor structure on $B\hat{}$ is that the correspondence $H \mapsto I_H^n$ is not clearly a sub-Mackey functor of B. However if H is a subgroup of G, and n a positive integer, let

$$J_n(H) = \{X \in B(H) \mid |X| = 0,\ \ \forall p \text{ prime },\ \forall P \subseteq H,\ P\ p\text{-group},\ |X^P| \equiv 0\ (p^n)\}$$

Then $J_n(H)$ is an ideal of $B(G)$, and one can show as in Lemma 6.4.5 that the topology defined by the ideals $J_n(H)$ is equivalent to the topology defined by the ideals I_H^n. It is easy to see moreover that the correspondence $H \mapsto J_n(H)$ is a sub-Mackey functor of B. Hence the completion $B(H)\hat{}$ is isomorphic to

$$B(H)\hat{} \simeq \lim_{\substack{\longleftarrow \\ n\to\infty}} B(H)/J_n(H)$$

and in this form, it has a natural structure of Green functor. □

Remark 6.4.8. For further results on completion of Green functors, see Luca ([**26**]).

Bibliography

[1] D. Benson. *Representations and cohomology I*, volume 30 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1991.

[2] D. Benson. *Representations and cohomology II*, volume 31 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1991.

[3] D. Benson. Stably splitting BG. *Bull. AMS*, 33:189–198, 1996.

[4] D. Benson and M. Feschbach. Stable splittings of classifying spaces of finite groups. *Topology*, 31:157–176, 1992.

[5] S. Bouc. Projecteurs dans l'anneau de Burnside, projecteurs dans l'anneau de Green, et modules de Steinberg généralisés. *J. of Algebra*, 139(2):395–445, June 1991.

[6] S. Bouc. Exponentielle et modules de Steinberg. *J. of Algebra*, 150(1):118–157, August 1992.

[7] S. Bouc. Construction de foncteurs entre catégories de G-ensembles. *J. of Algebra*, 183(0239):737–825, 1996.

[8] S. Bouc. Foncteurs d'ensembles munis d'une double action. *J. of Algebra*, 183(0238):664–736, 1996.

[9] S. Bouc. *Green-functors and G-sets*, volume 1671 of *Lecture Notes in Mathematics*. Springer-Verlag, October 1997.

[10] S. Bouc. Résolutions de foncteurs de Mackey. In *Group representations: Cohomology, Group Actions and Topology*, volume 63, pages 31–83. AMS Proceedings of Symposia in pure mathematics, January 1998.

[11] S. Bouc and J. Thévenaz. The group of endo-permutation modules. *Inventiones Mathematicae*, 1998. To appear.

[12] M. Broué. On Scott modules and p-permutation modules: an approach through the Brauer morphism. *Proceedings of the A.M.S.*, 93(3):401–408, 1985.

[13] K. Brown. Euler characteristics of groups: the p-fractional part. *Invent. Math.*, 29:1–5, 1975.

[14] K. Brown and J. Thévenaz. A generalization of Sylow's third theorem. *Journal of Algebra*, 115:414–430, 1988.

[15] W. Burnside. *Theory of groups of finite order*. Cambridge University Press, second edition, 1911.

[16] G. Carlsson. Equivariant stable homotopy and Segal's Burnside ring conjecture. *Annals of Math.*, 120:189–224, 1984.

[17] S. B. Conlon. Decompositions induced from the Burnside algebra. *J. Algebra*, 10:102–122, 1968.

[18] C. Curtis and I. Reiner. *Methods of representation theory with applications to finite groups and orders*, volume II of *Wiley classics library*. Wiley, 1990.

[19] A. Dress. A characterization of solvable groups. *Math. Zeit.*, 110:213–217, 1969.

[20] A. Dress. Notes on the theory of representations of finite groups. Bielefeld Notes, 1971.

[21] A. Dress. *Contributions to the theory of induced representations*, volume 342 of *Lecture Notes in Mathematics*, pages 183–240. Springer-Verlag, 1973.

[22] D. Gluck. Idempotent formula for the Burnside ring with applications to the p-subgroup simplicial complex. *Illinois J. Math.*, 25:63–67, 1981.

[23] J. Green. Axiomatic representation theory for finite groups. *J. Pure Appl. Algebra*, 1:41–77, 1971.

[24] G. Karpilovsky. *Group representations 4*, volume 182 of *Mathematics Studies*. NorthHolland, 1995.

[25] L. G. Lewis, J. P. May, and J. E. McClure. Classifying G-spaces and the Segal conjecture. In *Current trends in algebraic topology*, volume 2, pages 165–179. CMS Conference Proc., 1982.

[26] F. Luca. The defect of completion of a Green functor. In *Group representations: Cohomology, Group Actions and Topology*, volume 63, pages 369–381. AMS Proceedings of Symposia in pure mathematics, January 1998.

[27] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate texts in Mathematics*. Springer, 1971.

[28] J. Martino and S. B. Priddy. The complete stable splitting for the classifying space of a finite group. *Topology*, 31:143–156, 1992.

[29] T. Matsuda. On the unit group of Burnside rings. *Japan. J. Math.*, 8(1):71–93, 1982.

[30] T. Matsuda and T. Miyata. On the unit groups of the Burnside rings of finite groups. *J. Math. Soc. Japan*, 35:345–354, 1983.

[31] S. B. Priddy. On characterizing summands in the classifying space of a finite group I. *Amer. J. of Math.*, 112:737–748, 1990.

[32] D. Quillen. *Higher algebraic K-theory*, volume 341 of *Lecture Notes in Mathematics*, pages 85–147. Springer-Verlag, 1973.

[33] D. Quillen. Homotopy properties of the poset of non-trivial p-subgroups. *Adv. in Maths*, 28(2):101–128, 1978.

[34] R. Schwänzl. Koeffiziente in Burnside rings. *Arch. Math.*, 29:621–622, 1977.

[35] R. Schwänzl. On the spectrum of the Burnside ring. *J. pure and applied algebra*, 15:181–185, 1979.

[36] J.-P. Serre. *Représentations linéaires des groupes finis*. Collection Méthodes. Hermann, second edition, 1971.

[37] L. Solomon. The Burnside algebra of a finite group. *J. Combin. Theory*, 2:603–615, 1967.

[38] J. Thévenaz. *Idempotents de l'anneau de Burnside et caractéristique d'Euler*, volume 25 of *Séminaire Claude Chevalley sur les groupes finis, Tome 3*, pages 207–217. Publications mathématiques de l'Université Paris 7, 1985.

[39] J. Thévenaz. Permutation representation arising from simplicial complexes. *J.Combin. Theory*, 46:122–155, 1987. Ser.A.

[40] J. Thévenaz. Isomorphic Burnside rings. *Comm. Algebra*, 16(9):1945–1947, 1988.

[41] J. Thévenaz and P. Webb. Simple Mackey functors. In *Proceedings of the 2nd International group theory conference Bressanone 1989*, volume 23 of *Rend. Circ. Mat. Palermo*, pages 299–319, 1990. Serie II.

[42] J. Thévenaz and P. Webb. Homotopy equivalences of posets with a group action. *Journal of Combinatorial Theory*, 56(2):173–181, 1991.

[43] J. Thévenaz and P. Webb. The structure of Mackey functors. *Trans. Amer. Math. Soc.*, 347(6):1865–1961, June 1995.

[44] T. tom Dieck. The Burnside ring of a compact Lie group. *Math. Ann.*, 215:235–250, 1975.

[45] T. tom Dieck. Idempotent elements in the Burnside ring. *Journal of Pure an applied algebra*, 10:239–247, 1977.

[46] T. tom Dieck. *Transformation groups and representation theory*, volume 766 of *Lecture Notes in Mathematics*. Springer-Verlag, 1979.

[47] P. Webb. A split exact sequence for Mackey functors. *Comment. Math. Helv.*, 66:34–69, 1991.

[48] P. Webb. Two classification of simple Mackey functors with application to group cohomology and the decomposition of classifying spaces. *J. of Pure and App. Alg.*, 88:265–304, 1993.

[49] T. Yoshida. Idempotents in Burnside rings and Dress induction theorem. *J. Algebra*, 80:90–105, 1983.

[50] T. Yoshida. On the unit groups of Burnside rings. *J. Math. Soc. Japan*, 42(1):31–64, 1990.