Stabilizing bisets

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Abstract : Let G be a finite group and let R be a commutative ring. We analyse the (G,G)-bisets which stabilize an indecomposable RG-module. We prove that the minimal ones are unique up to equivalence. This result has the same flavor as the uniqueness of vertices and sources up to conjugation and resembles also the theory of cuspidal characters in the context of Harish-Chandra induction for reductive groups, but it is different and very general. We show that stabilizing bisets have rather strong properties and we explore two situations where they occur. Moreover, we prove some specific results for simple modules and also for p-groups.

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1. Introduction

It is an old idea to try to describe representations of a finite group G by means of induction from a subgroup A, as small as possible. Green's theory of vertices and sources is a classical instance of this procedure. A more general situation consists of starting from a subquotient A/B (where B is a normal subgroup of A), and apply first inflation from A/B to A, and then induction from A to G. This appears in Harish-Chandra induction for reductive groups (see for instance [DiDu]), and in a more general setting in Sections 5 and 6 of [Bo1]. Such theories also use operations going in the reverse direction, namely restriction from G to a subgroup S followed by deflation from S to S/T (where T is a normal subgroup of S).

In the situations just mentioned, there is always the procedure of allowing for a direct summand of the representation obtained by the operations of inflation and induction (respectively restriction and deflation). In this paper, we investigate the same ideas, but without allowing for direct summands. This appears naturally when the above operations are written in terms of bisets and when one simply requires that a biset stabilizes a representation.

More precisely, we let L be an indecomposable RG-module, where R is a commutative ring, and we require that a (G,G)-biset U stabilizes L, in the sense that $RU\otimes_{RG} L\cong L$. We can assume that U is transitive, hence of the form $U\cong \mathrm{Indinf}_{A/B}^G\,\mathrm{Iso}_\phi\,\mathrm{Defres}_{S/T}^G$, where A, B, S, T are as above and where $\phi: S/T\to A/B$ is an isomorphism. We then see that $L\cong \mathrm{Indinf}_{A/B}^G\big(\mathrm{Iso}_\phi(M)\big)$ where $M=\mathrm{Defres}_{S/T}^G(L)$ and $\mathrm{Iso}_\phi(M)$ denotes the module M transported by the isomorphism ϕ.

When U is minimal in the sense that $|S/T|$ is as small as possible, we prove a uniqueness result which has the same flavor as the uniqueness of vertices and sources up to conjugation but is a bit more complicated. In particular, the isomorphism types of the group S/T and the module M are unique. This immediately raises the question of the type of minimal group S/T which can be obtained, but this is not at all easy. In fact, it is not easy in general to obtain stabilizing bisets, although many examples show that they occur.

We prove various results which provide on the one hand some specific properties of stabilizing bisets and on the other hand partial information on the possibility of obtaining stabilizing bisets. In particular, we give two methods for constructing them in suitable cases. The first method is to obtain idempotent bisets, which obviously are stabilizing bisets, and we characterize them completely. The second source which yields stabilizing bisets occurs when a subgroup T is *expansive*, by which we mean that it has some special behaviour with respect to its conjugates (see Section 6 for a precise definition). If T is expansive and $S = N_G(T)$, then the biset $\mathrm{Indinf}_{S/T}^G \mathrm{Defres}_{S/T}^G$ is a stabilizing biset for suitable modules. Note that, more generally, expansive subgroups appear naturally in the study of biset functors (see [Bo4]).

Using the method of expansive subgroups, we prove that a simple module is stabilized by a biset of the form $\mathrm{Indinf}_{S/T}^G \mathrm{Defres}_{S/T}^G$, where T is expansive, $S = N_G(T)$, and S/T is a Roquette group, in the sense that all normal abelian subgroups of S/T are cyclic. As a corollary, it follows that any minimal biset stabilizing a simple module must go down to a Roquette group. This result has some analogy with the theory of cuspidal characters in the context of Harish-Chandra induction and restriction. However, there are no restrictions on the groups or the field, hence we cannot expect to obtain an extremely strong result in general (for instance, any non-abelian simple group is a Roquette group).

In the special case of p-groups in coprime characteristic, we have an essentially complete description of minimal stabilizing bisets by showing that they can be obtained by the method of expansive subgroups. So we do get a strong result in this case and we actually recover some of the results proved in [Bo2] (which originated in the work of Roquette, hence the terminology). In contrast, for a p-group in characteristic p, the minimal bisets stabilizing an indecomposable module are all obtained as idempotent bisets.

Let us end this introduction with a short description of the organization of the paper. In Section 2, we review some basic facts about bisets and introduce in particular the notion of a *butterfly*, which is a biset providing the passage from a section of a group to another. The main uniqueness result for minimal stabilizing bisets is proved in Section 3 and then a few elementary properties are gathered in Section 4. In Section 5 and 6, we discuss the two constructions of stabilizing bisets, namely idempotent bisets and bisets associated to expansive subgroups. The main theorem showing that, for a simple module, one can go down to a Roquette group is proved in Section 7. A few other results about stabilizing bisets for simple modules appear in Section 8. The case of p-groups in coprime characteristic is presented in Section 9, while Section 10 deals with p-groups in characteristic p. Finally, various examples are presented in Section 11, illustrating some of the previous results or providing answers to other natural questions.

2. Bisets

Throughout this paper, we let G and H denote finite groups and we let R be a commutative ring. Recall that a (G, H)-biset is a set U which is both a left G-set and a right H-set, such that both actions commute (that is, $(g{\cdot}u){\cdot}h = g{\cdot}(u{\cdot}h)$ for all $g \in G$, $h \in H$ and $u \in U$). If U is a (G, H)-biset, then RU denotes the free R-module with basis U. Clearly RU is an (RG, RH)-bimodule. Moreover, if U is a disjoint union of bisets $U = U_1 \cup U_2$, then $RU = RU_1 \oplus RU_2$.

If U is a (G,H)-biset and V an (H,J)-biset (where J is another finite group), then the *product* $U\times_H V$ denotes the (G,J)-biset defined by

$$U\times_H V := (U\times V)/\!\sim\,,$$

where $\sim$ is the equivalence relation defined by $(uh,v)\sim(u,hv)$ for all $u\in U$, $v\in V$, $h\in H$. The left action of G on $U\times_H V$ is induced by the left action on U and the right action of J is induced by the right action on V. Clearly, $R(U\times_H V)\cong RU\otimes_{RH} RV$. We often write simply UV instead of $U\times_H V$.

We now give a list of basic bisets which play an essential role. A *section* of a group H is a pair (S,T) of subgroups of H such that $T\trianglelefteq S$. The group S/T will be called the *subquotient* of H corresponding to the section (S,T). If (S,T) is a section of H, then the following bisets are defined:
- the $(S,S/T)$-biset $\mathrm{Inf}_{S/T}^S := S/T$ (inflation);
- the (H,S)-biset $\mathrm{Ind}_S^H := H$ (induction);
- the $(S/T,S)$-biset $\mathrm{Def}_{S/T}^S := T\backslash S$ (deflation);
- the (S,H)-biset $\mathrm{Res}_S^H := H$ (restriction);
- whenever $\alpha: H\to Q$ is a group isomorphism, the (Q,H)-biset $\mathrm{Iso}_\alpha := H$ (isomorphism) with left action of Q via α^{-1}. In particular, if $x\in H$, then conjugation by x is an isomorphism $c_x: S/T\to {}^xS/{}^xT$ and Conj_x denotes the corresponding $({}^xS/{}^xT, S/T)$-biset Iso_{c_x} (conjugation by x).
- the $(H,S/T)$-biset $\mathrm{Indinf}_{S/T}^H := \mathrm{Ind}_S^H\mathrm{Inf}_{S/T}^S = H\times_S (S/T)\cong H/T$;
- the $(S/T,H)$-biset $\mathrm{Defres}_{S/T}^H := \mathrm{Def}_{S/T}^S\mathrm{Res}_S^H = (T\backslash S)\times_S H\cong T\backslash H$.

Every biset decomposes uniquely as a disjoint union of transitive bisets. We recall the structure of transitive bisets.

2.1. Lemma. *Let U be a transitive (G,H)-biset. Then there exist a section (A,B) of G, a section (S,T) of H, and an isomorphism $\phi: S/T\to A/B$ such that*

$$U\cong G/B\times_{A/B}\mathrm{Iso}_\phi\times_{S/T} T\backslash H\cong G\times_A A/B\times_{A/B}\mathrm{Iso}_\phi\times_{S/T} S/T\times_S H\,.$$

In other words

$$U\cong \mathrm{Indinf}_{A/B}^G\,\mathrm{Iso}_\phi\,\mathrm{Defres}_{S/T}^H\,.$$

Moreover, the triple $\big((A,B),(S,T),\phi\big)$ is unique up to conjugation.

Proof : See [Bo1], Lemma 3, or [BoTh1], Lemma 7.4. □

We shall need to relate two different sections (S,T) and (C,D) of the same group G. First we say that a section (S',T') is a *subsection* of (S,T) if we have $T\le T'\le S'\le S$. Next we consider the following easy case.

2.2. Definition. Two sections (S,T) and (C,D) of a group G are said to be *linked* if the following two conditions hold :
- The inclusion $\alpha: S\cap C\to S$ induces an isomorphism

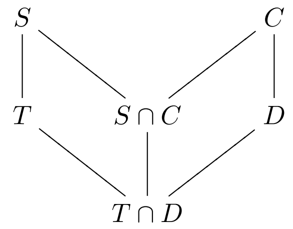
$$\overline{\alpha}:(S\cap C)/(T\cap D)\to S/T\,.$$

- The inclusion $\beta: S\cap C\to C$ induces an isomorphism

$$\overline{\beta}:(S\cap C)/(T\cap D)\to C/D\,.$$

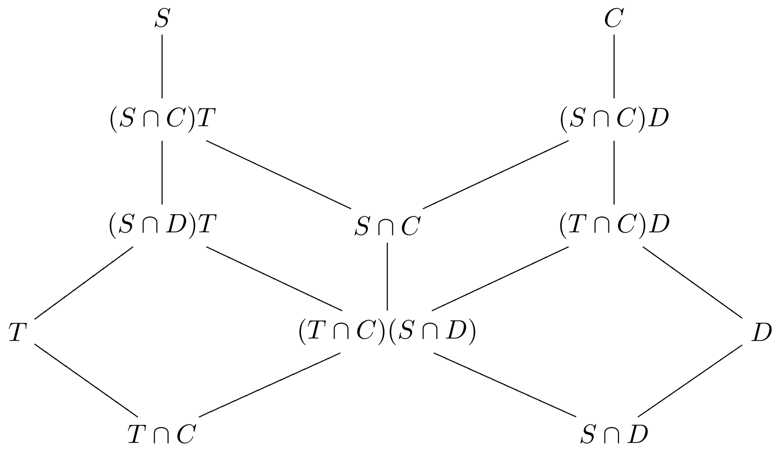
If (S,T) and (C,D) are linked, then the *isomorphism induced by the linking* is the composed isomorphism $\overline{\alpha}\,(\overline{\beta})^{-1} : C/D \to S/T$.

The linking is shown in the following diagram :



It is easy to see that (S,T) and (C,D) are linked if and only if $(S\cap C)T = S$, $(S\cap C)D = C$, and $S\cap D = T\cap C$.

Our next lemma is well-known (see for instance Chapter 4 in [La]) and is illustrated in the following diagram.



2.3. Lemma (Zassenhaus). *Let (S,T) and (C,D) be two sections of a group G. Then the subsection $\big((S\cap C)T, (S\cap D)T\big)$ of (S,T) is linked to the subsection $\big((S\cap C)D, (T\cap C)D\big)$ of (C,D). The isomorphism corresponding to the linking is the composite*

$$(S\cap C)D/(T\cap C)D \longrightarrow (S\cap C)/(T\cap C)(S\cap D) \longrightarrow (S\cap C)T/(S\cap D)T\,.$$

2.4. Definition. Let (S,T) and (C,D) be two sections of a group G. The *butterfly* associated to (S,T) and (C,D) is the $(S/T, C/D)$-biset defined as follows :

$$\mathrm{Btf}\,(S,T,C,D) := \mathrm{Indinf}_{(S\cap C)T/(S\cap D)T}^{S/T}\,\mathrm{Iso}_\psi\,\mathrm{Defres}_{(S\cap C)D/(T\cap C)D}^{C/D}\,,$$

where Iso_ψ is the biset corresponding to the isomorphism of the Zassenhaus lemma :

$$\psi : (S\cap C)D/(T\cap C)D \longrightarrow (S\cap C)T/(S\cap D)T\,.$$

We say that ψ is the isomorphism associated to the butterfly.

The Zassenhaus lemma is also called the butterfly lemma and this explains the terminology. In the special case where the two sections (S,T) and (C,D) are linked, the corresponding butterfly reduces to

$$\mathrm{Btf}\,(S,T,C,D) = \mathrm{Iso}_{\psi}\,,$$

where $\psi : C/D \to S/T$ is the isomorphism induced by the linking, passing through the middle group $(S\cap C)/(T\cap D)$.

We shall need the generalized Mackey formula. This formula appears as Proposition A.1 in [BoTh2] and is the following :

2.5. Lemma (generalized Mackey formula). *Let (A,B) and (S,T) be two sections of a finite group G. Then there is the following decomposition as a disjoint union :*

$$\mathrm{Defres}^G_{S/T}\,\mathrm{Indinf}^G_{A/B} \cong \bigcup_{g\in[S\backslash G/A]} \mathrm{Btf}\,(S,T,\,{}^gA,\,{}^gB)\,\mathrm{Conj}_g\,.$$

3. Stabilizing Bisets

Bisets act on modules as follows. If U is a (G,H)-biset and L is a (left) RH-module, then we define

$$U(L) = RU\otimes_{RH} L\,,$$

and this is clearly an RG-module. We also say that U is *applied* to L. This notation is consistent with the notion of biset functor, where bisets act on the left (see [Bo1], [Bo3], [Bo4]). If U is one of the basic bisets (inflation, induction, deflation, restriction, isomorphism), then $U(L)$ is obtained from L by applying the corresponding operation with the same name (hence the name of the basic bisets). We only recall here the operation of deflation, induced by the $(G/N,G)$-biset $\mathrm{Def}^G_{G/N} := N\backslash G$, where N is a normal subgroup of G. The deflation $\mathrm{Def}^G_{G/N}(L)$ is the R-module L_N of coinvariants under the action of N (that is, the largest quotient of L on which N acts trivially), viewed as an $R(G/N)$-module.

The action of bisets has some elementary properties. First if $U = U_1\cup U_2$ is a disjoint union of two (G,H)-bisets, then

$$U(L)\cong U_1(L)\oplus U_2(L)\,.$$

The composition of the action of bisets corresponds to the action of the product of bisets, as follows. If U is a (G,H)-biset, V is an (H,J)-biset, and M is an RJ-module, then

$$\begin{aligned} U\big(V(M)\big) &= RU\otimes_{RH}(RV\otimes_{RJ}M)\cong(RU\otimes_{RH}RV)\otimes_{RJ}M\\ &\cong R(U\times_H V)\otimes_{RJ}M = (U\times_H V)(M)\,. \end{aligned}$$

This explains why we often write $U\,V$ instead of $U\times_H V$.

3.1. Definition. Let U be a (G, G)-biset and let L be an RG-module. Then U is said to *stabilize* L if $U(L) \cong L$.

Note first that the identity biset $\mathrm{Iso}_{\mathrm{id}}$ stabilizes any module. More generally, for any automorphism ϕ of G, the biset Iso_ϕ stabilizes a kG-module L whenever L is invariant under ϕ. So the notion of stabilizing biset generalizes the well-known and widely used notion of invariance under an automorphism.

We are interested in bisets stabilizing an indecomposable module (and later a simple module). If $U = \bigcup_i U_i$ is a decomposition of U as a disjoint union of transitive bisets and if L is an indecomposable RG-module stabilized by U, then

$$L \cong U(L) \cong \bigoplus_i U_i(L)\,.$$

Therefore we must have $L \cong U_i(L)$ for some i. For this reason, we shall assume that the biset U is transitive, hence of the form (see Lemma 2.1)

$$U = \mathrm{Indinf}_{A/B}^G \, \mathrm{Iso}_\phi \, \mathrm{Defres}_{S/T}^G$$

for some sections (A, B) and (S, T) of G and some isomorphism $\phi : S/T \to A/B$.

3.2. Definition. Let $U = \mathrm{Indinf}_{A/B}^G \, \mathrm{Iso}_\phi \, \mathrm{Defres}_{S/T}^G$ be a (G, G)-biset stabilizing an indecomposable RG-module L. Then U is said to be *minimal* if, for any transitive (G, G)-biset $U' = \mathrm{Indinf}_{A'/B'}^G \, \mathrm{Iso}_{\phi'} \, \mathrm{Defres}_{S'/T'}^G$ stabilizing L, we have $|S/T| \leq |S'/T'|$ (or equivalently $|A/B| \leq |A'/B'|$, because $A/B \cong S/T$ and $A'/B' \cong S'/T'$).

Now we come to our main uniqueness result.

3.3. Theorem. *Consider two transitive (G, G)-bisets*

$$U = \mathrm{Indinf}_{A/B}^G \, \mathrm{Iso}_\phi \, \mathrm{Defres}_{S/T}^G \qquad \text{and} \qquad U' = \mathrm{Indinf}_{A'/B'}^G \, \mathrm{Iso}_{\phi'} \, \mathrm{Defres}_{S'/T'}^G$$

and assume that U and U' stabilize an indecomposable RG-module L. Let $M = \mathrm{Defres}_{S/T}^G(L)$ and $M' = \mathrm{Defres}_{S'/T'}^G(L)$.

1. *There exists a unique double coset $S'gA$ such that*

$$\mathrm{Btf}\,(S', T', {}^gA, {}^gB)\, \mathrm{Conj}_g \, \mathrm{Iso}_\phi(M) \neq \{0\}\,.$$

2. *Suppose that U is a minimal biset stabilizing L. Let g belong to the unique double coset of part (1). Then :*
   - *The section $({}^gA, {}^gB)$ is linked to the subsection $\big((S' \cap {}^gA)T', (S' \cap {}^gB)T'\big)$ of (S', T').*
   - $\mathrm{Btf}\,(S', T', {}^gA, {}^gB) = \mathrm{Indinf}_{(S' \cap {}^gA)T'/(S' \cap {}^gB)T'}^{S'/T'} \mathrm{Iso}_\beta$, *where $\beta : {}^gA/{}^gB \to (S' \cap {}^gA)T'/(S' \cap {}^gB)T'$ is the isomorphism corresponding to the linking.*
   - $M' \cong \mathrm{Indinf}_{(S' \cap {}^gA)T'/(S' \cap {}^gB)T'}^{S'/T'} \mathrm{Iso}_\beta \, \mathrm{Conj}_g \, \mathrm{Iso}_\phi(M)$.
   - *If $h \in G$ does not belong to the same double coset as g, the section $({}^hA, {}^hB)$ is not linked to a subsection of (S', T').*
3. *Suppose that U and U' are both minimal bisets stabilizing L. Let g belong to the unique double coset of part (1). Then :*
   - *The sections $({}^gA, {}^gB)$ and (S', T') are linked.*

- $\mathrm{Btf}\,(S', T', {}^gA, {}^gB) = \mathrm{Iso}_\beta$, *where* $\beta : {}^gA/{}^gB \to S'/T'$ *is the isomorphism corresponding to the linking.*
- $M' \cong \mathrm{Iso}_\beta\, \mathrm{Conj}_g\, \mathrm{Iso}_\phi(M)$.
- *If* $h \in G$ *does not belong to the same double coset as* g*, the section* $({}^hA, {}^hB)$ *is not linked to* (S', T').

Proof : (1) Applying successively U and U', we obtain

$$L \cong U'\big(U(L)\big) \cong \mathrm{Indinf}^G_{A'/B'}\, \mathrm{Iso}_{\phi'}\, \mathrm{Defres}^G_{S'/T'}\, \mathrm{Indinf}^G_{A/B}\, \mathrm{Iso}_\phi\, \mathrm{Defres}^G_{S/T}(L)\,.$$

The middle composition $\mathrm{Defres}^G_{S'/T'}\, \mathrm{Indinf}^G_{A/B}$ is applied to $\mathrm{Iso}_\phi(M)$ and we decompose it according to the generalized Mackey formula (Lemma 2.5) :

$$\mathrm{Defres}^G_{S'/T'}\, \mathrm{Indinf}^G_{A/B}\, \mathrm{Iso}_\phi(M) \cong \bigoplus_{g\in[S'\backslash G/A]} \mathrm{Btf}\,(S', T', {}^gA, {}^gB)\, \mathrm{Conj}_g\, \mathrm{Iso}_\phi(M)\,.$$

But this module is indecomposable because by applying $\mathrm{Indinf}^G_{A'/B'}\, \mathrm{Iso}_{\phi'}$ to it we obtain the indecomposable module L. Therefore, there exists a unique double coset $S'gA$ such that

$$\mathrm{Btf}\,(S', T', {}^gA, {}^gB)\, \mathrm{Conj}_g\, \mathrm{Iso}_\phi(M) \neq \{0\}\,,$$

proving (1). For later use, note that we have

$$\mathrm{Defres}^G_{S'/T'}\, \mathrm{Indinf}^G_{A/B}\, \mathrm{Iso}_\phi(M) \cong \mathrm{Btf}\,(S', T', {}^gA, {}^gB)\, \mathrm{Conj}_g\, \mathrm{Iso}_\phi(M)\,.$$

(2) Let g be as in (1). Let

$$\beta : (S' \cap {}^gA)\, {}^gB/(T' \cap {}^gA)\, {}^gB \longrightarrow (S' \cap {}^gA)T'/(S' \cap {}^gB)T'$$

denote the isomorphism associated to the butterfly $\mathrm{Btf}\,(S', T', {}^gA, {}^gB)$. Let (A'', B'') be the image of the section

$$\big((S' \cap {}^gA)T', (S' \cap {}^gB)T'\big) \quad \text{of the group } S'/T'$$

under the isomorphism $\phi' : S'/T' \to A'/B'$. Then ϕ' induces an isomorphism between the corresponding subquotients

$$\overline{\phi'} : (S' \cap {}^gA)T'/(S' \cap {}^gB)T' \longrightarrow A''/B''\,.$$

Similarly let (S'', T'') be the image of the section

$$\big((S' \cap {}^gA)\, {}^gB, (T' \cap {}^gA)\, {}^gB\big) \quad \text{of the group } {}^gA/{}^gB$$

under the isomorphism $\phi^{-1}\, c_{g^{-1}} : {}^gA/{}^gB \to S/T$ and let

$$\overline{\phi} : S''/T'' \longrightarrow (S'^g \cap A)B/(T'^g \cap A)B$$

be the isomorphism induced by $\phi : S/T \to A/B$. We then have

$$\begin{aligned}\mathrm{Indinf}^G_{A'/B'}\, &\mathrm{Iso}_{\phi'}\, \mathrm{Indinf}^{S'/T'}_{(S'\cap {}^gA)T'/(S'\cap {}^gB)T'} \\ &\cong \mathrm{Indinf}^G_{A'/B'}\, \mathrm{Indinf}^{A'/B'}_{A''/B''}\, \mathrm{Iso}_{\overline{\phi'}} \cong \mathrm{Indinf}^G_{A''/B''}\, \mathrm{Iso}_{\overline{\phi'}}\end{aligned}$$

and similarly

$$\mathrm{Defres}^{\,{}^gA/\,{}^gB}_{(S'\cap\,{}^gA)\,{}^gB/(T'\cap\,{}^gA)\,{}^gB}\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{S/T}$$
$$\cong \mathrm{Conj}_g\,\mathrm{Iso}_{\overline{\phi}}\,\mathrm{Defres}^{S/T}_{S''/T''}\,\mathrm{Defres}^G_{S/T} \cong \mathrm{Conj}_g\,\mathrm{Iso}_{\overline{\phi}}\,\mathrm{Defres}^G_{S''/T''}\,.$$

Using all these observations as well as part (1), we obtain

$$\begin{aligned}
L \;\cong\;& \mathrm{Indinf}^G_{A'/B'}\;\mathrm{Iso}_{\phi'}\,\mathrm{Defres}^G_{S'/T'}\,\mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{S/T}(L)\\
\cong\;& \mathrm{Indinf}^G_{A'/B'}\;\mathrm{Iso}_{\phi'}\,\mathrm{Btf}\,(S',T',\,{}^gA,\,{}^gB)\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{S/T}(L)\\
\cong\;& \mathrm{Indinf}^G_{A'/B'}\;\mathrm{Iso}_{\phi'}\,\mathrm{Indinf}^{S'/T'}_{(S'\cap\,{}^gA)T'/(S'\cap\,{}^gB)T'}\,\mathrm{Iso}_\beta\\
&\qquad\qquad \mathrm{Defres}^{\,{}^gA/\,{}^gB}_{(S'\cap\,{}^gA)\,{}^gB/(T'\cap\,{}^gA)\,{}^gB}\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{S/T}(L)\\
\cong\;& \mathrm{Indinf}^G_{A''/B''}\;\mathrm{Iso}_{\overline{\phi'}}\,\mathrm{Iso}_\beta\,\mathrm{Conj}_g\,\mathrm{Iso}_{\overline{\phi}}\,\mathrm{Defres}^G_{S''/T''}(L)\,.
\end{aligned}$$

Thus L is stabilized by a (G,G)-biset with corresponding subquotients $A''/B'' \cong S''/T''$. By minimality of U, we must have $(S'',T'')=(S,T)$, and so (via $c_g\phi$) :

$$\big((S'\cap\,{}^gA)\,{}^gB,(T'\cap\,{}^gA)\,{}^gB\big)=(\,{}^gA,\,{}^gB)\,.$$

Therefore the isomorphism β associated to the butterfly $\mathrm{Btf}\,(S',T',\,{}^gA,\,{}^gB)$ is actually an isomorphism

$$\beta:\,{}^gA/\,{}^gB\longrightarrow (S'\cap\,{}^gA)T'/(S'\cap\,{}^gB)T'\,.$$

Thus the section $(\,{}^gA,\,{}^gB)$ is linked to the subsection $\big((S'\cap\,{}^gA)T',(S'\cap\,{}^gB)T'\big)$ of (S',T'), and moreover

$$\mathrm{Btf}\,(S',T',\,{}^gA,\,{}^gB)=\mathrm{Indinf}^{S'/T'}_{(S'\cap\,{}^gA)T'/(S'\cap\,{}^gB)T'}\,\mathrm{Iso}_\beta\,.$$

Note also that if $h\in G$ does not belong to the same double coset as g, then the section $(\,{}^hA,\,{}^hB)$ cannot be linked to a subsection of (S',T'), otherwise we would have an isomorphism $\gamma:\,{}^hA/\,{}^hB\to(S'\cap\,{}^hA)T'/(S'\cap\,{}^hB)T'$ corresponding to the linking and we would obtain a non-zero module

$$\mathrm{Btf}\,(S',T',\,{}^hA,\,{}^hB)(\tilde{M})=\mathrm{Indinf}^{S'/T'}_{(S'\cap\,{}^hA)T'/(S'\cap\,{}^hB)T'}\mathrm{Iso}_\gamma(\tilde{M})\,,$$

where $\tilde{M}=\mathrm{Conj}_h\,\mathrm{Iso}_\phi(M)$.

Moreover, the equality of sections $(S'',T'')=(S,T)$ above, which follows from the minimality of U, also implies that $\overline{\phi}=\phi$. Therefore we obtain

$$\begin{aligned}
M' \;=\;& \mathrm{Defres}^G_{S'/T'}(L)\\
\cong\;& \mathrm{Defres}^G_{S'/T'}\,\mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{S/T}(L)\\
\cong\;& \mathrm{Defres}^G_{S'/T'}\,\mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,(M)\\
\cong\;& \mathrm{Btf}\,(S',T',\,{}^gA,\,{}^gB)\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi\,(M)\\
\cong\;& \mathrm{Indinf}^{S'/T'}_{(S'\cap\,{}^gA)T'/(S'\cap\,{}^gB)T'}\,\mathrm{Iso}_\beta\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi(M)\,.
\end{aligned}$$

This proves (2).

(3) We continue with the above analysis and use now the minimality of U'. Then we must have also $(A'', B'') = (A', B')$, and so (via ϕ'^{-1}) :

$$\big((S' \cap {}^gA)T', (S' \cap {}^gB)T'\big) = (S', T') \,.$$

It follows that the two sections (S', T') and $({}^gA, {}^gB)$ are linked and $\beta : {}^gA/{}^gB \to S'/T'$ is the isomorphism corresponding to the linking. Moreover the corresponding butterfly $\mathrm{Btf}(S', T', {}^gA, {}^gB)$ is simply Iso_β. Finally, we also obtain $M' \cong \mathrm{Iso}_\beta \, \mathrm{Conj}_g \, \mathrm{Iso}_\phi(M)$. □

Applying this theorem to the case where $U = U'$, we obtain the following special case.

3.4. Corollary. *Let $U = \mathrm{Indinf}^G_{A/B} \, \mathrm{Iso}_\phi \, \mathrm{Defres}^G_{S/T}$ be a minimal (G,G)-biset stabilizing an indecomposable RG-module L and let $M = \mathrm{Defres}^G_{S/T}(L)$.*

1. *There exists a single double coset SgA such that*

$$\mathrm{Btf}(S, T, {}^gA, {}^gB) \, \mathrm{Conj}_g \, \mathrm{Iso}_\phi(M) \neq \{0\} \,.$$

2. *Let g belong to the unique double coset of part (1). Then :*
   - *The sections (S,T) and $({}^gA, {}^gB)$ are linked.*
   - $\mathrm{Btf}(S, T, {}^gA, {}^gB) = \mathrm{Iso}_\beta$, *where $\beta : {}^gA/{}^gB \to S/T$ is the isomorphism corresponding to the linking.*
   - *The module M is invariant under $\rho = \beta \, c_g \, \phi$, where $c_g : A/B \to {}^gA/{}^gB$ denotes the conjugation isomorphism.*
   - *If $h \in G$ does not belong to the same double coset as g, the section $({}^hA, {}^hB)$ is not linked to (S,T).*

Proof : This follows immediately from the previous theorem. □

Consequently, if we replace the section (A,B) by a conjugate (and modify the middle isomorphism accordingly by composing with a conjugation), then we can assume that the two sections (A,B) and (S,T) are linked. Now we show that the middle isomorphism in a stabilizing biset can always be chosen to be the isomorphism induced by the linking.

3.5. Corollary. *Let $U = \mathrm{Indinf}^G_{A/B} \, \mathrm{Iso}_\phi \, \mathrm{Defres}^G_{S/T}$ be a minimal (G,G)-biset stabilizing an indecomposable RG-module L. There exists a section $(\widetilde{A}, \widetilde{B}) = ({}^gA, {}^gB)$ linked to (S,T) and such that L is stabilized by the biset*

$$\widetilde{U} = \mathrm{Indinf}^G_{\widetilde{A}/\widetilde{B}} \, \mathrm{Iso}_\sigma \, \mathrm{Defres}^G_{S/T} \,,$$

where $\sigma : S/T \to \widetilde{A}/\widetilde{B}$ is the isomorphism corresponding to the linking. Moreover, if $h \notin S\widetilde{A}$, the section $({}^h\widetilde{A}, {}^h\widetilde{B})$ is not linked to (S,T).

Proof : Let $M = \mathrm{Defres}^G_{S/T}(L)$. Let SgA be the unique double coset of Corollary 3.4 and let $\widetilde{A} = {}^gA$ and $\widetilde{B} = {}^gB$. We know that M is invariant under

$$\rho = \beta \, c_g \, \phi \,,$$

where $c_g : A/B \to \widetilde{A}/\widetilde{B}$ is the conjugation isomorphism and $\beta : \widetilde{A}/\widetilde{B} \to S/T$ is the isomorphism corresponding to the linking. Thus we have $\mathrm{Iso}_{\rho^{-1}}(M) \cong M$ and therefore

$$\begin{aligned} L &\cong U(L) \\ &\cong \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Defres}_{S/T}^G(L) \\ &\cong \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi(M) \\ &\cong \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Iso}_{\rho^{-1}}(M) \\ &\cong \mathrm{Indinf}_{A/B}^G \mathrm{Conj}_{g^{-1}} \mathrm{Iso}_{\beta^{-1}}(M) \\ &\cong \mathrm{Indinf}_{\widetilde{A}/\widetilde{B}}^G \mathrm{Iso}_{\beta^{-1}} \mathrm{Defres}_{S/T}^G(L) \\ &\cong \widetilde{U}(L)\,. \end{aligned}$$

The result follows because $\sigma = \beta^{-1}$. □

3.6. Remark. Let $U = \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Defres}_{S/T}^G$ be a minimal (G,G)-biset stabilizing an RG-module L. One can always modify U by inserting an automorphism Iso_ψ which leaves invariant the module $M = \mathrm{Defres}_{S/T}^G(L)$. It follows from Corollary 3.5 that one can always modify U in this way in order to obtain a middle isomorphism simply induced by the isomorphism corresponding to a linking between (A,B) and (S,T).

4. Elementary properties

We establish a few elementary properties of stabilizing bisets.

4.1. Lemma. *Let $U = \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Defres}_{S/T}^G$ be a (G,G)-biset stabilizing a KG-module L, where K is a field. Then we have*

$$\dim(L) = |G : A| \cdot \dim(L_T)\,,$$

where $L_T = \mathrm{Def}_{S/T}^S \mathrm{Res}_S^G(L)$ is the K-vector space of T-coinvariants under the action of T.

Proof : The dimension is fixed under Iso_ϕ and under $\mathrm{Inf}_{A/B}^A$, but it is multiplied by the index $|G : A|$ under Ind_A^G. □

In the special case where the two sections appearing in a stabilizing biset coincide, we have the following additional information.

4.2. Proposition. *Suppose that $U = \mathrm{Indinf}_{S/T}^G \mathrm{Iso}_\phi \mathrm{Defres}_{S/T}^G$ is a (not necessarily minimal) (G,G)-biset stabilizing an indecomposable RG-module L. Then $N_G(T) = S$.*

Proof : The biset $\mathrm{Btf}\,(S,T,S,T)$ acts as the identity and therefore

$$\mathrm{Btf}\,(S,T,S,T)\,\mathrm{Iso}_\phi(M) = \mathrm{Iso}_\phi(M) \neq \{0\}\,,$$

where $M = \mathrm{Defres}^G_{S/T}(L)$. Thus the double coset $S1S = S$ is the unique double coset as in part (1) of Theorem 3.3. Let $g \in N_G(T)$. Then the butterfly associated to (S,T) and $({}^gS, {}^gT)$ just consists of restriction to the subgroup $(S \cap {}^gS)/T$ followed by induction from this subgroup. Therefore we have

$$\begin{aligned} \mathrm{Btf}(S,T,{}^gS,{}^gT)\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi(M) &= \mathrm{Btf}(S,T,{}^gS,T)\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi(M) \\ &= \mathrm{Ind}^{S/T}_{(S\cap {}^gS)/T}\,\mathrm{Res}^{{}^gS/T}_{(S\cap {}^gS)/T}\,\mathrm{Conj}_g\,\mathrm{Iso}_\phi(M)\,, \end{aligned}$$

and this is non-zero since none of these operations can annihilate a module. Therefore the double coset SgS must be equal to S. Hence $g \in S$, as was to be shown. □

Another useful fact is the following.

4.3. Proposition. *Let $U = \mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{S/T}$ be a minimal (G,G)-biset stabilizing an RG-module L. Let $M = \mathrm{Defres}^G_{S/T}(L)$. Then M is a faithful $R(S/T)$-module and $\mathrm{Iso}_\phi(M)$ is a faithful $R(A/B)$-module.*

Proof : Let N/T be the kernel of the action of S/T on the module M. Then $M \cong \mathrm{Inf}^{S/T}_{S/N}\,\mathrm{Def}^{S/T}_{S/N}(M)$. It follows that L is stabilized by the biset

$$\mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Inf}^{S/T}_{S/N}\,\mathrm{Def}^{S/T}_{S/N}\,\mathrm{Defres}^G_{S/T} \cong \mathrm{Indinf}^G_{A/C}\,\mathrm{Iso}_{\phi'}\,\mathrm{Defres}^G_{S/N}\,,$$

where C/B is the image of N/T under the isomorphism ϕ and where $\phi' : S/N \to A/C$ denotes the isomorphism induced by ϕ. By minimality of U, we must have $|S/N| = |S/T|$, hence $N = T$. □

5. Idempotent bisets

In this section, we introduce a first situation which gives rise to stabilizing bisets. Among transitive (G,G)-bisets, the idempotent bisets turn out to be of special interest and they are necessarily stabilizing bisets (for modules which may not be indecomposable). We characterize idempotent bisets by means of a property of double cosets and of linking of sections.

A (G,G)-biset U is called *idempotent* if $U^2 \cong U$, where $U^2 = U \times_G U$.

5.1. Proposition. *Let $U = \mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{S/T}$ be a (G,G)-biset. Then U is idempotent if and only if the following three conditions hold:*

(a) There is a unique (S,A)-double coset, in other words $SA = G$.

(b) The sections (S,T) and (A,B) are linked.

(c) There exist $x \in N_G(A,B)$ and $y \in N_G(S,T)$ such that

$$\phi\sigma^{-1}\phi = \mathrm{conj}_x\,\phi\,\mathrm{conj}_y^{-1}\,,$$

where $\sigma : S/T \to A/B$ is the isomorphism induced by the linking. Here $\mathrm{conj}_x : A/B \to A/B$ and $\mathrm{conj}_y : S/T \to S/T$ are induced by conjugation by x and y respectively.

Proof : By the generalized Mackey formula (Lemma 2.5), we have

$$\begin{aligned} U^2 &\cong \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Defres}_{S/T}^G \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Defres}_{S/T}^G \\ &\cong \bigcup_{g\in[S\backslash G/A]} \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Btf}\,(S,T,\,{}^gA,\,{}^gB)\,\mathrm{Conj}_g \mathrm{Iso}_\phi \mathrm{Defres}_{S/T}^G\,. \end{aligned}$$

If $U^2 \cong U$, then U^2 must be transitive and therefore there can be only one term in this disjoint union. It follows that there is a unique (S, A)-double coset, that is $SA = G$. Since the butterfly $\mathrm{Btf}\,(S, T, A, B)$ factorizes by definition through a subsection of (S, T), while U cannot factorize through a proper subsection of (S, T), the two sections (S, T) and (A, B) have to be linked and the butterfly has to be induced by the isomorphism $\sigma^{-1} : A/B \to S/T$ corresponding to the linking. Therefore, we are left with

$$\mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Defres}_{S/T}^G = U \cong U^2 \cong \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Iso}_{\sigma^{-1}} \mathrm{Iso}_\phi \mathrm{Defres}_{S/T}^G\,.$$

Since two transitive bisets are isomorphic if and only if the corresponding stabilizers in $G \times G$ are conjugate, this isomorphism implies the existence of $(x, y) \in G \times G$ conjugating one stabilizer into the other. Here, x must normalize both A and B, and y must normalize both S and T, while the isomorphism $\phi\sigma^{-1}\phi : S/T \to A/B$ must differ from ϕ by the two conjugations conj_x and conj_y^{-1}. Thus we have $x \in N_G(A, B)$ and $y \in N_G(S, T)$ such that $\phi\sigma^{-1}\phi = \mathrm{conj}_x\, \phi\, \mathrm{conj}_y^{-1}$.

Conversely, assume that (a), (b), (c) hold. Then the computation of U^2 as above yields only one term, because of (a), with a butterfly $\mathrm{Btf}\,(S, T, A, B)$ equal to $\mathrm{Iso}_{\sigma^{-1}}$, because of (b). Therefore, using (c), we obtain

$$\begin{aligned} U^2 &\cong \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Iso}_{\sigma^{-1}} \mathrm{Iso}_\phi \mathrm{Defres}_{S/T}^G \\ &\cong \mathrm{Indinf}_{A/B}^G \mathrm{Conj}_x \mathrm{Iso}_\phi \mathrm{Conj}_{y^{-1}} \mathrm{Defres}_{S/T}^G \\ &\cong \mathrm{Conj}_x \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Defres}_{S/T}^G \mathrm{Conj}_{y^{-1}} \\ &\cong \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Defres}_{S/T}^G = U\,, \end{aligned}$$

because as (G, G)-bisets, Conj_x and Conj_y are isomorphic to the identity. □

By Corollary 3.5, we can always assume that a minimal stabilizing biset has the form $U = \mathrm{Indinf}_{A/B}^P \mathrm{Iso}_\sigma \mathrm{Defres}_{S/T}^P$, where the two sections are linked and where $\sigma : S/T \to A/B$ is the isomorphism induced by the linking. With this harmless assumption, we obtain the following corollary.

5.2. Corollary. *Suppose that the sections (S, T) and (A, B) are linked and let $\sigma : S/T \to A/B$ be the isomorphism induced by the linking. If there is a unique (S, A)-double coset (i.e. $SA = G$), then the (G, G)-biset $U = \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\sigma \mathrm{Defres}_{S/T}^G$ is idempotent.*

Proof : Choosing $x = y = 1$ in Proposition 5.1, condition (c) becomes $\sigma\sigma^{-1}\sigma = \mathrm{conj}_1\, \sigma\, \mathrm{conj}_1$, which is obviously satisfied. □

5.3. Example. As instances of this, we have the following special cases:

• If N is a normal subgroup of G, then $U = \mathrm{Inf}_{G/N}^G \mathrm{Def}_{G/N}^G$ is idempotent.

• If G is a semi-direct product $G = N \rtimes A$, then $U = \mathrm{Ind}_A^G \mathrm{Iso}_\phi \mathrm{Def}_{G/N}^G$ is idempotent, where $\phi : G/N \to A$ is the obvious isomorphism.

• Both cases can be unified by considering a normal subgroup N, a subgroup A such that $NA = G$ and $B = A \cap N$. Then $U = \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Def}_{G/N}^G$ is idempotent, where $\phi : G/N \to A/B$ is the obvious isomorphism.

Finally, we emphasize the following obvious result.

5.4. Proposition. *Let U be an idempotent (G,G)-biset. For any RG-module L', the RG-module $L = U(L')$ is stabilized by U.*

Note that if L' is indecomposable, $L = U(L')$ need not be indecomposable (it may also be zero), and that two non-isomorphic modules L' and L'' may yield isomorphic modules $U(L') \cong U(L'')$. In the last two sections 10 and 11, we shall see examples where such idempotents bisets appear.

6. Expansive subgroups

In this section, we introduce a second situation which yields stabilizing bisets. We consider the case where the two sections appearing in a stabilizing biset coincide and we describe one case where a biset of the form

$$U = \mathrm{Indinf}_{S/T}^G \mathrm{Iso}_\phi \mathrm{Defres}_{S/T}^G$$

can stabilize a module. By Corollary 3.5, we can replace ϕ by the isomorphism induced by the linking between (S,T) and itself, namely the identity, so we can assume that $U = \mathrm{Indinf}_{S/T}^G \mathrm{Defres}_{S/T}^G$. By Proposition 4.2, we must have $S = N_G(T)$, so we just need a condition on T. Recall that the G-*core* of a subgroup H of G is the largest normal subgroup of G contained in H, that is, the intersection of all the G-conjugates of H. The following definition appears in [Bo4].

6.1. Definition. A subgroup T of G is called *expansive in G* if, for every $g \notin N_G(T)$, the $N_G(T)$-core of the subgroup $\big({}^gT \cap N_G(T)\big)T$ contains T properly.

Note first that any normal subgroup of G is expansive in G. The role of expansive subgroups in the study of biset functors is explained in [Bo4], but we do not need this generality here. For our purposes, the use of expansive subgroups appears in the following result.

6.2. Proposition. *Let T be an expansive subgroup of G and let $S = N_G(T)$. Suppose that M is an $R(S/T)$-module such that, for any non-trivial normal subgroup N/T of S/T, we have $\mathrm{Def}_{S/N}^{S/T}(M) = \{0\}$. Let $L = \mathrm{Indinf}_{S/T}^G(M)$.*

1. $\mathrm{Defres}_{S/T}^G(L) \cong M$.
2. *The biset $U = \mathrm{Indinf}_{S/T}^G \mathrm{Defres}_{S/T}^G$ stabilizes L.*
3. $\mathrm{End}_{RG}(L) \cong \mathrm{End}_{R(S/T)}(M)$ *as R-algebras.*
4. *L is indecomposable if and only if M is indecomposable. In particular, if R is a field of characteristic prime to $|G|$, then L is simple if and only if M is simple.*

Proof : By the generalized Mackey formula (Lemma 2.5), we have

$$\begin{aligned}
\mathrm{Defres}^G_{S/T}(L) &= \mathrm{Defres}^G_{S/T}\,\mathrm{Indinf}^G_{S/T}(M)\\
&\cong \bigoplus_{g\in[S\backslash G/S]} \mathrm{Btf}\,(S,T,\,{}^gS,\,{}^gT)\,\mathrm{Conj}_g(M)\\
&\cong \bigoplus_{g\in[S\backslash G/S]} \mathrm{Conj}_g\,\mathrm{Btf}\,(S^g,T^g,S,T)(M)\\
&\cong M\oplus\Big(\bigoplus_{\substack{g\in[S\backslash G/S]\\ g\notin S}} \mathrm{Conj}_g\,\mathrm{Btf}\,(S^g,T^g,S,T)(M)\Big)\,.
\end{aligned}$$

Now we have

$$\mathrm{Btf}\,(S^g,T^g,S,T)=\mathrm{Indinf}^{S^g/T^g}_{(S^g\cap S)T^g/(S^g\cap T)T^g}\,\mathrm{Iso}_\psi\,\mathrm{Defres}^{S/T}_{(S^g\cap S)T/(T^g\cap S)T}$$

and we need to prove that $\mathrm{Defres}^{S/T}_{(S^g\cap S)T/(T^g\cap S)T}(M)=\{0\}$ whenever $g\notin S$. Since T is expansive and $g\notin S=N_G(T)$, the S-core N of the subgroup $(T^g\cap S)T$ contains T properly. In other words, N/T is a non-trivial normal subgroup of S/T contained in $(T^g\cap S)T/T$. But we have

$$\mathrm{Defres}^{S/T}_{(S^g\cap S)T/(T^g\cap S)T}=\mathrm{Defres}^{S/N}_{(S^g\cap S)T/(T^g\cap S)T}\,\mathrm{Def}^{S/T}_{S/N}\,.$$

Since $\mathrm{Def}^{S/T}_{S/N}(M)=\{0\}$ by assumption, $\mathrm{Defres}^{S/T}_{(S^g\cap S)T/(T^g\cap S)T}(M)=\{0\}$. This proves (1) and (2) follows immediately.

By adjunction properties of induction and inflation, we have isomorphisms of R-modules

$$\begin{aligned}
\mathrm{End}_{RG}(L) &\cong \mathrm{Hom}_{RG}\big(L,\mathrm{Indinf}^G_{S/T}(M)\big)\\
&\cong \mathrm{Hom}_{RS}\big(\mathrm{Res}^G_S(L),\mathrm{Inf}^S_{S/T}(M)\big)\\
&\cong \mathrm{Hom}_{R(S/T)}\big(\mathrm{Defres}^G_{S/T}(L),M\big)\\
&\cong \mathrm{Hom}_{R(S/T)}(M,M)\,.
\end{aligned}$$

It is elementary to check that if $\alpha\in\mathrm{End}_{R(S/T)}(M)$, then the corresponding endomorphism in $\mathrm{End}_{RG}(L)$ is just the induced homomorphism $\mathrm{Ind}^G_S(\alpha)$. It follows that the above isomorphism preserves products and hence is an isomorphism of R-algebras. This proves (3). Finally (4) follows from the fact that a module is indecomposable if and only if there are no non-trivial idempotents in its endomorphism algebra. □

6.3. Corollary. *Let T be an expansive subgroup of G and let $S=N_G(T)$. Let K be a field and suppose that M is a faithful simple $K(S/T)$-module. Then the KG-module $L=\mathrm{Indinf}^G_{S/T}(M)$ is indecomposable and the conclusions of Theorem 6.2 hold.*

Proof : Let N/T be a non-trivial normal subgroup of S/T. Since M is simple and faithful, the largest quotient of M with trivial action of N/T must be zero. Thus we have $\mathrm{Def}^{S/T}_{S/N}(M)=\{0\}$ and Proposition 6.2 applies. □

7. Simple modules and genetic subgroups

In this section, we analyze further the situation of the previous section in the case of a simple module. Thus we work with an expansive subgroup T and a biset $\mathrm{Indinf}_{S/T}^{G}\,\mathrm{Defres}_{S/T}^{G}$ where $S = N_G(T)$. We prove the existence of a suitable stabilizing biset of this form in the case of a simple module. We can work over a field K, because any simple RG-module is in fact a simple KG-module for some field K (a quotient of R).

The following definitions are inspired by [Bo2], [Bo3], [Bo4] and [Ba]. We shall see later that the second definition agrees with the one given in [Bo3] (see Remark 7.7).

7.1. Definition. *(a) A finite group H is called a* Roquette group *if all its normal abelian subgroups are cyclic. In other words, for any prime p, any normal elementary abelian p-subgroup of H has order 1 or p.*

(b) A subgroup T of a finite group G is called a genetic subgroup *if T is an expansive subgroup of G and $N_G(T)/T$ is a Roquette group.*

Before stating the main result, we first prove the following lemma.

7.2. Lemma. *Let G be a finite group, let N be a cyclic normal subgroup of G of square-free order, and let K be a field whose characteristic does not divide $|N|$. Let W be an indecomposable KG-module of the form $W = \mathrm{Indinf}_{S/T}^{G}(Y)$, where (S,T) is a section of G and Y is a faithful simple $K(S/T)$-module. If $T\cap N = \mathbf{1}$, then T centralizes N and $N \leq S$.*

Proof : Since the order of N is coprime to the characteristic of K, the restriction $\mathrm{Res}_N^G(W)$ is a semi-simple KN-module. Its isotypic components are permuted by the action of G, and since W is indecomposable, they are permuted transitively. In particular, the kernels of all simple summands of $\mathrm{Res}_N^G(W)$ are conjugate under G. But N is cyclic, so that any subgroup of N is characteristic in N, hence normal in G. It follows that all the simple summands of $\mathrm{Res}_N^G(W)$ have the same kernel, equal to the intersection $N \cap \mathrm{Ker}(W)$.

Now $\mathrm{Ker}(Y) = \mathbf{1}$ because Y is faithful, and so $\mathrm{Ker}(W) = \bigcap\limits_{g\in G} {}^gT$. Therefore

$$N \cap \mathrm{Ker}(W) = \bigcap_{g\in G} (N \cap {}^gT) = \bigcap_{g\in G} {}^g(N\cap T) = \mathbf{1}$$

by assumption. It follows that $\mathrm{Res}_N^G(W)$ is a direct sum of faithful simple KN-modules. For any such simple KN-module X and any subgroup C of N, the module $\mathrm{Def}_{N/C}^{N}(X)$ is a quotient of X, hence either $\{0\}$ or X itself. But it cannot be X if C is non-trivial because C acts trivially on $\mathrm{Def}_{N/C}^{N}(X)$ while X is faithful. It follows that $\mathrm{Def}_{N/C}^{N}(X) = \{0\}$, hence $\mathrm{Def}_{N/C}^{N}\,\mathrm{Res}_N^G(W) = \{0\}$, for any non-trivial subgroup C of N.

On the other hand, the restriction of W to N is equal to

$$\mathrm{Res}_N^G(W) = \bigoplus_{g\in[G/NS]} \mathrm{Ind}_{N\cap {}^gS}^{N}\,\mathrm{Conj}_g\,\mathrm{Res}_{N\cap S}^{S}\,\mathrm{Inf}_{S/T}^{S}(Y)\,.$$

But $N\cap {}^gS = {}^g(N\cap S) = N\cap S$ because N is cyclic. So $\mathrm{Res}_N^G(W) = \mathrm{Ind}_{N\cap S}^{N}(M)$, for a suitable $K\big(N\cap S\big)$-module M.

Since the order of N is square-free, the subgroup $N \cap S$ has a complement C in N. Thus
$$\mathrm{Def}^N_{N/C}\,\mathrm{Ind}^N_{N\cap S} = \mathrm{Iso}^{N/C}_{N\cap S}\,.$$
It follows that $\mathrm{Def}^N_{N/C}\,\mathrm{Res}^G_N(W) = \mathrm{Iso}^{N/C}_{N\cap S}(M) \neq \{0\}$, since $M \neq \{0\}$. Therefore $C = \mathbf{1}$, hence $N \cap S = N$ and $N \leq S$. Now N and T normalize each other and intersect trivially, so they centralize each other, as was to be shown. □

7.3. Theorem. *Let K be a field and let G be a finite group. If V is a simple KG-module, then there exist a genetic subgroup T of G and a faithful simple $K\big(N_G(T)/T\big)$-module Y such that*
$$V \cong \mathrm{Indinf}^G_{N_G(T)/T}(Y)\,.$$
We have $Y \cong \mathrm{Defres}^G_{N_G(T)/T}(V)$, so that V is stabilized by the biset
$$U = \mathrm{Indinf}^G_{N_G(T)/T}\,\mathrm{Defres}^G_{N_G(T)/T}\,.$$
Moreover $\mathrm{End}_{KG}(V) \cong \mathrm{End}_{K\left(N_G(T)/T\right)}(Y)$ as K-algebras.

Proof : We only need to prove the existence of T and Y such that $V \cong \mathrm{Indinf}^G_{N_G(T)/T}(Y)$, because all the other statements then follow from Corollary 6.3. We prove the existence of T and Y by induction on the order of G. Assume first that V is not faithful. Then $V = \mathrm{Inf}^G_{\overline{G}}(\overline{V})$, where $\overline{G} = G/N$ for some non-trivial normal subgroup of N of G and some simple $K\overline{G}$-module $\overline{V}$. Then there is a genetic subgroup $\overline{T} = T/N$ of $\overline{G}$ and a faithful simple $K(N_{\overline{G}}(\overline{T})/\overline{T})$-module Y such that
$$\overline{V} \cong \mathrm{Indinf}^{\overline{G}}_{N_{\overline{G}}(\overline{T})/\overline{T}}(Y)\,.$$
Moreover $N_{\overline{G}}(\overline{T})/\overline{T} \cong N_G(T)/T$ and it is straightforward to check that T is an expansive subgroup of G if $\overline{T}$ is an expansive subgroup of $\overline{G}$. It follows that T is a genetic subgroup of G. Moreover we have $V \cong \mathrm{Indinf}^G_{N_G(T)/T}(Y)$ in this case. Therefore we can now assume that V is faithful.

If all the abelian normal subgroups of G are cyclic, then G is a Roquette group and $\mathbf{1}$ is a genetic subgroup of G. In this case $V \cong \mathrm{Indinf}^G_{N_G(\mathbf{1})/\mathbf{1}}(V)$, and V is faithful, so there is nothing to prove.

Now let E be any non-trivial abelian normal subgroup of G and assume that E is non-cyclic. Replacing E by its socle (which is characteristic in E, hence normal in G), we can assume that E is a direct product of elementary abelian p-subgroups for various primes p.

Since $E \trianglelefteq G$, the restriction $\mathrm{Res}^G_E(V)$ is semi-simple. Let L be a simple summand of $\mathrm{Res}^G_E(V)$, let I be the stabilizer of L in G, and let $\widetilde{L}$ be the isotypic component of $\mathrm{Res}^G_E(V)$ containing L (that is, the sum of all submodules isomorphic to L). Then I acts on $\widetilde{L}$, which is a simple KI-module, and $V \cong \mathrm{Ind}^G_I(\widetilde{L})$.

Let F denote the kernel of L. Then $F \trianglelefteq E$, and E/F is cyclic, since it is isomorphic to a multiplicative subgroup of the field $\mathrm{End}_{KE}(L)$. (Note that $\mathrm{End}_{KE}(L)$ is a commutative field because $L \cong (KE)/M$ for some maximal ideal M of KE and $\mathrm{End}_{KE}(L) \cong (KE)/M$ as a K-algebra.) In particular F

is non-trivial because E is not cyclic. Note also that the cyclic group E/F has square-free order, because E is a product of elementary abelian groups.

Set $H = N_G(F)$. Then $I \leq H$, and $V = \mathrm{Ind}_H^G(\widetilde{W})$, where $\widetilde{W} = \mathrm{Ind}_I^H(\widetilde{L})$. Then $\widetilde{W}$ is a simple KH-module, since for any proper KH-submodule $\widetilde{W'}$ of $\widetilde{W}$, the induced module $\mathrm{Ind}_H^G(\widetilde{W'})$ is a proper KG-submodule of V, hence equal to $\{0\}$ because V is simple. Thus $\widetilde{W'} = \{0\}$. Moreover F acts trivially on $\widetilde{W}$, for F acts trivially on $\widetilde{L}$ and $F \trianglelefteq H$. Setting $W = \mathrm{Def}_{H/F}^H(\widetilde{W})$, we obtain that

$$V \cong \mathrm{Indinf}_{H/F}^G(W)\,.$$

Since F is non-trivial, the induction hypothesis implies that there exists a genetic subgroup T/F of H/F and a faithful simple $K\big(N_{H/F}(T/F)/(T/F)\big)$-module Y such that W is obtained from Y by inflation followed by induction to H/F from the group

$$N_{H/F}(T/F)/(T/F) \cong N_H(T)/T\,.$$

In other words

$$W \cong \mathrm{Indinf}_{N_H(T)/T}^{H/F}(Y)\,.$$

It follows that

$$\widetilde{W} \cong \mathrm{Indinf}_{N_H(T)/T}^{H}(Y)\,,$$

and that

$$V \cong \mathrm{Indinf}_{N_H(T)/T}^{G}(Y)\,.$$

7.4. Claim. *The following conditions hold :*

(a) $E \cap T = F$;

(b) $E \leq N_G(T)$;

(c) $N_G(T) \leq H$, that is, $N_G(T) = N_H(T)$;

(d) The characteristic of K does not divide $|E/F|$.

The kernel of $\widetilde{W}$ is equal to the intersection of the H-conjugates of the kernel of $\mathrm{Inf}_{N_H(T)/T}^{N_H(T)}(Y)$, which is equal to T since Y is a faithful $K\big(N_H(T)/T\big)$-module. Thus

$$\mathrm{Ker}(\widetilde{W}) = \bigcap_{h\in H} {}^hT\,.$$

It follows that

$$E \cap \mathrm{Ker}(\widetilde{W}) = \bigcap_{h\in H} (E \cap {}^hT) = \bigcap_{h\in H} {}^h(E\cap T)\,.$$

Now the group $(E\cap T)/F$ is a subgroup of the cyclic group E/F, hence it is a characteristic subgroup. Since $H = N_G(F)$, it follows that H normalizes $E\cap T$. Thus

$$E \cap \mathrm{Ker}(\widetilde{W}) = E\cap T\,.$$

On the other hand $\widetilde{W} = \mathrm{Ind}_I^H(\widetilde{L})$, so $\mathrm{Ker}(\widetilde{W})$ is the intersection of the H-conjugates of the kernel of $\widetilde{L}$. Thus

$$E \cap \mathrm{Ker}(\widetilde{W}) = \bigcap_{h\in H} \big(E \cap {}^h\mathrm{Ker}(\widetilde{L})\big) = \bigcap_{h\in H} {}^h\big(E\cap \mathrm{Ker}(\widetilde{L})\big)\,.$$

But $E \cap \mathrm{Ker}(\widetilde{L})$ is the kernel of the restriction $\mathrm{Res}^I_E(\widetilde{L})$, which is the L-isotypic component of $\mathrm{Res}^G_E(V)$. In particular, its kernel is equal to the kernel F of L. This shows that $E \cap \mathrm{Ker}(\widetilde{W}) = \bigcap_{h\in H} {}^hF = F$. Thus we finally get $E \cap T = F$, proving (a). It follows in particular that $N_G(T) \leq N_G(F) = H$, i.e. $N_G(T) = N_H(T)$, so (c) holds.

Now we prove that the characteristic p of K does not divide $|E/F|$. This is obvious if $p = 0$. If $p > 0$, any p-subgroup of the cyclic group E/F acts trivially on any simple $K(E/F)$-module. But since E/F admits a simple faithful module over K, it follows that p does not divide $|E/F|$. Thus (d) holds.

Finally, we can apply Lemma 7.2 to the normal cyclic subgroup $N = E/F$ of H/F, the section $(N_H(T)/F, T/F)$, and the simple module

$$W \cong \mathrm{Indinf}^{H/F}_{N_H(T)/T}(Y)\,.$$

Note that condition (d), which we have just proved, is part of the assumption of the lemma, and that condition (a) implies that $(E/F) \cap (T/F) = \mathbf{1}$. Note also that $|E/F|$ is square-free as mentioned before when F was introduced. Lemma 7.2 asserts that T/F centralizes E/F and that $E/F \leq N_H(T)/F$. In particular, $E \leq N_G(T)$ and (b) holds.

7.5. Claim. *The subgroup T is a genetic subgroup of G.*

We know that T/F is a genetic subgroup of H/F. First notice that $N_G(T)/T = N_H(T)/T \cong N_{H/F}(T/F)\big/(T/F)$ is a Roquette group. So we only have to show that T is an expansive subgroup of G. Let $x \in G$ such that $x \notin N_G(T)$. Assume first that $x \in H$. Then, since T/F is an expansive subgroup of H/F, there exists a normal subgroup M/F of $N_G(T)/F = N_H(T)/F$ such that

$$T/F < M/F \leq \big({}^xT \cap N_G(T)\big)T/F\,.$$

It follows that $T < M \leq \big({}^xT \cap N_G(T)\big)T$ and M is a normal subgroup of $N_G(T)$, as required. Assume now that $x \notin H$. We have ${}^xF \neq F$ because $x \notin H$, hence $F < {}^xF{\cdot}F \leq E$ and also $T \leq {}^xF{\cdot}T \leq ET$. But by Claim 7.4, $E \leq N_G(T)$ and $E \cap T = F$, so the normal cyclic subgroup E/F of H/F is isomorphic to the normal subgroup ET/T of $N_G(T)/T$. Consequently, ${}^xF{\cdot}T/T \cong {}^xF{\cdot}F/F \neq \mathbf{1}$. Since any subgroup of a cyclic normal subgroup is also normal, ${}^xF{\cdot}T$ is a normal subgroup of $N_G(T)$, contained in $\big({}^xT \cap N_G(T)\big)T$ and containing T properly, as was to be shown. □

7.6. Remark. Using the method of this proof, we can actually prove more and we briefly indicate the additional conclusions. We write $\Sigma(G)$ for the socle of G, that is, the product of all minimal normal subgroups of G. This decomposes as $\Sigma(G) \cong \Sigma_{\mathrm{na}}(G) \times \Sigma_{\mathrm{ab}}(G)$, where $\Sigma_{\mathrm{na}}(G)$ is isomorphic to a direct product of non-abelian simple groups and $\Sigma_{\mathrm{ab}}(G)$ is isomorphic to a direct product of groups of prime order.

Now the fact that T is expansive means that, for every $x \notin N_G(T)$, there exists a subgroup $M_x \leq T$ such that ${}^x(M_x) \leq N_G(T)$ and ${}^x(M_x)T$ has an $N_G(T)$-core containing T properly. We can obtain further that M_x is normal in T and that ${}^x(M_x)T/T$ is contained in $\Sigma_{\mathrm{ab}}\big(N_G(T)/T\big)$.

On the other hand, assuming that V is faithful, we can also obtain that $\Sigma(G)$ normalizes T, that $\Sigma_{\rm na}(G)\cap T$ is a direct factor of $\Sigma_{\rm na}(G)$, and that its complement centralizes T.

All these additional properties can be realized, but they do not seem to improve in any useful way our analysis of minimal stabilizing bisets.

7.7. Remark. Suppose that G is a p-group and $K=\mathbb{Q}$. In that case, the definition of a genetic section given in [Bo2] and [Bo3] is different from the one given here and requires that the conclusions of Theorem 7.3 are satisfied. But Proposition 4.4 in [Bo3] asserts exactly that T is a genetic subgroup in the sense given here if and only if $(N_G(T),T)$ is a genetic section in the sense of [Bo2] and [Bo3].

In Theorem 7.3, the two sections appearing in the stabilizing biset are the same section $(N_G(T),T)$. It is not clear if one can always find a minimal biset stabilizing a simple module with this additional property. However, the theorem has at least the following consequence for minimal bisets stabilizing a simple module.

7.8. Corollary. *Let $U=\mathrm{Indinf}_{A/B}^G\,\mathrm{Iso}_\phi\,\mathrm{Defres}_{S/T}^G$ be a minimal (G,G)-biset stabilizing a simple KG-module V. Then S/T is a Roquette group.*

Proof : Let $W=\mathrm{Defres}_{S/T}^G(V)$. It is clear that W is a simple $K(S/T)$-module, because V is simple by assumption and $V\cong\mathrm{Indinf}_{A/B}^G\,\mathrm{Iso}_\phi(W)$. By Theorem 7.3 applied to the group S/T and the simple module W, there exists a section (Q,R) of S/T such that Q/R is a Roquette group and

$$W\cong\mathrm{Indinf}_{Q/R}^{S/T}\,\mathrm{Defres}_{Q/R}^{S/T}(W)\,.$$

It follows that V is stabilized by the biset

$$\mathrm{Indinf}_{A/B}^G\,\mathrm{Iso}_\phi\,\mathrm{Indinf}_{Q/R}^{S/T}\,\mathrm{Defres}_{Q/R}^{S/T}\,\mathrm{Defres}_{S/T}^G\cong\mathrm{Indinf}_{A'/B'}^G\,\mathrm{Iso}_{\phi'}\,\mathrm{Defres}_{Q/R}^G\,,$$

where (A',B') is the image of (Q,R) under the isomorphism ϕ and where ϕ' denotes the restriction of ϕ to Q/R. By minimality of the biset U of the statement, we must have $|Q/R|=|S/T|$, hence $Q=S$ and $R=T$. Thus S/T is a Roquette group. □

8. Further results on simple modules

We now return to the case of an arbitrary stabilizing biset, but we continue to consider simple modules. Our purpose is to obtain results on the section (A,B) when there is a minimal biset $U=\mathrm{Indinf}_{A/B}^G\,\mathrm{Iso}_\phi\,\mathrm{Defres}_{S/T}^G$ stabilizing a simple KG-module V. We first obtain a inequality of sizes.

8.1. Proposition. *Let K be a field and let $U = \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\phi \mathrm{Defres}_{S/T}^G$ be a (not necessarily minimal) biset stabilizing a simple KG-module V. Then $|A| \geq |N_G(T)|$ and in particular $|A| \geq |S|$.*

Proof : We have $\mathrm{Defres}_{N_G(T)/T}^G(V) = V_T$, the largest quotient of V with trivial action of T (viewed as a module for $N_G(T)/T$). Therefore there is a surjective homomorphism

$$\psi : \mathrm{Res}_{N_G(T)}^G(V) \longrightarrow V_T\,,$$

where V_T is viewed as a module for $N_G(T)$ by inflation. It follows that there is a non-zero homomorphism of KG-modules

$$\tilde{\psi} : V \longrightarrow \mathrm{Ind}_{N_G(T)}^G(V_T)\,,$$

and this is injective by simplicity of V. Therefore

$$\dim(V) \leq |G : N_G(T)| \dim(V_T)\,.$$

By Lemma 4.1, we also have $\dim(V) = |G : A| \dim(V_T)$ (where V_T is actually restricted to S/T, but this does not change its dimension). The result follows. □

Now we want to obtain information on the section (A, B).

8.2. Lemma. *Let K be a field and let (A, B) be a section of G.*

1. *Assume that there exists a section $(\widehat{A}, \widehat{B})$ such that $A < \widehat{A}$ and the inclusion $A \to \widehat{A}$ induces an isomorphism $A/B \cong \widehat{A}/\widehat{B}$. Then $\mathrm{Indinf}_{A/B}^G(Y)$ is not simple, for any $K(A/B)$-module Y.*
2. *Assume that there exists a $K(A/B)$-module Y such that $\mathrm{Indinf}_{A/B}^G(Y)$ is simple (hence Y is simple too). For any subgroup H of G normalized by A, we have $H \cap A \leq B$ if and only if $H \leq B$.*

Proof : (1) Let $\widehat{Y}$ be the $K(\widehat{A}/\widehat{B})$-module obtained from Y via the isomorphism $A/B \cong \widehat{A}/\widehat{B}$. Then we have

$$\mathrm{Res}_A^{\widehat{A}} \mathrm{Inf}_{\widehat{A}/\widehat{B}}^{\widehat{A}}(\widehat{Y}) \cong \mathrm{Inf}_{A/B}^A(Y)\,.$$

Therefore

$$\begin{aligned}
\mathrm{Indinf}_{A/B}^G(Y) &= \mathrm{Ind}_A^G \mathrm{Inf}_{A/B}^A(Y) \\
&\cong \mathrm{Ind}_A^G \mathrm{Res}_A^{\widehat{A}} \mathrm{Inf}_{\widehat{A}/\widehat{B}}^{\widehat{A}}(\widehat{Y}) \\
&\cong \mathrm{Ind}_{\widehat{A}}^G \mathrm{Ind}_A^{\widehat{A}} \mathrm{Res}_A^{\widehat{A}} \mathrm{Inf}_{\widehat{A}/\widehat{B}}^{\widehat{A}}(\widehat{Y}) \\
&\cong \mathrm{Ind}_{\widehat{A}}^G \big(\mathrm{Inf}_{\widehat{A}/\widehat{B}}^{\widehat{A}}(\widehat{Y}) \otimes \mathrm{Ind}_A^{\widehat{A}}(K)\big)\,.
\end{aligned}$$

The module $\mathrm{Ind}_A^{\widehat{A}}(K)$ has a trivial submodule K, and this is a proper submodule of $\mathrm{Ind}_A^{\widehat{A}}(K)$ because $A < \widehat{A}$. Tensoring with $\mathrm{Inf}_{\widehat{A}/\widehat{B}}^{\widehat{A}}(\widehat{Y})$ and then inducing to G, we see that $\mathrm{Ind}_{\widehat{A}}^G \big(\mathrm{Inf}_{\widehat{A}/\widehat{B}}^{\widehat{A}}(\widehat{Y}) \otimes \mathrm{Ind}_A^{\widehat{A}}(K)\big)$ has a proper submodule isomorphic to $\mathrm{Indinf}_{\widehat{A}/\widehat{B}}^G(\widehat{Y})$. It follows that $\mathrm{Indinf}_{A/B}^G(Y)$ is not simple.

(2) One implication is clear, so assume that $H \cap A \leq B$. We have $A \leq AH$ and H is a normal subgroup of AH because A normalizes H by assumption. Now the inclusion $A \to AH$ induces an isomorphism $A/B \cong AH/BH$ because $H \cap A \leq B$, hence $BH \cap A = B$. Since $\mathrm{Indinf}_{A/B}^{G}(Y)$ is simple, (1) implies that $A = AH$ and $B = BH$, that is $H \leq B$. □

8.3. Proposition. *Let $U = \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{S/T}^{G}$ be a (not necessarily minimal) (G,G)-biset stabilizing a simple KG-module V. Then any non-trivial subgroup of $N_G(B)/B$ normalized by A/B intersects A/B non-trivially.*

Proof : If H/B is a non-trivial subgroup of $N_G(B)/B$ normalized by A/B, then $H \not\leq B$. Since $\mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{S/T}^{G}(V) \cong V$ is simple, Lemma 8.2 applies. Therefore $H \cap A \not\leq B$, that is, $(H/B) \cap (A/B)$ is non-trivial. □

For a stablizing biset where both sections coincide, we have seen that the top subgroup of the section is necessarily the normalizer of the bottom subgroup (see Proposition 4.2). Here is another case where this happens.

8.4. Proposition. *Let $U = \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{S/T}^{G}$ be a minimal (G,G)-biset stabilizing a simple KG-module V. If B is a normal subgroup of G, then $A = G$ (that is, A is the normalizer of B).*

Proof : Let $M = \mathrm{Defres}_{S/T}^{G}(V)$. Then $V \cong \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi}(M)$ and since B is a normal subgroup of G, it must act trivially on V. Therefore $B \cap S$ acts trivially on $\mathrm{Res}_{S}^{G}(V)$ and we have equality of the coinvariants

$$\mathrm{Res}_{S}^{G}(V)_T = \mathrm{Res}_{S}^{G}(V)_{(B\cap S)T}\,.$$

By definition, the left hand side is $M = \mathrm{Defres}_{S/T}^{G}(V)$ and so $(B \cap S)T/T$ acts trivially on M. But M is a faithful $K(S/T)$-module by Proposition 4.3, using the minimality of U. It follows that $(B \cap S)T/T$ is trivial, so that $B \cap S = B \cap T$.

Now we have $BT \cap S = (B \cap S)T = (B \cap T)T = T$ and therefore the inclusion $S \to BS$ induces an isomorphism $\alpha : S/T \to BS/BT$. Moreover, since B acts trivially on V, we have $V_{BT} = V_T$, hence

$$\mathrm{Defres}_{BS/BT}^{G}(V) \cong \mathrm{Iso}_{\alpha}\mathrm{Defres}_{S/T}^{G}(V)\,.$$

It follows that V is stabilized by the biset

$$U' = \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\psi} \mathrm{Defres}_{BS/BT}^{G}\,,$$

where $\psi = \phi\alpha^{-1}$. By Corollary 3.5, V is also stabilized by the biset

$$\widetilde{U} = \mathrm{Indinf}_{{}^{g}A/B}^{G} \mathrm{Iso}_{\sigma} \mathrm{Defres}_{BS/BT}^{G}\,,$$

where $({}^{g}A, B) = ({}^{g}A, {}^{g}B)$ is linked to (BS, BT) and $\sigma : BS/BT \to {}^{g}A/B$ is the isomorphism corresponding to the linking. But since BT contains B, the isomorphism of the linking $\sigma^{-1} : {}^{g}A/B \to BS/BT$ is induced by an inclusion ${}^{g}A \to BS$. It follows that $BT \cap {}^{g}A = B$.

Now we apply Proposition 8.3. The subgroup BT/B is normalized by ${}^{g}A/B$, because ${}^{g}A \leq BS$. Since BT/B intersects ${}^{g}A/B$ trivially, we must have $BT/B = \mathbf{1}$, hence $BT = B$. Finally, by Lemma 4.1, we have

$$\dim(V) = |G : {}^{g}A| \cdot \dim(V_{BT}) = |G : {}^{g}A| \cdot \dim(V_B) = |G : {}^{g}A| \cdot \dim(V)$$

because B acts trivially on V. Therefore $|G : {}^{g}A| = 1$, hence $A = G$. □

We end this section with an easy observation which is in the same vein as Lemma 8.2.

8.5. Proposition. *Let $U = \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{S/T}^{G}$ be a (G,G)-biset stabilizing a simple KG-module V and let $M = \mathrm{Iso}_{\phi} \mathrm{Defres}_{S/T}^{G}(V)$. If M is the trivial $K(A/B)$-module, then V is the trivial KG-module and $A = G$.*

Proof : We have $V \cong \mathrm{Ind}_A^G(K)$, where K denotes the trivial module of the group A (inflated from M). Since the trivial KG-module is always a submodule of $\mathrm{Ind}_A^G(K)$, this module can be simple only if $A = G$, and then V is the trivial module. □

9. p-groups in coprime characteristic

Suppose that G is a p-group and K is a field of characteristic different from p. In that case, we show that the stabilizing biset obtained in Theorem 7.3 is minimal. In fact, we recover one of the main results obtained by the first author [Bo2] when $K = \mathbb{Q}$ and generalized by Barker [Ba] when K has characteristic 0.

An important ingredient is the classification of all Roquette p-groups, which we first recall.

9.1. Lemma. *Let p be a prime and let P be a Roquette p-group of order p^n.*

1. *If p is odd, then P is cyclic.*
2. *If $p = 2$, then P is cyclic, generalized quaternion (with $n \geq 3$), dihedral with $n \geq 4$, or semi-dihedral (with $n \geq 4$).*
3. *If P is cyclic or generalized quaternion, there is a unique subgroup Z of order p. Any non-trivial subgroup contains Z.*
4. *If P is dihedral and $Z = Z(P)$, then any non-trivial subgroup contains Z, except for two conjugacy classes of non-central subgroups of order 2. If T is a non-central subgroup of order 2, then $S = N_P(T) = TZ$ is a Klein 4-group and $N_P(S)$ is a (dihedral) group of order 8.*
5. *If P is semi-dihedral and $Z = Z(P)$, then any non-trivial subgroup contains Z, except for one conjugacy class of non-central subgroups of order 2. If T is a non-central subgroup of order 2, then $S = N_P(T) = TZ$ is a Klein 4-group and $N_P(S)$ is a (dihedral) group of order 8.*

Proof : See Chapter 5, Section 4, in [Go]. □

We have seen in Corollary 7.8 that a minimal biset stabilizing a simple module factors through a subquotient which is a Roquette group. We now show conversely that Roquette groups are minimal for p-groups in characteristic different from p.

9.2. Theorem. *Let p be a prime, let P be a Roquette p-group, let K be a field of characteristic different from p, and let Y be a simple faithful KP-module. If Y is stabilized by a biset $U = \mathrm{Indinf}_{A/B}^{P}\,\mathrm{Iso}_{\phi}\,\mathrm{Defres}_{S/T}^{P}$, then $(A,B) = (S,T) = (P,\mathbf{1})$.*

Proof : There is nothing to prove if $P = \mathbf{1}$. We now assume that $P \neq \mathbf{1}$, so that Y is non-trivial. Let $M = \mathrm{Iso}_{\phi}\,\mathrm{Defres}_{S/T}^{P}(Y)$. Then $Y \cong \mathrm{Indinf}_{A/B}^{P}(M)$, and M is a simple $K(A/B)$-module. Moreover M is non-trivial, by Proposition 8.5.

Since B acts trivially on M, the group $B \cap Z(P)$ acts trivially on Y, thus $B \cap Z(P) = \mathbf{1}$ because Y is faithful. It follows from Lemma 9.1 that B is trivial, except possibly if $p = 2$, P is dihedral or semi-dihedral, and B is a non-central subgroup of order 2.

Similarly, setting $Z = T \cap Z(P)$, we have

$$\mathrm{Defres}_{S/T}^{P/Z}\,\mathrm{Def}_{P/Z}^{P}(Y) = \mathrm{Defres}_{S/T}^{P}(Y) = \mathrm{Iso}_{\phi^{-1}}(M) \neq \{0\}\,,$$

and therefore $Y_Z = \mathrm{Def}_{P/Z}^{P}(Y) \neq \{0\}$. But Y_Z is a quotient of Y and Z acts trivially on Y_Z. Since Y is simple and faithful, it follows that $Z = \mathbf{1}$. Thus $T = \mathbf{1}$, except possibly if $p = 2$, P is dihedral or semidihedral, and T is a non-central subgroup of order 2.

Assume first that $T = \mathbf{1}$. In this case $\dim_K(Y) = |P : A| \dim_K(Y)$ by Lemma 4.1, thus $A = P$. Therefore B is a normal subgroup of P and it cannot be a non-central subgroup of order 2. It follows that $B = \mathbf{1}$, and $(A,B) = (P,\mathbf{1})$. Since $S/T \cong A/B$, we also have $(S,T) = (P,\mathbf{1})$.

So we can assume that T is non-trivial and we need to show that this case is impossible. We have $p = 2$, P is dihedral or semidihedral, and T is a non central subgroup of order 2. Note that P has order at least 16, because the dihedral group of order 8 is not Roquette. Moreover, we have $N_P(T) = TZ$, where Z is the center of P, of order 2. Thus $S = T$ or S/T has order 2. But the first case is impossible because $S/T \cong A/B$ and M is a non-trivial $K(A/B)$-module. Hence S/T has order 2 and $S = N_P(T) = TZ$. Moreover A/B has order 2 as well and M must be the sign representation of A/B.

If $B = \mathbf{1}$, then A has order 2 and is necessarily contained in some Klein 4-group V. Now $Y \cong \mathrm{Ind}_A^P(M)$ is simple, hence $\mathrm{Ind}_A^V(M)$ is simple too. But this is impossible, because $\mathrm{Ind}_A^V(M)$ is a direct sum of two one-dimensional modules. Thus we can assume that B is a non-central subgroup of order 2. It follows that $A = N_P(B) = BZ$.

If B and T are conjugate, then in the biset U we can insert a conjugation and replace (A,B) by (S,T). Thus we can assume that $(A,B) = (S,T)$. We know from Theorem 3.3 that for any $g \notin S$, the section (S^g, T^g) cannot be linked to (S,T). Since $N_P(S)$ is (dihedral) of order 8 and P has order at least 16, we can choose $g \notin N_P(S)$. But then $S^g \cap S = Z$ (because $S^g \neq S$) and $T^g \cap T = \mathbf{1}$, so that (S^g, T^g) is linked to (S,T), a contradiction.

If B and T are not conjugate (so that in fact P must be dihedral), then $A \cap S = BZ \cap TZ = Z$ and $B \cap T = \mathbf{1}$ and we see that (A,B) is linked to (S,T). Now the double coset AS has cardinality 8 and we can choose $g \notin AS$. Then $B^g \cap T = \mathbf{1}$ and $A^g \cap S = B^g Z \cap TZ = Z$, so (A^g, B^g) is still linked to (S,T). This contradicts again Theorem 3.3. □

Now we come to the main result of this section, proved by the first author [Bo2] when $K = \mathbb{Q}$ and generalized by Barker [Ba] when K has characteristic 0.

9.3. Theorem. *Let p be a prime, let G be a finite p-group, let K be a field of characteristic different from p, and let V be a simple KG-module. There exists a genetic subgroup T of G such that the biset $U = \mathrm{Indinf}_{S/T}^G \mathrm{Defres}_{S/T}^G$ is a minimal biset stabilizing V, where $S = N_G(T)$. Moreover, if $Y = \mathrm{Defres}_{S/T}^G(V)$, then $\mathrm{End}_{KG}(V) \cong \mathrm{End}_{K(S/T)}(Y)$ as K-algebras.*

Proof : All the statements follow immediately from Theorem 7.3, except the minimality of U. Note in particular that Y is a faithful simple $K(S/T)$-module.

Let $U' = \mathrm{Indinf}_{A'/B'}^G \mathrm{Iso}_{\phi'} \mathrm{Defres}_{S'/T'}^G$ be a minimal biset stabilizing V. By minimality of U', we have $|S'/T'| \leq |S/T|$. Moreover, we have

$$\begin{aligned} Y &= \mathrm{Defres}_{S/T}^G(V) \cong \mathrm{Defres}_{S/T}^G U'(V) \\ &\cong \mathrm{Defres}_{S/T}^G U' \,\mathrm{Indinf}_{S/T}^G \mathrm{Defres}_{S/T}^G(V) \\ &\cong \mathrm{Defres}_{S/T}^G U' \,\mathrm{Indinf}_{S/T}^G(Y) \end{aligned}$$

and therefore $W := \mathrm{Defres}_{S/T}^G U' \,\mathrm{Indinf}_{S/T}^G$ is an $(S/T, S/T)$-biset stabilizing Y. Then W decomposes as a disjoint union of transitive $(S/T, S/T)$-bisets and one of them, say W_1, stabilizes Y (by indecomposability of Y). Moreover, since S'/T' is the subquotient corresponding to U' and since W factorizes through U', the subquotient S''/T'' corresponding to W_1 must be isomorphic to a subquotient of S'/T' (by applying the generalized Mackey formula). Thus we obtain $|S''/T''| \leq |S'/T'| \leq |S/T|$.

Now Theorem 9.2 asserts that the faithful simple module Y for the Roquette group S/T cannot be stabilized by a $(S/T, S/T)$-biset whose corresponding subquotient has cardinality strictly smaller than S/T. Thus $|S''/T''| = |S/T|$ and it follows that $|S'/T'| = |S/T|$. This shows that U is also a minimal biset stabilizing V. □

10. p-groups in characteristic p

Specific results can be proved involving a p-group in characteristic p. They are based on the following well-known phenomenon.

10.1. Lemma. *Let K be a field of characteristic p and let (S, P) be a section of G such that P is a p-subgroup. For any non-zero KG-module W, the $K(S/P)$-module $\mathrm{Defres}_{S/P}^G(W)$ is non-zero.*

Proof : Since the trivial module is the only simple KP-module, $\mathrm{Res}_P^G(W)$ must have a non-zero quotient with trivial action. In other words, $W_P \neq \{0\}$, hence $\mathrm{Defres}_{S/P}^G(W) \neq \{0\}$. □

By Corollary 3.5 (see also Remark 3.6), we can always assume that a minimal stabilizing biset has the form $U = \mathrm{Indinf}_{A/B}^P \mathrm{Iso}_\sigma \mathrm{Defres}_{S/T}^P$, where the two sections are linked and where $\sigma : S/T \to A/B$ is the isomorphism induced by the linking. We make this harmless assumption in the following result, which describes completely what happens with p-groups in characteristic p.

10.2. Proposition. *Let K be a field of characteristic p and let P be a p-group. Let $U = \mathrm{Indinf}_{A/B}^P \, \mathrm{Iso}_\sigma \, \mathrm{Defres}_{S/T}^P$ be a (P,P)-biset where the two sections are linked and where $\sigma : S/T \to A/B$ is the isomorphism induced by the linking.*

1. *If U is a minimal (P,P)-biset stabilizing an indecomposable KP-module L, then U is idempotent. In other words $SA = P$.*
2. *Suppose that $SA = P$, so that U is an idempotent biset. Let M be any $K(S/T)$-module and let $L = \mathrm{Indinf}_{A/B}^P \, \mathrm{Iso}_\sigma(M)$. Then $\mathrm{Defres}_{S/T}^P(L) \cong M$ and L is stabilized by U. Moreover, if K is algebraically closed, then L is indecomposable if and only if M is indecomposable.*

Proof : (1) By Corollary 3.4, there is a unique double coset SgA such that

$$\mathrm{Btf}\,(S,T,\,{}^gA,\,{}^gB)\,\mathrm{Conj}_g\,\mathrm{Iso}_\sigma(M) \neq \{0\}\,,$$

where $M = \mathrm{Defres}_{S/T}^P(L)$. But for any $h \in P$, we have

$$\mathrm{Btf}\,(S,T,\,{}^hA,\,{}^hB)\,\mathrm{Conj}_h\,\mathrm{Iso}_\sigma(M) \neq \{0\}\,,$$

because the deflation involved in $\mathrm{Btf}\,(S,T,\,{}^hA,\,{}^hB)$ does not annihilate any non-zero module, by Lemma 10.1. Therefore there is a unique (S,A)-double coset, that is, $SA = P$. By Corollary 5.2, this means that U is idempotent.

(2) Now we assume that $SA = P$ and that $L = \mathrm{Indinf}_{A/B}^P \, \mathrm{Iso}_\sigma(M)$. Then

$$\begin{aligned}
\mathrm{Defres}_{S/T}^P(L) &= \mathrm{Defres}_{S/T}^P \, \mathrm{Indinf}_{A/B}^P \, \mathrm{Iso}_\sigma(M) \\
&\cong \bigcup_{g\in[S\backslash P/A]} \mathrm{Btf}\,(S,T,\,{}^gA,\,{}^gB)\,\mathrm{Conj}_g\,\mathrm{Iso}_\sigma(M) \\
&\cong \mathrm{Btf}\,(S,T,A,B)\,\mathrm{Iso}_\sigma(M) \\
&\cong \mathrm{Iso}_{\sigma^{-1}}\,\mathrm{Iso}_\sigma(M) = M
\end{aligned}$$

and it follows that L is stabilized by U.

If $M = M_1 \oplus M_2$, then $L = \mathrm{Indinf}_{A/B}^P \, \mathrm{Iso}_\sigma(M_1) \oplus \mathrm{Indinf}_{A/B}^P \, \mathrm{Iso}_\sigma(M_2)$. Conversely, if M is indecomposable, then so is $L = \mathrm{Indinf}_{A/B}^P \, \mathrm{Iso}_\sigma(M)$ by Green's indecomposability theorem (which applies when K is algebraically closed). □

We know that stabilizing bisets occur with expansive subgroups (see Proposition 6.1 and Corollary 6.3) and this also has some relevance for p-groups in characteristic p. We show that, for an arbitrary finite group G, very few p-subgroups can be expansive.

10.3. Proposition. *Let K be a field of characteristic p and let P be a p-subgroup of G. Assume that there exists a faithful simple $K(N_G(P)/P)$-module. If P is an expansive subgroup of G, then $P = O_p(G)$.*

Proof : Let $S = N_G(P)$ and let M be a faithful simple $K(S/P)$-module. Suppose there exists $g \notin S$. Since P is expansive, the S-core N of the subgroup $(P^g \cap S)P$ contains P properly. Thus N/P is a non-trivial normal p-subgroup of S/P. By Lemma 10.1, $\mathrm{Def}_{S/N}^{S/P}(M) \neq \{0\}$, but the simple faithful module M cannot have a non-zero quotient with trivial action of N/P. Therefore g does not exist and so $S = G$, that is, P is a normal subgroup of G. Again the normal p-subgroup $O_p(G)/P$ acts trivially on M and since M is faithful, we must have $O_p(G) = P$. □

We know that any normal subgroup is always an expansive subgroup. Proposition 10.3 shows that the converse may happen under suitable hypotheses.

11. Examples

We illustrate various results of this paper by means of a few examples. They also allow us to answer some obvious questions. We first start with an easy case.

11.1. Example. Suppose that G is abelian. Let $U = \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\sigma} \mathrm{Defres}_{S/T}^{G}$ be a minimal (G, G)-biset stabilizing an indecomposable module L. By Corollary 3.5, we can assume that $\sigma : S/T \to A/B$ is induced by the linking between A/B and S/T. By Corollary 3.4, there is a unique double coset SgA such that

$$\mathrm{Btf}\,(S, T, {}^{g}\!A, {}^{g}\!B)\,\mathrm{Iso}_{\phi}\,\mathrm{Conj}_{g}(M) \neq \{0\}\,,$$

where $M = \mathrm{Defres}_{S/T}^{G}(L)$. But $\mathrm{Btf}\,(S, T, {}^{g}\!A, {}^{g}\!B) = \mathrm{Btf}\,(S, T, A, B)$ for all g, so there is a unique double coset $SA = G$ and it follows that U is idempotent. If we assume that L is simple, then we must have $A = G$ by Proposition 8.4 and so L is just inflated from G/B. Since the two sections are linked, we have $G/T \cong B/T \times S/T$ and B/T acts trivially. Hence the stabilizing biset is rather trivial.

If we work now with an indecomposable module in characteristic p and assume for simplicity that our abelian group G is a p-group, then the situation is fully described in Proposition 9.3.

11.2. Example. Let $G = S_3$ be the symmetric group on 3 letters, let C_3 be its normal subgroup of order 3, and let A be a subgroup of order 2. Then (S_3, C_3) is linked to $(A, \mathbf{1})$, via an isomorphism $\sigma : S_3/C_3 \to A$. Let K be a field of characteristic 3 and consider the indecomposable projective module $L = \mathrm{Ind}_{A}^{S_3}(M)$, where M is the sign representation of A. Then L is stabilized by the idempotent biset

$$U = \mathrm{Ind}_{A}^{S_3} \mathrm{Iso}_{\sigma} \mathrm{Def}_{S_3/C_3}^{S_3}\,,$$

which is easily seen to be minimal. This shows that Proposition 8.4 does not hold for non-simple modules, since here $B = \mathbf{1}$, but A is not equal to S_3. Also we have $|A| < |S_3|$, so we see that Proposition 8.1 does not hold for non-simple modules.

For simple modules, we often have minimal stabilizing bisets of the form $\mathrm{Indinf}_{S/T}^{G} \mathrm{Defres}_{S/T}^{G}$ (in particular for p-groups as in Section 9 and in several examples below), but we don't know if this happens or not in general. This certainly need not happen for non-simple modules, because this example shows that the two sections in any minimal biset stabilizing L are bound to be different.

If we consider the same example, but over a field of characteristic 0, then $L = \mathrm{Ind}_{A}^{S_3}(M)$ decomposes as $L = L_1 \oplus L_2$, where L_1 is the sign representation and L_2 is the two-dimensional simple module. Then $\mathrm{Def}_{S_3/C_3}^{S_3}(L_1) = M$ and $\mathrm{Def}_{S_3/C_3}^{S_3}(L_2) = \{0\}$. We see here that the idempotent biset U stabilizes a decomposable module L, but neither L_1 nor L_2 is stabilized by U.

11.3. Example. By Theorem 9.2, we know that Roquette p-groups are "minimal" for simple faithful modules. This does not hold anymore for solvable groups, as the following example shows. Let $G = GL_2(\mathbb{F}_3) \cong Q_8 \rtimes S_3$, where Q_8 denotes the quaternion group of order 8, and let $Z = Z(Q_8) = Z(G)$. Then G is Roquette and one can check that the subgroup S_3 is expansive. Now $N_G(S_3) = Z \times S_3$, so $N_G(S_3)/S_3 \cong Z$ is Roquette and S_3 is a genetic subgroup of G. Let Y be the sign representation of $(Z \times S_3)/S_3$ (over a field of characteristic different from 2 and 3) and let $V = \mathrm{Indinf}^G_{(Z\times S_3)/S_3}(Y)$. Then V is a 4-dimensional simple module and it is stabilized by $\mathrm{Indinf}^G_{(Z\times S_3)/S_3} \mathrm{Defres}^G_{(Z\times S_3)/S_3}$, by Corollary 6.3. Hence the group G is Roquette but is not minimal.

11.4. Example. Let $n \geq 2$, $1 \leq k \leq n-1$, and $q \geq 3$ a prime power. Let T be the subgroup of $G = \mathrm{GL}_n(\mathbb{F}_q)$ defined by

$$T = \begin{pmatrix} \mathrm{SL}_k(\mathbb{F}_q) & \mathrm{M}_{k,n-k}(\mathbb{F}_q) \\ 0 & \mathrm{GL}_{n-k}(\mathbb{F}_q) \end{pmatrix}.$$

We claim that T is a genetic subgroup of G, with $N_G(T)/T$ cyclic. Therefore, if we let Y be any faithful simple $K(S/T)$-module, where $S = N_G(T)$ and K is a field of characteristic 0 (for simplicity), then by Corollary 6.3 the KG-module $V = \mathrm{Indinf}^G_{S/T}(Y)$ is simple and stabilized by the biset $\mathrm{Indinf}^G_{S/T} \mathrm{Defres}^G_{S/T}$, which is clearly minimal since Y cannot be obtained from a proper subsection of S/T.

It is easy to see that the normaliser of T is the subgroup

$$S = N_G(T) = \begin{pmatrix} \mathrm{GL}_k(\mathbb{F}_q) & \mathrm{M}_{k,n-k}(\mathbb{F}_q) \\ 0 & \mathrm{GL}_{n-k}(\mathbb{F}_q) \end{pmatrix}.$$

In other words, S is the stabilizer of the subspace V generated by $\{v_1, \cdots, v_k\}$, where $\{v_1, \cdots, v_n\}$ is the canonical basis of $E = \mathbb{F}_q^n$. The quotient S/T is isomorphic to $\mathbb{F}_q^\times$, hence cyclic of order $q-1$.

We are left with the proof that T is an expansive subgroup of G. If $g \in G$, the S-core of $({}^gT \cap S)T$ is the subgroup $({}^gT \cap S)T$ itself because S/T is cyclic. We have to prove that ${}^gT = T$ whenever ${}^gT \cap S \leq T$.

The subgroup ${}^gT$ is contained in the stabilizer of $g(V)$. We choose another basis $\{w_1, \cdots, w_n\}$ of E such that $\{w_1, \cdots, w_h\}$ is a basis of $g(V) \cap V$ (for some $h \leq k$), $\{w_1, \cdots, w_k\}$ is a basis of V, and $\{w_1, \cdots, w_{2k-h}\}$ is a basis of $g(V)+V$. The subgroup ${}^gS \cap S$ is the intersection of the stabilizers of V and $g(V)$. With respect to this new basis, an element $x \in {}^gS \cap S$ can be written in the form

$$x = \begin{pmatrix} A & X & Y & Z \\ 0 & B & 0 & T \\ 0 & 0 & C & U \\ 0 & 0 & 0 & D \end{pmatrix},$$

where A, B, C, D are invertible square matrices of size h, $k-h$, $k-h$, $n-2k+h$ respectively. The action of x on $g(V)$ is given by the matrix

$$\begin{pmatrix} A & Y \\ 0 & C \end{pmatrix},$$

hence $x \in {}^gT$ if and only if $\det(A)\cdot\det(C) = 1$. On the other hand $x \in T$ if and only if $\det(A)\cdot\det(B) = 1$. The assumption ${}^gT \cap S \leq T$ means that

$$\det(A)\cdot\det(C) = 1 \;\Rightarrow\; \det(A)\cdot\det(B) = 1\,.$$

Since we can choose freely B, this implication can hold only if B is an empty matrix, i.e. if $k = h$. In that case $g(V) = V$, i.e. $g \in S = N_G(T)$. This proves that T is an expansive subgroup of G.

11.5. Example. For simplicity, let $K = \mathbb{C}$. We give a few examples of a simple group G with an expansive subgroup T of index 2 in its normalizer S (hence genetic). In each case, we take the sign representation Y of S/T and we let $V = \mathrm{Indinf}_{S/T}^G(Y)$. Then V is a simple KG-module, $Y \cong \mathrm{Defres}_{S/T}^G(V)$ and V is stabilized by $\mathrm{Indinf}_{S/T}^G \mathrm{Defres}_{S/T}^G$, by Corollary 6.3.

$G = PSL_2(\mathbb{F}_7)$, $S = S_4$, $T = A_4$.
$G = M_{11}$, $S = M_{10}$, and $T = A_6$ of index 2 in S.
$G = M_{11}$, S of order 144, and T of order 72.
$G = A_8$, S of order 576, and T of order 288.

However, no non-trivial expansive subgroup exists in A_5, A_6, A_7, $PSL_2(\mathbb{F}_{11})$, so no such example can occur.

11.6. Example. Idempotent bisets also occur in simple groups. For instance, let q be a power of 2 and consider the group $G = Sp_4(\mathbb{F}_q)$. Then G has a subgroup $B \cong \Omega_4^-(q) \cong SL_2(\mathbb{F}_{q^2})$ of index 2 in its normalizer $A \cong O_4^-(q)$, as well as a subgroup $T \cong Sp_2(\mathbb{F}_{q^2}) \cong SL_2(\mathbb{F}_{q^2})$ of index 2 in its normalizer S. By Theorem A in [LPS], this is an example of a factorization $G = SA$. Moreover, the sections (A, B) and (S, T) are linked (with $B \cap T$ dihedral of order $2(q^2+1)$), so we obtain an idempotent biset $U = \mathrm{Indinf}_{A/B}^G \mathrm{Iso}_\sigma \mathrm{Defres}_{S/T}^G$ where $\sigma : S/T \to A/B$ is the isomorphism induced by the linking. If Y is the sign representation of A/B and $V = \mathrm{Indinf}_{A/B}^G(Y)$, then V is stabilized by U. However, V is not simple.

11.7. Example. Both types of bisets studied in this paper can occur simultaneously. Let the p-group P be either dihedral of order 8 or extraspecial of order p^3 and exponent p with p odd. Let T be a non-central subgroup of order p and let S be its normalizer, hence $S = T \times Z$ where $Z = Z(P)$. Then T is easily seen to be expansive, hence genetic because S/T is cyclic. If Y is a one-dimensional faithful representation of S/T (in characteristic prime to p), then $V = \mathrm{Indinf}_{S/T}^P(Y)$ is simple and stabilized by $U = \mathrm{Indinf}_{S/T}^P \mathrm{Defres}_{S/T}^P$. On the other hand, we can also choose another subgroup B of order p, not conjugate to T, and let $A = N_P(B) = B \times Z$. Then $SA = P$ and the sections (A, B) and (S, T) are linked, so we obtain an idempotent biset $U' = \mathrm{Indinf}_{A/B}^P \mathrm{Iso}_\sigma \mathrm{Defres}_{S/T}^P$ where $\sigma : S/T \to A/B$ is the isomorphism induced by the linking. The simple module V is also stabilized by U', and both U and U' are minimal. Our main uniqueness theorem applies of course and tells us that the sections in U are linked to those in U', which is obvious in this case. But it should be emphasized that one of the bisets is obtained from an expansive subgroup, while the other is idempotent.

References

[Ba] L. Barker. Genotypes of irreducible representations of finite p-groups, *J. Algebra* 306 (2007), 655–681.

[Bo1] S. Bouc. Foncteurs d'ensembles munis d'une double action, *J. Algebra* 183 (1996), 664–736.

[Bo2] S. Bouc. The functor of rational representations for p-groups, *Adv. Math.* 186 (2004), 267–306.

[Bo3] S. Bouc. Biset functors and genetic sections for p-groups, *J. Algebra* 284 (2005), 179–202.

[Bo4] S. Bouc. Biset functors for finite groups, Springer Lecture Notes in Mathematics no. 1990 (2010).

[BoTh1] S. Bouc, J. Thévenaz. The group of endo-permutation modules, *Invent. Math.* 139 (2000), 275–349.

[BoTh2] S. Bouc, J. Thévenaz. Gluing torsion endo-permutation modules, *J. London Math. Soc.* 78 (2008), 477–501.

[DiDu] R. Dipper, J. Du. Harish-Chandra vertices and Steinberg's tensor product theorems for finite general linear groups, *Proc. London Math. Soc.* (3) 75 (1997), 559–599.

[Go] D. Gorenstein. Finite groups, Harper and Row, New-York, 1968.

[La] S. Lang. Algebra, Addison-Wesley, Reading Massachusetts, London, Sydney, 1965.

[LPS] M.W. Liebeck, C.E. Praeger, J. Saxl. The maximal factorizations of the finite simple groups and their automorphism groups, *Mem. Amer. Math. Soc.* 86 (1990), no. 432.

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