The primitive idempotents of the p-permutation ring

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Abstract : Let G be a finite group, let p be a prime number, and let K be a field of characteristic 0 and k be a field of characteristic p, both large enough. In this note we state explicit formulae for the primitive idempotents of $K \otimes_{\mathbb{Z}} pp_k(G)$, where $pp_k(G)$ is the ring of p-permutation kG-modules.

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1. Introduction

Let G be a finite group, let p be a prime number, and let K be a field of characteristic 0 and k be a field of characteristic p, both large enough. In this note we state explicit formulae for the primitive idempotents of $K \otimes_{\mathbb{Z}} pp_k(G)$, where $pp_k(G)$ is the ring of p-permutation kG-modules (also called the trivial source ring).

To obtain these formulae, we first use induction and restriction to reduce to the case where G is cyclic modulo p, i.e. G has a normal Sylow p-subgroup with cyclic quotient. Then we solve the easy and well known case where G is a cyclic p'-group. Finally we conclude by using the natural ring homomomorphism from the Burnside ring $B(G)$ of G to $pp_k(G)$, and the classical formulae for the primitive idempotents of $K \otimes_{\mathbb{Z}} B(G)$.

Our formulae are an essential tool in [2], where Cartan matrices of Mackey algebras are considered, and some invariants of these matrices (determinant, rank) are explicitly computed.

2. p-permutation modules

2.1. Notation.

• Throughout the paper, G will be a fixed finite group and p a fixed prime number. We consider a field k of characteristic p and we denote by kG the group algebra of G over k. We assume that k is large enough in the sense that it is a splitting field for every group algebra $k\big(N_G(P)/P\big)$, where P runs through the set of all p-subgroups of G.

• We let K be a field of characteristic 0 and we assume that K is large enough in the sense that it contains the values of all the Brauer characters of the groups $N_G(P)/P$, where P runs through the set of all p-subgroups of G.

We recall quickly how Brauer characters are defined. We let $\overline{k}$ be an extension of k containing all the n-th roots of unity, where n is the p'-part of the exponent of G. We choose an isomorphism $\theta : \mu_n(\overline{k}) \to \mu_n(\mathbb{C})$ from the group of n-th roots of unity in $\overline{k}$ and the corresponding group in $\mathbb{C}$. If V is an r-dimensional kH-module for the group $H = N_G(P)/P$ and if s is an element of the set $H_{p'}$ of all p'-elements of H, the matrix of the action of s on V has eigenvalues $(\lambda_1, \ldots, \lambda_r)$ in the group $\mu_n(\overline{k})$. The *Brauer character* ϕ_V of V is the central function defined on $H_{p'}$, with values in the field $\mathbb{Q}[\mu_n(\mathbb{C})]$, sending s to $\sum_{i=1}^{r} \theta(\lambda_i)$. The actual values of Brauer characters may lie in a subfield of $\mathbb{Q}[\mu_n(\mathbb{C})]$ and we simply require that K contains all these values.

2.2. Remark : Let V be as above and let W be a t-dimensional kH-module. If s has eigenvalues $(\mu_1, \ldots, \mu_t)$ on W, its eigenvalues for the diagonal action of H on $V \otimes_k W$ are $(\lambda_i \mu_j)_{1 \leq i \leq r,\, 1 \leq j \leq t}$. It follows that $\phi_{V \otimes_k W}(s) = \sum_{i=1}^{r} \sum_{j=1}^{t} \theta(\lambda_i \mu_j) = \phi_V(s)\phi_W(s)$.

• When H is a subgroup of G, and M is a kG-module, we denote by $\mathrm{Res}_H^G M$ the kH-module obtained by restricting the action of G to H. When L is a kH-module, we denote by $\mathrm{Ind}_H^G L$ the induced kG-module.

• When M is a kG-module, and P is a subgroup of G, the k-vector space of fixed points of P on M is denoted by M^P. When $Q \leq P$ are subgroups of G, the *relative trace* is the map $\mathrm{tr}_Q^P : M^Q \to M^P$ defined by $\mathrm{tr}_Q^P(m) = \sum_{x \in [P/Q]} x \cdot m$.

• When M is a kG-module, the *Brauer quotient* of M at P is the k-vector space

$$M[P] = M^P / \sum_{Q<P} \mathrm{tr}_Q^P M^Q \ .$$

This k-vector space has a natural structure of $k\overline{N}_G(P)$-module, where as usual $\overline{N}_G(P) = N_G(P)/P$. It is equal to zero if P is not a p-group.

• If P is a normal p-subgroup of G and M is a $k(G/P)$-module, denote by $\mathrm{Inf}_{G/P}^G M$ the kG-module obtained from M by inflation to G. Then there is an isomorphism

$$(\mathrm{Inf}_{G/P}^G M)[P] \cong M$$

of $k(G/P)$-modules.

• When G acts on a set X (on the left), and $x, y \in X$, we write $x =_G y$ if x and y are in the same G-orbit. We denote by $[G\backslash X]$ a set of representatives of G-orbits on X, and by X^G the set of fixed points of G on X. For $x \in X$, we denote by G_x its stabilizer in G.

2.3. Review of p-permutation modules. We begin by recalling some definitions and basic results. We refer to [3], and to [1] Sections 3.11 and 5.5 for details :

2.4. Definition. *A* permutation *kG-module is a kG-module admitting a G-invariant k-basis. A p*-permutation *kG-module M is a kG-module such that $\mathrm{Res}^G_S M$ is a permutation kS-module, where S is a Sylow p-subgroup of G.*

The p-permutation kG-modules are also called *trivial source modules*, because the indecomposable ones coincide with the indecomposable modules having a trivial source (see [3] 0.4). Moreover, the p-permutation modules also coincide with the direct summands of permutation modules (see [1], Lemma 3.11.2).

2.5. Proposition.

1. *If H is a subgroup of G, and M is a p-permutation kG-module, then the restriction $\mathrm{Res}^G_H M$ of M to H is a p-permutation kH-module.*
2. *If H is a subgroup of G, and L is a p-permutation kH-module, then the induced module $\mathrm{Ind}^G_H L$ is a p-permutation kG-module.*
3. *If N is a normal subgroup of G, and L is a p-permutation $k(G/N)$-module, the inflated module $\mathrm{Inf}^G_{G/N} L$ is a p-permutation kG-module.*
4. *If P is a p-group, and M is a permutation kP-module with P-invariant basis X, then the image of the set X^P in $M[P]$ is a k-basis of $M[P]$.*
5. *If P is a p-subgroup of G, and M is a p-permutation kG-module, then the Brauer quotient $M[P]$ is a p-permutation $k\overline{N}_G(P)$-module.*
6. *If M and N are p-permutation kG-modules, then their tensor product $M \otimes_k N$ is again a p-permutation kG-module.*

Proof : Assertions 1,2,3, and 6 are straightforward consequences of the same assertions for *permutation* modules. For Assertion 4, see [3] 1.1.(3). Assertion 5 follows easily from Assertion 4 (see also [3] 3.1). □

This leads to the following definition :

2.6. Definition. *The p*-permutation ring *$pp_k(G)$ is the Grothendieck group of the category of p-permutation kG-modules, with relations corresponding to direct sum decompositions, i.e. $[M] + [N] = [M \oplus N]$. The ring structure* *on $pp_k(G)$ is induced by the tensor product of modules over k. The identity element of $pp_k(G)$ is the class of the trivial kG-module k.*

As the Krull-Schmidt theorem holds for kG-modules, the additive group $pp_k(G)$ is a free (abelian) group on the set of isomorphism classes of indecomposable p-permutation kG-modules. These modules have the following properties :

2.7. Theorem. [[3] Theorem 3.2]

1. *The vertices of an indecomposable p-permutation kG-module M are the maximal p-subgroups P of G such that $M[P] \neq \{0\}$.*
2. *An indecomposable p-permutation kG-module has vertex P if and only if $M[P]$ is a non-zero projective $k\overline{N}_G(P)$-module.*
3. *The correspondence $M \mapsto M[P]$ induces a bijection between the isomorphism classes of indecomposable p-permutation kG-modules with vertex P and the isomorphism classes of indecomposable projective $k\overline{N}_G(P)$-modules.*

2.8. Notation. *Let $\mathcal{P}_{G,p}$ denote the set of pairs (P, E), where P is a p-subgroup of G, and E is an indecomposable projective $k\overline{N}_G(P)$-module. The group G acts on $\mathcal{P}_{G,p}$ by conjugation, and we denote by $[\mathcal{P}_{G,p}]$ a set of representatives of G-orbits on $\mathcal{P}_{G,p}$.*

For $(P, E) \in \mathcal{P}_{G,p}$, let $M_{P,E}$ denote the (unique up to isomorphism) indecomposable p-permutation kG-module such that $M_{P,E}[P] \cong E$.

2.9. Corollary. *The classes of the modules $M_{P,E}$, for $(P, E) \in [\mathcal{P}_{G,p}]$ form a $\mathbb{Z}$-basis of $pp_k(G)$.*

2.10. Notation. *The operations Res^G_H, Ind^G_H, $\mathrm{Inf}^G_{G/N}$ extend linearly to maps between the corresponding p-permutations rings, denoted with the same symbol.*

The maps Res^G_H and $\mathrm{Inf}^G_{G/N}$ are ring homomorphisms, whereas Ind^G_H is not in general. Similarly :

2.11. Proposition. *Let P be a p-subgroup of G. Then the correspondence $M \mapsto M[P]$ induces a ring homomorphism $\mathrm{Br}^G_P : pp_k(G) \to pp_k\big(\overline{N}_G(P)\big)$.*

Proof : Let M and N be p-permutation kG-modules. The canonical bilinear map $M \times N \to M \otimes_k N$ is G-equivariant, hence it induces a bilinear map $\beta_P : M[P] \times N[P] \to (M \otimes_k N)[P]$ (see [3] 1.2), which is $\overline{N}_G(P)$-equivariant. Now if X is a P-invariant k-basis of M, and Y a P-invariant k-basis of N, then $X\times Y$ is a P-invariant basis of $M\otimes_k N$. The images of the sets X^P, Y^P, and $(X\times Y)^P$ are bases of $M[P]$, $N[P]$, and $(M\otimes_k N)[P]$, respectively, and the restriction of β_P to theses bases is the canonical bijection $X^P\times Y^P\to (X\times Y)^P$. It follows that β_P induces an isomorphism $M[P]\otimes_k N[P]\to (M\otimes_k N)[P]$ of $k\overline{N}_G(P)$-modules. Proposition 2.11 follows. $\square$

2.12. Notation. *Let $\mathcal{Q}_{G,p}$ denote the set of pairs (P,s), where P is a p-subgroup of G, and s is a p'-element of $\overline{N}_G(P)$. The group G acts on $\mathcal{Q}_{G,p}$, and we denote by $[\mathcal{Q}_{G,p}]$ a set of representatives of G-orbits on $\mathcal{Q}_{G,p}$.*

If $(P,s)\in\mathcal{Q}_{G,p}$, we denote by $N_G(P,s)$ the stabilizer of (P,s) in G, and by $<Ps>$ the subgroup of $N_G(P)$ generated by Ps (i.e. the inverse image in $N_G(P)$ of the cyclic group $<s>$ of $\overline{N}_G(P)$).

2.13. Remarks :

- When H is a subgroup of G, there is a natural inclusion of $\mathcal{Q}_{H,p}$ into $\mathcal{Q}_{G,p}$, as $\overline{N}_H(P)\leq\overline{N}_G(P)$ for any p-subgroup P of H. We will consider $\mathcal{Q}_{H,p}$ as a subset of $\mathcal{Q}_{G,p}$.
- When $(P,s)\in\mathcal{Q}_{G,p}$, the group $N_G(P,s)$ is the set of elements g in $N_G(P)$ whose image in $\overline{N}_G(P)$ centralizes s. In other words, there is a short exact sequence of groups

  $$(\mathbf{2.14})\qquad \mathbf{1}\to P\to N_G(P,s)\to C_{\overline{N}_G(P)}(s)\to\mathbf{1}\ .$$

  In particular $N_G(P,s)$ is a subgroup of $N_G(<Ps>)$.

2.15. Notation. *Let $(P,s)\in\mathcal{Q}_{G,p}$. Let $\tau^G_{P,s}$ denote the additive map from $pp_k(G)$ to K sending the class of a p-permutation kG-module M to the value at s of the Brauer character of the $\overline{N}_G(P)$-module $M[P]$.*

2.16. Remarks :

- It is clear that $\tau^G_{P,s}(M)$ only depends on the restriction of M to the group $<Ps>$. In other words

  $$\tau^G_{P,s}=\tau^{<Ps>}_{P,s}\circ\mathrm{Res}^G_{<Ps>}\ .$$

  Furthermore, it is clear from the definition that

  $$(\mathbf{2.17})\qquad \tau^G_{P,s}=\tau^{<Ps>/P}_{\mathbf{1},s}\circ\mathrm{Br}^{<Ps>}_P\circ\mathrm{Res}^G_{<Ps>}\ .$$

- It is easy to see that $\tau^G_{P,s}$ only depends on the G-orbit of (P,s), that is, $\tau^G_{P^g,s^g}=\tau^G_{P,s}$ for every $g\in G$.

The following proposition is Corollary 5.5.5 in [1], but our construction of the species is slightly different (but equivalent, of course). For this reason, we sketch an independent proof :

2.18. Proposition.

1. *The map $\tau^G_{P,s}$ is a ring homomorphism $pp_k(G) \to K$ and extends to a K-algebra homomorphism (a species) $\tau^G_{P,s} : K \otimes_{\mathbb{Z}} pp_k(G) \to K$.*
2. *The set $\{\tau^G_{P,s} \mid (P,s) \in [\mathcal{Q}_{G,p}]\}$ is the set of all distinct species from $K \otimes_{\mathbb{Z}} pp_k(G)$ to K. These species induce a K-algebra isomorphism*

$$T = \prod_{(P,s)\in[\mathcal{Q}_{G,p}]} \tau^G_{P,s} : K \otimes_{\mathbb{Z}} pp_k(G) \to \prod_{(P,s)\in[\mathcal{Q}_{G,p}]} K \ .$$

Proof : By 2.17, to prove Assertion 1, it suffices to prove that $\tau_{\mathbf{1},s}^{<Ps>/P}$ is a ring homomorphism, since both $\mathrm{Res}^G_{<Ps>}$ and $\mathrm{Br}_P^{<Ps>}$ are ring homomorphisms. In other words, we can assume that $P = \mathbf{1}$. Now the value of $\tau^G_{\mathbf{1},s}$ on the class of a kG-module M is the value $\phi_M(s)$ of the Brauer character of M at s, so Assertion 1 follows from Remark 2.2.

For Assertion 2, it suffices to prove that T is an isomorphism. Since $[\mathcal{P}_{G,p}]$ and $[\mathcal{Q}_{G,p}]$ have the same cardinality, the matrix $\mathcal{M}$ of T is a square matrix. Let $(P,E) \in \mathcal{P}_{G,p}$, and $(Q,s) \in \mathcal{Q}_{G,p}$. Then $\tau_{Q,s}(M_{P,E})$ is equal to zero if Q is not contained in P up to G-conjugation, because in this case $M_{P,E}(Q) = \{0\}$ by Theorem 2.7. It follows that $\mathcal{M}$ is block triangular. As moreover $M_{P,E}[P] \cong E$, we have that $\tau_{P,s}(M_{P,E}) = \phi_E(s)$. This means that the diagonal block of $\mathcal{M}$ corresponding to P is the matrix of Brauer characters of projective $k\overline{N}_G(P)$-modules, and these are linearly independent by Lemma 5.3.1 of [1]. It follows that all the diagonal blocks of $\mathcal{M}$ are non singular, so $\mathcal{M}$ is invertible, and T is an isomorphism. □

2.19. Corollary. *The algebra $K \otimes_{\mathbb{Z}} pp_k(G)$ is a split semisimple commutative K-algebra. Its primitive idempotents $F^G_{P,s}$ are indexed by $[\mathcal{Q}_{G,p}]$, and the idempotent $F^G_{P,s}$ is characterized by*

$$\forall (R,u) \in \mathcal{Q}_{G,p}, \quad \tau^G_{R,u}(F^G_{P,s}) = \begin{cases} 1 & \text{if } (R,u) =_G (P,s) \\ 0 & \text{otherwise.} \end{cases}$$

3. Restriction and induction

3.1. Proposition. *Let $H \leq G$, and $(P,s) \in \mathcal{Q}_{G,p}$. Then*

$$\mathrm{Res}^G_H F^G_{P,s} = \sum_{(Q,t)} F^H_{Q,t} \ ,$$

where (Q,t) runs through a set of representatives of H-conjugacy classes of G-conjugates of (P,s) contained in H.

Proof : Indeed, as Res_H^G is an algebra homomorphism, the element $\mathrm{Res}_H^G F_{P,s}^G$ is an idempotent of $K \otimes_{\mathbb{Z}} pp_k(H)$, hence it is equal to a sum of some distinct primitive idempotents $F_{Q,t}^H$. The idempotent $F_{Q,t}^H$ appears in this decomposition if and only if $\tau_{Q,t}^H(\mathrm{Res}_H^G F_{P,s}^G) = 1$. By Remark 2.16

$$\begin{aligned}\tau_{Q,t}^H(\mathrm{Res}_H^G F_{P,s}^G) &= \tau_{Q,t}^{<Qt>}(\mathrm{Res}_{<Qt>}^H \mathrm{Res}_H^G F_{P,s}^G)\\ &= \tau_{Q,t}^{<Qt>}(\mathrm{Res}_{<Qt>}^G F_{P,s}^G)\\ &= \tau_{Q,t}^G(F_{P,s}^G) \ .\end{aligned}$$

Now $\tau_{Q,t}^G(F_{P,s}^G)$ is equal to 1 if and only if (Q,t) and (P,s) are G-conjugate. This completes the proof. □

3.2. Proposition. *Let $H \leq G$, and $(Q,t) \in \mathcal{Q}_{H,p}$. Then*

$$\mathrm{Ind}_H^G F_{Q,t}^H = |N_G(Q,t) : N_H(Q,t)| F_{Q,t}^G \ .$$

Proof : Since $K \otimes_{\mathbb{Z}} pp_k(G)$ is a split semisimple commutative K-algebra, any element X in $K \otimes_{\mathbb{Z}} pp_k(G)$ can be written

$$X = \sum_{(P,s)\in[\mathcal{Q}_{G,p}]} \tau_{P,s}^G(X) F_{P,s}^G \ , \tag{3.3}$$

and moreover for any $(P,s) \in \mathcal{Q}_{G,p}$

$$\tau_{P,s}^G(X) F_{P,s}^G = X \cdot F_{P,s}^G \ .$$

Setting $X = \mathrm{Ind}_H^G F_{Q,t}^H$ in this equation gives

$$\begin{aligned}\tau_{P,s}^G(\mathrm{Ind}_H^G F_{Q,t}^H) F_{P,s}^G &= (\mathrm{Ind}_H^G F_{Q,t}^H) \cdot F_{P,s}^G\\ &= \mathrm{Ind}_H^G (F_{Q,t}^H \cdot \mathrm{Res}_H^G F_{P,s}^G) \ .\end{aligned}$$

By Proposition 3.1, the element $\mathrm{Res}_H^G F_{P,s}^G$ is equal to the sum of the distinct idempotents F_{P^y,s^y}^H associated to elements y of G such that $<Ps>^y \leq H$. The product $F_{Q,t}^H \cdot F_{P^y,s^y}^H$ is equal to zero, unless (Q,t) is H-conjugate to (P^y, s^y), which implies that (Q,t) and (P,s) are G-conjugate. It follows that the only non zero term in the right hand side of Equation 3.3 is the term corresponding to (Q,t). Hence

$$\mathrm{Ind}_H^G F_{Q,t}^H = \tau_{Q,t}^G(\mathrm{Ind}_H^G F_{Q,t}^H) F_{Q,t}^G \ .$$

Now by Remark 2.16 and the Mackey formula

$$\begin{aligned}\tau_{Q,t}^G(\mathrm{Ind}_H^G F_{Q,t}^H) &= \tau_{Q,t}^{<Qt>}(\mathrm{Res}_{<Qt>}^G \mathrm{Ind}_H^G F_{Q,t}^H)\\ &= \tau_{Q,t}^{<Qt>}\Big(\sum_{x\in <Qt>\backslash G/H} \mathrm{Ind}_{<Qt>\cap {}^xH}^{<Qt>}\, {}^x\mathrm{Res}_{<Qt>^x\cap H}^H F_{Q,t}^H\Big)\ .\end{aligned}$$

By Proposition 3.1, the element $\mathrm{Res}_{<Qt>^x\cap H}^H F_{Q,t}^H$ is equal to the sum of the distinct idempotents $F_{Q^y,t^y}^{<Qt>^x\cap H}$ corresponding to elements $y\in H$ such that $<Qt>^y \leq <Qt>^x\cap H$. This implies $<Qt>^y = <Qt>^x$, i.e. $y\in N_G(<Qt>)x$, thus $x\in N_G(<Qt>)\cdot H$. This gives

$$\begin{aligned}\tau_{Q,t}^G(\mathrm{Ind}_H^G F_{Q,t}^H) &= \tau_{Q,t}^{<Qt>}\Big(\sum_{\substack{x\in N_G(<Qt>)H/H\\ y\in N_H(Q,t)\backslash N_G(<Qt>)x}} {}^xF_{Q^y,t^y}^{<Qt>}\Big)\\ &= \sum_{\substack{x\in N_G(<Qt>)/N_H(<Qt>)\\ y\in N_H(Q,t)\backslash N_G(<Qt>)x}} \tau_{Q,t}^{<Qt>}(F_{Q^{yx^{-1}},t^{yx^{-1}}}^{<Qt>})\\ &= \sum_{z\in N_H(Q,t)\backslash N_G(<Qt>)} \tau_{Q,t}^{<Qt>}(F_{Q^z,t^z}^{<Qt>})\ ,\end{aligned}$$

where $z = yx^{-1}$. Finally $\tau_{Q,t}^{<Qt>}(F_{Q^z,t^z}^{<Qt>})$ is equal to 1 if (Q^z,t^z) is conjugate to (Q,t) in $<Qt>$, and to zero otherwise.

If $u\in <Qt>$ is such that $(Q^z,t^z)^u = (Q,t)$, then $zu\in N_G(Q,t)$. But since $[<Qt>,t]\leq Q$, we have $<Qt>\leq N_G(Q,t)$, so $u\in N_G(Q,t)$, hence $z\in N_G(Q,t)$, and $(Q^z,t^z) = (Q,t)$. It follows that

$$\tau_{Q,t}^G(\mathrm{Ind}_H^G F_{Q,t}^H) = |N_G(Q,t):N_H(Q,t)|\ ,$$

which completes the proof of the proposition. □

3.4. Corollary. *Let $(P,s)\in \mathcal{Q}_{G,p}$. Then*

$$F_{P,s}^G = \frac{|s|}{|C_{\overline{N}_G(P)}(s)|}\mathrm{Ind}_{<Ps>}^G F_{P,s}^{<Ps>}\ .$$

Proof : Apply Proposition 3.2 with $(Q,t) = (P,s)$ and $H = <Ps>$. Then $N_H(Q,t) = <Ps>$, thus by Exact sequence 2.14

$$|N_G(Q,t):N_H(Q,t)| = \frac{|P||C_{\overline{N}_G(P)}(s)|}{|P||s|} = \frac{|C_{\overline{N}_G(P)}(s)|}{|s|}\ ,$$

and the corollary follows. □

4. Idempotents

It follows from Corollary 3.4 that, in order to find formulae for the primitive idempotents $F_{P,s}^G$ of $K\otimes_{\mathbb{Z}} pp_k(G)$, it suffices to find the formula expressing the idempotent $F_{P,s}^{<Ps>}$. In other words, we can assume that $G = <Ps>$, i.e. that G has a normal Sylow p-subgroup P with cyclic quotient generated by s.

4.1. A morphism from the Burnside ring. When G is an arbitrary finite group, there is an obvious ring homomorphism $\mathcal{L}_G$ from the Burnside ring $B(G)$ to $pp_k(G)$, induced by the *linearization* operation, sending a finite G-set X to the permutation module kX, which is obviously a p-permutation module. This morphism also commutes with restriction and induction : if $H \leq G$, then

$$(\mathbf{4.2}) \qquad \mathcal{L}_H \circ \mathrm{Res}_H^G = \mathrm{Res}_H^G \circ \mathcal{L}_G \ , \qquad \mathcal{L}_G \circ \mathrm{Ind}_H^G = \mathrm{Ind}_H^G \circ \mathcal{L}_H \ .$$

Indeed, for any G-set X, the kH-modules $k\mathrm{Res}_H^G X$ and $\mathrm{Res}_H^G(kX)$ are isomorphic, and for any H-set Y, the kG-modules $k\mathrm{Ind}_H^G Y$ and $\mathrm{Ind}_H^G(kY)$ are isomorphic.

Similarly, when P is a p-subgroup of G, the ring homomorphism $\Phi_P : B(G) \to B\big(\overline{N}_G(P)\big)$ induced by the operation $X \mapsto X^P$ on G-sets, is compatible with the Brauer morphism $\mathrm{Br}_P^G : pp_k(G) \to pp_k\big(\overline{N}_G(P)\big)$:

$$(\mathbf{4.3}) \qquad \mathcal{L}_{\overline{N}_G(P)} \circ \Phi_P = \mathrm{Br}_P^G \circ \mathcal{L}_G \ .$$

This is because for any G-set X, the $k\overline{N}_G(P)$-modules $k(X^P)$ and $(kX)[P]$ are isomorphic.

The ring homomorphism $\mathcal{L}_G$ extends linearly to a K-algebra homomorphism $K \otimes_{\mathbb{Z}} B(G) \to K \otimes_{\mathbb{Z}} pp_k(G)$, still denoted by $\mathcal{L}_G$. The algebra $K \otimes_{\mathbb{Z}} B(G)$ is also a split semisimple commutative K-algebra. Its species are the K-algebra maps

$$K \otimes_{\mathbb{Z}} B(G) \to K, \quad X \mapsto |X^H| \ ,$$

where H runs through the set of all subgroups of G up to conjugation. Here we denote by $|X^H|$ the number of H-fixed points of a G-set X and this notation is then extended K-linearly to any $X \in K \otimes_{\mathbb{Z}} B(G)$. The primitive idempotents e_H^G of $K \otimes_{\mathbb{Z}} B(G)$ are indexed by subgroups H of G, up to conjugation. They are given by the following formulae, found by Gluck ([4]) and later independently by Yoshida ([5]) :

$$(\mathbf{4.4}) \qquad e_H^G = \frac{1}{|N_G(H)|} \sum_{L \leq H} |L| \, \mu(L,H) \, G/L \ ,$$

where μ denotes the Möbius function of the poset of subgroups of G. The idempotent e_H^G is characterized by the fact that for any $X \in K \otimes_{\mathbb{Z}} B(G)$

$$X \cdot e_H^G = |X^H| e_H^G \ .$$

4.5. Remark : Since $|X^H|$ only depends on $\mathrm{Res}_H^G X$, it follows in particular that X is a scalar multiple of the "top" idempotent e_G^G if and only if $\mathrm{Res}_H^G X = 0$ for any proper subgroup H of G. In particular, if N is a normal subgroup of G, then

$$(e_G^G)^N = e_{G/N}^{G/N} \ . \tag{4.6}$$

This is because for any proper subgroup H/N of G/N

$$\mathrm{Res}_{H/N}^{G/N} (e_G^G)^N = (\mathrm{Res}_H^G e_G^G)^N = 0 \ .$$

So $(e_G^G)^N$ is a scalar multiple of $e_{G/N}^{G/N}$. As it is also an idempotent, it is equal to 0 or $e_{G/N}^{G/N}$. Finally

$$|\big((e_G^G)^N\big)^{G/N}| = |(e_G^G)^G| = 1 \ ,$$

so $(e_G^G)^N$ is non zero.

4.7. The case of a cyclic p'-group. Suppose that G is a cyclic p'-group, of order n, generated by an element s. In this case, there are exactly n group homomorphisms from G to the multiplicative group $k^\times$ of k. For each of these group homomorphisms φ, let k_φ denote the kG-module k on which the generator s acts by multiplication by $\varphi(s)$. As G is a p'-group, this module is simple and projective. The (classes of the) modules k_φ, for $\varphi \in \widehat{G} = \mathrm{Hom}(G, k^\times)$, form a basis of $pp_k(G)$.

Since moreover for $\varphi, \psi \in \widehat{G}$, the modules $k_\varphi \otimes_k k_\psi$ and $k_{\varphi\psi}$ are isomorphic, the algebra $K \otimes_{\mathbb{Z}} pp_k(G)$ is isomorphic to the group algebra of the group $\widehat{G}$. This leads to the following classical formula :

4.8. Lemma. *Let G be a cyclic p'-group. Then for any $t \in G$,*

$$F_{\mathbf{1},t}^G = \frac{1}{n} \sum_{\varphi \in \widehat{G}} \tilde{\varphi}(t^{-1}) k_\varphi \ ,$$

where $\tilde{\varphi}$ is the Brauer character of k_φ.

Proof : Indeed for $s, t \in G$

$$\tau^G_{\mathbf{1},t}\Big(\frac{1}{n}\sum_{\varphi\in\widehat{G}} \tilde{\varphi}(s^{-1})k_\varphi\Big) = \frac{1}{n}\sum_{\varphi\in\widehat{G}} \tilde{\varphi}(s^{-1})\tilde{\varphi}(t) = \delta_{s,t} \ ,$$

where $\delta_{s,t}$ is the Kronecker symbol. □

4.9. The case $G = <Ps>$. Suppose now more generally that $G = <Ps>$, where P is a normal Sylow p-subgroup of G and s is a p'-element. In this case, by Proposition 3.1, the restriction of $F^G_{P,s}$ to any proper subgroup of G is equal to zero. Moreover, since $N_G(P,t) = G$ for any $t \in G/P$, the conjugacy class of the pair (P,t) reduces to $\{(P,t)\}$.

4.10. Lemma. *Suppose $G = <Ps>$, and set $E^G_G = \mathcal{L}_G(e^G_G)$. Then*

$$E^G_G = \sum_{<t>=<s>} F^G_{P,t} \ .$$

Proof : By 4.2 and by Remark 4.5, the restriction of E^G_G to any proper subgroup of G is equal to zero. Let $(Q,t) \in \mathcal{Q}_{G,p}$, such that the group $L = <Qt>$ is a proper subgroup of G. By Proposition 3.2, there is a rational number r such that

$$F^G_{Q,t} = r\,\mathrm{Ind}_L^G F^L_{Q,t} \ .$$

It follows that

$$E^G_G \cdot F^G_{Q,t} = r\,\mathrm{Ind}_L^G\big((\mathrm{Res}^G_L E^G_G)\cdot F^L_{Q,t}\big) = 0 \ .$$

Now E^G_G is an idempotent of $K \otimes_{\mathbb{Z}} pp_k(G)$, hence is it a sum of some of the primitive idempotents $F^G_{Q,t}$ associated to pairs (Q,t) for which $<Qt> = G$. This condition is equivalent to $Q = P$ and $<t> = <s>$.

It remains to show that all these idempotents $F^G_{P,t}$ appear in the decomposition of E^G_G, i.e. equivalently that $\tau^G_{P,t}(E^G_G) = 1$ for any generator t of $<s>$. Now by 4.6 and Remark 2.16

$$\tau^G_{P,t}(E^G_G) = \tau^{G/P}_{\mathbf{1},t}\big(\mathrm{Br}^G_P(E^G_G)\big) = \tau^{<s>}_{\mathbf{1},t}(E^{<s>}_{<s>}) \ .$$

Now the value at t of the Brauer character of a permutation module kX is equal to the number of fixed points of t on X. By K-linearity, this gives

$$\tau^{<s>}_{\mathbf{1},t}(E^{<s>}_{<s>}) = |(e^{<s>}_{<s>})^t| \ ,$$

and this is equal to 1 if t generates $<s>$, and to 0 otherwise, as was to be shown. □

4.11. Proposition. *Let $(P, s) \in \mathcal{Q}_{G,p}$, and suppose that $G = {<}Ps{>}$. Then*

$$F^G_{P,s} = E^G_G \cdot \mathrm{Inf}^G_{G/P} F^{G/P}_{\mathbf{1},s} \ .$$

Proof : Set $E_s = E^G_G \cdot \mathrm{Inf}^G_{G/P} F^{G/P}_{\mathbf{1},s}$. Then E_s is an idempotent of $K \otimes_{\mathbb{Z}} pp_k(G)$, as it is the product of two (commuting) idempotents. Let $(Q, t) \in \mathcal{Q}_{G,p}$. If ${<}Qt{>} \neq G$, then $\tau^G_{Q,t}(E^G_G) = 0$ by Lemma 4.10, thus $\tau^G_{Q,t}(E_s) = 0$. And if ${<}Qt{>} = G$, then $Q = P$ and ${<}t{>} = {<}s{>}$. In this case

$$\tau^G_{Q,t}(E_s) = \tau^G_{P,t}(E^G_G) \cdot \tau^G_{P,t}(\mathrm{Inf}^G_{G/P} F^{G/P}_{\mathbf{1},s}) \ .$$

By Lemma 4.10, the first factor in the right hand side is equal to 1. The second factor is equal to

$$\begin{aligned} \tau^G_{P,t}(\mathrm{Inf}^G_{G/P} F^{G/P}_{\mathbf{1},s}) &= \tau^{G/P}_{\mathbf{1},t} \mathrm{Br}^G_P(\mathrm{Inf}^G_{G/P} F^{G/P}_{\mathbf{1},s}) \\ &= \tau^{G/P}_{\mathbf{1},t}(F^{G/P}_{\mathbf{1},s}) = \delta_{t,s} \ , \end{aligned}$$

where $\delta_{t,s}$ is the Kronecker symbol. Hence $\tau^G_{P,t}(E_s) = \delta_{t,s}$, and this completes the proof. □

4.12. Theorem. *Let G be a finite group, and let $(P, s) \in \mathcal{Q}_{G,p}$. Then the primitive idempotent $F^G_{P,s}$ of the p-permutation algebra $K \otimes_{\mathbb{Z}} pp_k(G)$ is given by the following formula :*

$$F^G_{P,s} = \frac{1}{|P||s||C_{\overline{N}_G(P)}(s)|} \sum_{\substack{\varphi \in \widehat{<s>} \\ L \leq <Ps> \\ PL = <Ps>}} \tilde{\varphi}(s^{-1})|L|\mu(L, {<}Ps{>})\, \mathrm{Ind}^G_L k^{<Ps>}_{L,\varphi} \ ,$$

where $k^{<Ps>}_{L,\varphi} = \mathrm{Res}^{<Ps>}_L \mathrm{Inf}^{<Ps>}_{<s>} k_\varphi$.

Proof : By Corollary 3.4, and Proposition 4.11

$$F^G_{P,s} = \frac{|s|}{|C_{\overline{N}_G(P)}(s)|} \mathrm{Ind}^G_{<Ps>}(E^{<Ps>}_{<Ps>} \cdot \mathrm{Inf}^{<Ps>}_{<s>} F^{<s>}_{\mathbf{1},s}) \ .$$

By Equation 4.4, this gives

$$F^G_{P,s} = \frac{|s|}{|C_{\overline{N}_G(P)}(s)|} \mathrm{Ind}^G_{<Ps>} \frac{1}{|P||s|} \sum_{L \leq <Ps>} |L|\mu(L, {<}Ps{>}) \mathrm{Ind}^{<Ps>}_L k \cdot \mathrm{Inf}^{<Ps>}_{<s>} F^{<s>}_{\mathbf{1},s} \ .$$

Moreover for each $L \leq <Ps>$

$$\begin{aligned}\mathrm{Ind}_L^{<Ps>} k \cdot \mathrm{Inf}_{<s>}^{<Ps>} F_{\mathbf{1},s}^{<s>} &\cong \mathrm{Ind}_L^{<Ps>}(\mathrm{Res}_L^{<Ps>}\mathrm{Inf}_{<s>}^{<Ps>} F_{\mathbf{1},s}^{<s>})\\ &\cong \mathrm{Ind}_L^{<Ps>}\mathrm{Inf}_{L/L\cap P}^{L}\mathrm{Iso}_{LP/P}^{L/L\cap P}\mathrm{Res}_{LP/P}^{<s>}F_{\mathbf{1},s}^{<s>} \ .\end{aligned}$$

Here we have used the fact that if L and P are subgroups of a group H, with $P \trianglelefteq H$, then there is an isomorphism of functors

$$\mathrm{Res}_L^H \circ \mathrm{Inf}_{H/P}^H \cong \mathrm{Inf}_{L/L\cap P}^L \circ \mathrm{Iso}_{LP/P}^{L/L\cap P} \circ \mathrm{Res}_{LP/P}^{H/P} \ ,$$

which follows from the isomorphism of $(L, H/P)$-bisets

$$H \times_H (H/P) \cong {}_L(H/P)_{H/P} \cong (L/L\cap P) \times_{L/L\cap P} (LP/P) \times_{LP/P} (H/P) \ .$$

Now Proposition 3.1 implies that $\mathrm{Res}_{LP/P}^{<s>}F_{\mathbf{1},s}^{<s>} = 0$ if $LP/P \neq <s>$, i.e. equivalently if $PL \neq <Ps>$. It follows that

$$F_{P,s}^G = \frac{1}{|P||C_{\overline{N}_G(P)}(s)|} \sum_{\substack{L \leq <Ps> \\ PL = <Ps>}} |L|\mu(L, <Ps>)\mathrm{Ind}_L^G(\mathrm{Res}_L^{<Ps>}\mathrm{Inf}_{<s>}^{<Ps>} F_{\mathbf{1},s}^{<s>}) \ .$$

By Lemma 4.8, this gives

$$F_{P,s}^G = \frac{1}{|P||s||C_{\overline{N}_G(P)}(s)|} \sum_{\substack{\varphi \in \widehat{<s>} \\ L \leq <Ps> \\ PL = <Ps>}} \tilde{\varphi}(s^{-1})|L|\mu(L, <Ps>)\mathrm{Ind}_L^G k_{L,\varphi}^{<Ps>} \ ,$$

where $k_{L,\varphi}^{<Ps>} = \mathrm{Res}_L^{<Ps>}\mathrm{Inf}_{<s>}^{<Ps>} k_\varphi$, as was to be shown. □

References

[1] D. Benson. *Representations and cohomology I*, volume 30 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1991.

[2] S. Bouc. On the Cartan matrix of Mackey algebras. Preprint, 2009.

[3] M. Broué. On Scott modules and p-permutation modules: an approach through the Brauer morphism. *Proc.AMS*, 93(3):401–408, March 1985.

[4] D. Gluck. Idempotent formula for the Burnside ring with applications to the p-subgroup simplicial complex. *Illinois J. Math.*, 25:63–67, 1981.

[5] T. Yoshida. Idempotents in Burnside rings and Dress induction theorem. *J. Algebra*, 80:90–105, 1983.

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