The primitive idempotents of the $p$-permutation ring

Serge Bouc and Jacques Thévenaz

Abstract: Let $G$ be a finite group, let $p$ be a prime number, and let $K$ be a field of characteristic 0 and $k$ be a field of characteristic $p$, both large enough. In this note we state explicit formulae for the primitive idempotents of $K \otimes_{\mathbb{Z}} pp_k(G)$, where $pp_k(G)$ is the ring of $p$-permutation $kG$-modules.

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1. Introduction

Let $G$ be a finite group, let $p$ be a prime number, and let $K$ be a field of characteristic 0 and $k$ be a field of characteristic $p$, both large enough. In this note we state explicit formulae for the primitive idempotents of $K \otimes_{\mathbb{Z}} pp_k(G)$, where $pp_k(G)$ is the ring of $p$-permutation $kG$-modules (also called the trivial source ring).

To obtain these formulae, we first use induction and restriction to reduce to the case where $G$ is cyclic modulo $p$, i.e. $G$ has a normal Sylow $p$-subgroup with cyclic quotient. Then we solve the easy and well known case where $G$ is a cyclic $p'$-group. Finally we conclude by using the natural ring homomorphism from the Burnside ring $B(G)$ of $G$ to $pp_k(G)$, and the classical formulae for the primitive idempotents of $K \otimes_{\mathbb{Z}} B(G)$.

Our formulae are an essential tool in [2], where Cartan matrices of Mackey algebras are considered, and some invariants of these matrices (determinant, rank) are explicitly computed.

2. $p$-permutation modules

2.1. Notation.

- Throughout the paper, $G$ will be a fixed finite group and $p$ a fixed prime number. We consider a field $k$ of characteristic $p$ and we denote by $kG$ the group algebra of $G$ over $k$. We assume that $k$ is large enough in the sense that it is a splitting field for every group algebra $k(N_G(P)/P)$, where $P$ runs through the set of all $p$-subgroups of $G$. 

• We let $K$ be a field of characteristic 0 and we assume that $K$ is large enough in the sense that it contains the values of all the Brauer characters of the groups $N_G(P)/P$, where $P$ runs through the set of all $p$-subgroups of $G$.

We recall quickly how Brauer characters are defined. We let $\overline{k}$ be an extension of $k$ containing all the $n$-th roots of unity, where $n$ is the $p'$-part of the exponent of $G$. We choose an isomorphism $\theta : \mu_n(\overline{k}) \to \mu_n(\mathbb{C})$ from the group of $n$-th roots of unity in $\overline{k}$ and the corresponding group in $\mathbb{C}$. If $V$ is an $r$-dimensional $kH$-module for the group $H = N_G(P)/P$ and if $s$ is an element of the set $H_{p'}$ of all $p'$-elements of $H$, the matrix of the action of $s$ on $V$ has eigenvalues $(\lambda_1, \ldots, \lambda_r)$ in the group $\mu_n(\overline{k})$. The Brauer character $\phi_V$ of $V$ is the central function defined on $H_{p'}$, with values in the field $\mathbb{Q}[\mu_n(\mathbb{C})]$, sending $s$ to $\sum_{i=1}^r \theta(\lambda_i)$. The actual values of Brauer characters may lie in a subfield of $\mathbb{Q}[\mu_n(\mathbb{C})]$ and we simply require that $K$ contains all these values.

2.2. Remark : Let $V$ be as above and let $W$ be a $t$-dimensional $kH$-module. If $s$ has eigenvalues $(\mu_1, \ldots, \mu_t)$ on $W$, its eigenvalues for the diagonal action of $H$ on $V \otimes_k W$ are $(\lambda_i \mu_j)_{1 \leq i \leq r, 1 \leq j \leq t}$. It follows that $\phi_V \otimes_k W(s) = \sum_{i=1}^r \sum_{j=1}^t \theta(\lambda_i \mu_j) = \phi_V(s) \phi_W(s)$.

• When $H$ is a subgroup of $G$, and $M$ is a $kG$-module, we denote by $\text{Res}_H^G M$ the $kH$-module obtained by restricting the action of $G$ to $H$. When $L$ is a $kH$-module, we denote by $\text{Ind}_H^G L$ the induced $kG$-module.

• When $M$ is a $kG$-module, and $P$ is a subgroup of $G$, the $k$-vector space of fixed points of $P$ on $M$ is denoted by $M^P$. When $Q \leq P$ are subgroups of $G$, the relative trace is the map $\text{tr}_P^Q : M^Q \to M^P$ defined by $\text{tr}_P^Q(m) = \sum_{x \in [P/Q]} x \cdot m$.

• When $M$ is a $kG$-module, the Brauer quotient of $M$ at $P$ is the $k$-vector space

$$M[P] = M^P / \sum_{Q \leq P} \text{tr}_P^Q M^Q.$$ 

This $k$-vector space has a natural structure of $k\overline{N}_G(P)$-module, where as usual $\overline{N}_G(P) = N_G(P)/P$. It is equal to zero if $P$ is not a $p$-group.

• If $P$ is a normal $p$-subgroup of $G$ and $M$ is a $k(G/P)$-module, denote by $\text{Inf}_{G/P}^G M$ the $kG$-module obtained from $M$ by inflation to $G$. Then there is an isomorphism

$$(\text{Inf}_{G/P}^G M)[P] \cong M$$

of $k(G/P)$-modules.

• When $G$ acts on a set $X$ (on the left), and $x, y \in X$, we write $x \equiv_G y$ if $x$ and $y$ are in the same $G$-orbit. We denote by $[G \backslash X]$ a set of representatives
of $G$-orbits on $X$, and by $X^G$ the set of fixed points of $G$ on $X$. For $x \in X$, we denote by $G_x$ its stabilizer in $G$.

2.3. Review of $p$-permutation modules. We begin by recalling some definitions and basic results. We refer to [3], and to [1] Sections 3.11 and 5.5 for details:

2.4. Definition. A permutation $kG$-module is a $kG$-module admitting a $G$-invariant $k$-basis. A $p$-permutation $kG$-module $M$ is a $kG$-module such that $\text{Res}_S^G M$ is a permutation $kS$-module, where $S$ is a Sylow $p$-subgroup of $G$.

The $p$-permutation $kG$-modules are also called trivial source modules, because the indecomposable ones coincide with the indecomposable modules having a trivial source (see [3] 0.4). Moreover, the $p$-permutation modules also coincide with the direct summands of permutation modules (see [1], Lemma 3.11.2).

2.5. Proposition.

1. If $H$ is a subgroup of $G$, and $M$ is a $p$-permutation $kG$-module, then the restriction $\text{Res}_H^G M$ of $M$ to $H$ is a $p$-permutation $kH$-module.

2. If $H$ is a subgroup of $G$, and $L$ is a $p$-permutation $kH$-module, then the induced module $\text{Ind}_H^G L$ is a $p$-permutation $kG$-module.

3. If $N$ is a normal subgroup of $G$, and $L$ is a $p$-permutation $k(G/N)$-module, the inflated module $\text{Inf}_{G/N}^G L$ is a $p$-permutation $kG$-module.

4. If $P$ is a $p$-group, and $M$ is a permutation $kP$-module with $P$-invariant basis $X$, then the image of the set $X^P$ in $M[P]$ is a $k$-basis of $M[P]$.

5. If $P$ is a $p$-subgroup of $G$, and $M$ is a $p$-permutation $kG$-module, then the Brauer quotient $M[P]$ is a $p$-permutation $kN_G(P)$-module.

6. If $M$ and $N$ are $p$-permutation $kG$-modules, then their tensor product $M \otimes_k N$ is again a $p$-permutation $kG$-module.

Proof: Assertions 1,2,3, and 6 are straightforward consequences of the same assertions for permutation modules. For Assertion 4, see [3] 1.1.(3). Assertion 5 follows easily from Assertion 4 (see also [3] 3.1).

This leads to the following definition:

2.6. Definition. The $p$-permutation ring $\text{pp}_k(G)$ is the Grothendieck group of the category of $p$-permutation $kG$-modules, with relations corresponding to direct sum decompositions, i.e. $[M] + [N] = [M \oplus N]$. The ring structure
on \( \text{pp}_k(G) \) is induced by the tensor product of modules over \( k \). The identity element of \( \text{pp}_k(G) \) is the class of the trivial \( kG \)-module \( k \).

As the Krull-Schmidt theorem holds for \( kG \)-modules, the additive group \( \text{pp}_k(G) \) is a free (abelian) group on the set of isomorphism classes of indecomposable \( p \)-permutation \( kG \)-modules. These modules have the following properties:

**2.7. Theorem.** [ [3] Theorem 3.2]

1. The vertices of an indecomposable \( p \)-permutation \( kG \)-module \( M \) are the maximal \( p \)-subgroups \( P \) of \( G \) such that \( M[P] \neq \{0\} \).
2. An indecomposable \( p \)-permutation \( kG \)-module has vertex \( P \) if and only if \( M[P] \) is a non-zero projective \( kN_G(P) \)-module.
3. The correspondence \( M \mapsto M[P] \) induces a bijection between the isomorphism classes of indecomposable \( p \)-permutation \( kG \)-modules with vertex \( P \) and the isomorphism classes of indecomposable projective \( kN_G(P) \)-modules.

**2.8. Notation.** Let \( \mathcal{P}_{G,p} \) denote the set of pairs \((P,E)\), where \( P \) is a \( p \)-subgroup of \( G \), and \( E \) is an indecomposable projective \( kN_G(P) \)-module. The group \( G \) acts on \( \mathcal{P}_{G,p} \) by conjugation, and we denote by \([\mathcal{P}_{G,p}]\) a set of representatives of \( G \)-orbits on \( \mathcal{P}_{G,p} \).

For \((P,E) \in \mathcal{P}_{G,p}\), let \( M_{P,E} \) denote the (unique up to isomorphism) indecomposable \( p \)-permutation \( kG \)-module such that \( M_{P,E}[P] \cong E \).

**2.9. Corollary.** The classes of the modules \( M_{P,E} \), for \((P,E) \in [\mathcal{P}_{G,p}]\) form a \( \mathbb{Z} \)-basis of \( \text{pp}_k(G) \).

**2.10. Notation.** The operations \( \text{Res}_H^G, \text{Ind}_H^G, \text{Inf}_{G/N}^G \) extend linearly to maps between the corresponding \( p \)-permutations rings, denoted with the same symbol.

The maps \( \text{Res}_H^G \) and \( \text{Inf}_{G/N}^G \) are ring homomorphisms, whereas \( \text{Ind}_H^G \) is not in general. Similarly:

**2.11. Proposition.** Let \( P \) be a \( p \)-subgroup of \( G \). Then the correspondence \( M \mapsto M[P] \) induces a ring homomorphism \( \text{Br}_P^G : \text{pp}_k(G) \to \text{pp}_k(N_G(P)) \).

**Proof:** Let \( M \) and \( N \) be \( p \)-permutation \( kG \)-modules. The canonical bilinear map \( M \times N \to M \otimes_k N \) is \( G \)-equivariant, hence it induces a bilinear map \( \beta_P : M[P] \times N[P] \to (M \otimes_k N)[P] \) (see [3] 1.2), which is \( N_G(P) \)-equivariant. Now if \( X \) is a \( P \)-invariant \( k \)-basis of \( M \), and \( Y \) a \( P \)-invariant \( k \)-basis of \( N \),
then $X \times Y$ is a $P$-invariant basis of $M \otimes_k N$. The images of the sets $X^P$, $Y^P$, and $(X \times Y)^P$ are bases of $M[P]$, $N[P]$, and $(M \otimes_k N)[P]$, respectively, and the restriction of $\beta_P$ to these bases is the canonical bijection $X^P \times Y^P \to (X \times Y)^P$. It follows that $\beta_P$ induces an isomorphism $M[P] \otimes_k N[P] \to (M \otimes_k N)[P]$ of $k\overline{N}_G(P)$-modules. Proposition 2.11 follows.

2.12. Notation. Let $Q_{G,p}$ denote the set of pairs $(P, s)$, where $P$ is a $p$-subgroup of $G$, and $s$ is a $p'$-element of $N_G(P)$. The group $G$ acts on $Q_{G,p}$, and we denote by $[Q_{G,p}]$ a set of representatives of $G$-orbits on $Q_{G,p}$.

If $(P, s) \in Q_{G,p}$, we denote by $N_G(P, s)$ the stabilizer of $(P, s)$ in $G$, and by $<Ps>$ the subgroup of $N_G(P)$ generated by $Ps$ (i.e. the inverse image in $N_G(P)$ of the cyclic group $<s>$ of $\overline{N}_G(P)$).

2.13. Remarks:
- When $H$ is a subgroup of $G$, there is a natural inclusion of $Q_{H,p}$ into $Q_{G,p}$, as $N_H(P) \leq N_G(P)$ for any $p$-subgroup $P$ of $H$. We will consider $Q_{H,p}$ as a subset of $Q_{G,p}$.
- When $(P, s) \in Q_{G,p}$, the group $N_G(P, s)$ is the set of elements $g$ in $N_G(P)$ whose image in $\overline{N}_G(P)$ centralizes $s$. In other words, there is a short exact sequence of groups

$$1 \to P \to N_G(P, s) \to C_{\overline{N}_G(P)}(s) \to 1.$$  

In particular $N_G(P, s)$ is a subgroup of $N_G(<Ps>)$.

2.15. Notation. Let $(P, s) \in Q_{G,p}$. Let $\tau_{G}^s$ denote the additive map from $pp_k(G)$ to $K$ sending the class of a $p$-permutation $kG$-module $M$ to the value at $s$ of the Brauer character of the $\overline{N}_G(P)$-module $M[P]$.

2.16. Remarks:
- It is clear that $\tau_{G,s}^s(M)$ only depends on the restriction of $M$ to the group $<Ps>$. In other words

$$\tau_{G,s}^s = \tau_{P,s}^{<Ps>} \circ \text{Res}_{<Ps>}^G.$$  

Furthermore, it is clear from the definition that

$$\tau_{G,s}^s = \tau_{1,s}^{<Ps>/P} \circ \text{Br}_{P}^{<Ps>} \circ \text{Res}_{<Ps>}^G.$$  

- It is easy to see that $\tau_{G,s}^g$ only depends on the $G$-orbit of $(P, s)$, that is, $\tau_{G,s}^{g} = \tau_{G,s}^{g}$ for every $g \in G$.

The following proposition is Corollary 5.5.5 in [1], but our construction of the species is slightly different (but equivalent, of course). For this reason, we sketch an independent proof:
2.18. Proposition.

1. The map $\tau^{G}_{P,s}$ is a ring homomorphism $\text{pp}_{k}(G) \to K$ and extends to a $K$-algebra homomorphism (a species) $\tau^{G}_{P,s} : K \otimes_{Z} \text{pp}_{k}(G) \to K$.

2. The set $\{\tau^{G}_{P,s} \mid (P, s) \in [Q_{G,p}]\}$ is the set of all distinct species from $K \otimes_{Z} \text{pp}_{k}(G)$ to $K$. These species induce a $K$-algebra isomorphism $T = \prod_{(P,s)\in [Q_{G,p}]} \tau^{G}_{P,s} : K \otimes_{Z} \text{pp}_{k}(G) \to \prod_{(P,s)\in [Q_{G,p}]} K$.

Proof: By 2.17, to prove Assertion 1, it suffices to prove that $\tau^{G}_{P,s}$ is a ring homomorphism, since both $\text{Res}^{G}_{P,s}$ and $\text{Br}^{G}_{P,s}$ are ring homomorphisms. In other words, we can assume that $P = 1$. Now the value of $\tau^{G}_{1,s}$ on the class of a $kG$-module $M$ is the value $\phi_{M}(s)$ of the Brauer character of $M$ at $s$, so Assertion 1 follows from Remark 2.2.

For Assertion 2, it suffices to prove that $T$ is an isomorphism. Since $[P_{G,p}]$ and $[Q_{G,p}]$ have the same cardinality, the matrix $M$ of $T$ is a square matrix. Let $(P, E) \in P_{G,p}$, and $(Q, s) \in Q_{G,p}$. Then $\tau^{G}_{Q,s}(M_{P,E})$ is equal to zero if $Q$ is not contained in $P$ up to $G$-conjugation, because in this case $M_{P,E}(Q) = \{0\}$ by Theorem 2.7. It follows that $M$ is block triangular. As moreover $M_{P,E}[P] \cong E$, we have that $\tau^{G}_{P,s}(M_{P,E}) = \phi_{E}(s)$. This means that the diagonal block of $M$ corresponding to $P$ is the matrix of Brauer characters of projective $kN_{G}(P)$-modules, and these are linearly independent by Lemma 5.3.1 of [1]. It follows that all the diagonal blocks of $M$ are non singular, so $M$ is invertible, and $T$ is an isomorphism.

2.19. Corollary. The algebra $K \otimes_{Z} \text{pp}_{k}(G)$ is a split semisimple commutative $K$-algebra. Its primitive idempotents $F^{G}_{P,s}$ are indexed by $[Q_{G,p}]$, and the idempotent $F^{G}_{P,s}$ is characterized by

$$\forall (R, u) \in Q_{G,p}: \tau^{G}_{R,u}(F^{G}_{P,s}) = \begin{cases} 1 & \text{if } (R, u) =_{G} (P, s) \\ 0 & \text{otherwise.} \end{cases}$$

3. Restriction and induction

3.1. Proposition. Let $H \leq G$, and $(P, s) \in Q_{G,p}$. Then

$$\text{Res}^{G}_{H} F^{G}_{P,s} = \sum_{(Q,t)} F^{H}_{Q,t},$$

where $(Q,t)$ runs through a set of representatives of $H$-conjugacy classes of $G$-conjugates of $(P,s)$ contained in $H$. 

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Proof : Indeed, as $Res^G_H$ is an algebra homomorphism, the element $Res^G_H F^G_{P,s}$ is an idempotent of $K \otimes_{\mathbb{Z}} pp_k(H)$, hence it is equal to a sum of some distinct primitive idempotents $F^G_{H,P,y,s}$. The idempotent $F^H_{Q,t}$ appears in this decomposition if and only if $\tau^H_{Q,t}(Res^G_H F^G_{P,s}) = 1$. By Remark 2.16

$$\tau^H_{Q,t}(Res^G_H F^G_{P,s}) = \tau^{<Q,t>}_{Q,t}(Res^G_{<Q,t>}F^G_{P,s}) = \tau^{<Q,t>}_{Q,t}(Res^G_{<Q,t>}F^G_{P,s}) = \tau^G_{Q,t}(F^G_{P,s}).$$

Now $\tau^G_{Q,t}(F^G_{P,s})$ is equal to 1 if and only if $(Q,t)$ and $(P,s)$ are $G$-conjugate. This completes the proof.

3.2. Proposition. Let $H \leq G$, and $(Q,t) \in Q_{H,p}$. Then

$$Ind^G_H F^H_{Q,t} = |N_G(Q,t) : N_H(Q,t)| F^G_{Q,t}.$$

Proof : Since $K \otimes_{\mathbb{Z}} pp_k(G)$ is a split semisimple commutative $K$-algebra, any element $X$ in $K \otimes_{\mathbb{Z}} pp_k(G)$ can be written

$$X = \sum_{(P,s) \in [Q_{G,p}]} \tau^G_{P,s}(X) F^G_{P,s}, \tag{3.3}$$

and moreover for any $(P,s) \in Q_{G,p}$

$$\tau^G_{P,s}(X) F^G_{P,s} = X \cdot F^G_{P,s}.$$ 

Setting $X = Ind^G_H F^H_{Q,t}$ in this equation gives

$$\tau^G_{P,s}(Ind^G_H F^H_{Q,t}) F^G_{P,s} = (Ind^G_H F^H_{Q,t}) \cdot F^G_{P,s} = Ind^G_H (F^H_{Q,t} \cdot Res^G_{H,F^G_{P,s}}).$$

By Proposition 3.1, the element $Res^G_H F^G_{P,s}$ is equal to the sum of the distinct idempotents $F^H_{P,y,s} \cdot F^H_{P,y,s}$ associated to elements $y$ of $G$ such that $<P,s>y \leq H$. The product $F^H_{Q,t} \cdot F^H_{P,y,s}$ is equal to zero, unless $(Q,t)$ is $H$-conjugate to $(P,y, s^y)$, which implies that $(Q,t)$ and $(P,s)$ are $G$-conjugate. It follows that the only non zero term in the right hand side of Equation 3.3 is the term corresponding to $(Q,t)$. Hence

$$Ind^G_H F^H_{Q,t} = \tau^G_{Q,t}(Ind^G_H F^H_{Q,t}) F^G_{Q,t}.$$
Now by Remark 2.16 and the Mackey formula
\[
\tau^G_{Q,t}(\text{Ind}_H^G F^H_{Q,t}) = \tau^{<Qt>}_{Q,t} (\text{Res}^G_{<Qt>^y} \text{Ind}_H^G F^H_{Q,t})
\]
\[
= \tau^{<Qt>}_{Q,t} \left( \sum_{x \in <Qt> \cap H} \text{Ind}_{<Qt>^x \cap H}^H \text{Res}^H_{<Qt>^x \cap H} F^H_{Q,t} \right).
\]

By Proposition 3.1, the element \( \text{Res}^H_{<Qt>^y} F^H_{Q,t} \) is equal to the sum of the distinct idempotents \( F^Q_{<Qt>^x} \) corresponding to elements \( y \in H \) such that \( <Qt>^y \leq <Qt>^x \cap H \). This implies \( <Qt>^y = <Qt>^x \), i.e. \( y \in N_G(<Qt>)x \), thus \( x \in N_G(<Qt>) \cdot H \). This gives
\[
\tau^G_{Q,t}(\text{Ind}_H^G F^H_{Q,t}) = \tau^{<Qt>}_{Q,t} \left( \sum_{x \in N_G(<Qt>) \cap H/H} \text{Ind}_{<Qt>^x}^H \text{Res}_{<Qt>^x}^H F^H_{Q,t} \right)
\]
\[
= \sum_{x \in N_G(<Qt>) \cap H/H} \tau^{<Qt>}_{Q,t} (F^{<Qt>}_{Q^y,t^y})
\]
\[
= \sum_{z \in N_H(Q,t)/N_G(<Qt>)} \tau^{<Qt>}_{Q,t} (F^{<Qt>}_{Q^z,t^z}),
\]
where \( z = yx^{-1} \). Finally \( \tau^{<Qt>}_{Q,t} (F^{<Qt>}_{Q^z,t^z}) \) is equal to 1 if \( (Q^z, t^z) \) is conjugate to \( (Q, t) \) in \( <Qt> \), and to zero otherwise.

If \( u \in <Qt> \) is such that \( (Q^z, t^z)^u = (Q, t) \), then \( zu \in N_G(Q, t) \). But since \( [<Qt>, t] \leq Q \), we have \( <Qt> \leq N_G(Q, t) \), so \( u \in N_G(Q, t) \), hence \( z \in N_G(Q, t) \), and \( (Q^z, t^z) = (Q, t) \). It follows that
\[
\tau^G_{Q,t}(\text{Ind}_H^G F^H_{Q,t}) = |N_G(Q, t) : N_H(Q, t)|,
\]
which completes the proof of the proposition.

3.4. Corollary. Let \((P, s) \in \mathcal{Q}_{G,p}\). Then
\[
F^G_{P,s} = \frac{|s|}{|C_{\mathcal{K}_G(P)}(s)|} \text{Ind}_{<Ps>}^G F_{<Ps>},
\]

Proof: Apply Proposition 3.2 with \((Q, t) = (P, s)\) and \( H = <Ps> \). Then \( N_H(Q, t) = <Ps> \), thus by Exact sequence 2.14
\[
|N_G(Q, t) : N_H(Q, t)| = \frac{|P| |C_{\mathcal{K}_G(P)}(s)|}{|P||s|} = \frac{|C_{\mathcal{K}_G(P)}(s)|}{|s|},
\]
and the corollary follows.
4. Idempotents

It follows from Corollary 3.4 that, in order to find formulae for the primitive idempotents $F_{G<P,s}$ of $K\otimes_{\mathbb{Z}} \text{pp}_k(G)$, it suffices to find the formula expressing the idempotent $F_{< Ps>}$. In other words, we can assume that $G = <Ps>$, i.e. that $G$ has a normal Sylow $p$-subgroup $P$ with cyclic quotient generated by $s$.

4.1. A morphism from the Burnside ring. When $G$ is an arbitrary finite group, there is an obvious ring homomorphism $L_G$ from the Burnside ring $B(G)$ to $\text{pp}_k(G)$, induced by the linearization operation, sending a finite $G$-set $X$ to the permutation module $kX$, which is obviously a $p$-permutation module. This morphism also commutes with restriction and induction: if $H \leq G$, then

$$L_H \circ \text{Res}^G_H = \text{Res}^G_H \circ L_G, \quad L_G \circ \text{Ind}^G_H = \text{Ind}^G_H \circ L_H.$$  \hspace{1cm} (4.2)

Indeed, for any $G$-set $X$, the $kH$-modules $k\text{Res}^G_H X$ and $\text{Res}^G_H(kX)$ are isomorphic, and for any $H$-set $Y$, the $kG$-modules $k\text{Ind}^G_H Y$ and $\text{Ind}^G_H(kY)$ are isomorphic.

Similarly, when $P$ is a $p$-subgroup of $G$, the ring homomorphism $\Phi_P : B(G) \to B(\text{N}_G(P))$ induced by the operation $X \mapsto X^P$ on $G$-sets, is compatible with the Brauer morphism $\text{Br}^P : \text{pp}_k(G) \to \text{pp}_k(\text{N}_G(P))$:

$$L_{\text{N}_G(P)} \circ \Phi_P = \text{Br}^P \circ L_G.$$  \hspace{1cm} (4.3)

This is because for any $G$-set $X$, the $k\text{N}_G(P)$-modules $k(X^P)$ and $(kX)[P]$ are isomorphic.

The ring homomorphism $L_G$ extends linearly to a $K$-algebra homomorphism $K \otimes_{\mathbb{Z}} B(G) \to K \otimes_{\mathbb{Z}} \text{pp}_k(G)$, still denoted by $L_G$. The algebra $K \otimes_{\mathbb{Z}} B(G)$ is also a split semisimple commutative $K$-algebra. Its species are the $K$-algebra maps

$$K \otimes_{\mathbb{Z}} B(G) \to K, \quad X \mapsto |X^H|,$$

where $H$ runs through the set of all subgroups of $G$ up to conjugation. Here we denote by $|X^H|$ the number of $H$-fixed points of a $G$-set $X$ and this notation is then extended $K$-linearly to any $X \in K \otimes_{\mathbb{Z}} B(G)$. The primitive idempotents $e^G_H$ of $K \otimes_{\mathbb{Z}} B(G)$ are indexed by subgroups $H$ of $G$, up to conjugation. They are given by the following formulae, found by Gluck ([4]) and later independently by Yoshida ([5]):

$$e^G_H = \frac{1}{|\text{N}_G(H)|} \sum_{L \leq H} |L| \mu(L, H) G/L,$$  \hspace{1cm} (4.4)
where $\mu$ denotes the Möbius function of the poset of subgroups of $G$. The idempotent $e^G_H$ is characterized by the fact that for any $X \in K \otimes \mathbb{Z} B(G)$

$$X \cdot e^G_H = |X^H|e^G_H.$$  

4.5. Remark: Since $|X^H|$ only depends on $\text{Res}^G_H X$, it follows in particular that $X$ is a scalar multiple of the “top” idempotent $e^G_G$ if and only if $\text{Res}^G_H X = 0$ for any proper subgroup $H$ of $G$. In particular, if $N$ is a normal subgroup of $G$, then

$$\text{(4.6)} \quad (e^G_G)^N = e^{G/N}_{G/N}.$$  

This is because for any proper subgroup $H/N$ of $G/N$

$$\text{Res}^{G/N}_{H/N}(e^G_G)^N = (\text{Res}^G_H e^G_G)^N = 0.$$  

So $(e^G_G)^N$ is a scalar multiple of $e^{G/N}_{G/N}$. As it is also an idempotent, it is equal to 0 or $e^{G/N}_{G/N}$. Finally

$$|((e^G_G)^N)^{G/N}| = |(e^G_G)^G| = 1,$$

so $(e^G_G)^N$ is non zero.

4.7. The case of a cyclic $p'$-group. Suppose that $G$ is a cyclic $p'$-group, of order $n$, generated by an element $s$. In this case, there are exactly $n$ group homomorphisms from $G$ to the multiplicative group $k^\times$ of $k$. For each of these group homomorphisms $\varphi$, let $k_\varphi$ denote the $kG$-module $k$ on which the generator $s$ acts by multiplication by $\varphi(s)$. As $G$ is a $p'$-group, this module is simple and projective. The (classes of the) modules $k_\varphi$, for $\varphi \in \widehat{G} = \text{Hom}(G, k^\times)$, form a basis of $pp_k(G)$.

Since moreover for $\varphi, \psi \in \widehat{G}$, the modules $k_\varphi \otimes_k k_\psi$ and $k_{\varphi \psi}$ are isomorphic, the algebra $K \otimes \mathbb{Z} pp_k(G)$ is isomorphic to the group algebra of the group $\widehat{G}$.

This leads to the following classical formula:

4.8. Lemma. Let $G$ be a cyclic $p'$-group. Then for any $t \in G$,

$$F^G_{1,t} = \frac{1}{n} \sum_{\varphi \in \widehat{G}} \tilde{\varphi}(t^{-1})k_\varphi,$$

where $\tilde{\varphi}$ is the Brauer character of $k_\varphi$. 

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Proof: Indeed for $s, t \in G$

$$\tau_{1,t}^G \left( \frac{1}{n} \sum_{\varphi \in \hat{G}} \hat{\varphi}(s^{-1}) k_{\varphi} \right) = \frac{1}{n} \sum_{\varphi \in \hat{G}} \hat{\varphi}(s^{-1}) \hat{\varphi}(t) = \delta_{s,t},$$

where $\delta_{s,t}$ is the Kronecker symbol.

4.9. The case $G = \langle Ps \rangle$. Suppose now more generally that $G = \langle Ps \rangle$, where $P$ is a normal Sylow $p$-subgroup of $G$ and $s$ is a $p'$-element. In this case, by Proposition 3.1, the restriction of $F_{P,s}^G$ to any proper subgroup of $G$ is equal to zero. Moreover, since $N_G(P, t) = G$ for any $t \in G/P$, the conjugacy class of the pair $(P, t)$ reduces to $\{(P, t)\}$.

4.10. Lemma. Suppose $G = \langle Ps \rangle$, and set $E_G^G = L_G(e_G^G)$. Then

$$E_G^G = \sum_{\langle t \rangle = \langle s \rangle} F_{P,t}^G.$$

Proof: By 4.2 and by Remark 4.5, the restriction of $E_G^G$ to any proper subgroup of $G$ is equal to zero. Let $(Q, t) \in Q_{G,p}$, such that the group $L = \langle Qt \rangle$ is a proper subgroup of $G$. By Proposition 3.2, there is a rational number $r$ such that

$$F_{Q,t}^G = r \text{Ind}_L^G F_{L,t}^L.$$

It follows that

$$E_G^G \cdot F_{Q,t}^G = r \text{Ind}_L^G \left( (\text{Res}_L^G E_G^G) \cdot F_{L,t}^L \right) = 0.$$

Now $E_G^G$ is an idempotent of $K \otimes_{\mathbb{Z}} pp_k(G)$, hence it is a sum of some of the primitive idempotents $F_{Q,t}^G$ associated to pairs $(Q, t)$ for which $\langle Qt \rangle = G$. This condition is equivalent to $Q = P$ and $\langle t \rangle = \langle s \rangle$.

It remains to show that all these idempotents $F_{P,t}^G$ appear in the decomposition of $E_G^G$, i.e. equivalently that $\tau_{P,t}^G(E_G^G) = 1$ for any generator $t$ of $\langle s \rangle$. Now by 4.6 and Remark 2.16

$$\tau_{P,t}^G(E_G^G) = \tau_{1,t}^G(\text{Br}_P^G(E_G^G)) = \tau_{1,t}^{\langle s \rangle}(E_{\langle s \rangle}^{\langle s \rangle}).$$

Now the value at $t$ of the Brauer character of a permutation module $kX$ is equal to the number of fixed points of $t$ on $X$. By $K$-linearity, this gives

$$\tau_{1,t}^{\langle s \rangle}(E_{\langle s \rangle}^{\langle s \rangle}) = |(e_{\langle s \rangle}^{\langle s \rangle})^t|,$$

and this is equal to 1 if $t$ generates $\langle s \rangle$, and to 0 otherwise, as was to be shown. $\square$
4.11. Proposition. Let \((P, s) \in \mathcal{Q}_{G,p}\), and suppose that \(G = \langle Ps \rangle\). Then
\[
F_{P,s}^G = E_G^G \cdot \text{Inf}_{G/p}^G F_{1,s}^{G/P}.
\]

Proof: Set \(E_s = E_{G}^G \cdot \text{Inf}_{G/P} F_{1,s}^{G/P}\). Then \(E_s\) is an idempotent of \(K \otimes_{\mathbb{Z} p p k} (G)\), as it is the product of two (commuting) idempotents. Let \((Q, t) \in \mathcal{Q}_{G,p}\). If \(\langle Qt \rangle \neq G\), then \(\tau_{Q,t}^G(E_G^G) = 0\) by Lemma 4.10, thus \(\tau_{Q,t}^G(E_s) = 0\). And if \(\langle Qt \rangle = G\), then \(Q = P\) and \(\langle t \rangle = \langle s \rangle\). In this case
\[
\tau_{Q,t}^G(E_s) = \tau_{P,t}^G(\text{Inf}_{G/P} F_{1,s}^{G/P}) \cdot \tau_{P,t}^G(\text{Inf}_{G/P} F_{1,s}^{G/P}) = \delta_{t,s},
\]
where \(\delta_{t,s}\) is the Kronecker symbol. Hence \(\tau_{P,t}^G(E_s) = \delta_{t,s}\), and this completes the proof.

4.12. Theorem. Let \(G\) be a finite group, and let \((P, s) \in \mathcal{Q}_{G,p}\). Then the primitive idempotent \(F_{P,s}^G\) of the \(p\)-permutation algebra \(K \otimes_{\mathbb{Z} p p k} (G)\) is given by the following formula:
\[
F_{P,s}^G = \frac{1}{|P||s||C_{\mathcal{N}_G(P)}(s)|} \sum_{\varphi \in \langle s \rangle, \overline{L} \leq \langle Ps \rangle, P L = \langle Ps \rangle} \tilde{\varphi}(s^{-1}) |L| \mu(L, <Ps>) \text{Ind}_{L}^G k_{L,\varphi}^{<Ps>},
\]
where \(k_{L,\varphi}^{<Ps>} = \text{Res}_{L}^{<Ps>} \text{Inf}_{<s>}^{<Ps>} k_{\varphi}\).

Proof: By Corollary 3.4, and Proposition 4.11
\[
F_{P,s}^G = \frac{|s|}{|C_{\mathcal{N}_G(P)}(s)|} \text{Ind}_{<Ps>}^G (E_{<Ps>} \cdot \text{Inf}_{<s>}^{<Ps>} F_{1,s}^{<s>}).
\]
By Equation 4.4, this gives
\[
F_{P,s}^G = \frac{|s|}{|C_{\mathcal{N}_G(P)}(s)|} \text{Ind}_{<Ps>}^G \frac{1}{|P||s||L \leq <Ps>|} |L| \mu(L, <Ps>) \text{Ind}_{L}^{<Ps>} k \cdot \text{Inf}_{<s>}^{<Ps>} F_{1,s}^{<s>}.
\]
Moreover for each $L \leq <Ps>$

$$\text{Ind}_{<ss>}^{<Ps>} k \cdot \text{Ind}_{<ss>}^{<Ps>} F_{<ss>}^{<ss>} \cong \text{Ind}_{<ss>}^{<Ps>} (\text{Res}_{<ss>}^{<Ps>} \text{Ind}_{<ss>}^{<Ps>} F_{<ss>}^{<ss>})$$

$$\cong \text{Ind}_{<ss>}^{<Ps>} \text{Inf}_{L/L \cap P}^{L/P} \text{Iso}_{L/P}^{L/L \cap P} \text{Res}_{L/P}^{L/P} F_{<ss>}^{<ss>}.$$ 

Here we have used the fact that if $L$ and $P$ are subgroups of a group $H$, with $P \leq H$, then there is an isomorphism of functors

$$\text{Res}_{H}^{H/P} \circ \text{Inf}_{H/P}^{H} \cong \text{Inf}_{L/L \cap P}^{L/P} \circ \text{Iso}_{L/P}^{L/L \cap P} \circ \text{Res}_{H/P}^{H/P},$$

which follows from the isomorphism of $(L, H/P)$-bisets

$$H \times_H (H/P) \cong L(H/P) \cong (L/L \cap P) \times_{L/L \cap P} (L/P) \times_{L/P} (H/P).$$

Now Proposition 3.1 implies that $\text{Res}_{L/P}^{L/P} F_{<ss>}^{<ss>} = 0$ if $LP/P \neq <ss>$, i.e. equivalently if $PL \neq <Ps>$. It follows that

$$F_{P,s}^{G} = \frac{1}{|P||C_{N_G(P)}(s)|} \sum_{L \leq <Ps>: PL = <Ps>} |L| \mu(L, <Ps>) \text{Ind}_{L}^{G} (\text{Res}_{L/P}^{L/P} \text{Inf}_{<ss>}^{<Ps>} F_{<ss>}^{<ss>}).$$

By Lemma 4.8, this gives

$$F_{P,s}^{G} = \frac{1}{|P||s||C_{N_G(P)}(s)|} \sum_{\varphi \in <ss>: L \leq <Ps> \atop PL = <Ps>} \hat{\varphi}(s^{-1}) |L| \mu(L, <Ps>) \text{Ind}_{L}^{G} k_{L,\varphi}^{<Ps>},$$

where $k_{L,\varphi}^{<Ps>} = \text{Res}_{L}^{<Ps>} \text{Inf}_{<ss>}^{<Ps>} k_{\varphi}$, as was to be shown.

References


Serge Bouc, CNRS-LAMFA, Université de Picardie - Jules Verne, 
33, rue St Leu, F-80039 Amiens Cedex 1, France.  
serge.bouc@u-picardie.fr

Jacques Thévenaz, Institut de Géométrie, Algèbre et Topologie,  
EPFL, Bâtiment BCH, CH-1015 Lausanne, Switzerland.  
Jacques.Thevenaz@epfl.ch