Bisets as categories, and tensor product of induced bimodules

Serge Bouc

Abstract : Bisets can be considered as categories. This note uses this point of view to give a simple proof of a Mackey-like formula expressing the tensor product of two induced bimodules.

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1. Introduction

Let R be a commutative ring, let G and H be finite groups, let X be a subgroup of $H\times G$, and M be an RX-module. If $m\in M$ and $(h,g)\in X$, set $h\cdot m\cdot g^{-1}=(h,g)\cdot m$: this is a slight extension of the usual correspondence between $R(H\times G)$-modules and (RH,RG)-bimodules.

The object of this note is to give a simple proof of the following result :

1.1. Theorem : *Let R be a commutative ring, let G, H, and K be finite groups, let X be a subgroup of $H\times G$ and Y be a subgroup of $K\times H$. Let M be an RX-module, and N be an RY-module. Then there is an isomorphism of (RK,RG)-bimodules*

$$(\mathrm{Ind}_Y^{K\times H}N)\otimes_{RH}(\mathrm{Ind}_X^{H\times G}M)\cong\bigoplus_{t\in[p_2(Y)\backslash H/p_1(X)]}\mathrm{Ind}_{Y*^{(t,1)}X}^{K\times G}(N\otimes_{k_2(Y)\cap{}^tk_1(X)}{}^{(t,1)}M)\ \ .$$

where the notation is as follows (cf. [1]) :

$$p_1(X)=\{h\in H\mid \exists g\in G,\ (h,g)\in X\},\ \ k_1(X)=\{h\in H\mid (h,1)\in X\}$$

$$p_2(Y)=\{h\in H\mid \exists k\in K,\ (k,h)\in Y\},\ \ k_2(Y)=\{h\in H\mid (1,h)\in Y\}$$

$$Y*^{(t,1)}X=\{(k,g)\in K\times G\mid \exists h\in H,\ (k,h)\in Y,\ (h^t,g)\in X\}\ \ .$$

The action of $(k,g)\in Y*^{(t,1)}X$ on $N\otimes_{k_2(Y)\cap{}^tk_1(X)}{}^{(t,1)}M$ is given by

$$k\cdot(n\otimes m)\cdot g^{-1}=(k\cdot n\cdot h^{-1})\otimes(h^t\cdot m\cdot g^{-1})\ \ ,$$

if $h\in H$ is chosen such that $(k,h)\in Y$ and $(h^t,g)\in X$.

2. Functors over bisets

Recall that when G and H are groups, *an* (H,G)*-biset* U is a set equipped with a left action of H and a right action of G which commute, i.e. such that $(hu)g = h(ug)$ for any $h \in H$, $u \in U$, and $g \in G$.

2.1. Notation : *Let G and H be groups. When U is an (H,G)-biset, let $\langle U \rangle$ denote the following category :*

- *The objects of $\langle U \rangle$ are the elements of U.*
- *If $u, v \in U$, then*

$$\mathrm{Hom}_{\langle U \rangle}(u, v) = \{(h, g) \in H \times G \mid hu = vg\} \enspace .$$

- *If $u, v, w \in U$, the composition of the morphisms $(h, g) : u \to v$ and $(h', g') : v \to w$ is the morphism $(h'h, g'g) : u \to w$.*
- *If $u \in U$, the identity morphism $\mathrm{Id}_u : u \to u$ is the pair $(1, 1) \in G \times G$.*

Note that the category $\langle U \rangle$ is a groupoid (any morphism is an isomorphism), and that for any $u \in U$, the group

$$A(u) = \mathrm{Hom}_{\langle U \rangle}(u, u) = \{(h, g) \in H \times G \mid hu = ug\}$$

is a subgroup of $H \times G$.

A functor M from $\langle U \rangle$ to a category $\mathcal{C}$ consists of a collection of objects $M(u)$ of $\mathcal{C}$, for $u \in U$, together with morphisms

$$M(h, g) : M(u) \to M(hug^{-1})$$

in the category $\mathcal{C}$, for $(h, g) \in H \times G$, fulfilling the usual functorial conditions. In particular, for each $u \in U$, there is a group homomorphism $A(u) \to \mathrm{Aut}_{\mathcal{C}} M(u)$.

Functors from $\langle U \rangle$ to $\mathcal{C}$ are the objects of a category $\mathsf{Fun}(\langle U \rangle, \mathcal{C})$, in which the morphisms are natural transformation of functors.

2.2. Notation : *When $\mathcal{C}$ is a subcategory of the category* Sets *of sets, and M is a functor $\langle U \rangle \to \mathcal{C}$, the image of $m \in M(u)$ by the map $M(h, g) : M(u) \to M(hug^{-1})$, for $(h, g) \in H \times G$, will be denoted by hmg^{-1}.*

In this case, a functor $M : \langle U \rangle \to \mathcal{C}$ is a collection of objects $M(u)$ of $\mathcal{C}$, for $u \in U$, together with morphisms $m \mapsto hmg^{-1} : M(u) \to M(hug^{-1})$ in $\mathcal{C}$, for $(h,g) \in H \times G$, such that $h'(hmg^{-1})g'^{-1} = (h'h)m(g'g)^{-1}$ and $1m1 = m$, for any (h,g), (h',g') in $H \times G$, any $u \in U$, and any $m \in M(u)$.

2.3. Example : Suppose that $\mathcal{C} = \mathsf{Sets}$. Then the disjoint union $\bigsqcup M = \bigsqcup\limits_{u \in U} M(u)$ becomes an (H,G)-biset, and the map $\bigsqcup M \to U$ sending elements of $M(u)$ to u, for $u \in U$, is a map of (H,G)-bisets. Conversely, if $\pi : S \to U$ is a map of (H,G)-bisets, then the assignment $u \mapsto \pi^{-1}(u)$ is a functor from $\langle U \rangle$ to Sets.

In other words, a functor $\langle U \rangle \to \mathsf{Sets}$ is just an (H,G)-biset over U. More precisely, the category $\mathsf{Fun}(\langle U \rangle, \mathsf{Sets})$ is equivalent to the category of (H,G)-bisets over U.

2.4. Example : Let R be a commutative ring. In the remainder of this note, the category $\mathcal{C}$ will be the category R-Mod of (left) R-modules. If M is functor from $\langle U \rangle$ to R-Mod, then for each $u \in U$, the R-module $M(u)$ has a natural structure of $RA(u)$-module.

Conversely, let $[H\backslash U/G]$ be a set of representatives of (H,G)-orbits on U. Equivalently $[H\backslash U/G]$ is a set of representatives of isomorphism classes in the category $\langle U \rangle$. Since $\langle U \rangle$ is a groupoid, it is equivalent to its full subcategory $[H\backslash U/G]$. In particular, this yields an equivalence of categories

$$\mathsf{Fun}(\langle U \rangle, R\text{-}\mathsf{Mod}) \cong \prod_{u \in [H\backslash U/G]} RA(u)\text{-}\mathsf{Mod} \quad . \tag{2.5}$$

2.6. Remark : In the situation of Example 2.4, the direct sum

$$\Sigma(M) = \mathop{\oplus}_{u \in U} M(u)$$

has a natural structure of (RH,RG)-bimodule, i.e. using the usual group isomorphism $(h,g) \mapsto (h,g^{-1})$ from $H \times G^{op}$ to $H \times G$, of left $R(H \times G)$-module.

Moreover, is is easy to see that there is an isomorphism of (RH,RG)-bimodules

$$\Sigma(M) \cong \mathop{\oplus}_{u \in [H\backslash U/G]} \mathrm{Ind}_{A(u)}^{H \times G} M(u) \quad .$$

3. Product of bisets, and product of functors

Let G, H and K be groups. If U is an (H,G)-biset and V is a (K,H)-biset, recall that the product (or *composition*) of V and U is the set

$$V \times_H U = (V \times U)/H \quad ,$$

where the right action of H on $(V \times U)$ is defined by $(v,u)\cdot h = (vh, h^{-1}u)$, for $v \in V$, $u \in U$, and $h \in H$. The set $V \times_H U$ is a (K,G)-biset for the following action

$$\forall z \in K,\ \forall x \in G,\ \forall v \in V,\ \forall u \in U,\ \ z\cdot(v,_{_H} u)\cdot x = (zv,_{_H} ux) \ \ ,$$

where $(v,_{_H} u)$ denotes the H-orbit of (v,u).

3.1. Definition : *Let G, H, and K be finite groups. Let U be an (H,G)-biset, and V be a (K,H)-biset. If M is a functor $\langle U\rangle \to R\text{-}\mathsf{Mod}$ and N is a functor $\langle V\rangle \to R\text{-}\mathsf{Mod}$, the tensor product $N \otimes_H M$ is the functor $\langle V \times_H U\rangle \to R\text{-}\mathsf{Mod}$ defined by*

$$(N\otimes_H M)(v,_{_H} u) = \big(\mathop{\oplus}_{h\in H} N(vh)\otimes_R M(h^{-1}u)\big)/\mathcal{I}_{v,u} \ \ ,$$

where $\mathcal{I}_{v,u}$ is the R-submodule generated by the elements of the form

$$[ny\otimes y^{-1}m]_{hy} - [n\otimes m]_h \ \ ,$$

where $y \in H$, and where $[n\otimes m]_h$ denotes the element $n\otimes m$ of the component indexed by $h \in H$ in the direct sum, for $n \in N(vh)$, and $m \in M(h^{-1}u)$.

If $(k,g) \in K \times G$, then by definition

$$k\,[n\otimes m]_h\, g = [kn\otimes mg]_h \ \ .$$

3.2. Remark : It follows from this definition that

$$(N\otimes_H M)\big((v,_{_H} u)\big) \cong N(v)\otimes_{RH_{v,u}} M(u) \ \ ,$$

where $H_{v,u}$ is the set of elements $h \in H$ such that $vh = v$ and $hu = u$.

3.3. Lemma : *There is an isomorphism of (RK,RG)-bimodules*

$$\Sigma(N)\otimes_{RH}\Sigma(M) \cong \Sigma(N\otimes_H M) \ \ ,$$

sending (from right to left) the element $[n\otimes m]_h$ to $n\otimes_{RH} m$.

Proof : To be more precise, the map α from

$$\Sigma(N\otimes_H M) = \mathop{\oplus}_{(v,_{_H} u)\in V\times_H U} \big(\oplus_{h\in H} N(vh)\otimes_R M(h^{-1}u)\big)/\mathcal{I}_{v,u}$$

sending the element $[n \otimes m]_h$ in the component indexed by $(v,_H u)$ to the element $n \otimes m$ of the tensor product

$$\Sigma(N) \otimes_{RH} \Sigma(M) = \big(\mathop{\oplus}_{v \in V} N(v) \big) \otimes_{RH} \big(\mathop{\oplus}_{u \in U} M(u) \big)$$

is well defined. To show that it is an isomorphism, define a map

$$\beta : \Sigma(N) \otimes_{RH} \Sigma(M) \to \Sigma(N \otimes_H M)$$

in the following way : choose a set S of representatives of the classes $(v,_H u)$. Now map the element $n \otimes_{RH} m \in N(v) \otimes M(u) \subseteq \Sigma(N) \otimes_{RH} \Sigma(M)$ to $[n \otimes m]_h$, where $h \in H$ is chosen such that $(vh^{-1}, hu) \in S$. Again, it is easy to see that this map is well defined, and that the maps α and β are mutual inverse isomorphisms of (RK, RG)-bimodules. □

3.4. Corollary : *Let G, H, and K be finite groups. Let X be a subgroup of $H \times G$ and Y be a subgroup of $K \times H$. Let M be an RX-module, and N be an RY-module. Then there is an isomorphism of (RK, RG)-bimodules*

$$(\mathrm{Ind}_Y^{K \times H} N) \otimes_{RH} (\mathrm{Ind}_X^{H \times G} M) \cong \mathop{\oplus}_{t \in p_2(Y) \backslash H / p_1(X)} \mathrm{Ind}_{Y * {}^{(t,1)}X}^{K \times G} (N \otimes_{k_2(Y) \cap {}^t k_1(X)} {}^{(t,1)}M) \quad .$$

Proof : Set $U = (H \times G)/X$. Then U is an (H, G)-biset by $h \cdot (t, s)X \cdot g = (ht, g^{-1}s)X$, and this biset is transitive. If u is the point X of U, then $A(u) = X$, and the equivalence of categories 2.5 reads

$$\mathsf{Fun}(\langle U \rangle, R\text{-}\mathsf{Mod}) \cong RX\text{-}\mathsf{Mod} \quad .$$

More precisely, for an RX-module M, this equivalence yields a functor $\tilde{M} : \langle U \rangle \to R\text{-}\mathsf{Mod}$ in the following way : for any $(h, g) \in H \times G$, set

$$\tilde{M}\big((h, g)X\big) = M \quad .$$

Next, fix a set S of representatives of elements of U, i.e. X-cosets in $H \times G$. For $(t, s) \in S$, and $(h, g) \in H \times G$, define a map

$$\tilde{M}(h, g) : \tilde{M}\big((t, s)X\big) = M \to \tilde{M}\big((ht, gs)X\big) = M$$

by $\tilde{M}(h, g)(m) = (y, x)m$, where (y, x) is the unique element of X such that $(ht, gs)(y, x)^{-1} \in S$.

Then it is easy to check that $\tilde{M}$ is indeed a functor, and that there is an isomorphism of (RH, RG)-bimodules

$$\Sigma(\tilde{M}) \cong \mathrm{Ind}_X^{H \times G} M \quad .$$

Similarly, set $V = (K \times H)/Y$, and define a functor $\tilde{N}\langle V\rangle \to R\text{-}\mathsf{Mod}$, using the RY-module N. Then the corollary is a straightforward consequence of the lemma, applied to the functors $\tilde{M}$ and $\tilde{N}$, using Remark 2.6 and Remark 3.2. □

References

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Serge Bouc
CNRS-LAMFA
Université de Picardie
33 rue St Leu
80039 - Amiens Cedex 1
FRANCE
`email : serge.bouc@u-picardie.fr`