Biset functors

Serge Bouc

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1 Groups and morphisms
Overview

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   - Group homomorphisms
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2 The biset category
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Group homomorphisms and representations

When $G$ and $H$ are (finite) groups, the natural notion of morphism $f : G \to H$ is the notion of group homomorphism. It leads to the category of groups: objects are groups, and morphisms are group homomorphisms. With this definition, many objects naturally attached to groups become contravariant functors: e.g. any $K_H$-module can be viewed as a $K_G$-module, by restriction along $f$. This yields a map $R^K(f) : R^K(H) \to R^K(G)$ between representation groups. Any $H$-set can be viewed as a $G$-set, and this yields a map $B(f) : B(H) \to B(G)$ between Burnside groups.

Examples:
- If $f$ is the inclusion map $G \leq H$, then $R^K(f)$ and $B(f)$ are ordinary restriction maps, denoted by $\text{Res}^H_G$.
- If $N \unlhd G$, and $f : G \to H = G/N$ is the projection map, then $R^K(f)$ and $B(f)$ are inflation maps, denoted by $\text{Inf}^G_H$. 
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It leads to the category of groups: objects are groups, and morphisms are group homomorphisms.

With this definition, many objects naturally attached to groups become contravariant functors:

- For any $K$-$H$-module, it becomes a $K$-$G$-module by restriction along $f$.
- Any $H$-set becomes a $G$-set through $f$.

Examples:
- If $f$ is the inclusion map $G \leq H$, then $R_{K,H}(f)$ and $B(f)$ are ordinary restriction maps, denoted by $\text{Res}_{H,G}$.
- If $N \trianglelefteq G$ and $f : G \rightarrow H = G/N$ is the projection map, then $R_{K,H}(f)$ and $B(f)$ are inflation maps, denoted by $\text{Inf}_{G,H}$. 
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- any $\mathbb{K}H$-module can be viewed as a $\mathbb{K}G$-module, by restriction along $f$. This yields a map $R_\mathbb{K}(f) : R_\mathbb{K}(H) \to R_\mathbb{K}(G)$ between representation groups.
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- If $f$ is the inclusion map $G \leq H$, then $R_\mathbb{K}(f)$ and $B(f)$ are ordinary restriction maps,
Group homomorphisms and representations

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  - any $KH$-module can be viewed as a $KG$-module, by restriction along $f$. This yields a map $R_K(f) : R_K(H) \to R_K(G)$ between representation groups.
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- **Examples:**
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From homomorphisms to bisets

- There are other natural covariant operations:

If $G \leq H$, there is an induction map $\text{Ind}^H_G : R^K(G) \to R^K(H)$, and also $\text{Ind}^H_G : B(G) \to B(H)$.

If $H = G/N$, there is a deflation map $\text{Def}^G_H : B(G) \to B(H)$, induced by $X \mapsto N \backslash X$.

If $\text{char}(K) \nmid |N|$, there is $\text{Def}^G_H : R^K(G) \to R^K(H)$, induced by $V \mapsto V N = V / [N, V]$.

It is often useful to compose these two different kinds of operations. This gives a list or formulae, e.g. the celebrated Mackey formula $\text{Res}^G_K \circ \text{Ind}^G_H = \sum_{x \in K \setminus G/H}$.

Question: Is there a way to encode all these operations ($\text{Res}^H_G$, $\text{Ind}^H_G$, $\text{Def}^G_{G/N}$, $\text{Def}^G_{G/N}$) in a single formalism? Is there a category in which all these would appear as "morphisms"?

Answer: Yes. The biset category for finite groups.
There are other natural covariant operations:

- If $G \leq H$, there is an induction map $\text{Ind}^H_G : R_K(G) \to R_K(H)$.

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There are other natural **covariant** operations:

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**Answer**: Yes.
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**Answer**: Yes: the biset category for finite groups.
Bisets

Definition

Let $G$ and $H$ be groups.
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Bisets

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$$(h \cdot u) \cdot g = h \cdot (u \cdot g).$$
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Let $\mathbb{K}$ be a field. The vector space $\mathbb{K}U$ is a $(\mathbb{K}H, \mathbb{K}G)$-bimodule.
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[ Equivalently, an $(H, G)$-biset is an $(H \times G^{\text{op}})$-set. ]

Let $\mathbb{K}$ be a field. The vector space $\mathbb{K}U$ is a $(\mathbb{K}H, \mathbb{K}G)$-bimodule. This gives a functor $L \mapsto \mathbb{K}U \otimes_{\mathbb{K}G} L$ from $\mathbb{K}G\text{-Mod}$ to $\mathbb{K}H\text{-Mod}$,
Bisets

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Let $G$ and $H$ be groups. An $(H, G)$-biset $U$ is a (finite) set with a left $H$-action and a right $G$-action, which commute i.e.

$$\forall h \in H, \forall u \in U, \forall g \in G, \ (h \cdot u) \cdot g = h \cdot (u \cdot g).$$

[ Equivalently, an $(H, G)$-biset is an $(H \times G^{\text{op}})$-set. ]

Let $\mathbb{K}$ be a field. The vector space $\mathbb{K}U$ is a $(\mathbb{K}H, \mathbb{K}G)$-bimodule. This gives a functor $L \mapsto \mathbb{K}U \otimes_{\mathbb{K}G} L$ from $\mathbb{K}G\text{-Mod}$ to $\mathbb{K}H\text{-Mod}$, which induces a map $R_{\mathbb{K}}(U) : R_{\mathbb{K}}(G) \to R_{\mathbb{K}}(H)$, if $\text{char} \mathbb{K} \nmid |G_u| \ \forall u \in U \ldots$
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- If $H \leq G$, let $\text{Res}^G_H = _H G_G$. Then $R_{\mathbb{K}}(\text{Res}^G_H)$ is the restriction map $R_{\mathbb{K}}(G) \rightarrow R_{\mathbb{K}}(H)$. 

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- If $H \leq G$, let $\text{Res}_H^G = _HG_G$. Then $R_\mathbb{K}(\text{Res}_H^G)$ is the restriction map $R_\mathbb{K}(G) \to R_\mathbb{K}(H)$.

- Let $\text{Ind}_H^G = _GG_H$. 

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Bisets

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- If $H \leq G$, let $\text{Res}^G_H = _HG$. Then $R_\mathbb{K}(\text{Res}^G_H)$ is the restriction map $R_\mathbb{K}(G) \rightarrow R_\mathbb{K}(H)$.
- Let $\text{Ind}^G_H = _GG$. Then $R_\mathbb{K}(\text{Ind}^G_H)$ is the induction map $R_\mathbb{K}(H) \rightarrow R_\mathbb{K}(G)$. 

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If $N \trianglelefteq G$, and $H = G/N$, let $\text{Inf}_H^G = gH_H$. 
If \( N \trianglelefteq G \), and \( H = G/N \), let \( \text{Inf}^G_H = gHg \). Then \( R_K(\text{Inf}^G_H) \) is the inflation map \( R_K(H) \to R_K(G) \).
Bisets

- If $N \trianglelefteq G$, and $H = G/N$, let $\text{Inf}^G_H = gH$. Then $R_K(\text{Inf}^G_H)$ is the inflation map $R_K(H) \rightarrow R_K(G)$.
- Let $\text{Def}^G_H = hHg$.
If $N \trianglelefteq G$, and $H = G/N$, let $\text{Inf}_H^G = {}_G H_\cdot$. Then $R_K(\text{Inf}_H^G)$ is the inflation map $R_K(H) \to R_K(G)$.

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If \( f : G \to H \), let \( \text{Iso}(f) = h H_f g \).
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If $f : G \xrightarrow{\cong} H$, let $\text{Iso}(f) = hH_{fG}$. Then $R_K(\text{Iso}(f))$ is the transport by isomorphism $R_K(G) \to R_K(H)$. 
If $N \trianglelefteq G$, and $H = G/N$, let $\text{Inf}_G^H = {}_GH_H$. Then $R^K(\text{Inf}_G^H)$ is the inflation map $R^K(H) \rightarrow R^K(G)$.

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When $H = G$ and $f = \text{Id}$, set $\text{Id}_G = \text{Iso}(f)$. Thus $\text{Id}_G$ is the set $G$, for its $(G, G)$-biset structure by multiplication.
If $N \trianglelefteq G$, and $H = G/N$, let $\text{Inf}^G_H = \lambda_H^G$. Then $R_K(\text{Inf}^G_H)$ is the inflation map $R_K(H) \to R_K(G)$.

Let $\text{Def}^G_H = \kappa_H^G$. Then $R_K(\text{Def}^G_H)$ is the deflation map $R_K(G) \to R_K(H)$.

If $f : G \to H$, let $\text{Iso}(f) = \kappa_f^G$. Then $R_K(\text{Iso}(f))$ is the transport by isomorphism $R_K(G) \to R_K(H)$.

When $H = G$ and $f = \text{Id}$, set $\text{Id}_G = \text{Iso}(f)$. Thus $\text{Id}_G$ is the set $G$, for its $(G, G)$-biset structure by multiplication. The map $R_K(\text{Id}_G)$ is the identity map of $R_K(G)$. 
Let $G$, $H$, and $K$ be groups. Let $U$ be an $(H, G)$-biset, and $V$ be a $(K, H)$-biset.
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**Definition**

The composition of $V$ and $U$ is the set $V \circ U = (V \times U) / \langle (vh, u) = (v, hu) \mid \forall v \in V, \forall h \in H, \forall u \in U \rangle = V \times H U$. It is a $(K, G)$-biset by $k(v, H u) g = (kv, H u g)$, for $k \in K$, $v \in V$, $u \in U$, and $g \in G$. 

$K V \otimes K H K U \sim \Rightarrow R_K(V) \circ R_K(U) = R_K(V \circ U)$. 

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$$K V \otimes_{K H} K U \cong K (V \times_H U) \Rightarrow R^K (V) \circ R^K (U) = R^K (V \circ U).$$
Properties of $R_K$

1. If $G$ is a group, then $F(G)$ is an abelian group.
2. If $U$ is an $(H,G)$-biset, then $F(U) : F(G) \to F(H)$.
3. If $U$ and $U'$ are isomorphic $(H,G)$-bisets, then $F(U) = F(U')$.
4. If $U$ and $U'$ are $(H,G)$-bisets, then $F(U \sqcup U') = F(U) + F(U')$.
5. If $U$ is an $(H,G)$-biset and $V$ is a $(K,H)$-biset, then $F(V) \circ F(U) = F(V \circ U)$.
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Remark: Conditions 3 and 4 allow to define a map $F(\alpha) : F(G) \to F(H)$, for any $\alpha$ in the Burnside group $B(H,G)$ of $(H,G)$-bisets.
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Remark: Conditions 3 and 4 allow to define a map $F(\alpha) : F(G) \to F(H)$, for any $\alpha$ in the Burnside group $B(H, G)$ of $(H, G)$-bisets.
A biset functor $F$ consists of the following data:

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3. If $U$ and $U'$ are isomorphic $(H, G)$-bisets, then $F(U) = F(U')$.
4. If $U$ and $U'$ are $(H, G)$-bisets, then $F(U \sqcup U') = F(U) + F(U')$.
5. If $U$ is an $(H, G)$-biset and $V$ is a $(K, H)$-biset, then $F(V) \circ F(U) = F(V \circ U)$.
6. If $G$ is a group, then $F(Id_G) = Id_{F(G)}$.

Remark: Conditions 3 and 4 allow to define a map $F(\alpha) : F(G) \to F(H)$, for any $\alpha$ in the Burnside group $B(H, G)$ of $(H, G)$-bisets.
The biset category

Let $p$ be a prime number.

Definition

The biset category $C_p$ for finite $p$-groups is defined as follows:

- The objects are finite $p$-groups.
- If $G$, $H$ are finite $p$-groups, then $\text{Hom}_{C_p}(G, H) = B(H, G)$.
- The composition of morphisms $G \to H \to K$ is obtained by linearly extending the product $(V, U) \mapsto V \times HU$ of bisets.
- The identity morphism of $G$ is the (class of) the $(G, G)$-biset $\text{Id}_G$.

Definition

A $p$-biset functor is an additive functor from $C_p$ to $\mathbb{Z}$-Mod.

$p$-Biset functors form an abelian category $\mathcal{F}_p$.

Remark:

One can also consider only some types of bisets (e.g. left or right free).
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**Remark**: One can define similarly biset functors with values in $R\text{-Mod}$ (where $R$ is a commutative ring).
Some properties of the biset category

- The biset category $\mathcal{C}$ is a pre-additive category (cf. Mc Lane).
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$\mathcal{C}$ is also an $\ast$-category:

- $\alpha \mapsto \alpha^{\text{op}}$ from $B(H, G)$ to $B(G, H)$.
- The functor $\ast : \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ is an equivalence of categories $\mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$.
- This gives a duality in the category $\mathcal{F}$: if $\mathcal{F}$ is a biset functor, the dual functor $\mathcal{F}^{\ast}$ is defined as the composition $\mathcal{C}^{\ast} \rightarrow \mathcal{C}^{\text{op}} \mathcal{F} \rightarrow (\mathcal{Z}-\text{Mod})^{\text{op}} \rightarrow \mathcal{Z}-\text{Mod}$.

In other words $\{\mathcal{F}^{\ast}(G) = \text{Hom}_{\mathcal{Z}}(\mathcal{F}(G), \mathcal{Z})\}$. $\mathcal{F}^{\ast}(\alpha) = t^{\mathcal{F}(\alpha^{\text{op}})}$. 

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Remarks

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- Any \((H, G)\)-biset is a disjoint union of transitive ones. The classes of transitive \((H, G)\)-bisets form a \(\mathbb{Z}\)-basis of \(B(H, G)\).
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Any transitive \((H, G)\)-biset is isomorphic to a composition

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where \(A \trianglelefteq B \leq G\), \(C \trianglelefteq D \leq H\), and \(f : B/A \xrightarrow{\cong} D/C\).

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Let $F$ be a biset functor, and $G$ be a group. The set $\partial F(G)$ of faithful elements in $F(G)$ is defined by

$$\partial F(G) = \bigcap_{1 < N \trianglelefteq G} \text{Ker Def}_{G/G/N}.$$ 

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The map $\bigoplus_{N \trianglelefteq G} \text{Inf}_{G/G/N} : \bigoplus_{N \trianglelefteq G} \partial F(G/N) \to F(G)$ is an isomorphism.

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**Example**: If $F = R_K$, then $\partial F(G)$ is the set of linear combinations of faithful irreducible $K^G$-modules.

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There is an explicit one-to-one correspondence

Simple biset functors with values in $\mathbb{R}$-$\text{Mod}$ up to isomorphism $\leftrightarrow$ Pairs $(H, L) \{ H \text{ finite group} \}

$L$ simple $\text{ROut}(H)$-$\text{module}$ up to isomorphism

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Problem: Describe the $p$-biset functor $R^Q$?

One can describe the full lattice of subfunctors of $R^Q$.

Main tool: the linearization morphism $\chi: B \to R^Q$, induced by $X \mapsto QX$.

Look at the rational representations of $p$-groups. . .

Revisit a theorem of Roquette (1958), to obtain a description of the rational irreducible representations of a $p$-group $P$ in purely combinatorial terms.

**Definition**

A finite $p$-group has normal $p$-rank 1 if all its normal abelian subgroups are cyclic.

If $Q$ has normal $p$-rank 1, then $Q$ is cyclic, generalized quaternion, dihedral of order at least 16, or semi-dihedral.

**Lemma**

If $Q$ has normal $p$-rank 1, then $Q$ has a unique faithful rational irreducible representation $\Phi_Q$. In other words $\partial R_Q(Q) = \langle \Phi_Q \rangle \cong Z$. 
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Notation

If $S \leq P$, let $Z_P(S)$ be the subgroup of $\frac{N_P(S)}{S}$ defined by $Z_P(S) = \frac{Z(N_P(S))}{S}$.

Definition

The subgroup $S$ is a genetic subgroup of $P$ if the following two conditions hold:

1. The group $\frac{N_P(S)}{S}$ has normal $p$-rank 1.
2. If $x \in P$ is such that $Sx \cap Z_P(S) \leq S$, then $Sx = S$.

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If $S$ is a genetic subgroup of $P$, set $V(S) = \text{Ind}_P^{N_P(S)} \text{Inf}_{N_P(S)} N_P(S)/S \Phi_{N_P(S)/S}$.
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Define a relation $\triangleleft_P$ on the set of subgroups of $P$ by

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Genetic bases of $p$-groups

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**Theorem**

1. If $S$ is a genetic subgroup of $P$, then $V(S)$ is a simple $\mathbb{Q}P$-module.

In particular, $\triangleleft_P$ is an equivalence relation on the set of genetic subgroups of $P$. A genetic basis of $P$ is a set of representatives of equivalence classes. A genetic basis of $P$ is in one-to-one correspondence with Irr$_\mathbb{Q}(P)$. 

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Biset functors

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2. If $V$ is a simple $\mathbb{Q}P$-module, then there exists a genetic subgroup $S$ of $P$ such that $V \cong V(S)$.

A genetic basis of $P$ is a set of representatives of equivalence classes. A genetic basis of $P$ is in one-to-one correspondence with $\text{Irr}_{\mathbb{Q}}(P)$. 

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S \triangleleft_P T \iff \exists x \in P, \; S^x \cap Z_P(T) \leq T \text{ and } T \cap Z_P(S^x) \leq S^x.
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**Theorem**

1. If \( S \) is a genetic subgroup of \( P \), then \( V(S) \) is a simple \( \mathbb{Q}P \)-module.
2. If \( V \) is a simple \( \mathbb{Q}P \)-module, then there exists a genetic subgroup \( S \) of \( P \) such that \( V \cong V(S) \).
3. If \( S \) and \( T \) are genetic subgroups of \( P \), then \( V(S) \cong V(T) \) if and only if \( S \triangleleft_P T \).

In particular \( \triangleleft_P \) is an equivalence relation on the set of genetic subgroups of \( P \).

A genetic basis of \( P \) is a set of representatives of equivalence classes.

A genetic basis of \( P \) is in one-to-one correspondence with \( \text{Irr} \mathbb{Q}P \).
Genetic bases of $p$-groups

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1. If $S$ is a genetic subgroup of $P$, then $V(S)$ is a simple $\mathbb{Q}P$-module.
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**Definition**
Define a relation $\unrhd_P$ on the set of subgroups of $P$ by
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1. If $S$ is a genetic subgroup of $P$, then $V(S)$ is a simple $\mathbb{Q}P$-module.
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In particular $\unrhd_P$ is an equivalence relation on the set of genetic subgroups of $P$. A genetic basis of $P$ is a set of representatives of equivalence classes. A genetic basis of $P$ is in one-to-one correspondence with $\text{Irr}_{\mathbb{Q}}(P)$. 
Let $P$ be a $p$-group, and $G$ be a genetic basis of $P$. The map
\[ \bigoplus_{S \in G} \text{Ind}_{G}^{P} (N_{P}(S)/S) \rightarrow R_{Q}(P) \]
is an isomorphism.

The functor $F$ is called rational if $I_{P,G}$ is an isomorphism, for any $P$ and $G$.
Let $P$ be a $p$-group, and $\mathcal{G}$ be a genetic basis of $P$. Then the map

$$I_{P,\mathcal{G}} = \bigoplus_{S \in \mathcal{G}} \text{Ind}_{\mathcal{N}_P(S)/S}^G : \bigoplus_{S \in \mathcal{G}} \partial R_\mathbb{Q}(\mathcal{N}_P(S)/S) \to R_\mathbb{Q}(P)$$

The functor $F$ is called rational if $I_{P,\mathcal{G}}$ is an isomorphism, for any $P$ and $\mathcal{G}$. A rational $p$-biset functor is determined by its values at $p$-groups of normal $p$-rank 1, which are generally (hopefully) easy to compute.
Let $P$ be a $p$-group, and $G$ be a genetic basis of $P$. Then the map

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**Theorem**

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**Theorem**

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Units of Burnside rings

Let $B \times (G)$ denote the group of units of the Burnside ring $B(G)$. It is an elementary abelian 2-group.

Problem:
Describe $B \times (G)$: find its order, and generators.

In general, this is a hard problem: e.g. (tom Dieck):
"If $|G|$ is odd, then $B \times (G) = \{ \pm 1 \}$" ⇔ the Odd Order Theorem.

The correspondence $G \mapsto B \times (G)$ has a natural structure of biset functor, denoted by $B \times$.

The restriction of $B \times$ to $p$-groups is a rational $p$-biset functor.

Theorem (2007)
If $P$ is a $p$-group, then $B \times (P) \simeq \left( \mathbb{Z}/2\mathbb{Z} \right)^{d_P}$, where $d_P$ is the number of subgroups $S$ in a given genetic basis of $P$ for which $N_P(S)/S$ has order 1 or 2, or is a dihedral 2-group.
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Linearly isomorphic permutation representations

Let $H = \text{PGL}_3(F_p)$. Then $H$ acts on the set $\Pi$ of points and the set $\Lambda$ of lines of the projective plane. The incidence matrix of the geometry yields an isomorphism of $\mathbb{Q}H$-modules $\mathbb{Q}\Pi \cong \mathbb{Q}\Lambda$. Still $\Pi \not\cong \Lambda$ as $H$-sets.

Problem: More generally, when $G$ is a finite group, describe all pairs $(X, Y)$ of $G$-sets such that $\mathbb{Q}X \cong \mathbb{Q}Y$. In other words, let $K$ denote the kernel of the linearization morphism $\chi : B \to \text{R}Q$. Describe $K(G)$?
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The kernel $K$ of $B \to R_Q$

Recall that $H = \text{PGL}_3(F_p)$, that $\Pi$ is the set of points and $\Lambda$ the set of lines of the projective plane, and that $\Pi - \Lambda \in K(H)$.


1. Let $S$ be a Sylow $p$-subgroup of $H$, and set $\delta = \text{Res}^H_S(\Pi - \Lambda) \in K(S)$.
   Then $K$ is generated as a $p$-biset subfunctor of $B$ by the single element $\delta$.

2. Let $P$ be a $p$-group. Then $K(P)$ is the set of linear combinations of elements $\text{Ind}_{\text{inf}}^P R/Q \eta$, where $R/Q \sim (\mathbb{Z}/p\mathbb{Z})^2$ or $R/Q \sim \mathbb{S}$, and $\eta$ is a specific element in each case.

Proof: show that $K/\langle \delta \rangle$ is a rational $p$-biset functor.

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The Dade group of a $p$-group

Let $k$ be a field of characteristic $p$, and $P$ be a $p$-group. In order to classify endo-permutation $kP$-modules, Dade (1978) introduced an abelian group $D(P) = D(kP)$ (now called the Dade group of $P$), and determined its structure when $P$ is abelian. Later, Puig (≤1990) showed that $D(P)$ is always finitely generated. The Dade group comes with natural operations of restriction, (tensor) induction, inflation, deflation, and transport by isomorphism. Thus, if $P$ and $Q$ are $p$-groups, and $U$ is a $(Q,P)$-biset, one can define a map $D(U) : D(P) \to D(Q)$ [with J. Thévenaz (2000)]. Unfortunately, in general $D(V) \circ D(U) \neq D(V \circ U)$. Nevertheless, some closely related objects are genuine $p$-biset functors: The $Q$-linear extension $P \mapsto Q \otimes Z D(P)$. The correspondence $P \mapsto D\Omega(P)$, where $D\Omega(P) \leq D(P)$ is the subgroup of relative syzygies.
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Serge Bouc (CNRS - Université de Picardie)
The Dade group of a $p$-group

- Let $k$ be a field of characteristic $p$, and $P$ be a $p$-group. In order to classify endo-permutation $kP$-modules, Dade (1978) introduced an abelian group $D(P) = D_k(P)$ (now called the Dade group of $P$), and determined its structure when $P$ is abelian. Later, Puig ($\leq$ 1990) showed that $D(P)$ is always finitely generated.

- The Dade group comes with natural operations of restriction, (tensor) induction, inflation, deflation, and transport by isomorphism. Thus, if $P$ and $Q$ are $p$-groups, and $U$ is a $(Q, P)$-biset, one can define a map $D(U) : D(P) \rightarrow D(Q)$ [with J. Thévenaz (2000)].

- Unfortunately, in general $D(V) \circ D(U) \neq D(V \circ U)$.

- Nevertheless, some closely related objects are genuine $p$-biset functors:

\[ Q \text{-linear extension } P \mapsto Q \otimes \mathbb{Z} D(P) \]
\[ \text{The correspondence } P \mapsto D(\Omega(P)), \text{ where } D(\Omega(P)) \leq D(P) \text{ is the subgroup of relative syzygies.} \]
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- The correspondence $P \mapsto D^\Omega(P)$, where $D^\Omega(P) \leq D(P)$ is the subgroup of relative syzygies.
The Dade group of a $p$-group

- [with J. Thévenaz (2000)] As a $p$-biset functor with values in $\mathbb{Q}$-vector spaces, the functor $\mathbb{Q}D$ is \textbf{simple}, isomorphic to $S(\mathbb{Z}/p\mathbb{Z})^2, \mathbb{Q}$. 

There is a surjective morphism of $p$-biset functors $\Theta : B^* \to D\Omega$, which yields an exact sequence $0 \to R^* \chi^* \to B^* \to D\Omega / D\Omega_{\text{tors}} \to 0$. 

In particular $D\Omega / D\Omega_{\text{tors}} \sim \mathbb{K}^*$. 

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**Theorem (2006)**

Suppose $p > 2$. Then $D = D_\Omega$.

If $P$ is a $p$-group, then

$$D(P) \cong \mathbb{Z}^{nc_P} \oplus \left(\mathbb{Z}/2\mathbb{Z}\right)^{c'_P},$$

where $nc_P$ is the number of conjugacy classes of non cyclic subgroups of $P$, and $c'_P$ is the number of conjugacy classes of non trivial cyclic subgroups of $P$.

Suppose $p = 2$. Then $D \neq D_\Omega$.

If $P$ is a $2$-group, then

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**Proof:** $D_{tors}$ is a rational $p$-biset functor.
Using results proved by J. Carlson and J. Thévenaz (2000-2004-2005) in their classification of endo-trivial modules,

\[\text{Theorem (2006)}\]

Suppose \( p > 2 \). Then \( D = D_{\Omega} \).

If \( P \) is a \( p \)-group, then

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**Proof:** $D_{tors}$ “is” a rational $p$-biset functor...
Let $k$ be a field of positive characteristic $p$, and $P$ be a finite $p$-group. If $X$ is a non empty finite $P$-set, the relative syzygy $\Omega_X$ of the trivial module relative to $X$ is the kernel of the augmentation map $kX \rightarrow k$.

**Theorem (Alperin)**

If $X^P = \emptyset$, then $\Omega_X$ is a cep $kP$-module (cep=capped endo-permutation).

The group of relative syzygies $D^\Omega(P)$ is the subgroup of $D(P)$ generated by the classes of $\Omega_X$, where $X$ is a non empty finite $P$-set with $X^P = \emptyset$. 
Let $F$ be a $p$-biset functor. Then $F$ is rational if and only if the following two conditions hold:

1. If $P$ is a $p$-group with non cyclic centre, then $\partial F(P) = \{0\}$.

2. If $P$ is a $p$-group, if $E \trianglelefteq P$ with $E \cong (\mathbb{Z}/p\mathbb{Z})^2$, if $Z \leq E \cap Z(P)$ with $|Z| = p$, then the map

$$\text{Res}^P_{C_P(E)} \oplus \text{Def}^P_{P/Z} : F(P) \rightarrow F(C_P(E)) \oplus F(P/Z)$$

is injective.