Gluing endo-permutation modules

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(joint work with J. Thévenaz [BoTh2], [BoTh3], [BoTh4])

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   - The Dade group
   - Functorial structure
   - Gluing data

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   - The general case
   - Construction of the maps
3. A sectional characterization of the Dade group

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Let k be a field of characteristic $p > 0$, and P be a finite p-group. A finitely generated kP-module M is an endo-permutation module if $End_k(M)$ is a permutation module, i.e. admits a P-invariant k-basis.

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- Endo-permutation modules appear in many different places in representation theory of finite groups (e.g. sources of simple modules, or equivalences of blocks).
- This notion was introduced by E.C. Dade in 1978, as a generalization of the notion of endo-trivial module.
- Other examples are the relative syzygies of the trivial module (Alperin) : $0 \to \Omega_X \to kX \to k \to 0$, where X is a non empty finite P-set.

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- Two important subgroups of $D(P)$ are $T(P)$ and $D^\Omega(P)$.
- The description of the structure of $D(P)$ for an arbitrary finite p-group P has been completed recently (2006).

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  If $p > 2$, the correspondence $P \mapsto D(P)$ is a biset functor.

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If (T,S) is a section of the finite p-group P, denote by $Defres^P_{T/S}$ the composition $D(P) \xrightarrow{Res^P_T} D(T) \xrightarrow{Def^T_{T/S}} D(T/S)$

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- Let $\mathcal{S}$ be a family of subgroups of P. If $u \in D(P)$, define a sequence $r_P(u) = (v_Q)_{Q \in \mathcal{S}}$, where $v_Q \in D\big(N_P(Q)/Q\big)$, by

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- This problem was initially raised by Puig, who solved it when P is abelian.

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- All but possibly one of the connected components of $\mathcal{A}_{\geq 2}(P)$ consist of isolated points (maximal elementary abelian subgroups of rank 2).
- (BoTh4) The poset $\mathcal{A}_{\geq 2}(P)$ has the homotopy type of a wedge of spheres (of possibly different dimensions).

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Theorem (BoTh2)

Let P be a non cyclic p-group, for $p > 2$. Then there is an exact sequence of abelian groups

$$0 \to D_t(P) \xrightarrow{r_P} \varprojlim_{1<Q\leq P} D_t\big(N_P(Q)/Q\big) \xrightarrow{\widetilde{d}_P} \widetilde{H}^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big)^P \to 0 \enspace ,$$

where $D_t(P)$ is the torsion subgroup of $D(P)$.

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where $D_t(P)$ is the torsion subgroup of $D(P)$.

In particular, if $\mathcal{A}_{\geq 2}(P)$ is not connected, then the gluing problem for torsion elements doesn't always have a solution in the torsion subgroup $D_t(P)$.

The exact sequence in the general case

Theorem

Let P be a finite p-group, for $p > 2$. Then there is an exact sequence of abelian groups

$$0 \to T(P) \to D(P) \xrightarrow{r_P} \varprojlim_{1<Q\leq P} D\big(N_P(Q)/Q\big) \xrightarrow{h_P} H^1\big(\mathcal{A}_{\geq 2}(P), \mathbb{Z}\big)^{(P)} \ .$$

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- It follows that the gluing problem for a torsion gluing data sequence always has a solution, which may be a non torsion element.
- The map h_P is not surjective in general. In all the examples I have considered, it has finite cokernel.

The map $\widetilde{d}_P : \varprojlim_{1<Q\leq P} D_t\Big(N_P(Q)/Q\Big)\to\widetilde{H}^0\Big(\mathcal{A}_{\geq 2}(P),\mathbb{F}_2\Big)^P$

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3. Fix a subgroup Z of order p in $Z(P)$. Define $d_P(v)(E) = Res^{EZ}_Z r^{-1}_{EZ} \mathrm{res}^P_{EZ}(v) \in D(Z) \cong \mathbb{Z}/2\mathbb{Z} \cong \mathbb{F}_2$.

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Sections and subquotients

Definition

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Example : the family of elementary abelian p-groups is closed under taking subquotients.

Sectional limits

Notation

Let F be a biset functor. Denote by $\varprojlim_{(T,S)\in\mathcal{X}(P)} F(T/S)$ *the set of sequences* $(u_{T,S})_{(T,S)\in\mathcal{X}(P)}$, *where* $u_{T,S} \in F(T/S)$, *such that :*

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Remark : There is a natural deflation-restriction map

$$\varepsilon_P : F(P) \to \varprojlim_{(T,S)\in\mathcal{X}(P)} F(T/S)$$

Elementary abelian sections

Theorem (BoTh2)

Let $\mathcal{E}$ denote the class of finite elementary abelian p-groups. Let F be a biset functor, and P be a finite p-group. Let

$$\varepsilon_P : F(P) \to \varprojlim_{(T,S)\in\mathcal{E}(P)} F(T/S)$$ *denote the deflation-restriction map.*

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Corollary

Suppose $p > 2$. Then the map $\varepsilon_P : D_t(P) \to \varprojlim_{(T,S)\in\mathcal{E}(P)} D_t(T/S)$ *is an isomorphism.*

A sectional characterization of $D(P)$

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Let $\mathcal{X}_3$ denote the class of finite p-groups which are either elementary abelian of rank at most 3, or extraspecial of order p^3 and exponent p.

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Theorem (BoTh3)

Let P be a p-group for $p > 2$. Then the deflation-restriction map

$$D(P) \to \lim_{\substack{\longleftarrow \\ (T,S)\in\mathcal{X}_3(P)}} D(T/S)$$

is an isomorphism.